Orientability of moduli spaces of
Spin(7)-instantons

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Abstract

Suppose \((X, \Omega, g)\) is a compact Spin(7)-manifold, for instance a Riemannian 8-manifold with holonomy Spin(7), or a Calabi–Yau 4-fold. Let \(G = U(m)\) or \(SU(m)\), and \(P \to X\) be a principal \(G\)-bundle. We prove that the moduli space \(M_P^{\text{Spin}(7)}\) of irreducible Spin(7)-instanton connections on \(P\) modulo gauge, as a manifold or derived manifold, is orientable.

This improves theorems of Cao and Leung [5, Th. 2.1] and Muñoz and Shahbazi [25]. Our results have applications to the programme of defining Donaldson–Thomas type invariants counting (semi)stable coherent sheaves on a Calabi–Yau 4-fold, as in Donaldson and Thomas [11], Cao and Leung [4], and Borisov and Joyce [3].

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1 Introduction

Suppose $X$ is a compact 8-manifold with a Spin(7)-structure $(\Omega, g)$, and $G$ is a Lie group, and $P \to X$ a principal $G$-bundle. As in Donaldson and Thomas [11], a Spin(7)-instanton on $P$ is a connection $\nabla_P$ on $P$ with $\pi^2_7(F^{\nabla_P}) = 0$, where $\pi^2_7(F^{\nabla_P})$ is a certain component of the curvature $F^{\nabla_P}$ of $\nabla_P$. The deformation theory of Spin(7)-instantons is elliptic, and therefore moduli spaces $M^\text{Spin(7)}_P$ of irreducible Spin(7)-instantons on $P$ modulo gauge are derived manifolds, which for non-flat connections are smooth manifolds if $\Omega$ is generic. This is a very similar story to moduli spaces $M^\text{asd}_P$ of anti-self-dual instantons on compact, oriented Riemannian 4-manifolds $(X, g)$, as studied in Donaldson theory [9].

In this paper we will prove that when $G = \text{U}(m)$ or $\text{SU}(m)$ all such moduli spaces $M^\text{Spin(7)}_P$ are orientable, extending results by Cao and Leung [5, Th. 2.1] and Muñoz and Shahbazi [25].

The analogous problem of orienting instanton moduli spaces $M^\text{asd}_P$ on 4-manifolds $X$ was solved by Donaldson [7–9], and our proof is based on his techniques. However, the 8-dimensional case is considerably more difficult. This is because orientability for $M_P$ depends on phenomena happening on submanifolds $Z \subset X$ of codimension 3 in $X$. When $X$ is a 4-manifold, such $Z$ are just circles, which are simple. But when $X$ is an 8-manifold, $Z$ is a 5-manifold, and so is much more complicated. Our proof uses the classification of compact, simply-connected 5-manifolds [6].

Calabi–Yau 4-folds $(X, J, g, \theta)$ are examples of compact Spin(7)-manifolds. Because of this, our theory also solves the problem of ‘orientability’ for moduli spaces of (semi)stable coherent sheaves on Calabi–Yau 4-folds due to Cao and Leung [4], and Borisov and Joyce [3].

Sections 1.1–1.3 summarize background material on the general theory of orientations in gauge theory from Joyce, Tanaka and Upmeier [19, §1–2], and on Spin(7)-manifolds and Spin(7)-instantons. The main results are stated in §1.4 and applications to Calabi–Yau 4-folds discussed in §1.5. The proof of the main theorem is given in §2.

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1.1 Connection moduli spaces $M_P$ and orientations

The following definitions are taken from Joyce, Tanaka and Upmeier [19, §1–2].

Definition 1.1. Suppose we are given the following data:

(a) A compact, connected manifold $X$, of dimension $n > 0$. 

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(b) A Lie group $G$, with $\dim G > 0$, and centre $Z(G) \subseteq G$, and Lie algebra $\mathfrak{g}$.

c) A principal $G$-bundle $\pi : P \to X$. We write $\text{Ad}(P) \to X$ for the vector bundle with fibre $\mathfrak{g}$ defined by $\text{Ad}(P) = (P \times \mathfrak{g})/G$, where $G$ acts on $P$ by the principal bundle action, and on $\mathfrak{g}$ by the adjoint action.

Write $\mathcal{A}_P$ for the set of connections $\nabla_P$ on the principal bundle $P \to X$. This is a real affine space modelled on the infinite-dimensional vector space $\Gamma^\infty(\text{Ad}(P))$, and we make $\mathcal{A}_P$ into a topological space using the $C^\infty$ topology on $\Gamma^\infty(\text{Ad}(P))$. Here if $E \to X$ is a vector bundle then $\Gamma^\infty(E)$ denotes the vector space of smooth sections of $E$. Note that $\mathcal{A}_P$ is contractible.

We write $\mathcal{G} = \text{Map}_{C^\infty}(X,G)$ for the infinite-dimensional Lie group of smooth maps $\gamma : X \to G$. Then $\mathcal{G}$ acts on $\mathcal{A}_P$, and hence on $\mathcal{A}_P$ by gauge transformations, and the action is continuous for the topology on $\mathcal{A}_P$.

There is a natural inclusion $Z(G) \hookrightarrow \mathcal{G}$ mapping $z \in Z(G)$ to the constant map $\gamma : X \to G$ with value $z$. As $X$ is connected, this identifies $Z(G)$ with the centre $Z(\mathcal{G})$ of $\mathcal{G}$, so we may take the quotient group $\mathcal{G}/Z(G)$. The action of $Z(G) \subseteq \mathcal{G}$ on $\mathcal{A}_P$ is trivial, so the $\mathcal{G}$-action on $\mathcal{A}_P$ descends to a $\mathcal{G}/Z(G)$-action.

Each $\nabla_P \in \mathcal{A}_P$ has a (finite-dimensional) stabilizer group $\text{Stab}_\mathcal{G}(\nabla_P) \subset \mathcal{G}$ under the $\mathcal{G}$-action on $\mathcal{A}_P$, with $Z(G) \subseteq \text{Stab}_\mathcal{G}(\nabla_P)$. As $X$ is connected, $\text{Stab}_\mathcal{G}(\nabla_P)$ is isomorphic to a closed Lie subgroup $H$ of $G$ with $Z(G) \subseteq H$. As in [9, p. 133] we call $\nabla_P$ irreducible if $\text{Stab}_\mathcal{G}(\nabla_P) = Z(G)$, and reducible otherwise.

Write $\mathcal{A}^{\text{irr}}_P, \mathcal{A}^{\text{red}}_P$ for the subsets of irreducible and reducible connections in $\mathcal{A}_P$. Then $\mathcal{A}^{\text{irr}}_P$ is open and dense in $\mathcal{A}_P$, and $\mathcal{A}^{\text{red}}_P$ is closed and of infinite codimension in the infinite-dimensional affine space $\mathcal{A}_P$. Hence the inclusion $\mathcal{A}^{\text{irr}}_P \hookrightarrow \mathcal{A}_P$ is a weak homotopy equivalence, and $\mathcal{A}^{\text{irr}}_P$ is weakly contractible.

We write $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(G))$ for the moduli space of gauge equivalence classes of connections on $P$, and $\mathcal{M}^{\text{irr}}_P = \mathcal{A}^{\text{irr}}_P/(\mathcal{G}/Z(G))$ for the subspace $\mathcal{M}^{\text{irr}}_P \subseteq \mathcal{M}_P$ of irreducible connections. We take $\mathcal{M}^{\text{irr}}_P$ to be a topological space, with the quotient topology. However, as explained in [19, Rem. 2.1], we should regard $\mathcal{M}_P$ as a topological stack in the sense of Metzler [22] and Noohi [26, 27], rather than just as a topological space.

The inclusion $\mathcal{A}^{\text{irr}}_P \hookrightarrow \mathcal{A}_P$ is a weak homotopy equivalence, so the inclusion $\mathcal{M}^{\text{irr}}_P \hookrightarrow \mathcal{M}$ is a weak homotopy equivalence of topological stacks in the sense of Noohi [27]. Therefore, for the algebraic topological questions that concern us, working on $\mathcal{M}^{\text{irr}}$ and on $\mathcal{M}$ are essentially equivalent, so we could just restrict our attention to the topological space $\mathcal{M}^{\text{irr}}$, and not worry about topological stacks at all, following most other authors in the area.

The main reason we do not do this in [19] is that to relate orientations on different moduli spaces we consider direct sums of connections, which are generally reducible, so restricting to irreducible connections would cause problems.

We define orientation bundles $O^\mathcal{P}_\bullet$ on moduli spaces $\mathcal{M}_P$:

**Definition 1.2.** Work in the situation of Definition 1.1 with the same notation. Suppose we are given real vector bundles $E_0, E_1 \to X$, of the same rank $r$, and a linear elliptic partial differential operator $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$, of degree
d. As a shorthand we write $E_e = (E_0, E_1, D)$. With respect to connections $\nabla_{E_0}$ on $E_0 \otimes \bigotimes T^*X$ for $0 \leq i < d$, when $e \in \Gamma^\infty(E_0)$ we may write

$$D(e) = \sum_{i=0}^d a_i \cdot \nabla_{E_0}^i e,$$

(1.1)

where $a_i \in \Gamma^\infty(E_0 \otimes E_1 \otimes S^i T^*X)$ for $i = 0, \ldots, d$. The condition that $D$ is elliptic is that $a_d|_x \cdot \otimes^d \xi : E_0|_x \to E_1|_x$ is an isomorphism for all $x \in X$ and $0 \neq \xi \in T^*_x X$, and the symbol $\sigma(D)$ of $D$ is defined using $a_d$.

Let $\nabla_P \in \mathcal{A}_P$. Then $\nabla_P$ induces a connection $\nabla_{\text{Ad}(P)}$ on the vector bundle $\text{Ad}(P) \to X$. Thus we may form the twisted elliptic operator

$$D_{\nabla_{\text{Ad}(P)}} : \Gamma^\infty(\text{Ad}(P) \otimes E_0) \longrightarrow \Gamma^\infty(\text{Ad}(P) \otimes E_1),$$

$$D_{\nabla_{\text{Ad}(P)}} : e \longmapsto \sum_{i=0}^d (\text{id}_{\text{Ad}(P)} \otimes a_i) \cdot \nabla_{\text{Ad}(P)}^i e,$$

(1.2)

where $\nabla_{\text{Ad}(P) \otimes E_0}$ are the connections on $\text{Ad}(P) \otimes E_0 \otimes \bigotimes T^*X$ for $0 \leq i < d$ induced by $\nabla_{\text{Ad}(P)}$ and $\nabla_E$.

Since $D_{\nabla_{\text{Ad}(P)}}$ is a linear elliptic operator on a compact manifold $X$, it has finite-dimensional kernel $\text{Ker}(D_{\nabla_{\text{Ad}(P)}})$ and cokernel $\text{Coker}(D_{\nabla_{\text{Ad}(P)}})$, where the index of $D_{\nabla_{\text{Ad}(P)}}$ is $\text{ind}(D_{\nabla_{\text{Ad}(P)}}) = \dim \text{Ker}(D_{\nabla_{\text{Ad}(P)}}) - \dim \text{Coker}(D_{\nabla_{\text{Ad}(P)}})$. This index is independent of $\nabla_P \in \mathcal{M}_P$, so we write $\text{ind}_{\text{Ad}}^\bullet := \text{ind}(D_{\nabla_{\text{Ad}(P)}})$.

The determinant $\text{det}(D_{\nabla_{\text{Ad}(P)}})$ is the 1-dimensional real vector space

$$\text{det}(D_{\nabla_{\text{Ad}(P)}}) = \text{det}(D_{\nabla_{\text{Ad}(P)}}) \otimes (\text{det}(D_{\nabla_{\text{Ad}(P)}}))^\ast,$$

where if $V$ is a finite-dimensional real vector space then $\text{det} V = \Lambda^{\dim V} V$.

These operators $D_{\nabla_{\text{Ad}(P)}}$ vary continuously with $\nabla_P \in \mathcal{A}_P$, so they form a family of elliptic operators over the base topological space $\mathcal{A}_P$. Thus as in Atiyah and Singer [1], there is a natural real line bundle $L^\text{Ad}^\bullet_P \to \mathcal{A}_P$ with fibre $L^\text{Ad}^\bullet_P|_{\Delta_P} = \text{det}(D_{\nabla_{\text{Ad}(P)}})$ at each $\Delta_P \in \mathcal{A}_P$. It is naturally equivariant under the action of $\mathcal{G}/Z(\mathcal{G})$ on $\mathcal{A}_P$, and so pushes down to a real line bundle $L^\text{Ad}^\bullet_P \to \mathcal{M}_P$ on the topological stack $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(\mathcal{G}))$. We call $L^\text{Ad}^\bullet_P$ the determinant line bundle of $\mathcal{M}_P$. The restriction $L^\text{Ad}^\bullet_P|_{\mathcal{M}_P^\text{irr}}$ is a topological real line bundle in the usual sense on the topological space $\mathcal{M}_P^\text{irr}$.

Define the orientation bundle $O^\text{Ad}^\bullet_P$ of $\mathcal{M}_P$ by $O^\text{Ad}^\bullet_P = (L^\text{Ad}^\bullet_P \setminus 0(\mathcal{M}_P))/\{0, \infty\}$. That is, we take the complement $L^\text{Ad}^\bullet_P \setminus 0(\mathcal{M}_P)$ of the zero section $0(\mathcal{M}_P)$ in $L^\text{Ad}^\bullet_P$, and quotient by the action of $\{0, \infty\}$ on the fibres of $L^\text{Ad}^\bullet_P \setminus 0(\mathcal{M}_P)$ by multiplication. The projection $L^\text{Ad}^\bullet_P \to \mathcal{M}_P$ descends to $\pi : O^\text{Ad}^\bullet_P \to \mathcal{M}_P$, which is a fibre bundle with fibre $(\mathbb{R} \setminus \{0\})/(0, \infty) \cong \{1, -1\} = \mathbb{Z}_2$, since $L^\text{Ad}^\bullet_P \to \mathcal{M}_P$ is a fibration with fibre $\mathbb{R}$. That is, $\pi : O^\text{Ad}^\bullet_P \to \mathcal{M}_P$ is a principal $\mathbb{Z}_2$-bundle, in the sense of topological stacks. The fibres of $O^\text{Ad}^\bullet_P \to \mathcal{M}_P$ are orientations on the real line fibres of $L^\text{Ad}^\bullet_P \to \mathcal{M}_P$. The restriction $O^\text{Ad}^\bullet_P|_{\mathcal{M}_P^\text{irr}}$ is a principal $\mathbb{Z}_2$-bundle on the topological space $\mathcal{M}_P^\text{irr}$, in the usual sense.
We say that \( \mathcal{M}_P \) is orientable if \( \hat{O}_P^{E^*} \) is isomorphic to the trivial principal \( \mathbb{Z}_2 \)-bundle \( \mathcal{M}_P \times \mathbb{Z}_2 \rightarrow \mathcal{M}_P \). An orientation \( \omega \) on \( \mathcal{M}_P \) is an isomorphism \( \omega: \hat{O}_P^{E^*} \xrightarrow{\sim} \mathcal{M}_P \times \mathbb{Z}_2 \) of principal \( \mathbb{Z}_2 \)-bundles. If \( \omega \) is an orientation, we write \(-\omega\) for the opposite orientation. As \( \mathcal{M}_P \) is connected, if \( \mathcal{M}_P \) is orientable it has exactly two orientations.

