Lagrangian Mean Curvature Flow in Calabi–Yau manifolds and the Thomas–Yau Conjecture. II

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Based on *'Conjectures on Bridgeland stability for Fukaya categories of Calabi–Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow'*, EMS Surv. Math. Sci. 2 (2015), 1-62. arXiv:1401.4949.
See also Thomas and Yau math.DG/0104196, math.DG/0104197.

These slides available at `http://people.maths.ox.ac.uk/~joyce/`.
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Plan of talk, continuing from yesterday:

4. (Lagrangian) Mean Curvature Flow

Let (M,g) be a Riemannian manifold, and L_0 a compact submanifold in M. Write $\nu_0 \to L_0$ for the normal bundle of L_0 in M. The *mean curvature* of L_0 is H_0, a smooth section of ν_0. It is the negative gradient at L_0 of the volume functional

$$\mathrm{Vol} : \{\text{compact submanifolds of } M\} \longrightarrow (0,\infty).$$

Mean Curvature Flow (*MCF*) is the o.d.e. $\frac{\mathrm{d}}{\mathrm{d}t}L_t = H_t$ for a smooth 1-parameter family of submanifolds L_t, $t \in [0,T)$ starting at L_0. It is the negative gradient flow of Vol, so following MCF is a way to try to minimize the volume of L_t, and get a minimal submanifold. For any compact L_0 the MCF L_t, $t \in [0,T)$ exists for some maximal time $T > 0$. If $T < \infty$ then $L_t \to L_T$ as $t \to T_-$ where L_T is a *singular* submanifold of (M,g).

Lagrangian Mean Curvature Flow

Let (M,J,g,Ω) be a Calabi–Yau manifold, and $L_0 \subset M$ be a compact oriented Lagrangian, with mean curvature H_0, and phase function $\Phi : L_0 \to \mathrm{U}(1)$. As L_0 is Lagrangian there is a canonical isomorphism $\nu_0 \cong T^*L_0$, where ν_0 is the normal bundle of L_0 in M. Under this isomorphism we have

$$\Gamma^\infty(\nu_0) \ni H_0 \cong -i\Phi^{-1}\mathrm{d}\Phi \in \Gamma^\infty(T^*L_0).$$

Here $-i\Phi^{-1}\mathrm{d}\Phi$ is a closed 1-form on L_0 with $[-i\Phi^{-1}\mathrm{d}\Phi] = 2\pi\mu_{L_0}$ in $H^1_{\mathrm{dR}}(L_0;\mathbb{R})$, for μ_{L_0} the Maslov class. If L_0 is graded, so we have $\phi : L_0 \to \mathbb{R}$ with $\Phi = e^{i\pi\phi}$, then $-i\Phi^{-1}\mathrm{d}\Phi = \pi\mathrm{d}\phi$ is exact. Now closed 1-forms in $\Gamma^\infty(T^*L_0)$ are identified with infinitesimal deformations in $\Gamma^\infty(\nu_0)$ of L_0 as a Lagrangian, and exact 1-forms with infinitesimal deformations of L_0 in its Hamiltonian isotopy class. Smoczyk proved that for the MCF L_t, $t \in [0,T)$ of L_0, the L_t are Lagrangians, and if L_0 is Maslov zero/graded then the L_t remain in the Hamiltonian isotopy class of L_0.

Lagrangian Mean Curvature Flow

Thus LMCF of graded Lagrangians also remains in a fixed isomorphism class in $D^bF(M,\omega)$.
A Lagrangian L_0 is a fixed point of LMCF if $\mathrm{d}\Phi = 0$, that is, if Φ is constant, $\Phi \equiv e^{i\pi\phi_0}$, so that L_0 is special Lagrangian of phase $e^{i\pi\phi_0}$. Thus, for the functional

$$\mathrm{Vol}_{\mathrm{Lag}} : \{\text{compact Lagrangians in } M\} \longrightarrow (0,\infty),$$

the only critical points are special Lagrangians, which are also the absolute minima of $\mathrm{Vol}_{\mathrm{Lag}}$ – there are no higher critical points. So naïvely we might hope (with Thomas and Yau) that LMCF, the negative gradient flow of $\mathrm{Vol}_{\mathrm{Lag}}$, exists for all time, and retracts

$$\{\text{compact Lagrangians in } M\} \longrightarrow \{\text{special Lagrangians in } M\}.$$

This is false – there are many finite time singularities of LMCF – but I will suggest that a more sophisticated version may be true.

