# Complex manifolds and Kähler Geometry

Lecture 1 of 16: Complex manifolds

Dominic Joyce, Oxford University
September 2019

2019 Nairobi Workshop in Algebraic Geometry

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`





Plan of talk:

1 Complex manifolds

1.1 Complex manifolds

1.2 Holomorphic functions and holomorphic maps

1.3 Complex submanifolds

1.4 Projective complex manifolds

## 1.1. Complex manifolds

We will give two definitions of complex manifolds. This lecture, we use complex charts and holomorphic transition functions. Next lecture, in a more Differential Geometric style, we use (almost) complex structures on a real manifold. The two points of view are equivalent, by the Newlander–Nirenberg Theorem.

Recall the definition of a (smooth, real) manifold: a topological space $X$ with an atlas of charts $(U_i, \phi_i)$ with transition functions $\phi_{ij}$ diffeomorphisms between open sets in $\mathbb{R}^n$. We can instead require other conditions on $\phi_{ij}$, e.g. $\phi_{ij}$ continuous gives you *topological manifolds*, or we could require $\phi_{ij}$ to be $C^k$, or real analytic. Requiring the $\phi_{ij}$ to be holomorphic gives you *complex manifolds*.





### Definition

Let $X$ be a topological space, and fix $n \geqslant 0$. A (*complex*) *chart* on $X$ is $(U, \phi)$, where $U \subseteq \mathbb{C}^n$ is open and $\phi : U \to X$ is a homeomorphism from $U$ to an open subset $\phi(U)$ in $X$. Let $(U, \phi), (V, \psi)$ be charts. The *transition function* between them is

$$\psi^{-1} \circ \phi : \phi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr) \longrightarrow \psi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr).$$

It is automatically a homeomorphism between open subsets of $\mathbb{C}^n$. We call $(U, \phi), (V, \psi)$ *compatible* if $\psi^{-1} \circ \phi$ is a *biholomorphism* between open subsets of $\mathbb{C}^n$, i.e. holomorphic with holomorphic inverse.

A (*complex*) *atlas* on $X$ is a system $\{(U_i, \phi_i) : i \in I\}$ of pairwise compatible charts on $X$ with $X = \bigcup_{i \in I} \phi_i(U_i)$. We may write $\phi_{ij}$ for the transition function $\phi_j^{-1} \circ \phi_i$.

## Definition (Continued)

An atlas is called *maximal* if it is not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas, the set of all charts $(U, \phi)$ compatible with $(U_i, \phi_i)$ for all $i \in I$.

An ($n$*-dimensional*) *complex manifold* is a paracompact, Hausdorff topological space $X$ together with a maximal atlas $\{(U_i, \phi_i) : i \in I\}$ of $n$-dimensional complex charts $(U_i, \phi_i)$. Here *paracompact* is to avoid pathological examples from topology; sometimes one asks for *second countable* instead.

Usually we refer to $X$ as the complex manifold, and suppress the atlas. Taking the atlas *maximal* makes it independent of choices.





What a complex atlas on $X$ gives you is a notion of *local holomorphic coordinates*. Let $x \in X$. Then we can choose a chart $(U_i, \phi_i)$ with $x \in \phi_i(U)$, since $X = \bigcup_{i \in I} \phi_i(U_i)$. Then we think of $\phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}^n$ as holomorphic coordinates $(z_1, \ldots, z_n)$ defined on an open neighbourhood $\phi_i(U_i)$ of $x$. We can do a lot of definitions and proofs using local holomorphic coordinates.

## Example

The simplest complex manifold is $\mathbb{C}^n$. $(U, \phi) = (\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})$ is a chart on $\mathbb{C}^n$, and $\{(\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})\}$ is an atlas on $\mathbb{C}^n$. This is contained in a unique maximal atlas, which makes $\mathbb{C}^n$ into a complex manifold.