We also define the normalized orientation bundle \( \tilde{\hat{O}}_P^{E^*} \rightarrow \mathcal{M}_P \) by

\[
\tilde{\hat{O}}_P^{E^*} = O_P^{E^*} \otimes_{\mathbb{Z}_2} O_{X \times G}[|\nabla^0|].
\]

That is, we tensor the orientation bundle with the orientation torsor \( O_{X \times G}[|\nabla^0|] \) of the trivial principal \( G \)-bundle \( X \times G \rightarrow X \) at the trivial connection \( \nabla^0 \). Then \( \tilde{\hat{O}}_P^{E^*} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 \) is canonically trivial. Since we have natural isomorphisms

\[
\text{Ker}(D^0_{\text{Ad}(P)}) \cong \mathfrak{g} \otimes \text{Ker} D, \quad \text{Coker}(D^0_{\text{Ad}(P)}) = \mathfrak{g} \otimes \text{Coker} D,
\]

we see that (using an orientation convention) there is a natural isomorphism

\[
O_{X \times G}[|\nabla^0|] \cong \text{Or}(\det D)^{\otimes \dim \mathfrak{g}} \otimes_{\mathbb{Z}_2} \text{Or}(\mathfrak{g})^{\otimes \dim D},
\]

where \( \text{Or}(\det D) \), \( \text{Or}(\mathfrak{g}) \) are the \( \mathbb{Z}_2 \)-torsors of orientations on \( \det D \) and \( \mathfrak{g} \). Thus, choosing orientations for \( \det D \) and \( \mathfrak{g} \) gives an isomorphism \( \tilde{\hat{O}}_P^{E^*} \cong \hat{O}_P^{E^*} \).

Normalized orientation bundles are convenient because they behave nicely under the Excision Theorem, Theorem 2.1 below. Note that \( \hat{O}_P^{E^*} \) is trivializable if and only if \( \tilde{\hat{O}}_P^{E^*} \) is, so for questions of orientability there is no difference.

**Remark 1.3.** (i) Up to continuous isotopy, and hence up to isomorphism, \( \hat{L}_P^{E^*}, \tilde{\hat{L}}_P^{E^*} \) in Definition 1.2 depend on the elliptic operator \( D: \Gamma^\infty(E_0) \rightarrow \Gamma^\infty(E_1) \) up to continuous deformation amongst elliptic operators, and hence only on the symbol \( \sigma(D) \) of \( D \) (essentially, the highest order coefficients \( a_d \) in \( \sum_{i} a_i x_i \), up to deformation).

(ii) For orienting moduli spaces of ‘instantons’ in gauge theory, as in [1.2] we usually start not with an elliptic operator on \( X \), but with an elliptic complex

\[
0 \rightarrow \Gamma^\infty(E_0) \xrightarrow{D_0} \Gamma^\infty(E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{k-1}} \Gamma^\infty(E_k) \rightarrow 0. \quad (1.3)
\]

If \( k > 1 \) and \( \nabla_P \) is an arbitrary connection on a \( G \)-principal bundle \( P \rightarrow X \) then twisting \( 1.3 \) by \( (\text{Ad}(P), \nabla_{\text{Ad}(P)}) \) as in \( 1.2 \) may not yield a complex (that is, we may have \( D_{i+1}^{\nabla_{\text{Ad}(P)}} \circ D_i^{\nabla_{\text{Ad}(P)}} \neq 0 \)), so the definition of \( \det(D^{\nabla_{\text{Ad}(P)}}_{\mathbb{Z}_2}) \) does not work, though it does work if \( \nabla_P \) satisfies the appropriate instanton-type curvature condition. To get round this, we choose metrics on \( X \) and the \( E_i \), so that we can take adjoints \( D_i^* \), and replace \( 1.3 \) by the elliptic operator

\[
\Gamma^\infty\left( \bigoplus_{0 \leq i \leq k/2} E_{2i} \right) \xrightarrow{\sum_{i}(D_{2i} + D_{2i-1}^*)} \Gamma^\infty\left( \bigoplus_{0 \leq i < k/2} E_{2i+1} \right), \quad (1.4)
\]

and then Definition 1.2 works with \( 1.4 \) in place of \( E_* \).
1.2 Orienting moduli spaces in gauge theory

In gauge theory one studies moduli spaces \( \mathcal{M}_P^{\text{sa}} \) of (irreducible) connections \( \nabla_P \) on a principal bundle \( P \to X \) (perhaps plus some extra data, such as a Higgs field) satisfying a curvature condition. Under suitable genericity conditions, these moduli spaces \( \mathcal{M}_P^{\text{sa}} \) will be smooth manifolds, and the ideas of [19] can often be used to prove \( \mathcal{M}_P^{\text{sa}} \) is orientable, and construct a canonical orientation on \( \mathcal{M}_P^{\text{sa}} \). These orientations are important in defining enumerative invariants such as Casson invariants, Donaldson invariants, and Seiberg–Witten invariants. We illustrate this with the example of instantons on 4-manifolds. [9]:

**Example 1.4.** Let \((X, g)\) be a compact, oriented Riemannian 4-manifold, and \( G \) a Lie group (e.g. \( G = SU(2) \)), and \( P \to X \) a principal \( G \)-bundle. For each connection \( \nabla_P \) on \( P \), the curvature \( F^P = F^P = \nabla^P \) is a section of \( \text{Ad}(P) \otimes \Lambda^2 T^*X \). We have \( \Lambda^2 T^*X = \Lambda^2_+ T^*X \oplus \Lambda^2_+ T^*X \), where \( \Lambda^2_\pm T^*X \) are the subbundles of 2-forms \( \alpha \) on \( X \) with \( *\alpha = \pm \alpha \). Thus \( F^P = F^P_+ \oplus F^P_\pm \) with \( F^P_\pm \) the component in \( \text{Ad}(P) \otimes \Lambda^2_\pm T^*X \). We call \( \nabla_P \) an (anti-self-dual) instanton if \( F^P_+ = 0 \).

Write \( \mathcal{M}_P^{\text{sad}} \) for the moduli space of gauge isomorphism classes \([\nabla_P]\) of irreducible instanton connections \( \nabla_P \) on \( P \). The deformation theory of \([\nabla_P]\) in \( \mathcal{M}_P^{\text{sad}} \) is governed by the Atiyah–Hitchin–Singer complex [2]:

\[
0 \longrightarrow \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^0 T^*X) \xrightarrow{d^\nabla_P} \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^1 T^*X) \xrightarrow{d^\nabla_P} \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^2 T^*X) \xrightarrow{d^\nabla_P} 0,
\]

(1.5)

where \( d^\nabla_P \circ d^\nabla_P = 0 \) as \( F^\nabla_P = 0 \). Write \( \mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2_\pm \) for the cohomology groups of (1.5). Then \( \mathcal{H}^0 \) is the Lie algebra of \( \text{Aut}(\nabla_P) \), so \( \mathcal{H}^0 = Z(\mathfrak{g}) \), the Lie algebra of the centre \( Z(G) \) of \( G \), as \( \nabla_P \) is irreducible. Also \( \mathcal{H}^1 \) is the Zariski tangent space of \( \mathcal{M}_P^{\text{sad}} \) at \([\nabla_P]\), and \( \mathcal{H}^2_\pm \) is the obstruction space. If \( g \) is generic then as in [9, §4.3], for non-flat connections \( \mathcal{H}^2_\pm = 0 \) for all \( \nabla_P \), and \( \mathcal{M}_P^{\text{sad}} \) is a smooth manifold, with tangent space \( T_{[\nabla_P]} \mathcal{M}_P^{\text{sad}} = \mathcal{H}^1 \). Note that \( \mathcal{M}_P^{\text{sad}} \subset \mathcal{M}_P \) is a subspace of the topological stack \( \mathcal{M}_P \) from Definition 1.1.

Take \( E_* \) to be the elliptic operator on \( X \)

\[
D = d + d^*_+ : \Gamma^\infty(\Lambda^0 T^*X \otimes \Lambda^2_+ T^*X) \longrightarrow \Gamma^\infty(\Lambda^1 T^*X).
\]

Turning the complex (1.5) into a single elliptic operator as in Remark 1.3 ii) yields the twisted operator \( D^\text{Ad}(P) \) from (1.2). Hence the line bundle \( E^\text{Ad}(P)_{\mathcal{M}_P} \) has fibre at \([\nabla_P]\) the determinant line of (1.5), which (after choosing an isomorphism \( \det Z(\mathfrak{g}) \cong \mathbb{R} \)) is \( \det(\mathcal{H}^1)^* = \det T_{[\nabla_P]} \mathcal{M}_P^{\text{sad}} \). It follows that \( O_{\mathcal{M}_P^{\text{sad}}} \big|_{\mathcal{M}_P^{\text{sad}}} \) is the orientation bundle of the manifold \( \mathcal{M}_P^{\text{sad}} \), and an orientation on \( \mathcal{M}_P \) in Definition 1.2 restricts to an orientation on the manifold \( \mathcal{M}_P^{\text{sad}} \) in the usual sense of differential geometry. This is a very useful way of defining orientations on \( \mathcal{M}_P^{\text{sad}} \), first used by Donaldson [7, 9].

There are several other important classes of gauge-theoretic moduli spaces \( \mathcal{M}_P^{\text{sa}} \) which have elliptic deformation theory, and so are generically smooth mani-
Definition 1.7. Let \( (P, \omega) \) be a principal \( P \)-bundle. A Spin(7)-structure \((\Omega, g)\) on \( X \) is a 4-form \( \Omega \) and a Riemannian metric \( g \) on \( X \), such that for all \( x \in X \) there exist isomorphisms \( T_x X \cong \mathbb{R}^8 \) identifying \( \Omega|_x \cong \Omega_0 \) and \( g|_x \cong g_0 \). We call \((\Omega, g)\) torsion-free if \( d\Omega \equiv 0 \). This implies that \( \text{Hol}(g) \subseteq \text{Spin}(7) \). A Spin(7)-structure \((\Omega, g)\) induces a splitting \( \Lambda^2 T^* X = \Lambda^{2,0} T^* X \oplus \Lambda^{2,1} T^* X \) into vector subbundles of ranks 7, 21, the eigenspaces of \( \alpha \mapsto +(*) \wedge \Omega \).

A Spin(7)-manifold \((X, \Omega, g)\) is an 8-manifold \( X \) with a torsion-free Spin(7)-structure \((\Omega, g)\). Examples of compact Spin(7)-manifolds with holonomy \( \text{Spin}(7) \) were constructed by Joyce [13, §13–§15].

A Calabi–Yau 4-fold \((X, J, g, \theta)\) is a compact complex manifold \((X, J)\) with trivial canonical bundle \( K_X \), equipped with a Ricci-flat Kähler metric \( g \) with \( \text{Hol}(g) = \text{SU}(4) \), and a holomorphic \((4,0)\)-form \( \theta \) with \( 2\omega^4 = 3\theta \wedge \bar{\theta} \), where \( \omega \) is the Kähler form of \( g \). Many examples of Calabi–Yau 4-folds may be produced using complex algebraic geometry and the Calabi Conjecture. As \( \text{SU}(4) \subset \text{Spin}(7) \), any Calabi–Yau 4-fold has a torsion-free Spin(7)-structure \((\Omega, g)\), with \( \Omega = \frac{1}{2} \omega \wedge \omega + \text{Re} \theta \), so \((X, \Omega, g)\) is a Spin(7)-manifold.

Definition 1.7. Let \((X, \Omega, g)\) be a compact Spin(7)-manifold, \( G \) a Lie group, and \( P \to X \) a principal \( G \)-bundle. A Spin(7)-instanton on \( P \) is a connection \( \nabla_P \)
on $P$, whose curvature satisfies $\pi_2^P(F^\nabla) = 0$ in $\Gamma^\infty(\text{Ad}(P) \otimes \Lambda^2 T^* X)$. Write $\mathcal{M}^{\text{Spin}(7)}_P$ for the moduli space of irreducible Spin(7)-instantons on $P$, modulo gauge transformations of $P$. Then $\mathcal{M}^{\text{Spin}(7)}_P$ is a derived manifold, which is a manifold for non-flat connections if $\Omega$ is generic. (We do not need $d\Omega = 0$ here.)

Donaldson and Thomas [11] discussed Spin(7)-instantons, proposing research directions, and examples of Spin(7)-instantons on compact Spin(7)-manifolds with holonomy Spin(7) were given by Lewis [21], Tanaka [29], and Walpuski [30].

To apply §1.2 to Spin(7)-instantons, we replace (1.5) by the complex:

$$0 \rightarrow \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^0 T^* X) \xrightarrow{d_{\nabla_P}} \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^1 T^* X) \xrightarrow{d_{\nabla_P}} \Gamma^\infty(\text{Ad}(P) \otimes \Lambda^2 T^* X) \rightarrow 0.$$ 

The orientation bundle of $\mathcal{M}^{\text{Spin}(7)}_P$ is the pullback of $O^{E\bullet}_P$ in Definition 1.2 under the inclusion $\mathcal{M}^{\text{Spin}(7)}_P \hookrightarrow \mathcal{M}_P$, where $E\bullet$ is the elliptic operator $D = d + d^*_\partial : \Gamma^\infty(\Lambda^0 T^* X \oplus \Lambda^2 T^* X) \rightarrow \Gamma^\infty(\Lambda^1 T^* X)$.

The symbol of $E\bullet$ is that of the positive Dirac operator $D_+ : \Gamma^\infty(S_+) \rightarrow \Gamma^\infty(S_-)$ on $X$, which makes sense on general oriented, spin Riemannian 8-manifolds, not just Spin(7)-manifolds.

1.4 The main results

Here is our main result. It will be proved in §2 using a wide range of ideas and techniques, including much of the general theory of orientations in [19,20], some surgery theory, some special geometry of SU(4), and the classification of compact simply-connected 5-manifolds in Crowley [6].

**Theorem 1.8.** Let $X$ be a compact, oriented, spin Riemannian 8-manifold, and $\mathcal{M}_P$ be the positive Dirac operator $D_+ : \Gamma^\infty(S_+) \rightarrow \Gamma^\infty(S_-)$ on $X$ in Definition 1.2. Suppose $P \rightarrow X$ is a principal $G$-bundle for $G = \text{U}(m)$ or $\text{SU}(m)$. Then $\mathcal{M}_P$ is orientable, that is, $O^{E\bullet}_P \rightarrow \mathcal{M}_P$ is a trivial principal $\mathbb{Z}_2$-bundle.