Finite time singularities of Lagrangian MCF

Finite time singularities of MCF L_t, $t \in [0,T)$ as $t \to T_-$ are divided into Type I (quickly forming) and Type II (slowly forming). In a Type I singularity, part of the submanifold L_t shrinks homothetically to a point x in M with rate $(T-t)^{1/2}$, and the flow near x is modelled on an *MCF shrinker* in $\mathbb{R}^n \cong T_xM$. Type II singularities are more difficult to describe, and less well understood.
An oriented Lagrangian L_0 in a Calabi–Yau m-fold (M,J,g,Ω) is called *almost calibrated* if the phase function $\Phi : L_0 \to \mathrm{U}(1)$ has $\mathrm{Re}(e^{-i\pi\phi_0}\Phi) > 0$ for some $\phi_0 \in \mathbb{R}$, that is, the phase variation of L_0 is less than π. Wang 2001 proved that LMCF starting from an almost calibrated Lagrangian L_0 remains almost calibrated, and does not develop a Type I singularity. Neves 2006 proved that LMCF starting from a graded Lagrangian L_0 does not develop a Type I singularity. Basically this is because there are no graded LMCF shrinkers in $\mathbb{C}^m$.

An important result for any Thomas–Yau type programme is:

Theorem (Neves 2010)

Let (M, J, g, Ω) be a Calabi–Yau m-fold, and L_0 a compact Lagrangian in (M, ω). Then there exists a Hamiltonian perturbation $\tilde{L}_0$ of L_0 such that the Lagrangian MCF $\tilde{L}_t$, $t \in [0, T)$ starting from $\tilde{L}_0$ develops a finite time singularity at $t = T$.

In particular, no notion of 'stability' of Lagrangians L_0 which depends only on the Hamiltonian isotopy class can ensure that LMCF L_t, $t \in [0, \infty)$ exists for all time. So any revision of the Thomas–Yau Conjecture must cope with finite time singularities of LMCF, presumably by continuing the flow after a surgery.

5. The Thomas–Yau Conjecture, version 2.0

I'll now explain a series of conjectures aiming to drag the Thomas–Yau Conjecture into the 20^{th} century. Please take these in the spirit they are intended – as provisional, probably wrong in detail.

Conjecture 1 (Folklore.)

Let (M, J, g, Ω) be a Calabi–Yau m-fold. Then there exists a Bridgeland stability condition $\mathcal{S}_\Omega = (Z, \mathcal{P})$ on $D^bF(M, \omega)$, with central charge Z given by the composition

$$K^{\mathrm{num}}(D^bF(M,\omega)) \xrightarrow{(L,b)\mapsto[L]} H^m(M;\mathbb{Z})/\mathit{torsion} \xrightarrow{\cdot[\Omega]} \mathbb{C},$$

and if (L, b) is an object in $D^bF(M, \omega)$ with L special Lagrangian with grading $\phi \equiv \phi_0$ then $(L, b) \in \mathcal{P}(\phi_0)$.

Enlarging $D^bF(M,\omega)$ by immersed or singular Lagrangians

Conjecture 1 says that all special Lagrangians in $D^bF(M,\omega)$ are $\mathcal{S}_\Omega$-semistable. We would like to claim the converse, that every $\mathcal{S}_\Omega$-semistable object in $D^bF(M,\omega)$ is isomorphic to a special Lagrangian object in $D^bF(M,\omega)$. But there is a problem: $D^bF(M,\omega)$ may not have enough objects. If the special Lagrangian exists, it must be unique in its isomorphism class in $D^bF(M,\omega)$, by an argument of Thomas. If this unique special Lagrangian is immersed, but $D^bF(M,\omega)$ is defined using only embedded Lagrangians, then there will be no special Lagrangian representative in $D^bF(M,\omega)$. If this unique special Lagrangian has singularities (say mild ones, for which Lagrangian Floer theory still works?) but $D^bF(M,\omega)$ is defined using only nonsingular Lagrangians, again no special Lagrangian representative exists.
Our solution is to enlarge $D^bF(M,\omega)$, adding immersed and singular Lagrangians, probably not changing $D^bF(M,\omega)$ up to equivalence.

Enlarging $D^bF(M,\omega)$ by immersed or singular Lagrangians

Conjecture 2 (Dubious, needs work.)