### Example

Complex projective space $\mathbb{CP}^n$ is a compact $n$-dimensional complex manifold. We use homogeneous coordinates $[z_0, \ldots, z_n]$ on $\mathbb{CP}^n$. For $i = 0, \ldots, n$, define a chart $(U_i, \phi_i)$ on $\mathbb{CP}^n$ by $U_i = \mathbb{C}^n$ and $\phi_i : \mathbb{C}^n \to \mathbb{CP}^n$ given by

$$\phi_i : (w_1, \ldots, w_n) \longmapsto [w_1, \ldots, w_i, 1, w_{i+1}, \ldots, w_n].$$

This is a homeomorphism with the open subset $\phi_i(U_i) = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : z_i \neq 0\}$ in $\mathbb{CP}^n$.
For $0 \leqslant i < j \leqslant n$, the transition function $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ maps $\{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_j \neq 0\}$ to $\{(y_1, \ldots, y_n) \in \mathbb{C}^n : y_{i+1} \neq 0\}$ by

$$(x_1, \ldots, x_n) \longmapsto \left(\tfrac{x_1}{x_j}, \ldots, \tfrac{x_i}{x_j}, \tfrac{1}{x_j}, \tfrac{x_{i+1}}{x_j}, \ldots, \tfrac{x_{j-1}}{x_j}, \tfrac{x_{j+1}}{x_j}, \ldots, \tfrac{x_n}{x_j}\right).$$

This is a biholomorphism. So $(U_i, \phi_i), (U_j, \phi_j)$ are compatible, and $\{(U_i, \phi_i) : i = 0, \ldots, n\}$ is an atlas. It is contained in a unique maximal atlas, which makes $\mathbb{CP}^n$ into a complex manifold.





## 1.2. Holomorphic functions and holomorphic maps

Let $X$ be a complex manifold, and $f : X \to \mathbb{C}$ a function. We call $f$ *holomorphic* if for all charts $(U, \phi)$ in the (maximal) atlas on $X$, $f \circ \phi$ is a holomorphic function $U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^n$ is open. It is enough to check this on the charts of any atlas on $X$.
Let $X, Y$ be complex manifolds of dimensions $m, n$, and $f : X \to Y$ a continuous function. We call $f$ *holomorphic* if whenever $(U, \phi)$ and $(V, \psi)$ are charts from the atlases on $X, Y$, the map

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}\big(f(\phi(U)) \cap \psi(V)\big) \longrightarrow V$$

is a holomorphic map from an open subset of $\mathbb{C}^m$ to an open subset of $\mathbb{C}^n$. Complex manifolds and holomorphic maps form a *category*.
A *biholomorphism* $f : X \to Y$ is a holomorphic map with a holomorphic inverse.

## 1.3. Complex submanifolds

Let $X$ be a complex manifold of dimension $n$, and $Y \subseteq X$. We call $Y$ an (*embedded*) *complex submanifold of* $X$ *of dimension* $k$, for $0 \leqslant k \leqslant n$, if for each $y \in Y$ there exist local holomorphic coordinates $(z_1, \ldots, z_n)$ on $X$ such that $Y$ is locally of the form $z_{k+1} = \cdots = z_n = 0$. That is, we have a chart $(U, \phi)$ on $X$ with $y \in \phi(U)$ such that $Y \cap \phi(U) = \phi(\mathbb{C}^k \cap U)$, where $\mathbb{C}^k = \{(z_1, \ldots, z_k, 0, \ldots, 0) \in \mathbb{C}^n\}$. Usually we want $Y$ closed in $X$.

We can give a complex submanifold $Y$ of $X$ the structure of a complex $k$-manifold: for $(U, \phi)$ as above, $(\mathbb{C}^k \cap U, \phi|_{\mathbb{C}^k \cap U})$ is a $k$-dimensional chart on $Y$, and the set of such charts is an atlas on $Y$. The inclusion $i_Y : Y \hookrightarrow X$ is holomorphic.

Conversely, a holomorphic map $f : Y \to X$ is called an *embedding* if it is injective, locally closed, and on tangent spaces $\mathrm{d}f|_y : T_y Y \to T_{f(y)} X$ is injective for all $y \in Y$. If $f$ is an embedding then $f(Y)$ is a complex submanifold of $X$ biholomorphic to $Y$.