This was previously proved by Cao and Leung [5, Th. 2.1] in the special case that $G = \text{U}(m)$ and $H_\text{odd}(X,\mathbb{Z}) = 0$, and by Munoz and Shahbazi [25] in the special case that $G = \text{SU}(m)$ and $\text{Hom}(H^3(X,\mathbb{Z}),\mathbb{Z}_2) = 0$.

As in [13] if $(X,\Omega,g)$ is a Spin(7)-manifold and $P \rightarrow X$ a principal $G$-bundle then orientations on $\mathcal{M}_P$ restrict to orientations on $\mathcal{M}^{\text{Spin}(7)}_P$, giving:

**Corollary 1.9.** Let $(X,\Omega,g)$ be a compact Spin(7)-manifold. Then for any principal $G$-bundle $P \rightarrow X$ for $G = \text{U}(m)$ or $\text{SU}(m)$, the moduli space $\mathcal{M}^{\text{Spin}(7)}_P$ of Spin(7)-instantons on $P$ is orientable, as a manifold or derived manifold.

Corollary 1.9 will be an important ingredient in any future programme to define Donaldson-invariant-style enumerative invariants of Spin(7)-manifolds $(X,\Omega,g)$ by 'counting' suitably compactified moduli spaces $\mathcal{M}^{\text{Spin}(7)}_P$, as in Donaldson and Thomas [11] and Donaldson and Segal [10].
Remark 1.10. In a companion paper, Joyce and Upmeier \[20\] prove that if \((X, g)\) is a compact, oriented, spin Riemannian 7-manifold, and \(E_\bullet\) is the Dirac operator on \(X\), and we choose an orientation on \(\text{det} \, D\) and a flag structure on \(X\) (an algebro-topological structure on odd-dimensional manifolds defined in Joyce \[15, §3.1\]), then we can construct canonical orientations on \(M_P\) for all principal \(U(m)\) or \(SU(m)\)-bundles \(P \to X\). Thus if \((X, \varphi, g)\) is a compact torsion-free \(G_2\)-manifold, we can construct canonical orientations on moduli spaces \(M_{G_2}^P\) of \(G_2\)-instantons on \(P\).

The authors know how to define an analogue of flag structures for compact, spin 8-manifolds \(X\), such that if we choose one of these structures on \(X\) and an orientation of \(\text{det} \, D^+\), then we can improve Theorem 1.8 and Corollary 1.9 to construct canonical orientations on \(M_P\) and \(M_{\text{Spin}(7)}^P\). However, these analogues of flag structures are more complicated and less attractive than flag structures, and we have decided not to write them up for the present.

1.5 Applications to Calabi–Yau 4-folds

As in Definition 1.6 a Calabi–Yau 4-fold \((X, J, g, \theta)\) may be regarded as a \(\text{Spin}(7)\)-manifold, so we can also consider \(\text{Spin}(7)\)-instantons on Calabi–Yau 4-folds, as in Donaldson and Thomas \[11\] (who called them \(\text{SU}(4)\)-instantons).

In terms of complex geometry, for a connection \(\nabla_E\) on a complex vector bundle \(E \to X\), the \(\text{Spin}(7)\)-instanton equations on a Calabi–Yau 4-fold may be written \(F_{\nabla_E} \wedge \omega^3 = 0\), \((F_{\nabla_E})^2_{+} = 0\), where \((F_{\nabla_E})^2_{+}\) is a ‘real half’ of the \((2, 0)\)-component \((F_{\nabla_E})^2_{+}\) of \(F_{\nabla_E}\).

Suppose that \(E \to X\) is a \(\text{rank } m\) polystable holomorphic vector bundle with \(c_1(E) = 0\). Then by the Hitchin–Kobayashi correspondence, \(E\) admits a natural connection \(\nabla_E\) with \(F_{\nabla_E} \wedge \omega^3 = 0\) and \((F_{\nabla_E})^2_{+} = 0\). These are called the Hermitian–Einstein equations, in the case \(c_1(E) = 0\).

Now the \(\text{Spin}(7)\)-instanton equations are a subset of the Hermitian–Einstein equations, so polystable holomorphic vector bundles yield examples of \(\text{Spin}(7)\)-instantons. As in \[11\] §2 and \[24\] Prop. 11, if the Chern classes of a complex vector bundle \(E \to X\) satisfy an identity which is automatic if \(E\) admits a holomorphic structure, any \(\text{Spin}(7)\)-instanton on \(E\) satisfies the Hermitian–Einstein equations, so the \(\text{Spin}(7)\)-instanton moduli space \(M_{\text{Spin}(7)}^E\) is (at least as a set) the moduli space of polystable holomorphic structures on \(E \to X\).

This is important, because the Hermitian–Einstein equations are overdetermined elliptic, so one would not expect their moduli space \(M_{\text{He-Ei}}^E\) to be a (derived) manifold, or have a virtual cycle which one could use to define enumerative invariants. However, the \(\text{Spin}(7)\)-instanton equations are elliptic modulo gauge-fixing, so their moduli space \(M_{\text{Spin}(7)}^E\) is a (derived) manifold, and if it is compact and oriented it has a virtual cycle. So as \(M_{\text{Spin}(7)}^E = M_{\text{He-Ei}}^E\), it seems reasonable to try and define Donaldson–Thomas type invariants ‘counting’ polystable vector bundles on Calabi–Yau 4-folds.

To define enumerative invariants, one needs compact moduli spaces, but moduli spaces of vector bundles are generally noncompact. To compactify mod-
uli spaces $\mathcal{M}_{E}^{\text{He-Ei}}$, there are two obvious approaches:

(a) Using gauge theory, by understanding the limiting behaviour of sequences $(\nabla_{E_{i}})_{i=1}^{\infty}$ of Hermitian–Einstein equations on $E \to X$ by ‘bubbling’ on 4-submanifolds (or worse) in $X$, in a similar way to compactified moduli spaces of instantons on 4-manifolds in Donaldson theory [9].

(b) Using algebraic geometry, by considering $\mathcal{M}_{E}^{\text{He-Ei}}$ as an open subset in a compact moduli space $\mathcal{M}_{\alpha}^{\text{coh}}$ of semistable torsion-free coherent sheaves in class $\alpha = [E]$ in $K^0(X)$.

Approach (a) is formidably difficult in any dimension greater than 4. But in approach (b) we get compact moduli spaces $\mathcal{M}_{\alpha}^{\text{coh}}$ for free by standard results in algebraic geometry. We still need to extend the derived manifold structure on $\mathcal{M}_{\text{Spin}(7)} = \mathcal{M}_{E}^{\text{He-Ei}}$ to the compactification $\mathcal{M}_{\alpha}^{\text{coh}}$. This was solved by Borisov and Joyce [3] for moduli spaces of stable coherent sheaves on Calabi–Yau 4-folds, using Derived Algebraic Geometry and Pantev–Toën–Vaquié–Vezzosi’s theory of $k$-shifted symplectic structures [28]. (See also Cao and Leung [4] for an approach using gauge theory on vector bundles.)

Borisov–Joyce [3] and Cao–Leung [4] propose defining Donaldson–Thomas style ‘DT4 invariants’ of Calabi–Yau 4-folds, using approach (b). An essential ingredient is an ‘orientation’ on the moduli spaces $\mathcal{M}_{\alpha}^{\text{coh}}$, in the sense of [3, §2.4]. In a sequel [12] by Gross and Joyce we will use Theorem 1.8 to prove orientability for all such moduli spaces $\mathcal{M}_{\alpha}^{\text{coh}}$, contributing to the programme of [3,4].

## 2 Proof of Theorem 1.8

Let $X$ be a compact, oriented, spin Riemannian 8-manifold, $E_{\bullet}$ be the positive Dirac operator on $X$, and $P \to X$ be a principal $G$-bundle for $G = \text{U}(m)$ or $\text{SU}(m)$. We must prove the orientation bundle $O_{P}^{\bullet} \to \mathcal{M}_{\alpha}$ in Definition 1.2 is trivial. As in Definition 1.2, this is equivalent to the normalized orientation bundle $\check{O}_{P}^{\bullet} \to \mathcal{M}_{\alpha}$ being trivial. We will do this in the following steps:

**Step 1.** Use results of Joyce, Tanaka and Upmeier [19] to show that $\mathcal{M}_{\alpha}$ is orientable for any principal $\text{U}(m)$ or $\text{SU}(m)$-bundle $P \to X$ if and only if this holds when $P = X \times \text{SU}(4)$ is the trivial $\text{SU}(4)$-bundle over $X$.

**Step 2.** Let $P = X \times \text{SU}(4) \to X$ be the trivial $\text{SU}(4)$-bundle and $\nabla^{0}$ the trivial connection on $P$, so that $[\nabla^{0}]$ is a base-point in $\mathcal{M}_{\alpha}$. The fundamental group $\pi_{1}(\mathcal{M}_{\alpha})$ is the set of isotopy classes $[\gamma]$ of loops $\gamma : S^{1} \to \mathcal{M}_{\alpha}$ with $\gamma(1) = [\nabla^{0}]$. As in [19] §2 there is a group morphism $\Theta : \pi_{1}(\mathcal{M}_{\alpha}) \to \mathbb{Z}_{2} = \{\pm 1\}$ such that $\Theta([\gamma])$ is the monodromy of the principal $\mathbb{Z}_{2}$-bundle $\check{O}_{P}^{\bullet} \to \mathcal{M}_{\alpha}$ around $\gamma$, and $\mathcal{M}_{\alpha}$ is orientable if and only if $\Theta \equiv 1$.

We establish (already known) natural 1-1 correspondences between:

(a) Elements $[\gamma] \in \pi_{1}(\mathcal{M}_{\alpha})$.

(b) Isomorphism classes $[Q,q]$ of pairs $(Q,q)$, where $Q \to X \times S^{1}$ is a principal $\text{SU}(4)$-bundle and $q : Q|_{X \times \{1\}} \to (X \times \{1\}) \times \text{SU}(4) = P$ is a trivialization of $Q$ over $X \times \{1\}$. 

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(c) Isotopy classes $[\Phi]$ of smooth maps $\Phi : X \to SU(4)$.

Write $[X, SU(4)]$ for the set of isotopy classes $[\Phi]$ of smooth maps $\Phi : X \to SU(4)$. Then the 1-1 correspondence gives a bijection $\pi_1(M_P) \cong [X, SU(4)]$. This is an isomorphism of groups, where $[X, SU(4)]$ has group operation $[\Phi] \cdot [\Phi'] = [\mu(\Phi, \Phi')]$, for $\mu : SU(4) \times SU(4) \to SU(4)$ the multiplication map. In fact $\pi_1(M_P), [X, SU(4)]$ are abelian, as $SU(4)$ is in the stable range for 8-manifolds.

Let $\hat{\Theta} : [X, SU(4)] \to \mathbb{Z}_2$ be identified with $\Theta$ under $\pi_1(M_P) \cong [X, SU(4)]$. We must prove that $\hat{\Theta} \equiv 1$.

**Step 3.** Define subsets $Y_k \subset SU(4)$ for $k = 0, \ldots, 3$ by

$$Y_k = \{ A \in SU(4) : \dim_{\mathbb{C}} \{ x = (x_1, x_2, x_3, 0)^T \in \mathbb{C}^4 : A x = -x \} = k \}, \quad (2.1)$$

so that $SU(4) = Y_0 \amalg \cdots \amalg Y_3$. We prove that:

(i) $Y_k$ is a connected, simply-connected, oriented, embedded submanifold of $SU(4)$ (which is also oriented) of real codimension $k(k + 2)$. Hence $Y_0$ is open in $SU(4)$, and $Y_1, Y_2, Y_3$ have codimensions $3, 8, 15$.

(ii) The closure of $Y_k$ in $SU(4)$ is $Y_k = Y_k \amalg Y_{k+1} \amalg \cdots \amalg Y_3$.

(iii) There is a smooth family of smooth maps $\Psi_t : Y_0 \to SU(4)$ for $t \in [0, 1]$ with $\Psi_0$ the inclusion $Y_0 \hookrightarrow SU(4)$, and $\Psi_1 \equiv \text{Id}$ the constant map with value $\text{Id} \in SU(4)$. That is, $Y_0$ retracts to $\{\text{Id} \}$ in $SU(4)$.

(iv) We may define a smooth map $\phi : Y_1 \to \mathbb{CP}^2$ by $\phi(A) = [x_1, x_2, x_3]$ if $A x = -x$ for $x = (x_1, x_2, x_3, 0)^T$. The normal bundle $\nu$ of $Y_1$ in $SU(4)$ is isomorphic to $\mathbb{R} \oplus \phi^*(O(1))$, for $O(1) \to \mathbb{CP}^2$ the standard line bundle.

It is known that the cohomology of $SU(4)$ may be written as a graded ring

$$H^*(SU(4), \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[p_3, p_5, p_7], \quad (2.2)$$

where $p_3, p_5, p_7$ are odd generators in degrees 3, 5, 7, which satisfy

$$\mu^*(p_k) = p_k \boxtimes 1 + 1 \boxtimes p_k \quad \text{in} \quad H^*(SU(4) \times SU(4), \mathbb{Z}) \cong H^*(SU(4), \mathbb{Z}) \otimes H^*(SU(4), \mathbb{Z}). \quad (2.3)$$

Under Poincaré duality $Pd : H^k(SU(4), \mathbb{Z}) \xrightarrow{\cong} H_{15-k}(SU(4), \mathbb{Z})$ we have

$$Pd(p_3) = [\mathbb{F}], \quad Pd(p_3 \cup p_5) = [\mathbb{F}]. \quad (2.4)$$

**Step 4.** Using the notation of Steps 2–3, define maps $\lambda_k : [X, SU(4)] \to H^k(X, \mathbb{Z})$ for $k = 3, 5, 7$ by $\lambda_k([\Phi]) = \Phi^*(p_k)$. Equation (2.3) and $[\Phi] \cdot [\Phi'] = [\mu(\Phi, \Phi')]$ imply that $\lambda_3, \lambda_5, \lambda_7$ are group morphisms. We can also define a map $\kappa : [X, SU(4)] \to \mathbb{Z}$ by

$$\kappa : [\Phi] \mapsto (\lambda_3([\Phi]) \cup \lambda_5([\Phi])) : [X]. \quad (2.5)$$

Note that this is not a group morphism, but is quadratic in $\Phi$. 

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We prove that for any \(\alpha \in H^5(X, \mathbb{Z})\) we can construct \([\Phi'] \in [X, SU(4)]\) with 
\[\lambda_3([\Phi']) = 0\] and \(\lambda_5([\Phi']) = \alpha\).