Suppose we enlarge the definition of $D^bF(M,\omega)$ to $D^b\tilde{F}(M,\omega)$ so that it contains 'as many Lagrangians L as possible for which HF^* can be defined', including in particular immersed Lagrangians, and some classes of singular Lagrangians.
Then for the stability condition $\mathcal{S}_\Omega$ in Conjecture 1, any object in $\mathcal{P}(\phi_0)$ is isomorphic in $D^b\tilde{F}(M,\omega)$ to some (L,b), where L is special Lagrangian with grading $\phi\equiv\phi_0$, and is unique.
A weaker but more credible claim: for any small $\epsilon>0$, any object in $\mathcal{P}(\phi_0)$ is isomorphic to some (L,b), where L has grading $\phi: L\to(\phi_0-\epsilon,\phi_0+\epsilon)$, i.e. L is close to special Lagrangian.

The kinds of singularity needed get worse with increasing dimension — for $\dim_{\mathbb{C}} M=2$, immersed Lagrangians may be enough.

HF^* for immersed Lagrangians

Akaho and Joyce J.D.G. 86 (2010) extend Fukaya–Oh–Ohta–Ono's HF^* theory from embedded to immersed Lagrangians. In the Akaho–Joyce theory, obstructions to HF^* of L come from both ordinary J-holomorphic discs D with $\partial D \subset L$, and from 'teardrops':

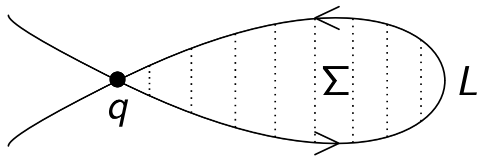

Figure 3: J-holomorphic 'teardrop' making immersed HF^* obstructed.

These 'teardrops' will be important in the study of LMCF.
A nice thing about the Akaho–Joyce immersed version of $D^bF_{\rm im}(M,\omega)$ is that we no longer need twisted complexes: every object of $D^bF_{\rm im}(M,\omega)$ is a single Lagrangian (L,b), with L immersed (e.g. L can be a finite union of embedded Lagrangians intersecting each other, from a twisted complex in $D^bF(M,\omega)$).

Thomas–Yau as an attempt to prove Conjectures 1 and 2

We can now understand the Thomas–Yau Conjecture as a strategy for proving Conjectures 1 and 2. Starting with any object (L_0,b_0) in $D^b\tilde F(M,\omega)$, we should follow LMCF L_t, $t\in[0,\infty)$ uniquely for all time (if this is possible), and extend b_0 by 'parallel translation' to bounding cochains $b_t : t\in[0,\infty)$ with $(L_t,b_t)\cong(L_0,b_0)$ in $D^b\tilde F(M,\omega)$. Then we consider the limit $L_\infty=\lim_{t\to\infty}L_t$. If all goes very well we should have $L_\infty=L^{\phi_1}\cup\cdots\cup L^{\phi_n}$, where $\phi_1<\phi_2<\cdots<\phi_n$ in $\mathbb{R}$ and L^{ϕ_i} is special Lagrangian with grading $\phi\equiv\phi_i$. This should correspond to a Harder–Narasimhan type filtration in $D^b\tilde F(M,\omega)$ of (L_∞,b_∞) into $(L^{\phi_1},b^{\phi_1}),\ldots,$ (L^{ϕ_n},b^{ϕ_n}), where $(L^{\phi_i},b^{\phi_i})\in\mathcal{P}(\phi_i)$, and this is the decomposition of $(L_0,b_0)\cong(L_\infty,b_\infty)$ in the Bridgeland stability condition $\mathcal{S}_\Omega$.

Lagrangian MCF with surgeries

As I said yesterday, finite time singularities in LMCF are unavoidable. So we should not expect the LMCF L_t, $t \in [0, \infty)$ to exist for all time without singularities. However, as in Perelman's proof of the Poincaré conjecture, it may be more realistic to hope that LMCF L_t, $t \in [0, \infty)$ exists with a series of *singular times* $0 < T_1 < T_2 < T_3 < \cdots$, such that L_t for $t \in (T_{i-1}, T_i)$ is nonsingular (or possibly has nice singularities) and satisfies LMCF, and L_{T_i} is singular with $\lim_{t \to T_i -} L_t = \lim_{t \to T_i +} L_t = L_{T_i}$. We do not require L_{T_i} to exist in $D^b\tilde{F}(M, \omega)$, but we require (L_t, b_t) to be isomorphic in $D^b\tilde{F}(M, \omega)$ for $t \in (T_{i-1}, T_i)$ and $t \in (T_i, T_{i+1})$, so the isomorphism class in $D^b\tilde{F}(M, \omega)$ does not change as we pass through the singular time. As we pass through $t = T_i$, the topology of L_t may change by some kind of surgery, e.g. a 'neck pinch'. We call this 'LMCF with surgeries'.