## 1.4. Projective complex manifolds

Let $\mathbb{CP}^n$ have homogeneous coordinates $[z_0, \ldots, z_n]$. Let $p(z_0, \ldots, z_n)$ be a complex polynomial in $n+1$ variables, which is homogeneous of order $k$. Then $p(\lambda z_0, \ldots, \lambda z_n) = \lambda^k p(z_0, \ldots, z_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Hence $p(\lambda z_0, \ldots, \lambda z_n) = 0$ if and only if $p(z_0, \ldots, z_n) = 0$. Thus, for $[z_0, \ldots, z_n] \in \mathbb{CP}^n$, the condition $p(z_0, \ldots, z_n) = 0$ is independent of the choice of representative $(z_0, \ldots, z_n)$ for $[z_0, \ldots, z_n]$.

A *projective variety* is a subset $X$ of $\mathbb{CP}^n$ which is defined by the vanishing of finitely many homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_d(z_0, \ldots, z_n)$, that is,

$$X = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : p_i(z_0, \ldots, z_n) = 0, \quad i = 1, \ldots, d\}.$$

Then $X$ is closed in $\mathbb{CP}^n$, and so compact. We call $X$ a *projective complex manifold* if $X$ is also a complex submanifold of $\mathbb{CP}^n$.

### Example

Let $p(z_0,\ldots,z_n)$ be a nonzero homogeneous complex polynomial, and define

$$X=\bigl\{[z_0,\ldots,z_n]\in\mathbb{CP}^n : p(z_0,\ldots,z_n)=0\bigr\}.$$

Then $X$ is a complex submanifold of $\mathbb{CP}^n$, of dimension $n-1$, provided the following condition holds: let $(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\}$ with $p(z_0,\ldots,z_n)=0$. Then $\frac{\partial p}{\partial z_i}(z_0,\ldots,z_n)\neq 0$ for some $i=0,\ldots,n$. This holds for generic homogeneous polynomials $p$.

### Example

For $d=1,2,\ldots$, $X=\bigl\{[z_0,z_1,z_2]\in\mathbb{CP}^2 : z_0^d+z_1^d+z_2^d=0\bigr\}$ is a projective complex 1-manifold, a Riemann surface of genus $g=\frac{1}{2}(d-1)(d-2)$.





### Example

$X=\bigl\{[z_0,z_1,z_2,z_3]\in\mathbb{CP}^3 : z_0^2+\cdots+z_3^2=0\bigr\}$ is a projective complex 2-manifold biholomorphic to $\mathbb{CP}^1\times\mathbb{CP}^1$.

### Example

Let $p_1,\ldots,p_k(z_0,\ldots,z_n)$ be homogeneous polynomials for $k\leqslant n$. Suppose that whenever $(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\}$ with $p_i(z_0,\ldots,z_n)=0$ for all $i$, then $\mathrm{d}p_1(z_0,\ldots,z_n),\ldots,\mathrm{d}p_k(z_0,\ldots,z_n)$ are linearly independent in $(\mathbb{C}^{n+1})^*$. Then

$$X=\bigl\{[z_0,\ldots,z_n]\in\mathbb{CP}^n : p_i(z_0,\ldots,z_n)=0,\ \ i=1,\ldots,k\bigr\}$$

is a projective complex manifold of dimension $n-k$, called a *complete intersection.*
Most projective complex manifolds are not complete intersections.

Projective complex manifolds give a huge number of interesting examples of complex manifolds. As they are defined using polynomials, one can study and classify them using algebraic techniques – Complex Algebraic Geometry.
Also, under some conditions one can guarantee that a compact complex manifold $X$ has an embedding $X \hookrightarrow \mathbb{CP}^n$ making it into a projective complex manifold. This is due to two important results, Chow's Theorem and the Kodaira Embedding Theorem.