Therefore any \([\Phi] \in [X, SU(4)]\) may be written \([\Phi] = [\Phi'] + [\Phi'']\) with 
\[\lambda_3([\Phi']) = 0\] and \(\lambda_5([\Phi'']) = 0\), since we can take \([\Phi']\) as above with \(\lambda_3([\Phi']) = 0\) and \(\lambda_5([\Phi'']) = \alpha = \lambda_5([\Phi])\), and \([\Phi''] = [\Phi] - [\Phi']\), and use the fact that \(\lambda_3, \lambda_5\) are group morphisms. Note that \(\kappa([\Phi']) = \kappa([\Phi'']) = 0\). Since \(\hat{\Theta}([\Phi]) = \hat{\Theta}([\Phi']) \cdot \hat{\Theta}([\Phi''])\), we see from Step 2 that it is sufficient to prove that \(\hat{\Theta}([\Phi]) = 1\) for all \([\Phi] \in [X, SU(4)]\) with \(\kappa([\Phi]) = 0\).

**Step 5.** Suppose \(X\) is connected, and \([\Phi] \in [X, SU(4)]\) with \(\kappa([\Phi]) = 0\). Choose a generic representative \(\Phi : X \to SU(4)\) for \([\Phi]\). Then \(\Phi\) is an embedding, and \(\Phi(X)\) intersects \(Y_k\) transversely in \(SU(4)\) for \(k = 1, 2, 3\) by genericness. Thus \(\Phi(X) \cap Y_1\) is an oriented 5-manifold, and \(\Phi(X) \cap Y_2\) an oriented 0-manifold, and \(\Phi(X) \cap Y_3 = \emptyset\), so \(\Phi(X) \cap Y_2\) is compact as \(Y_2 = Y_2 \sqcup Y_3\).

From (2.4)–(2.5) we see that the number of points in \(\Phi(X) \cap Y_2\), counted with signs, is \(\kappa([\Phi]) = 0\). Using this we show that we can perturb \(\Phi\) in its isotopy class to make \(\Phi(X) \cap Y_2 = \emptyset\). Then \(\Phi(X) \cap Y_1\) is compact, as \(Y_1 = Y_1 \sqcup Y_2 \sqcup Y_3\).

Define \(Z = \{x \in X : \Phi(x) \in Y_1\}\). Then \(Z\) is a compact, oriented, embedded 5-manifold in \(X\) diffeomorphic to \(\Phi(X) \cap Y_1\). Define \(\psi : Z \to \mathbb{CP}^2\) by \(\psi = \phi \circ \Phi|_Z\), for \(\phi\) as in Step 3(iv). The normal bundle \(\nu_Z\) of \(Z\) in \(X\) satisfies

\[
\nu_Z \cong \Phi|_Z^* (\nu) \cong \mathbb{R} \oplus \psi^*(\mathcal{O}(1)). \quad (2.6)
\]

As \(TX|_Z = TZ \oplus \nu_Z\), and \(X\) is spin so that \(w_2(TX) = 0\), we see that the second Stiefel–Whitney class \(w_2(Z) \in H^2(Z, \mathbb{Z}_2)\) satisfies

\[
w_2(Z) = w_2(TZ) = w_2(T) = w_2(\mathbb{R} \oplus \psi^*(\mathcal{O}(1))) = \psi^*(c_1(\mathcal{O}(1))) \mod 2.
\]

That is, \(w_2(Z)\) is the image in \(H^2(Z, \mathbb{Z}_2)\) of the integral class \(\psi^*(c_1(\mathcal{O}(1)))\) in \(H^2(Z, \mathbb{Z})\). This implies that \(Z\) admits a \(\text{Spin}^c\)-structure, and simplifies the classification of possible 5-manifolds \(Z\) up to diffeomorphism.

If \(X\) is connected, or simply-connected, we show that we can perturb \(\Phi\) in its isotopy class to make \(Z\) connected, or simply-connected, respectively.

The importance of \(Z\) is that \(\Phi\) maps \(X \setminus Z \to Y_0 \subset SU(4)\), where \(Y_0\) retracts to \(\{1\} \subset SU(4)\) by Step 3(iii). Hence \(\Phi|_{X \setminus Z}\) is isotopic to the constant map \(1\), and if \([Q, q]\) corresponds to \(\Phi\) as in Step 2, then \((Q, q)\) is trivial over \((X \setminus Z) \times S^1\). This allows us to use excision techniques in Steps 6 and 7.

**Step 6.** Suppose \(X\) is connected and simply-connected, and \([\gamma] \in \pi_1(\mathcal{M}_P)\) corresponds to \([\Phi] \in [X, SU(4)]\) as in Step 2 with \(\kappa([\Phi]) = 0\) as in Step 4, and define \(\Phi, Z, \psi, \nu_Z\) with \(Z\) connected and simply-connected as in Step 5.

Using the classification of compact, simply-connected 5-manifolds in Crowley [6], we show we can choose a tubular neighbourhood \(U\) of \(Z\) in \(X\), an explicit compact, oriented, spin Riemannian 8-manifold \(X'\) with \(H^\text{odd}(X', \mathbb{Z}) = 0\), and an embedding \(\iota : U \to X'\) of \(U\) as an open submanifold of \(X'\), where \(\iota\) preserves orientations and spin structures.

Using the Excision Theorem, Theorem 2.1 we show that the monodromy \(\Theta([\gamma])\) of \(\tilde{O}_P^\text{c} \to \mathcal{M}_P\) around \(\gamma\) equals the monodromy of \(\tilde{O}_P^{c'} \to \mathcal{M}_P\) around
some loop $\gamma'$ in $M_{P'}$, where $P' = X' \times SU(4)$. Since $H^{\text{odd}}(X', \mathbb{Z}) = 0$, $M_{P'}$ is orientable by [19, §2], so $\Theta([\gamma]) = \Theta([\Phi]) = 1$. As in Step 4 it is sufficient to prove this for $[\Phi]$ with $\kappa([\Phi]) = 0$, so $M_P$ is orientable. This proves Theorem \[1.8\] in the case $X$ is simply-connected.

**Step 7.** For $X$ not simply-connected, by doing surgeries on finitely many disjoint embedded circles $L_1, \ldots, L_k$ in $X$ we can modify $X$ to a simply-connected, compact, oriented, spin Riemannian 8-manifold $X'$, with open covers $X = U \cup V$, $X' = U' \cup V'$ for $V$ a small tubular neighbourhood of $L_1 \cup \cdots \cup L_k$ in $X$, and a diffeomorphism $\iota: U \to U'$ preserving orientations and spin structures.

Let $[\gamma] \in \pi_1(M_P)$ correspond to $(Q, q)$ as in Step 2. Then $Q \to X \times S^1$ is trivial over $(L_1 \cup \cdots \cup L_k) \times S^1$, as any SU(4)-bundle over a 2-manifold is trivial, so $Q$ is trivial over $V \times S^1$ as $V$ retracts onto $L_1 \cup \cdots \cup L_k$, and we can choose this trivialization compatible with $q$ on $V \times \{1\}$.

Using the Excision Theorem as in Step 6, we find that $\Theta([\gamma]) = \Theta([\gamma'])$ for some loop $\gamma'$ in $M_{P'}$, where $P' = X' \times SU(4)$. But $\Theta([\gamma']) = 1$ by Step 6, as $X'$ is simply-connected, so $\Theta([\gamma]) = 1$. Thus $M_P$ is orientable, completing the proof of Theorem \[1.8\].

We will give more details on Steps 1–6 in §2.1. Step 7 is very similar to Step 6, and we leave it as an exercise for the reader.

### 2.1 Step 1: Reduction to the case $P = X \times SU(4)$

We first recall the material in [19, 20] we will need in the rest of the proof. Let $X$ be a compact $n$-manifold and $E_\bullet$ an elliptic complex on $X$, and use the notation of Definitions 1.1–1.2. Joyce, Tanaka and Upmeier [19, §2] explain:

(i) If $P \to X$ is a principal $U(m)$-bundle, define a principal $SU(m+1)$-bundle $Q \to X$ by $Q = (P \times SU(m+1))/U(m)$, using the inclusion $U(m) \to SU(m+1)$ mapping $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}$. There is a natural morphism $\Xi^Q_P: M_P \to M_Q$, and an isomorphism of principal $\mathbb{Z}_2$-bundles $\xi^Q_P: \tilde{O}^P_{E^\bullet} \to (\Xi^Q_P)^*(\tilde{O}^Q_{E^\bullet})$. Hence, if $M_Q$ is orientable, then $M_P$ is orientable.

(ii) If $Q \to X$ is a principal $SU(m)$-bundle, define $R = (Q \times U(m))/SU(m)$, so that $R \to X$ is a principal $U(m)$-bundle. There is a natural morphism $K^R_Q: M_Q \to M_R$, and an isomorphism of principal $\mathbb{Z}_2$-bundles $\kappa^R_Q: \tilde{O}^Q_{E^\bullet} \to (K^R_Q)^*(\tilde{O}^R_{E^\bullet})$. Hence, if $M_R$ is orientable, then $M_Q$ is orientable.

(iii) If $P \to X$ is a principal $U(m)$-bundle and $k \geq 1$ we can define a principal $(P \times U(m+k))/U(m)$ over $X$, which we write as $P \oplus \mathbb{C}^k \to X$. There is a natural morphism $\Psi^P_{P \oplus \mathbb{C}^k}: M_P \to M_{P \oplus \mathbb{C}^k}$, and an isomorphism of principal $\mathbb{Z}_2$-bundles $\psi^P_{P \oplus \mathbb{C}^k}: \tilde{O}^P_{E^\bullet} \to (\Psi^P_{P \oplus \mathbb{C}^k})^*(\tilde{O}^P_{E^\bullet})$. Hence, if $M_{P \oplus \mathbb{C}^k}$ is orientable, then $M_P$ is orientable.

(iv) If $2m \geq n$ then the morphisms $\Xi^Q_P, K^R_Q, \Psi^P_{P \oplus \mathbb{C}^k}$ in (i)–(iii) are homotopy equivalences, and so identify normalized orientations on both sides.
Because of this, for any principal U(m)-bundle $P \to X$ we can take the direct limit $\lim_{k \to \infty} \mathcal{M}_{P \oplus \mathbb{C}^k}$, as a topological space or stack up to homotopy, using the morphisms $\mathcal{M}_{P \oplus \mathbb{C}^k} \to \mathcal{M}_{P \oplus \mathbb{C}^{k+1}}$ in (iii). We do not need to take $k \to \infty$, only to take $k$ large enough that $2m + 2k \geq n$. The limiting orientation bundle $\mathcal{O}_{P \oplus \mathbb{C}^\infty}^E \to \mathcal{M}_{P \oplus \mathbb{C}^\infty}$ also makes sense. This is called ‘stabilization’.

(v) Every principal U(m)-bundle $P \to X$ has a K-theory class $[P] \in K^0(X)$, the class of the complex vector bundle $(P \times \mathbb{C}^m)/U(m)$. If $\alpha \in K^0(X)$ with $2 \text{rank} \alpha \geq n$ (the ‘stable range’) then there exists a principal U(m)-bundle $P \to X$ with $[P] = \alpha$, and $P$ is unique up to isomorphism.

(vi) For each $\alpha \in K^0(X)$, choose $N_\alpha$ in $\mathbb{Z}$ with $2(\text{rank} \alpha + N_\alpha) \geq n + 1$. Set $m_\alpha = \text{rank} \alpha + N_\alpha$, and choose a principal U($m_\alpha$)-bundle $P_\alpha \to X$ with $[P_\alpha] = \alpha + N_\alpha 1_X$ in $K^0(X)$, where $1_X \in K^0(X)$ is the class $[X \times \mathbb{C}]$ of the trivial line bundle $X \times \mathbb{C} \to X$. As in (v), this determines $P_\alpha$ uniquely up to isomorphism.

Using stabilization as in (iv), define a topological stack $\mathcal{M}_u^U_\alpha$ by $\mathcal{M}_u^U = \lim_{k \to \infty} \mathcal{M}_{P_\alpha \oplus \mathbb{C}^k}$, taking the direct limit using $\Psi_{P_\alpha \oplus \mathbb{C}^k}^{P_\alpha \oplus \mathbb{C}^{k+1}}: \mathcal{M}_{P_\alpha \oplus \mathbb{C}^k} \to \mathcal{M}_{P_\alpha \oplus \mathbb{C}^{k+1}}$. Then $\mathcal{M}_u^U$ is independent of the choices of $N_\alpha, P_\alpha$ up to isomorphism, and the isomorphisms are unique up to isotopy. We also define a principal $\mathbb{Z}_2$-bundle $\mathcal{O}_{\alpha}^{E\bullet} \to \mathcal{M}_u^U$ by $\mathcal{O}_{\alpha}^{E\bullet} = \lim_{k \to \infty} \mathcal{O}_{P_\alpha \oplus \mathbb{C}^k}^{E\bullet}$.

(vii) For any principal U(m)-bundle $P \to X$ with $[P] = \alpha \in K^0(X)$, we have a morphism $\Sigma_P^U: \mathcal{M}_P \to \mathcal{M}_u^U$, natural up to isotopy, and an isomorphism of principal $\mathbb{Z}_2$-bundles $\sigma_P^U: \mathcal{O}_{\alpha}^{E_{\bullet}} \cong (\Sigma_P^U)^*(\mathcal{O}_{\alpha}^{E\bullet})$. Hence, if $\mathcal{M}_u^U$ is orientable, then $\mathcal{M}_P$ is orientable. If $2m \geq n$ then $\Sigma_P^U$ is a homotopy equivalence, so $\mathcal{M}_P$ is orientable if and only if $\mathcal{M}_u^U$ is orientable.

(viii) By considering direct sums of bundles we can show that for any $\alpha$ in $K^0(X)$, there is a homotopy equivalence $\mathcal{M}_u^U \simeq \mathcal{M}_1^U$, and $\mathcal{M}_u^U$ is orientable if and only if $\mathcal{M}_1^U$ is orientable.

(ix) Let $Q = X \times U(k)$ be the trivial U(k)-bundle, for any $k$ with $2k \geq n$, and suppose $\mathcal{M}_Q$ is orientable. Then (vii) implies $\mathcal{M}_1^U$ is orientable, so (viii) implies $\mathcal{M}_u^U$ is orientable for any $\alpha \in K^0(X)$, and (vii) implies that $\mathcal{M}_P$ is orientable for any principal U(m)-bundle $P \to X$.

(x) We have $\pi_1(\mathcal{M}_u^U) \cong K^1(X)$, the odd complex K-theory group of $X$. Hence if $K^1(X) = 0$ then any principal $\mathbb{Z}_2$-bundle over $\mathcal{M}_u^U$ is trivial, and $\mathcal{M}_u^U$ is orientable, so $\mathcal{M}_P$ is orientable for any principal U(m)- or SU(m)-bundle $P \to X$ by (ix) and (ii).

There is an Atiyah–Hirzebruch spectral sequence $H^{\text{odd}}(X, \mathbb{Z}) \Rightarrow K^1(X)$. Thus if $H^{\text{odd}}(X, \mathbb{Z}) = 0$ then $K^1(X) = 0$.