We state this as:

Conjecture 3 (Dubious, needs work.)

Let (L_0, b_0) be an object in $D^b\tilde{F}(M, \omega)$. Then there exists a unique family L_t, $t \in [0, \infty)$ satisfying LMCF with surgeries at singular times $0 < T_1 < T_2 < T_3 < \cdots$, and bounding cochains b_t for L_t for $t \in [0, \infty) \setminus \{T_1, T_2, \ldots\}$ unique up to equivalence such that $(L_t, b_t) \cong (L_0, b_0)$ in $D^b\tilde{F}(M, \omega)$.
Taking the limit of (L_t, b_t) as $t \to \infty$ enables us to construct the Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^b\tilde{F}(M, \omega)$.

It is essential that $D^b\tilde{F}(M, \omega)$ should contain immersed Lagrangians, and some kinds of singular Lagrangians (how bad the singularities are depends on dimension), for Conjecture 3 to hold, as otherwise the flow could cross from embedded to immersed, or from immersed to singular, at $t = T_i$, and then (L_t, b_t) would not exist in $D^b\tilde{F}(M, \omega)$ for $t > T_i$.

6. Possible surgeries during the flow

In my paper I describe (without proof) some of the surgeries I think are possible in LMCF at singular times T_i, in a feeble attempt to make Conjecture 3 sound more credible.
I will explain three of these:

(a) 'Neck pinch' by shrinking a Lawlor neck, giving an immersed Lagrangian for $t > T_i$.

(b) 'Opening a neck' by gluing in a Joyce–Lee–Tsui expander at an immersed point – roughly, the inverse to (a).

(c) 'Collapsing a zero object', when a connected component L' of L shrinks to a point, but $(L', b') \cong 0$ in $D^b\tilde{F}(M, \omega)$, so the isomorphism class of (L, b) is not changed by deleting (L', b').

(a) 'Neck pinch' by shrinking a Lawlor neck

Let Π_0, Π_1 be special Lagrangian planes in $\mathbb{C}^m$ of the same phase, intersecting transversely at 0, and satisfying an angle condition.
Lawlor 1989 defined an explicit SL m-fold N in $\mathbb{C}^m$ diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$ and asymptotic to $\Pi_0 \cup \Pi_1$ at infinity – a 'Lawlor neck'. As a manifold it is the connect sum of Π_0 and Π_1 at 0.
I claim that a possible Type II finite time singularity of LMCF L_t, $t \in [0, T)$ in a Calabi–Yau m-fold (M, J, g, Ω) is when, near some $x \in M$, L_t in M looks like $c_t \cdot N$ in $T_xM \cong \mathbb{C}^m$ for some $c_t \in (0, \infty)$ with $c_t \to 0$ as $t \to T_-$. Since $\lim_{c \to 0} c \cdot N = \Pi_0 \cup \Pi_1$, the limit $L_T = \lim_{t \to T_-}$ is actually a *nonsingular*, *immersed* Lagrangian, topologically different to L_t for $t \in [0, T)$.
I claim this is a *generic* singularity, in that if LMCF starting from L_0 has such a neck pinch, then so does LMCF starting from $\tilde{L}_0$ for any sufficiently small Hamiltonian perturbation $\tilde{L}_0$ of L_0.

Work in progress with Yng-Ing Lee shows that such neck pinches happen in examples of $\mathrm{SO}(m)$-equivariant Lagrangian MCF in $\mathbb{C}^m$. Since L_T is a compact, nonsingular, immersed Lagrangian, we can continue the flow L_t, $t \in [T, T')$ by LMCF in immersed Lagrangians. This neck pinch process can cut one connected component of L_t for $t < T$ into two components for $t > T$. This is important for the Bridgeland stability condition picture. As in Conjecture 3, we hope to construct LMCF with surgeries L_t, $t \in [0, \infty)$ such that $\lim_{t\to\infty} L_t = L_\infty = L^{\phi_1} \cup \cdots \cup L^{\phi_n}$ is a union of special Lagrangian components of different phases. Thus, if L_0 is connected, but (L_0, b_0) is not semistable, then the flow L_t, $t \in [0, \infty)$ has to cut L_0 into $n > 1$ components for $t \gg 0$. I believe this 'neck pinch' mechanism is how this happens.