**Theorem 1.1 (Chow's Theorem)**

*Suppose $X$ is a compact complex submanifold in $\mathbb{CP}^n$. Then $X$ is a projective complex manifold, that is, $X$ may be defined as a subset of $\mathbb{CP}^n$ by the vanishing of homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_k(z_0, \ldots, z_n)$.*





Thus, compact submanifolds of $\mathbb{CP}^n$ are algebraic objects. For a proof, see Griffiths and Harris, *Principles of Algebraic Geometry*.
As $\mathbb{CP}^n$ is compact, $X$ compact is equivalent to $X$ closed.
We will cover the Kodaira Embedding Theorem later in the course.
In brief, it says that if $X$ is a compact complex manifold and $L \to X$ is an 'ample line bundle' then we can use $L$ to construct an embedding $f : X \hookrightarrow \mathbb{CP}^n$ for some $n \gg 0$. Then $X$ is biholomorphic to $f(X)$, which is a compact complex submanifold of $\mathbb{CP}^n$, so by Chow's Theorem, $f(X)$ is algebraic, and $X$ is biholomorphic to a projective complex manifold.

Projective complex manifolds are also closely connected to compact Kähler manifolds (next week).

Every projective complex manifold is Kähler. But also, if $X$ is a compact Kähler manifold, then under mild topological conditions on $X$ one can show that $X$ possesses many ample line bundles $L \hookrightarrow X$, and then the Kodaira Embedding Theorem applies, and $X$ is biholomorphic to a projective complex manifold.





# Complex manifolds and Kähler Geometry

Lecture 2 of 16: Complex manifolds as real manifolds; almost complex structures

Dominic Joyce, Oxford University
September 2019

2019 Nairobi Workshop in Algebraic Geometry

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

2 Complex manifolds as real manifolds; almost complex structures

2.1 Almost complex structures

2.2 The Nijenhuis tensor

2.3 Another definition of complex manifolds

2.4 More on almost complex geometry





## 2.1. Almost complex structures

We now explain a second way to define complex manifolds. To see the point simply, suppose $V$ is a complex vector space, of complex dimension $n$. Underlying $V$ is a real vector space $V_{\mathbb{R}}$, of real dimension $2n$. Given $V_{\mathbb{R}}$, what extra information do we need to reconstruct $V$? The only thing we are missing is multiplication by $i \in \mathbb{C}$. This induces a real linear map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ with $J^2 = -\mathrm{id}_{V_{\mathbb{R}}}$.

Conversely, given a real vector space $V_{\mathbb{R}}$ and $J \in \mathrm{End}(V_{\mathbb{R}})$ with $J^2 = -\mathrm{id}_{V_{\mathbb{R}}}$, we make $V_{\mathbb{R}}$ into a complex vector space by setting $(a + ib) \cdot v = a \cdot v + b \cdot J(v)$, for $a, b \in \mathbb{R}$ and $v \in V_{\mathbb{R}}$; note that $\dim_{\mathbb{R}} V_{\mathbb{R}}$ must be even. So, complex vector spaces are equivalent to real vector spaces with an endomorphism $J$ with $J^2 = -\mathrm{id}$.

If $X$ is a complex $n$-manifold in the sense of §1, then underlying $X$ is a real $2n$-manifold $X_{\mathbb{R}}$. It has a tangent bundle $TX_{\mathbb{R}}$, whose fibres $T_xX_{\mathbb{R}}$ for $x \in X$ are real vector spaces of real dimension $2n$. Since $X$ is a complex $n$-manifold, they are also complex vector spaces of dimension $n$. So they have $J_x \in \mathrm{End}(T_xX_{\mathbb{R}})$ with $J_x^2 = -\mathrm{id}_{T_xX_{\mathbb{R}}}$. Over all $x \in X_{\mathbb{R}}$, these $J_x$ form a tensor $J_a^b$ with $J_a^b J_b^c = -\delta_a^c$, using index notation.