We can now prove Step 1. Suppose that $\mathcal{M}_P$ is orientable for $P = X \times SU(4)$ the trivial SU(4)-bundle. By (ii), (iv) this implies $\mathcal{M}_Q$ is orientable for $Q = X \times U(4)$ the trivial U(4)-bundle, so by (ix) $\mathcal{M}_R$ is orientable for any principal U(m) or SU(m)-bundle $R \to X$, as we have to prove.
Then we have a canonical identification of $\gamma$ and will be used in Steps 6 and 7.

**Theorem 2.1** (Excision Theorem). Suppose we are given the following data:

(a) Compact n-manifolds $X^+, X^-.$

(b) Elliptic complexes $E^*_{\bullet}$ on $X^\pm.$

(c) A Lie group $G,$ and principal $G$-bundles $P^\pm \to X^\pm$ with connections $\nabla_{P^\pm}.$

(d) Open covers $X^+ = U^+ \cup V^+, X^- = U^- \cup V^-.$

(e) A diffeomorphism $\iota : U^+ \to U^-$, such that $E^*_{\bullet}|_{U^+}$ and $\iota^*(E^*_{\bullet}|_{U^-})$ are isomorphic elliptic complexes on $U^+.$

(f) An isomorphism $\sigma : P^+|_{U^+} \to \iota^*(P^-|_{U^-})$ of principal $G$-bundles over $U^+,$ which identifies $\nabla_{P^+}|_{U^+}$ with $\iota^*(\nabla_{P^-}|_{U^-}).$

(g) Trivializations of principal $G$-bundles $\tau^\pm : P^\pm|_{V^\pm} \to V^\pm \times G$ over $V^\pm,$ which identify $\nabla_{P^\pm}|_{V^\pm}$ with the trivial connections, and satisfy
$$\iota|_{U^+ \cap V^+}^* (\tau^-) \circ \sigma|_{U^+ \cap V^+} = \tau^+|_{U^+ \cap V^+}.$$ 

Then we have a canonical identification of $\mathbb{Z}_2$-torsors
$$\Omega^{+-} : \bar{\Omega}^*_{\pi^+} \overset{\sim}{\to} \bar{\Omega}^*_{\pi^-}.$$ 

The isomorphisms (2.7) are functorial in a very strong sense. For example:

(i) If we vary any of the data in (a)–(g) continuously in a family over $t \in [0, 1],$ then the isomorphisms $\Omega^{+-}$ also vary continuously in $t \in [0, 1].$

(ii) The isomorphisms $\Omega^{+-}$ are unchanged by shrinking the open sets $U^\pm, V^\pm$ such that $X^\pm = U^\pm \cup V^\pm$ still hold, and restricting $\iota, \sigma, \tau^\pm.$

(iii) If we are also given a compact n-manifold $X^\times,$ elliptic complex $E^\times_{\bullet},$ bundle $P^\times \to X^\times,$ connection $\nabla_{P^\times},$ open cover $X^\times = U^\times \cup V^\times,$ diffeomorphism $\iota^\times : U^\times \to U^+,$ and isomorphisms $\sigma^\times : P^\times|_{U^-} \to \iota^\times|U^\times (P^\times|_{U^+})$, $\tau^\times : P^\times|_{V^\times} \to V^\times \times G$ satisfying the analogues of (a)–(g), then $\Omega^{\times\times}$ is defined using $\iota^\times \circ \iota : U^+ \to U^\times$ and $\iota^\times (\sigma^\times) \circ \sigma : P^\times|_{U^+} \to (\iota \circ \iota)^\times(P^\times|_{U^\times}).$

**2.2 Step 2: Alternative descriptions of $\pi_1(M_P)$**

We will justify the 1-1 correspondences between (a),(b),(c) in Step 2. Let $P = X \times SU(4) \to X$ be the trivial SU(4)-bundle and $\nabla^0$ the trivial connection on $P,$ so that $[\nabla^0] \in \mathcal{M}_P.$ As in (a), let $\gamma : S^1 \to \mathcal{M}_P$ be a smooth path with $\gamma(1) = [\nabla^0].$ Then $\gamma$ is a smooth path of connections on $P$ modulo gauge. We can think of $\gamma$ as a smooth family of pairs $(P_z, \nabla_{P_z})_{z \in S^1},$ where $P_z \to X$ is a principal SU(4)-bundle which is isomorphic to $P,$ but not canonically isomorphic to $P$ (since we quotient by the gauge group $\mathcal{G} = Map_C(\infty(X, SU(4))/\mathbb{Z}_4),$ and $\nabla_{P_z}$ is a connection on $P_z,$ with $P_z = P$ and $\nabla_{P_z} = \nabla^0$. 

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We can assemble the \((P_z)_{z \in S^1}\) into a principal SU(4)-bundle \(Q \to X \times S^1\) with \(Q|_{X \times \{z\}} = P_z\), and then \(Q|_{X \times \{1\}} = P_1 = P\) gives a trivialization \(q : Q|_{X \times \{1\}} \congto (X \times \{1\}) \times SU(4)\) as required. The connections \((\nabla P_z)_{z \in S^1}\) assemble into a partial connection \(\nabla_Q^X\) on \(Q\) in the \(X\) directions in \(X \times S^1\). Note that although each \(P_z\) is (noncanonically) trivial, \(Q\) need not be a trivial bundle on \(X \times S^1\), as it can have nontrivial topological twisting in the \(S^1\) directions.

Changing the loop \(\gamma\) by smooth isotopies deforms \(Q,q,\nabla_Q^X\) smoothly, and so preserves the pair \((Q,q)\) up to isomorphism. This gives a well-defined map \([\gamma] \mapsto [Q,q]\) from objects (a) to objects (b).

Conversely, given \([Q,q]\) choose a representative \((Q,q)\) and a partial connection \(\nabla_Q^X\) on \(Q\) in the \(X\) directions in \(X \times S^1\) with \(\nabla_Q^X|_{X \times \{1\}} = \nabla^0\), and define \(\gamma : S^1 \to M_P\) by \(\gamma(z) = [\nabla_Q^X|_{X \times \{z\}}]\). This is well defined as \(Q|_{X \times \{z\}}\) is noncanonically isomorphic to \(P\), since \(Q|_{X \times \{1\}} \cong P\). Then \(\gamma\) is a smooth loop in \(M_P\) with \(\gamma(1) = [\nabla^0]\), so \([\gamma] \in \pi_1(M_P)\). The space of partial connections \(\nabla_Q^X\) on \(Q\) is an infinite-dimensional affine space, so any two choices \(\nabla_Q^X,\nabla_Q^X\) are joined by a smooth path, and the corresponding loops \(\gamma,\tilde{\gamma}\) are smoothly isotopic, giving \([\gamma] = [\tilde{\gamma}]\). Hence the inverse map \([Q,q] \mapsto [\gamma]\) is defined, and (a),(b) are in 1-1 correspondence.

Now let \((Q,q)\) be as in (b). Choose a connection \(\nabla_Q\) on \(Q \to X \times S^1\). For each \(x \in X\), consider the path \(\delta_x : [0,2\pi] \to X \times S^1\) mapping \(\delta_x : \theta \mapsto (x,e^{i\theta})\). The holonomy of \(\nabla_Q\) around \(\delta_x\) is a smooth map \(\text{Hol}_{\delta_x}(\nabla_Q) : Q|_{\delta_x(0)} \to Q|_{\delta_x(2\pi)}\) which is equivariant under the \(SU(4)\)-actions on \(Q|_{\delta_x(0)},\ nabla|_{\delta_x(2\pi)}\) from the principal SU(4)-bundle. In this case \(\delta_x(0) = \delta_x(2\pi) = (x,1)\), and \(q\) identifies \(Q|_{\delta_x(0)} = Q|_{\delta_x(2\pi)} \cong SU(4)\), where \(SU(4)\) acts by left multiplication on itself.

Hence \(q\) identifies \(\text{Hol}_{\delta_x}(\nabla_Q)\) with a smooth map \(SU(4) \to SU(4)\) equivariant under left multiplication by \(SU(4)\), which must be right multiplication by some \(\Phi(x) \in SU(4)\). This defines the map \(\Phi : X \to SU(4)\) in (c), which is smooth as \(\nabla_Q\) is smooth. Any two connections \(\nabla_Q,\nabla_Q'\) on \(Q\) are smoothly isotopic, so \(\Phi\) is unique up to isotopy, and \([\Phi]\) is unique. This defines the map \([Q,q] \mapsto [\Phi]\) from objects (b) to objects (c).

Conversely, let \(\Phi : X \to SU(4)\) be a smooth map. Let \(\sim\) be the equivalence relation \(0 \sim 2\pi\) on \([0,2\pi]\), and identify \([0,2\pi]/\sim\) with \(S^1\) by \(\theta \mapsto e^{i\theta}\). Hence we also identify \(X \times [0,2\pi]/\sim\) with \(X \times S^1\). Define \(Q' \to X \times S^1\) to be the principal SU(4)-bundle \((X \times [0,2\pi] \times SU(4))/\sim\), where \(\approx\) is the equivalence relation \((x,0,e) \approx (x,2\pi,e\Phi(x))\) for \(x \in X\) and \(e \in SU(4)\), where the projection \(Q' \to X \times S^1\) maps \([x,\theta,e] \mapsto [x,\theta] \in X \times [0,2\pi]/\sim \cong X \times S^1\), and the \(SU(4)\)-action on \(Q'\) is by left multiplication on the \(SU(4)\) factor. Define \(q' : Q'|_{X \times \{1\}} \congto (X \times \{1\}) \times SU(4)\) to map \([x,0,e] \mapsto [x,e]\). Changing \(\Phi\) by smooth isotopy changes \((Q',q')\) by smooth isotopy, and hence by isomorphism, so we have a map \([\Phi] \mapsto [Q,q]\) from objects (b) to objects (c). It is easy to see this is inverse to the map \([Q,q] \mapsto [\Phi]\) above, so (b),(c) are in 1-1 correspondence.

The rest of Step 2 is clear.
2.3 Step 3: The geometry of SU(4)

Let $Y_k \subset \text{SU}(4)$ for $k = 0, \ldots, 3$ be as in (2.1). Write $\text{Gr}(\mathbb{C}^k, \mathbb{C}^3)$ for the Grassmannian of vector subspaces $V \subset \mathbb{C}^3$ with $V \cong \mathbb{C}^k$, a compact complex manifold of dimension $k(3 - k)$. Define a map $\phi_k : Y_k \to \text{Gr}(\mathbb{C}^k, \mathbb{C}^3)$ by

$$
\phi_k(A) = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : Ax = -x \quad \text{for} \quad x = (x_1, x_2, x_3, 0)^T\},
$$

where the right hand side is a $k$-dimensional subspace of $\mathbb{C}^3$ by (2.1), and thus a point of $\text{Gr}(\mathbb{C}^k, \mathbb{C}^3)$. Note that $\text{Gr}(\mathbb{C}^4, \mathbb{C}^3) = \mathbb{CP}^2$ and $\phi_1$ is $\phi$ in Step 3(iv).

The fibre of $\phi_k$ over $\{ (x_1, \ldots, x_k, 0, \ldots, 0) : x_j \in \mathbb{C} \}$ is

$$
\left\{ \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & (-1)^k B \\
\end{pmatrix} : B \in \text{SU}(4 - k), \; B \text{ has no} \\
eigenvalues \text{ in } \mathbb{C}^{3-k} \subset \mathbb{C}^{4-k} \right\}. \quad (2.8)
$$

This is diffeomorphic to an open subset of $\text{SU}(4 - k)$, the complement of a codimension 3 subset, and so is connected. By considering the action of $\text{SU}(3) \subset \text{SU}(4)$ on $\text{SU}(4)$ by conjugation, which preserves $Y_k$ and acts on $\text{Gr}(\mathbb{C}^3, \mathbb{C}^3)$, we see that $Y_k$ is an embedded submanifold of $\text{SU}(4)$ and $\phi_k$ is a fibre bundle with fibre (2.8). Hence

$$
\dim Y_k = \dim \text{Gr}(\mathbb{C}^k, \mathbb{C}^3) + \dim \text{SU}(4 - k) = 2k(3 - k) + (4 - k)^2 - 1
= 15 - k(k + 2) = \dim \text{SU}(4) - k(k + 2),
$$

so the codimension of $Y_k$ is $k(k + 2)$. As $\text{Gr}(\mathbb{C}^k, \mathbb{C}^3)$ and (2.8) are connected, simply-connected and oriented, $Y_k$ is embedded, connected, simply-connected and oriented. This proves Step 3(i).

Part (ii) is obvious. For (iii), for each $A \in Y_0 \subset \text{SU}(4)$ define vectors $e_j^1(A) \in \mathbb{C}^4$ for $j = 1, 2, 3$ and $t \in [0, 1]$ by

$$
e_j^1(A) = te_j + (1 - t)Ae_j,
$$

where $e_1 = (1000)^T$, $e_2 = (1000)^T$, and $e_3 = (1000)^T$. We claim that for each $A \in Y_0$ and $t \in [0, 1]$, the vectors $e_1^1(A), e_2^1(A), e_3^1(A)$ are $\mathbb{C}$-linearly independent in $\mathbb{C}^4$. For if not, there would exist $0 \neq (x_1, x_2, x_3) \in \mathbb{C}^3$ such that

$$
(1-t)A(x_1e_1 + x_2e_2 + x_3e_3) = -t(x_1e_1 + x_2e_2 + x_3e_3),
$$

so that $-t/(1-t)$ is an eigenvalue of $A$. As eigenvalues of $A$ have norm 1, this forces $t = \frac{1}{2}$, so $A x = \pm x$ for $0 \neq x = (x_1, x_2, x_3, 0)^T$, contradicting (2.1).

Next define vectors $f_j^1(A) \in \mathbb{C}^4$ for $j = 1, 2, 3$ and $t \in [0, 1]$ by

$$
f_j^1(A) = \frac{e_j^1(A)}{\|e_j^1(A)\|}, \quad f_k^2(A) = \frac{e_k^2(A) = \langle e_k^2(A), f_1^1(A) \rangle e_1^1(A) - \langle e_k^2(A), f_1^1(A) \rangle f_1^1(A)}{\|e_k^2(A)\|},
$$

$$
f_j^3(A) = \frac{e_j^3(A) - \langle e_j^3(A), f_1^1(A) \rangle e_1^1(A) - \langle e_j^3(A), f_2^1(A) \rangle e_2^1(A) - \langle e_j^3(A), f_3^1(A) \rangle e_3^1(A)}{\|e_j^3(A)\|}.
$$
That is, we use the Gram–Schmidt process to make \( f_i^1(A), f_i^2(A), f_i^3(A) \) Hermitian orthonormal in \( \mathbb{C}^4 \). There is then a unique \( f_i^t(A) \) such that the matrix \( \Psi_i(A) = (f_i^1(A) \cdots f_i^t(A)) \) with columns \( f_i^1(A), \ldots, f_i^t(A) \) lies in \( \text{SU}(4) \). Then \( \Psi_i(A) \in \text{SU}(4) \) depends smoothly on \( A \in Y_0 \) and \( t \in [0,1] \), with \( \Psi^0(A) = A \) and \( \Psi^1(A) = \text{Id} \), so \( \Psi_t : Y_0 \to \text{SU}(4) \) for \( t \in [0,1] \) satisfies Step 3(iii).