### Definition

Let $X$ be a real $2n$-manifold. An *almost complex structure* $J$ on $X$ is a tensor $J_a^b$ in $C^\infty(T^*X \otimes TX)$ with $J_a^b J_b^c = -\delta_a^c$. For a vector field $v \in C^\infty(TX)$, define $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so $J$ makes the tangent spaces $T_xX$ into *complex vector spaces*.

Any complex manifold in the sense of §1 yields a real manifold $X$ with an almost complex structure $J$. But not all $(X, J)$ come from complex manifolds: we must impose extra conditions on $J$.





## Holomorphic functions

### Definition

Suppose $X$ is a $2n$-manifold, and $J$ an almost complex structure on $X$. Let $f : X \to \mathbb{C}$ be smooth, and write $f = u + iv$. Then $\mathrm{d}u, \mathrm{d}v$ are 1-forms on $X$, so in index notation $\mathrm{d}u = \mathrm{d}u_a$, $\mathrm{d}v = \mathrm{d}v_b$. We call $f$ *holomorphic* if $\mathrm{d}u_a = J_a^b \mathrm{d}v_b$. Since $J^2 = -\mathrm{id}$, this is equivalent to $\mathrm{d}v_a = -J_a^b \mathrm{d}u_b$. Hence in complex 1-forms we have

$$J_a^b(\mathrm{d}u_b + i\mathrm{d}v_b) = i(\mathrm{d}u_a + i\mathrm{d}v_a),$$

that is, $J_a^b \mathrm{d}f_b = i\,\mathrm{d}f_a$.

**Example**

Let $\mathbb{R}^2$ have coordinates $(x, y)$, and let $J = \mathrm{d}x \otimes \frac{\partial}{\partial y} - \mathrm{d}y \otimes \frac{\partial}{\partial x}$ in $C^\infty(T^*\mathbb{R}^2 \otimes T\mathbb{R}^2)$. Then the equation $\mathrm{d}u_a = J_a^b \mathrm{d}v_b$ becomes

$$\frac{\partial u}{\partial x} \cdot \mathrm{d}x + \frac{\partial u}{\partial y} \cdot \mathrm{d}y = -\frac{\partial v}{\partial x} \cdot \mathrm{d}y + \frac{\partial v}{\partial y} \cdot \mathrm{d}x,$$

or equivalently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the *Cauchy–Riemann equations* for $u(x, y) + iv(x, y)$ to be a holomorphic function of $x + iy$.





## 2.2. The Nijenhuis tensor

It turns out that when $n > 1$, for some almost complex structures on $X$ there may be few holomorphic functions locally on $X$ — in extreme cases, all holomorphic functions are constant. This is because the equations are *overdetermined*: there are $2n$ equations on 2 functions. We can express this in terms of an *obstruction* to the existence of holomorphic functions locally on $X$, called the *Nijenhuis tensor*.

**Definition**

Write $[v, w]$ for the *Lie bracket* of vector fields $v, w$ on $X$. The *Nijenhuis tensor* $N = N^a_{bc}$ of $J$ satisfies

$$N^a_{bc} v^b w^c = \big([v, w] + J\big([Jv, w] + [v, Jw]\big) - [Jv, Jw]\big)^a \qquad (2.1)$$

for all $v, w \in C^\infty(TX)$.

The point is that the r.h.s. of $(2.1)$ is *pointwise* linear in $v, w$ (exercise): if we replace $v, w$ by $f \cdot v$, $g \cdot w$ for smooth $f, g : X \to \mathbb{R}$, then the r.h.s. is multiplied by $fg$, with no terms in derivatives of $f, g$.
Let $s + it : X \to \mathbb{C}$ be holomorphic. Then using $(2.1)$ one can show that for all vector fields $v, w$ we have $N^a_{bc} v^b w^c \mathrm{d}s_a \equiv N^a_{bc} v^b w^c \mathrm{d}t_a \equiv 0$ (exercise). Hence $N^a_{bc} \mathrm{d}s_a \equiv N^a_{bc} \mathrm{d}t_a \equiv 0$ in $C^\infty(\Lambda^2 T^*X)$. Thus, the Nijenhuis tensor constrains the possible first derivatives of holomorphic functions.





For $(X, J)$ to be a complex manifold, we want there to exist a system of holomorphic coordinates $(z_1, \ldots, z_n)$ near each point $x$ in $X$, that is, $(z_1, \ldots, z_n)$ are complex coordinates defined on open $x \in U \subseteq X$, and $z_j : U \to \mathbb{C}$ is holomorphic. If $z_j = s_j + it_j$ then $\mathrm{d}s_1, \ldots, \mathrm{d}s_n, \mathrm{d}t_1, \ldots, \mathrm{d}t_n$ span $T^*X$ on $U$. So $N^a_{bc}(\mathrm{d}s_j)_a \equiv N^a_{bc}(\mathrm{d}t_j)_a \equiv 0$ imply that $N \equiv 0$. Thus, holomorphic coordinates $(z_1, \ldots, z_n)$ can exist locally on $X$ only if the Nijenhuis tensor $N \equiv 0$.

The converse is the difficult *Newlander–Nirenberg Theorem*:

**Theorem 2.1 (Newlander–Nirenberg)**

*Suppose $J$ is an almost complex structure on $X$ with Nijenhuis tensor $N \equiv 0$. Then near each $x \in X$ there exist holomorphic coordinates $(z_1, \ldots, z_n)$.*

The point is to show that the first derivatives of holomorphic functions near $x$ span $T_x^*X$; then choosing any $(z_1, \ldots, z_n)$ whose derivatives span $T_x^*X$, they will be holomorphic coordinates in a small open neighbourhood of $x$.
Think of the Nijenhuis tensor $N$ as being like the 'curvature' of $J$, and the condition $N \equiv 0$ as a 'flatness condition'. If $g = g_{ab}$ is a Riemannian metric, the Riemann curvature $R^i_{jkl}$ is a tensor defined using $g$ and its derivatives, in a similar way to $N^a_{bc}$, and $R^i_{jkl} \equiv 0$ if $g$ is flat. (Actually, $N$ is a *torsion* rather than a curvature, as it depends on one derivative of $J$, not two.)





## 2.3. Another definition of complex manifolds

Here is our second definition of complex manifold:

**Definition**

Let $X$ be a $2n$-manifold, and $J$ an almost complex structure on $X$ with Nijenhuis tensor $N$. We call $J$ an *integrable almost complex structure*, or just a *complex structure*, if $N \equiv 0$, and then we call $(X, J)$ a *complex manifold*.

This is equivalent to the definition of complex manifolds using complex atlases in §1. Here is why.

Suppose $(X, J)$ is a complex manifold in the sense above. Then by the Newlander–Nirenberg theorem, there exist holomorphic coordinates $(z_1, \ldots, z_n)$ near each $x \in X$. Using these we define an atlas of charts $(U, \phi)$ on $X$. The transition functions are automatically holomorphic. Extending to the unique maximal atlas defines a complex structure on $X$ in the sense of §1.
Conversely, given a complex manifold $X_{\mathbb{C}}$ in the sense of §1, there is a natural underlying real manifold $X_{\mathbb{R}}$, and a unique almost complex structure $J$ on $X_{\mathbb{R}}$ for which all local coordinate functions $z_j$ are holomorphic, and $N \equiv 0$, so $J$ is a complex structure.





## Holomorphic maps

### Definition

Let $(X, I)$ and $(Y, J)$ be complex manifolds, and $f : X \to Y$ a smooth map. We call $f$ *holomorphic* if for all $x \in X$ with $y = f(x) \in Y$, so that $\mathrm{d}f|_x : T_xX \to T_yY$ is a linear map, we have $\mathrm{d}f|_x \circ I|_x = J|_y \circ \mathrm{d}f|_x$. That is, $\mathrm{d}f|_x : T_xX \to T_yY$ is a complex linear map, regarding $T_xX, T_yY$ as complex vector spaces using $I|_x, J|_y$.