For (iv), for simplicity let \( A \in Y_1 \) with \( \phi(A) = [1,0,0] \in \mathbb{CP}^2 \). Then

\[
A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Let \( \delta A \in T_A \text{SU}(4) \), so that we think of \( \delta A \) as a small \( 4 \times 4 \) complex matrix, with \( A + \delta A \) an infinitesimal perturbation of \( A \) in \( \text{SU}(4) \). Then

\[
(A + \delta A) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 + \delta A_{11} \\ \delta A_{21} \\ \delta A_{31} \\ \delta A_{41} \end{pmatrix},
\]

for \( \delta A_{11} \in \mathbb{C} \) small, with \( \delta A_{11} \in i\mathbb{R} \) as \( (A+\delta A) \) preserves lengths. Then the fibre \( \nu|_A \subset T_A \text{SU}(4) \) at \( A \) of the normal bundle \( \nu \) of \( Y_1 \) in \( \text{SU}(4) \) may be identified with \( \mathbb{R} \oplus \mathbb{C} \) with coordinates \((\text{Im}(\delta A_{11}), \delta A_{41})\).

Here \( \text{Im}(\delta A_{11}) \) measures the tangent direction in \( \text{SU}(4) \) which varies the eigenvalue \(-1\) of \( A \) to \( e^{i\theta} \) in \( S^1 \) close to \(-1\), so we should think of \( \delta A_{11} \) as lying in \( T_{-1}S^1 = i\mathbb{R} \). And \( \delta A_{41} \) measures the tangent directions in \( \text{SU}(4) \) which vary the eigenspace \([x_1, x_2, x_3, 0] \in \mathbb{CP}^3\) of \( A \) normal to \( \mathbb{CP}^2 = \{[y_1, y_2, y_3, 0] \in \mathbb{CP}^3\} \) in \( \mathbb{CP}^3 \), where \( A \in Y_1 \) must have a \(-1\)-eigenvector in \( \mathbb{CP}^2 \), so we should think of \( \delta A_{41} \) as lying in \( \tilde{\nu}|_{[1,0,0,0]} \), where \( \tilde{\nu} \) is the normal bundle of \( \mathbb{CP}^2 \) in \( \mathbb{CP}^3 \). Note that \( \tilde{\nu} \cong \mathcal{O}(1) \), so \( \tilde{\nu}|_{[1,0,0,0]} \cong \phi^*(\mathcal{O}(1)|A) \).

More generally, if \( A \in Y_1 \) with \( \phi(A) = [x_1, x_2, x_3] \in \mathbb{CP}^2 \) for \( x_1, x_2, x_3 \in \mathbb{C} \) with \( |x_1| + |x_2|^2 + |x_3|^2 = 1 \) and \( \delta A \in T_A \text{SU}(4) \), then we can identify \( \nu|_A \) with \( \mathbb{R} \oplus \mathbb{C} \) with coordinates \((y, z)\), where in matrix notation

\[
y = -i \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \delta A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \delta A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

Multiplying the representative \((x_1, x_2, x_3)\) for \([x_1, x_2, x_3] \in \mathbb{CP}^2\) by \( e^{i\theta} \) fixes \( y \), but multiplies \( z \) by \( e^{i\theta} \). So the invariant thing is to regard \( z \) as lying in \( \tilde{\nu}|_{[x_1, x_2, x_3, 0]} = \phi^*(\mathcal{O}(1)|A) \). This defines an isomorphism \( \nu \cong \mathbb{R} \oplus \phi^*(\mathcal{O}(1)) \), proving (iv).

To prove (2.2), by a well known calculation for all \( m \geq 2 \) we show that

\[
H^*(\text{SU}(m), \mathbb{Z}) \cong \Lambda_2[p_3, p_5, \ldots, p_{2m-1}]
\]

by induction on \( m \), where the first step \( m = 2 \) follows from \( \text{SU}(2) \cong S^3 \), and the inductive step from the Leray–Serre spectral sequence for the fibration.
SU(m − 1) ⊂ SU(m) → S^{2m−1}. For [2.3], the K"unneth Theorem gives
\[ \mu^*(p_k) \in H^k(SU(4) \times SU(4), \mathbb{Z}) = \langle p_k \boxtimes 1, 1 \boxtimes p_k \rangle, \]
so \( \mu^*(p_k) = a_k \cdot (p_k \boxtimes 1) + b_k \cdot (1 \boxtimes p_k) \) for \( a_k, b_k \in \mathbb{Z}, \) and \( a_k = b_k = 1 \) follows by restricting \( \mu \) to \( SU(4) \times \{Id\} \) and \( \{Id\} \times SU(4) \) in \( SU(4) \times SU(4). \)

For [2.4], note that as \( H^3(SU(4), \mathbb{Z}) = \langle p_3 \rangle_\mathbb{Z} \) and \( H^8(SU(4), \mathbb{Z}) = \langle p_3 \cup p_5 \rangle_\mathbb{Z}, \)
we have \[ \{Y_1\} = c \cdot Pd(p_3) \text{ and } \{Y_2\} = d \cdot Pd(p_3 \cup p_5) \] for some \( c, d \in \mathbb{Z}. \) From [2.9] for \( m = 2, 3, 4 \) we see that under the embeddings \( \iota : SU(2) \hookrightarrow SU(4) \) and \( j : SU(3) \hookrightarrow SU(4) \) given by
\[ \iota : B \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B \end{pmatrix}, \quad j : C \mapsto \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}, \]
we have \( p_3 \cdot \iota_*([SU(2)]) = 1 \) and \( (p_3 \cup p_5) \cdot j_*([SU(3)]) = 1. \) So using the intersection product \( \bullet \) on \( H_* (SU(4), \mathbb{Z}) \)
we have
\[ c = [Y_1] \bullet \iota_*([SU(2)]) \quad \text{and} \quad d = [Y_2] \bullet j_*([SU(3)]). \]
But \( Y_1 \) intersects \( \iota(SU(2)) \) transversely in one point \( \text{diag}(1, 1, -1, -1) \) in \( Y_1, \)
and \( Y_2 \) intersects \( j(SU(3)) \) transversely in one point \( \text{diag}(1, -1, -1, 1) \) in \( Y_2, \)
so these intersection numbers are \( \pm 1, \) and choosing orientations on \( Y_1, Y_2 \) appropriately
we can ensure that \( c = d = 1, \) proving [2.4].

2.4 Step 4: Reduction to the case \( \kappa([\Phi]) = 0 \)

Define maps \( \lambda_3, \lambda_5, \lambda_7 \) and \( \kappa \) on \( [X, SU(4)] \) as in Step 4. Let \( \alpha \in H^5(X, \mathbb{Z}), \)
so that \( Pd(\alpha) \in H_5(X, \mathbb{Z}). \) We can choose a compact, oriented, embedded 3-submanifold \( W \subset X \) with \( [W] = Pd(\alpha). \) Write \( \nu \to W \) for the normal bundle of \( W \) in \( X. \)
Now \( W \) admits a spin structure, as any oriented 3-manifold does, and \( X \) is spin, so \( \nu \) admits a spin structure on its fibres. Hence \( \nu \) is trivial, as any Spin(5)-bundle on a 3-manifold is trivial. Thus we may choose a tubular neighbourhood \( T \) of \( W \) in \( X \) and a diffeomorphism \( T \cong W \times B^5, \) where \( B^5 \subset \mathbb{R}^5 \)
is the open unit ball.

By Mimura and Toda [23] we have \( \pi_5(SU(4)) \cong \mathbb{Z}, \)
and the natural map \( \pi_5(SU(4)) \to H_5(SU(4), \mathbb{Z}) \) is an isomorphism. Thus there exists a smooth map \( \Psi : S^5 - \{\infty\} \to SU(4) \) with \( \Psi_*[p_5] \cdot [S^5] = 1. \) We may choose \( \Psi \) with \( \Psi \equiv \text{Id} \) outside the ball \( B^5_{1/2} \) of radius \( \frac{1}{2} \) in \( \mathbb{R}^5 \subset S^5. \)
Define \( \Phi' : X \to SU(4) \)
by \( \Phi'|_{X \setminus T} \equiv \text{Id}, \)
and \( \Phi'|_T \) is identified under \( T \cong W \times B^5 \) with the map \( W \times B^5 \to SU(4), (w, b) \mapsto \Psi(b). \)
As \( \Psi \equiv 1 \) on \( B^5 \setminus B^5_{1/2}, \) this \( \Phi' \) is smooth.

Since \( \Psi_*[p_5] \cdot [S^5] = 1, \) it follows that \( Pd \circ \lambda_5([\Phi']) = Pd \circ \Phi'^*(p_5) = [Y] = Pd(\alpha), \)
so \( \lambda_5([\Phi']) = \alpha. \) As \( \Phi'|_{X \setminus T} \equiv \text{Id}, \)
and the morphism \( H_3(X \setminus T, \mathbb{Z}) \to H_3(X, \mathbb{Z}) \)
induced by the inclusion \( X \setminus T \to X \) is an isomorphism for dimensional reasons, we see that \( \lambda_3([\Phi']) = 0, \) as we want. The rest of Step 4 is clear.
2.5 Step 5: A 5-submanifold $Z \subset X$ with $\Phi \approx 1$ on $X \setminus Z$

Suppose $X$ is connected, and $[\Phi] \in [X, SU(4)]$ with $\kappa([\Phi]) = 0$. Choose a generic representative $\Phi : X \to SU(4)$ for $[\Phi]$. Then as in Step 5, $\Phi$ is an embedding, and $\Phi(X) \cap Y_2$ is a compact, oriented 0-manifold, that is, a finite set of points with signs $\pm 1$, and the number of points in $\Phi(X) \cap Y_2$ counted with signs is $\kappa([\Phi]) = 0$. Thus we may write $\Phi(X) \cap Y_2 = \{r_1, \ldots, r_k, s_1, \ldots, s_k\}$, where the $r_i$ have positive orientation and the $s_i$ negative orientation. There are unique disjoint $p_1, \ldots, p_k, q_1, \ldots, q_k \in X$ with $\Phi(p_i) = r_i$ and $\Phi(q_i) = s_i$.

As $X$ is connected we may choose smooth embedded paths $\gamma_i : [0, 1] \to X$ with $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_i$ for $i = 1, \ldots, k$, and as $\dim X > 2$ we may choose $\gamma_1([0, 1]), \ldots, \gamma_k([0, 1])$ to be disjoint. Then $\Phi \circ \gamma_i : [0, 1] \to SU(4)$ are smooth embedded paths in $SU(4)$ with $\Phi \circ \gamma_i(0) = r_i$ and $\Phi \circ \gamma_i(1) = s_i$, with $r_i, s_i$ the only intersection points of $\Phi \circ \gamma_i([0, 1])$ with $Y_2$.

As $Y_2$ is connected we may choose smooth embedded paths $\delta_i : [0, 1] \to Y_2$ with $\delta_i(0) = r_i$ and $\delta_i(1) = s_i$. Then $\Phi \circ \gamma_i$ and $\delta_i$ are both smooth paths $[0, 1] \to SU(4)$ with end points $r_i, s_i$. Since $SU(4)$ is simply-connected we may choose smooth isotopies $\epsilon_i : [0, 1]^2 \to SU(4)$ with $\epsilon_i(0, t) = \Phi \circ \gamma_i(t)$, $\epsilon_i(1, t) = \delta_i(t)$, $\epsilon_i(s, 0) = r_i$, $\epsilon_i(s, 1) = s_i$ for all $s, t \in [0, 1]$, where we may take $\epsilon_i$ to be an embedding on $[0, 1] \times (0, 1)$, and to map to $Y_2$ only at $(s, 0), (s, 1)$ and $(1, t)$.

We now use the ‘Whitney trick’ (as used in the proof of the Whitney Embedding Theorem): we modify $\Phi$ in small open neighbourhoods of the paths $\gamma_1([0, 1]), \ldots, \gamma_k([0, 1])$ in $X$, deforming $\Phi$ along the disks $\epsilon_i([0, 1]^2)$ in $SU(4)$, so as to eliminate the intersection points $r_i, s_i$ of $\Phi(X) \cap Y_2$ in pairs. Thus we may perturb $\Phi$ in its isotopy class so that $\Phi(X) \cap Y_2 = \emptyset$. We can also suppose $\Phi$ is an embedding, and $\Phi(X)$ intersects $Y_1$ transversely.

As in Step 5, it now follows that $Z = \{x \in X : \Phi(x) \in Y_1\}$ is a compact, oriented, embedded 5-submanifold in $X$ diffeomorphic to $\Phi(X) \cap Y_1$, and defining $\psi : Z \to \mathbb{CP}^2$ by $\psi = \phi \circ \Phi|_Z$, the normal bundle $\nu_Z$ of $Z$ in $X$ satisfies $\nu_Z \cong \mathbb{R} \oplus \psi^*(O(1))$ as in [2.6], and $w_2(Z)$ is the image in $H^2(Z, \mathbb{Z})$ of the integral class $\psi^*(c_1(O(1)))$ in $H^2(Z, \mathbb{Z})$.

Next suppose $X$ is connected. We will show that we can perturb $\Phi$ in its isotopy class to make $Z$ connected. Suppose first that $Z$ has two connected components $Z_0$ and $Z_1$. As $X$ is connected we can choose points $z_0 \in Z_0$, $z_1 \in Z_1$ and a smooth path $\gamma : [0, 1] \to X$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$, where we suppose that $\gamma([0, 1])$ is embedded and meets $Z$ transversely only at $z_0, z_1$. Then $\Phi \circ \gamma : [0, 1] \to SU(4)$ is a smooth path in $SU(4)$ with endpoints $\Phi(z_0), \Phi(z_1)$ in $Y_1$. As $Y_1$ is connected we can choose a smooth path $\delta : [0, 1] \to Y_1$ with $\delta(0) = \Phi(z_0)$, $\delta(1) = \Phi(z_1)$, where we suppose that $\delta([0, 1])$ is embedded and meets $\Phi(X) \cap Y_1 = \Phi(Z)$ transversely only at $\Phi(z_0), \Phi(z_1)$.

Then $\Phi \circ \gamma([0, 1]) \cup \delta([0, 1])$ are two paths from $\Phi(z_0)$ to $\Phi(z_1)$ in $SU(4)$, so $\Phi \circ \gamma([0, 1]) \cup \delta([0, 1])$ is a piecewise-smooth embedded circle in $SU(4)$. As $SU(4)$ is simply-connected we may choose a smooth embedded 2-discs $D$ in $SU(4)$, with boundary $\Phi \circ \gamma([0, 1]) \cup \delta([0, 1])$, and corners at $\Phi(z_0), \Phi(z_1)$.