This agrees with the definition of holomorphic maps in §1, under the correspondence between the two definitions of complex manifold. If $g : Y \to \mathbb{C}$ is a holomorphic function then $g \circ f : X \to \mathbb{C}$ is a holomorphic function. In fact, a smooth map $f : X \to Y$ is holomorphic if and only if for all local holomorphic functions $g : V \to \mathbb{C}$ for $V \subseteq Y$ open, $g \circ f : U = f^{-1}(V) \to \mathbb{C}$ is a local holomorphic function on $X$.

## Complex submanifolds

**Definition**

Let $(X, J)$ be a complex manifold, and $Y$ a submanifold of $X$. We call $Y$ a *complex submanifold* if for each $y \in Y$ we have $J(T_y Y) = T_y Y$, as subspaces of $T_y X$.
Then $J_Y = J|_{TY}$ is an almost complex structure on $Y$. The Nijenhuis tensor $N_Y$ of $J_Y$ is the restriction to $Y$ of the Nijenhuis tensor $N$ of $J$, so it is zero, $J_Y$ is a complex structure, and $(Y, J_Y)$ is a complex manifold.





## Real dimension two

Let $J$ be an almost complex structure on $X$, with Nijenhuis tensor $N = N^a_{bc}$. Then $N$ has natural symmetries $N^a_{bc} = -N^a_{cb}$, and $J^d_b J^e_c N^a_{de} = -N^a_{bc}$ (exercise). Using these one can show that $N \equiv 0$ when $\mathrm{dim}\, X = 2$. So almost complex 2-manifolds are complex, that is, they are Riemann surfaces. This corresponds to the fact that for $f : X \to \mathbb{C}$ to be holomorphic is $2n$ equations on 2 functions, which is overdetermined when $n > 1$, but determined when $n = 1$.

## 2.4. More on almost complex geometry

Consider the question: how much of complex geometry also works for non-integrable almost complex structures $J$ on $X$ with $\dim X > 2$?
We already know there are few holomorphic functions $f : X \to \mathbb{C}$ even locally. There are also few complex submanifolds $Y \subset X$ with $2 < \dim Y < \dim X$. However, 2-dimensional complex submanifolds $Y$ in $X$ (*J-holomorphic curves*) are well-behaved. This is important in *Symplectic Geometry*.

**Definition**

Let $X$ be a $2n$-manifold. A *symplectic form* $\omega$ on $X$ is a 2-form $\omega$ with $\mathrm{d}\omega \equiv 0$, such that $\omega|_x^n$ is nonzero in $\Lambda^{2n} T_x^* X$ for all $x \in X$. Then $(X, \omega)$ is a *symplectic manifold*.





## Symplectic manifolds

*Darboux' Theorem* says that near each point $x$ in a symplectic manifold $(X, \omega)$ we can choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $X$ with $\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$. So all symplectic manifolds are locally the same as the standard model $(\mathbb{R}^{2n}, \omega_0)$.

Similarly, the Newlander–Nirenberg Theorem shows that if $J$ is an almost complex structure on $X$ with Nijenhuis tensor $N \equiv 0$, then near each $x \in X$ we can choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $X$ with $J = \sum_{j=1}^n \mathrm{d}x_j \otimes \frac{\partial}{\partial y_j} - \mathrm{d}y_j \otimes \frac{\partial}{\partial x_j}$.
Thus, all complex manifolds are locally the same as the standard model $(\mathbb{R}^{2n}, J_0)$.

Let $(X, \omega)$ be symplectic. An almost complex structure $J$ on $X$ is *compatible with* $\omega$ if $\omega(Jv, Jw) = \omega(v, w)$ for all vector fields $v, w$ on $X$, and $\omega(v, Jv) > 0$ if $v \neq 0$. Every symplectic manifold admits compatible almost complex structures.

Many important areas of Symplectic Geometry — Gromov-Witten invariants, Lagrangian Floer cohomology, Fukaya categories, $\ldots$ — depend on choosing a compatible $J$ on $(X, \omega)$ and then 'counting' $J$-holomorphic curves in $X$. Often one can make the 'number' independent of the choice of $J$.