In a similar way to the use of the ‘Whitney trick’ above, we may modify $\Phi : X \to SU(4)$ in a small open neighbourhood of $\gamma([0, 1])$ in $X$ to a new
\(\Phi' : X \to \text{SU}(4)\), where we deform \(\Phi\) near \(\gamma([0,1])\) along the disc \(D\) in \(\text{SU}(4)\), so that \(\Phi'\) near \(\gamma([0,1])\) is close to the path \(\delta([0,1])\) in \(\text{SU}(4)\). We can arrange that \(Z' = \Phi'^{-1}(Y_1)\) near \(\gamma([0,1])\) is a tube \([0,1] \times S^4\), where \(S^4\) is a small 4-sphere. That is, we replace \(Z = Z_1 \amalg Z_2\) by the connected sum \(Z' = Z_1 \# Z_2\), joining \(Z_1, Z_2\) by a narrow neck \([0,1] \times S^4\) close to \(\gamma([0,1])\) in \(X\), and making \(Z'\) connected. If \(Z\) has \(k > 2\) connected components, we use the trick above \(k - 1\) times to make \(Z'\) connected.

Finally, suppose \(X\) is simply-connected. We will show that we can perturb \(\Phi\) in its isotopy class to make \(Z\) simply-connected. By surgery theory, as \(Z\) is a compact, oriented, simply-connected 5-manifold \(\hat{Z}\) is a sphere. That is, we replace \(\Phi\) near \(\gamma([0,1])\) is close to the path \(\delta([0,1])\) in \(\text{SU}(4)\). We can arrange that \(Z' = \Phi'^{-1}(Y_1)\) near \(\gamma([0,1])\) is a tube \([0,1] \times S^4\), where \(S^4\) is a small 4-sphere. That is, we replace \(Z = Z_1 \amalg Z_2\) by the connected sum \(Z' = Z_1 \# Z_2\), joining \(Z_1, Z_2\) by a narrow neck \([0,1] \times S^4\) close to \(\gamma([0,1])\) in \(X\), and making \(Z'\) connected. If \(Z\) has \(k > 2\) connected components, we use the trick above \(k - 1\) times to make \(Z'\) connected.

As \(X\) is simply-connected, each circle \(L_i\) in \(Z \subset X\) may be written as \(L_i = \partial D_i\), for \(D_i \subset X\) a 2-disc in \(X\). By perturbing \(D_i\) generically we can suppose that \(D_i\) is embedded, that it intersects \(Z\) transversely only at \(\partial D_i = L_i\), and that \(D_1, \ldots, D_k\) are disjoint.

Also \(\Phi(L_i)\) is an embedded circle in \(Y_1\). As \(Y_1\) is simply-connected, we may write \(\Phi(L_i) = \partial E_i\) for \(E_i \subset Y_1\) a 2-disc in \(Y_1\). By perturbing \(E_i\) generically we can suppose that \(E_i\) is embedded, that it intersects \(\Phi(Z) = \Phi(X) \cap Y_1\) transversely only at \(\partial E_i = \Phi(L_i)\), and that \(E_1, \ldots, E_k\) are disjoint.

We now have embedded 2-discs \(\Phi(D_i), E_i\) in \(\text{SU}(4)\) with common boundary \(\Phi(L_i)\), so \(\Phi(D_i) \cup E_i\) is a piecewise-smooth \(S^2\) in \(\text{SU}(4)\). Since \(\pi_2(\text{SU}(4)) = 0\) by \([23]\), we may choose smooth embedded 3-discs \(F_1, \ldots, F_k\) in \(\text{SU}(4)\), with boundary \(\partial F_i = \Phi(D_i) \cup E_i\), and a codimension 2 corner along \(\Phi(L_i)\).

Again, in a similar way to the use of the ‘Whitney trick’ above, we may modify \(\Phi : X \to \text{SU}(4)\) in small open neighbourhoods of \(D_1, \ldots, D_k\) in \(X\) to a new \(\Phi' : X \to \text{SU}(4)\), where we deform \(\Phi\) near \(D_i\) along the 3-disc \(F_i\) in \(\text{SU}(4)\), so that \(\Phi'\) near \(D_i\) is close to the disc \(E_i\) in \(Y_1 \subset \text{SU}(4)\). We also need to consider the (trivial) normal bundles of \(L_i\) and \(D_i\) in \(X\), and their images as subbundles of the (trivial) normal bundles of \(\Phi(L_i)\) and \(\Phi(D_i)\) in \(\text{SU}(4)\), and how these subbundles deform along \(F_i\) to \(E_i\).

If we only deform \(\Phi\) along \(F_i\) to \(\Phi'\) such that \(\Phi'(D_i) = E_i\), then \(\Phi'(X)\) may intersect \(Y_1\) non-transversely along \(E_i\), and \(Z' = \Phi'^{-1}(Y_1)\) will not be a submanifold of \(X\). However, if we deform \(\Phi'\) a little way further, pushing \(\Phi'(D_i)\) a little way beyond the boundary of \(F_i\) at \(E_i\), then \(Z' = \Phi'^{-1}(Y_1)\) becomes a 5-submanifold of \(X\) locally modelled near \(D_i\) on \(D_i \times S^3\), where the \(S^3\) factors are small spheres in a (trivial) rank 4 subbundle of the normal bundle of \(E_i\) in \(Y_1\). That is, \(Z'\) is diffeomorphic to the 5-manifold \(\tilde{Z}\) constructed above by surgery on \(L_1, \ldots, L_k\), so \(Z'\) is simply-connected.

Therefore if \(X\) is simply-connected, we can perturb \(\Phi\) in its isotopy class to make \(\tilde{Z}\) simply-connected, completing Step 5.
2.6 Step 6: \( \mathcal{M}_p \) is orientable if \( X \) is simply-connected

Suppose \( X \) is connected and simply-connected, let \( [\gamma] \in \pi_1(\mathcal{M}_p) \) correspond to \( [\Phi] \) in \([X, \text{SU}(4)]\) as in Step 2 with \( \kappa([\Phi]) = 0 \) as in Step 4, and choose \( \Phi, Z, \psi, \nu_Z \) with \( Z \) connected and simply-connected as in Step 5. Then \( Z \) is a compact, oriented 5-manifold, \( \phi : Z \to \mathbb{CP}^3 \) is smooth, the normal bundle \( \nu_Z \) of \( Z \) in \( X \) is \( \nu_Z \cong \mathbb{R} \oplus \psi^*(\mathcal{O}(1)) \), and \( w_2(Z) \) is the image in \( H^2(Z, \mathbb{Z}_2) \) of the integral class \( \psi^*(c_1(\mathcal{O}(1))) \) in \( H^2(Z, \mathbb{Z}) \).

In the next proposition, using results of Crowley [6] on the diffeomorphism classification of compact, simply-connected 5-manifolds, we will construct a compact, oriented, spin 8-manifold \( X \) such that the boundaries \( \nu_Z \) of \( Z \) is identified with \( \mathbb{CP}^3 \). Composing this with the obvious embedding \( \iota : \mathbb{CP}^1 \to \mathbb{R}^4 \) gives an embedding \( j : Z \hookrightarrow X' \) with \( \nu_Z \) as the trivial normal bundle.

Proposition 2.2. Suppose \( Z \) is a compact, connected, simply-connected, oriented 5-manifold, and \( L \to Z \) is a complex line bundle, such that the second Stiefel–Whitney class \( w_2(Z) \) is the image of \( c_1(L) \) under the projection \( H^2(Z, \mathbb{Z}) \to H^2(Z, \mathbb{Z}_2) \). Then there exist group isomorphisms

\[
H^2(Z, \mathbb{Z}) \cong \mathbb{Z}^r, \quad H^3(Z, \mathbb{Z}) \cong \mathbb{Z}^r \oplus G \oplus G,
\]

for some \( r \geq 0 \) and finite abelian group \( G \), such that the pairing \( H^2(Z, \mathbb{Z}) \times H^3(Z, \mathbb{Z}) \to \mathbb{Z} \) maps \((a_1, \ldots, a_r) \cdot (b_1, \ldots, b_r, g_1, g_2) \mapsto a_1 b_1 + \cdots + a_r b_r \), and \( c_1(L) \) is identified with \((k, 0, \ldots, 0)\) for some \( k \in \mathbb{Z} \). Furthermore:

(a) If \( k = 0 \), so that \( c_1(L) = 0 \) and \( L \) is trivial, and \( w_2(Z) = 0 \) so \( Z \) is spin, there exists an embedding \( \iota : Z \hookrightarrow \mathbb{S}^6 \) with trivial normal bundle \( \mathbb{R} \). Composing this with the obvious embedding \( \mathbb{S}^6 \hookrightarrow \mathbb{S}^8 \) mapping \((x_1, \ldots, x_7) \mapsto (x_1, \ldots, x_7, 0, 0)\) gives an embedding \( j : Z \hookrightarrow X' = \mathbb{S}^8 \) with trivial normal bundle \( \nu_Z' \cong \mathbb{R}^3 \cong \mathbb{R} \oplus L \).

(b) If \( k \neq 0 \) is even, so that \( r \geq 1 \), and \( L \) is nontrivial, and \( w_2(Z) = 0 \) so \( Z \) is spin, there exists an embedding \( \iota : Z \hookrightarrow \mathbb{CP}^1 \times \mathbb{S}^4 \) with trivial normal bundle \( \mathbb{R} \), such that \( L \cong (\pi_{\mathbb{CP}^1} \circ \iota)^*(\mathcal{O}(k)) \), for \( \mathcal{O}(1) \to \mathbb{CP}^1 \) the standard line bundle, \( \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k} \), and \( \pi_{\mathbb{CP}^1} : \mathbb{CP}^1 \times \mathbb{S}^4 \to \mathbb{CP}^1 \) the projection. Define \( X' = \mathbb{CP}^1 \times \mathbb{S}^4 \) to be the \( \mathbb{CP}^1 \)-bundle \( \mathbb{P}(\pi_{\mathbb{CP}^1}^*(\mathcal{O}(0) \oplus \mathcal{O}(k))) \). This bundle has a natural section \([1, 0] : \mathbb{CP}^1 \times \mathbb{S}^4 \to X' \) embedding \( \mathbb{CP}^1 \times \mathbb{S}^4 \) as a submanifold of \( X' \) with normal bundle \( \pi_{\mathbb{CP}^1}^*(\mathcal{O}(k)) \). Hence \( j = [1, 0] \circ \iota \) is an embedding \( j : Z \hookrightarrow X' \) with normal bundle \( \nu'_Z \cong \mathbb{R} \oplus L \).

(c) Let \( k \) be odd, so that \( r \geq 1 \), and \( L \) is nontrivial, and \( w_2(Z) \neq 0 \), so \( Z \) is not spin. Write \( \pi_{\mathbb{CP}^1} : Y \to \mathbb{CP}^1 \) for the nontrivial \( S^4 \) bundle, constructed by writing \( \mathbb{CP}^1 = D^+ \cup_{S^1} D^- \) for the union of closed 2-discs \( D^\pm \) along their boundary \( S^1 \), and defining \( Y = (D^+ \times S^4) \cup_{S^1 \times S^4} (D^- \times S^4) \), where the boundaries \( S^1 \times S^4 \) of \( D^\pm \times S^4 \) are glued using a map \( S^1 \to SO(5) \) representing the nontrivial element of \( \pi_1(SO(5)) \cong \mathbb{Z}_2 \).

Then there exists an embedding \( \iota : Z \hookrightarrow Y \) with trivial normal bundle \( \mathbb{R} \), such that \( L \cong (\pi_{\mathbb{CP}^1} \circ \iota)^*(\mathcal{O}(k)) \).
Define $X' \to Y$ to be the $\mathbb{CP}^1$-bundle $\mathbb{P} \left( \pi^*_{\mathbb{CP}^1} (\mathcal{O}(0) \oplus \mathcal{O}(k)) \right)$. This bundle has a natural section $[1,0] : Y \to X'$ embedding $\mathbb{CP}^1 \times S^4$ as a submanifold of $X'$ with normal bundle $\pi^*_{\mathbb{CP}^1} (\mathcal{O}(k))$. Hence $j = [1,0] \circ \iota$ is an embedding $j : Z \hookrightarrow X'$ with normal bundle $\nu'_Z \cong \mathbb{R} \oplus L$.

In each of (a)–(c), $X'$ is compact, oriented and spin, with $H^{\text{odd}}(X',Z) = 0$.

Proof. The Universal Coefficient Theorem implies that the torsions of $H_1(Z,\mathbb{Z})$ and $H^2(Z,\mathbb{Z})$ are isomorphic. But $H_1(Z,\mathbb{Z}) = 0$ as $Z$ is simply-connected, so $H^2(Z,\mathbb{Z})$ is torsion-free, and thus $H^2(Z,\mathbb{Z}) \cong \mathbb{Z}^r$ for $r = b_2(Z)$. We may choose the isomorphism $H^2(Z,\mathbb{Z}) \cong \mathbb{Z}^r$ to identify $c_1(L)$ with $(k,0,\ldots,0)$ for some $k \in \mathbb{Z}$, as any element of $\mathbb{Z}^r$ is conjugate to some $(k,0,\ldots,0)$ under $\text{SL}(k,\mathbb{Z})$. Then $H_2(Z,\mathbb{Z}) \cong \mathbb{Z}^r \oplus K$ for some finite group $K$, such that the pairing $H^2(Z,\mathbb{Z}) \times H_2(Z,\mathbb{Z}) \to \mathbb{Z}$ maps $(a_1,\ldots,a_r) \cdot (b_1,\ldots,b_r,k) \to a_1b_1 + \cdots + a_rb_r$. The results of Crowley [6] discussed next imply that $K$ is of the form $G \oplus G$. This proves the first part of the proposition.

Crowley [6] describes the classification of compact, connected, simply-connected 5-manifolds $Z$ up to diffeomorphism. To each such $Z$ we associate a pair $(\Gamma,w)$ of a finitely generated abelian group $\Gamma$ and a morphism $w : \Gamma \to \mathbb{Z}_2$, by $\Gamma = H_2(Z,\mathbb{Z})$ and $w(\gamma) = w_2(Z) \cdot \gamma$. Using results of Smale and Barden, Crowley notes that the map from diffeomorphism classes of 5-manifolds $Z$ to isomorphism classes of pairs $(\Gamma,w)$ is injective. He then characterizes which pairs $(\Gamma,w)$ lie in the image of this map, and in some cases gives an embedding $\iota : Z \to Y$ into an explicit 6-manifold $Y$, and thus writes $Z = \partial Y$ for a 6-manifold with boundary $Y$.

Any finitely generated abelian group $\Gamma$ is of the form $\mathbb{Z}^r \oplus K$ for $K$ a finite abelian group, and $w : \Gamma \to \mathbb{Z}_2$ is the sum of morphisms $\mathbb{Z}^r \to \mathbb{Z}_2$ and $K \to \mathbb{Z}_2$. If $w_2(Z)$ lies in the image of $\mathbb{Z}^r \cong H^2(Z,\mathbb{Z}) \to H^2(Z,\mathbb{Z}_2)$, as in our situation, the morphism $K \to \mathbb{Z}_2$ is zero. This excludes many cases in Crowley’s classification. In particular, Crowley allows either $K \cong G \oplus G$ or $K \cong G \oplus G \oplus \mathbb{Z}_2$ for $G$ a finite abelian group, but $K \cong G \oplus G \oplus \mathbb{Z}_2$ occurs only if $w|_K \neq 0$, and so does not happen in our case. This justifies $K \cong G \oplus G$.

Note too that Crowley’s classification is compatible with connected sums: if $Z$ corresponds to $(\Gamma,w)$ with $(\Gamma,w) \cong (\Gamma_1,w_1) \oplus (\Gamma_2,w_2)$ for $(\Gamma_i,w_i)$ corresponding to $Z_i$, $i = 1,2$, then $Z \cong Z_1 \# Z_2$.

For (a), given any $r \geq 0$ and finite abelian group $G$, Crowley [6, §2.1] constructs a compact, connected, simply-connected, spin 5-manifold $Z'$ with $H_2(Z') \cong \mathbb{Z}^r \oplus G \oplus G$ as follows: starting from $\mathbb{Z}^r \oplus G \oplus G$ he constructs a finite CW-complex $C$ with only 2- and 3-cells, chooses an embedding $C \hookrightarrow \mathbb{R}^6$, takes $D$ to be a regular open neighbourhood of $C$ in $\mathbb{R}^6$, so that $\partial D$ is a compact 6-manifold with boundary, and defines $Z' = \partial D$. The result of Smale referred to above implies that for $Z$ as in the proposition with $k = 0$, there is a diffeomorphism $\iota : Z \to Z'$. Then $\iota : Z \to \mathbb{RP}^5 \subset S^6$ is an embedding with trivial normal bundle $\mathbb{R}$. The rest of (a) is immediate.

For (b)–(c), for $Z$ as in the proposition with $k \neq 0$ so $r \geq 1$, we may split

$$(H_2(Z,\mathbb{Z}),w_2(Z) \cdot) \cong (\mathbb{Z},k \mod 2) \oplus (\mathbb{Z}^{r-1} \oplus G \oplus G,0).$$
Then as above we have $Z \cong Z_1 \# Z_2$ for $Z_1, Z_2$ with $H_2(Z_1, \mathbb{Z}) \cong \mathbb{Z}$, $w_2(Z_1) = k \mod 2$, $H_2(Z_2, \mathbb{Z}) \cong \mathbb{Z}^{r-1} \oplus G \oplus G$, $w_2(Z_2) = 0$. This is only valid if $Z_1, Z_2$ exist with these invariants, but we justify this shortly.

In case (b), when $w_1(Z_1) = 0$ as $k$ is even, we may take $Z_1 = \mathbb{C}P^1 \times S^3$, with an embedding $\iota_1 : Z_1 \to \mathbb{C}P^1 \times S^4$ from the identity on $\mathbb{C}P^1$ and the equator embedding $S^3 \hookrightarrow S^4$. Part (a) gives a 5-manifold $Z_2$ with $H_2(Z_2, \mathbb{Z}) \cong \mathbb{Z}^{r-1} \oplus G \oplus G$ and $w_2(Z_2) = 0$, and an embedding $\iota_2 : Z_2 \hookrightarrow S^6$. Taking connected sums of both 5- and 6-manifolds gives an embedding

$$Z_1 \# Z_2 \hookrightarrow (\mathbb{C}P^1 \times S^4) \# S^6 \cong \mathbb{C}P^1 \times S^4.$$ 

For $Z$ as in part (b), the diffeomorphism $Z \cong Z_1 \# Z_2$ gives an embedding $\iota : Z \hookrightarrow \mathbb{C}P^1 \times S^4$, with trivial normal bundle $\mathbb{R}$. The pullback $\iota^*$ in

$$\iota^* : H^2(\mathbb{C}P^1 \times S^4) \cong \mathbb{Z} \longrightarrow H^2(Z, \mathbb{Z}) \cong H^2(Z_1, \mathbb{Z}) \oplus H^2(Z_2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^{r-1}$$

acts by $a \mapsto (a, 0, \ldots, 0)$. Hence $\iota^*(\pi_{\mathbb{C}P^1}^*(O(k))) = (k, 0, \ldots, 0) = c_1(L)$, which implies that $(\pi_{\mathbb{C}P^1} \circ \iota)^*(O(k)) \cong L$. The rest of (b) is immediate.

In case (c), when $w_1(Z_1) = 1 \mod 2$ as $k$ is odd so $Z_1$ is not spin, as in Crowley \cite{Crowley}*{§2} we take $\pi_{\mathbb{C}P^1} : Z_1 \to \mathbb{C}P^1$ to be the nontrivial $S^3$-bundle over $\mathbb{C}P^1$ (this is $X_{\infty}$ in Crowley’s notation), defined as in the proposition with $S^3, \text{SO}(4)$ in place of $S^4, \text{SO}(5)$. For $\pi_{\mathbb{C}P^1} : Y \to \mathbb{C}P^1$ as in the proposition, there is a natural embedding $\iota_1 : Z_1 \to Y$ with $\pi_{\mathbb{C}P^1} \circ \iota_1 = \pi_{\mathbb{C}P^1}$, which embeds the $S^3$ fibres of $\pi_{\mathbb{C}P^1} : Z_1 \to \mathbb{C}P^1$ as equators in the $S^4$ fibres of $\pi_{\mathbb{C}P^1} : Y \to \mathbb{C}P^1$. The rest of (c) follows (b), replacing $\pi_{\mathbb{C}P^1} : \mathbb{C}P^1 \times S^4 \to \mathbb{C}P^1$ by $\pi_{\mathbb{C}P^1} : Y \to \mathbb{C}P^1$.

For the last part, in (a) we have $X' = S^8$, which is compact, oriented and spin, with $H^{\text{odd}}(X', \mathbb{Z}) = 0$. In (b) we have a fibration $\mathbb{C}P^1 \hookrightarrow X' \to \mathbb{C}P^1 \times S^4$, so $X'$ is compact and oriented as $\mathbb{C}P^1, \mathbb{C}P^1 \times S^4$ are, and $H^{\text{odd}}(X', \mathbb{Z}) = 0$ by the Leray–Serre spectral sequence as $H^{\text{odd}}(\mathbb{C}P^1, \mathbb{Z}) = H^{\text{odd}}(\mathbb{C}P^1 \times S^4, \mathbb{Z}) = 0$. We can also show $w_2(X') = 0$ as $k$ is even, so $X'$ is spin.

In case (c) we have fibrations $\mathbb{C}P^1 \hookrightarrow X' \to Y$ and $S^4 \to Y \to \mathbb{C}P^1$, so $X'$ is compact and oriented with $H^{\text{odd}}(X', \mathbb{Z}) = 0$ as in (b). It is less obvious that $X'$ is spin, since $Y$ is not. The composition $X' \to Y \to \mathbb{C}P^1$ induces a pullback map $(0, 1) = H^2(\mathbb{C}P^1, \mathbb{Z}_2) \to H^2(X', \mathbb{Z}_2)$. We can compute $w_2(X')$, and we find that $Y$ being non-spin, and $k$ being odd, both contribute the image of $1 \in H^2(\mathbb{C}P^1, \mathbb{Z}_2)$ to $w_2(X') \in H^2(X', \mathbb{Z}_2)$, but the sum of these contributions is 0, so $w_2(X') = 0$ and $X'$ is spin. This completes the proof.

Now let us return to the situation of Step 6, with $Z \subset X$ a compact, connected, simply-connected, oriented, embedded 5-submanifold, and $\psi : Z \to \mathbb{C}P^2$ a smooth map, such that the normal bundle $\nu_Z$ of $Z$ in $X$ is $\nu_Z \cong \mathbb{R} \oplus \psi^*(\mathcal{O}(1))$, and $w_2(Z)$ is the image of $c_1(\psi^*(\mathcal{O}(1)))$ in $H^2(Z, \mathbb{Z}_2)$. Proposition \ref{2.2} with $L = \psi^*(\mathcal{O}(1))$ constructs a compact, oriented, spin 8-manifold $X'$ with $H^{\text{odd}}(X', \mathbb{Z}) = 0$ and an embedding $j : Z \to X'$ with normal bundle $\nu'_{X'} \cong \mathbb{R} \oplus \psi^*(\mathcal{O}(1))$, so the normal bundles of $Z$ in $X$ and $X'$ agree. Hence we can choose tubular neighbourhoods $U, U'$ of $Z$ in $X, X'$ and an orientation-preserving diffeomorphism $\iota : U \to U'$.
Choose Riemannian metrics $g, g'$ on $X, X'$ such that $i$ identifies $g|U$ with $g'|U'$, and let $E_0, E_0'$ be the positive Dirac operators of $g, g'$. Since Z and hence $U, U'$ are simply-connected, the spin structures on $U, U'$ are unique, and so $i$ is also spin-preserving, and thus identifies $E_0|U$ and $E_0'|U'$.

As in Step 5 we have $\Phi : X \to \text{SU}(4)$ with $\Phi^{-1}(Y_1) = Z$ and $\Phi^{-1}(Y_2) = \Phi^{-1}(Y_3) = \emptyset$, so that $X \setminus Z = \Phi^{-1}(Y_0)$. But $Y_0$ retracts to $\{\text{Id}\}$ in $\text{SU}(4)$ by Step 3(iii). Thus we may deform $\Phi$ in its isotopy class to make $\Phi \equiv \text{Id}$ except close to $Z$, so we can choose an open set $V$ in $X$ such that $X = U \cup V$, and $Z \cap V = \emptyset$, and deform $\Phi$ so that $\Phi|_V \equiv \text{Id}$.

Define $V' = i(U \cap V) \cup (X' \setminus U')$. Then $V' \subset X'$ is open with $X' = U' \cup V'$, and $i$ identifies $U \cap V$ with $U' \cap V'$. Define $\Phi' : X' \to \text{SU}(4)$ by $\Phi'|_{U'} = \Phi \circ i^{-1}$ and $\Phi'|_{V'} \equiv \text{Id}$. Then $\Phi'$ is smooth, and $i$ identifies $\Phi|_U$ and $\Phi'|_{U'}$.

Let $Q \to X \times S^1, q : Q|_{X \times \{1\}} \cong (X \times \{1\}) \times SU(4) = P$ and $Q' \to X' \times S^1, q' : Q'|_{X' \times \{1\}} \cong (X' \times \{1\}) \times SU(4) = P'$ correspond to $\Phi : X \to SU(4)$ and $\Phi' : X' \to SU(4)$ by the 1-1 correspondence between (b),(c) in Step 2. Then the diffeomorphism $i : U \to U'$ identifying $\Phi|_U$ and $\Phi'|_{U'}$ induces an isomorphism $i \times \text{id}_{S^1} : U \times S^1 \to U' \times S^1$, and an isomorphism $\sigma : Q|_{U \times S^1} \to (i \times \text{id}_{S^1})^*Q'|_{U \times S^1}$ of principal SU(4)-bundles over $U \times S^1$ compatible with $q|_{U \times \{1\}}, q'|_{U' \times \{1\}}$. Also $\Phi|_V \equiv \text{Id}$ and $\Phi'|_{V'} \equiv \text{Id}$ induce trivializations $\tau : Q|_{V \times S^1} \to V \times S^1 \times SU(4), \tau' : Q'|_{V' \times S^1} \to V' \times S^1 \times SU(4)$, compatible with $q|_{V \times \{1\}}, q'|_{V' \times \{1\}}$ and $\sigma|_{(U \times V) \times S^1}$.

Choose a partial connection $\nabla_Q^X$ on $Q \to X \times S^1$ in the $X$ directions, such that $\nabla_Q^X|_{X \times \{1\}}$ is identified with $\nabla^0$ under $q$, and $\nabla_Q^X|_{V \times S^1}$ is identified with $\nabla^0|_{V \times S^1}$ under $\tau$. Then there is a unique partial connection $\nabla_Q^X$ on $Q' \to X' \times S^1$ in the $X'$ directions, such that $\nabla_Q^X|_{X' \times \{1\}}$ is identified with $\nabla^0$ under $q'$, and $\nabla_Q^X|_{V' \times S^1}$ is identified with $\nabla^0|_{V' \times S^1}$ under $\tau'$, and $\nabla_Q^X|_{U \times S^1}$ is identified with $\nabla^0|_{U \times S^1}$ under $\sigma$.

The 1-1 correspondence between (a),(b) in Step 2 identifies $Q, q, \nabla_Q^X$ and $Q', q', \nabla_Q^{X'}$ with loops $\gamma : S^1 \to M_P$ and $\gamma' : S^1 \to M_{P'}$ based at $[\nabla^0]$, where $P = X \times SU(4)$ and $P' = X' \times SU(4)$ are the trivial SU(4)-bundles over $X, X'$.

For each $z \in S^1$ we have principal SU(4)-bundles $Q|_{X \times \{z\}} \to X, Q'|_{X' \times \{z\}} \to X'$ with connections $\nabla_Q^X|_{X \times \{z\}}, \nabla_Q^{X'}|_{X' \times \{z\}}$ representing points $\gamma(z) \in M_P$ and $\gamma'(z) \in M_{P'}$. Apply the Excision Theorem, Theorem 2.1, to these, with $X, X', E_0, E_0', SU(4), Q|_{X \times \{z\}}, Q'|_{X' \times \{z\}}, U, V, U', V', \sigma|_{U \times \{z\}}, \tau|_{V \times \{z\}}, \tau'|_{V' \times \{z\}}$ in place of $X^+, X^-, E_0^+, E_0^-, G, P^+, P^-, U^+, U^-, V^+, V^-, \sigma^+, \sigma^-, \tau^+, \tau^-$. This gives an isomorphism of $\text{Z}_2$-torsors $\Omega_\gamma : \hat{\Omega}_P^E|_{\gamma(z)} \cong \hat{\Omega}_{P'}^{E'}|_{\gamma'(z)}$.

Theorem 2.1(i) implies that $\Omega_\gamma$ varies continuously with $z \in S^1$. Hence the monodromy of $\hat{\Omega}_{P'}^{E'}$ around $\gamma$ in $M_P$, which is $\Theta([\gamma])$ in the notation of Step 2, equals the monodromy $\Theta([\gamma'])$ of $\hat{\Omega}_P^E$ around $\gamma'$ in $M_{P'}$. But $M_{P'}$ is orientable by $\text{2.1}(x)$ as $H^{\text{odd}}(X', Z) = 0$, so $\Theta([\gamma']) = 1$, and thus $\Theta([\gamma]) = \Theta([\Phi]) = 1$. Since this holds for all $[\Phi] \in [X, SU(4)]$ with $\kappa([\Phi]) = 0$, Step 4 implies that $M_P$ is orientable when $X$ is simply-connected, completing Step 6.
References