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(b) 'Opening a neck'

LMCF of *immersed* Lagrangians $L_t : t \in [0, T)$ only changes L_t by Hamiltonian isotopy in a weak, local sense: the flow can slide two sheets of L_t over one another, introduce extra self-intersection points, etc. In the Akaho–Joyce immersed HF^* theory, this kind of weak Hamiltonian isotopy can move you from Lagrangians with HF^* unobstructed to Lagrangians with HF^* obstructed.

The typical problem is if we have J-holomorphic curves C_t, D_t like this:

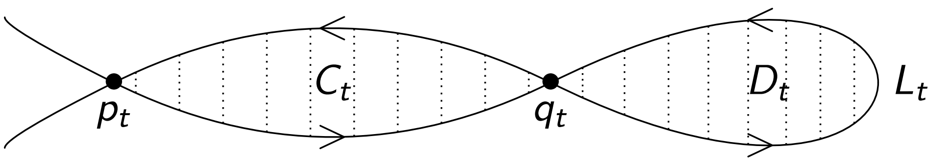


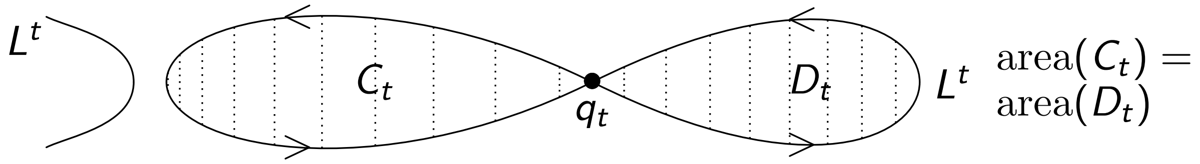
Figure 4: Wall-crossing for immersed HF^* unobstructed/obstructed.

then HF^* is unobstructed when $\mathrm{area}(C_t) < \mathrm{area}(D_t)$ and obstructed when $\mathrm{area}(C_t) > \mathrm{area}(D_t)$. But flowing to obstructed Lagrangians L_t is bad, as no bounding cochain b_t exists.

Joyce–Lee–Tsui J.D.G. 84 (2010) find explicit LMCF expanders N in $\mathbb{C}^m$ asymptotic to a union of Lagrangian planes $\Pi_0 \cup \Pi_1$, very like Lawlor necks. At the time T when $\mathrm{area}(C_T) = \mathrm{area}(D_T)$, we do a surgery, gluing in a JLT expander N at p_T asymptotic to $T^+_{p_T} L_T \cup T^-_{p_T} L_T$ in $T_{p_T} M \cong \mathbb{C}^m$.

A calculation in my paper shows that the angle conditions for existence of the JLT expander hold iff $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{area}(C_t) - \mathrm{area}(D_t)) > 0$, that is, iff we are crossing from HF^* unobstructed to obstructed.

For $t > T$ the J-holomorphic curves look like this:



As $\mathrm{area}(C_t) = \mathrm{area}(D_t)$, the contributions of C_t, D_t to obstructing HF^* of L_t cancel, and HF^* is unobstructed.

Begley–Moore arXiv:1501.07823 prove my conjecture that LMCF $L_t : t \in [T, T+\epsilon)$ gluing in the JLT expander at p_T exists.

(c) 'Collapsing a zero object'

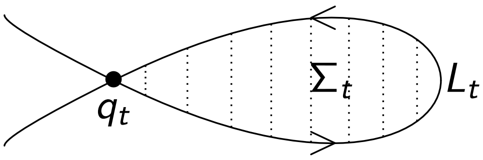
Let L_0 be a compact Lagrangian in $\mathbb{C}^m$. If L_0 is contained in a ball of radius R, then LMCF $L_t : t \in [0, T)$ starting from L_0 must shrink to a point in $\mathbb{C}^m$ within time $T = R^{1/2}$, unless it becomes singular first. Similarly, any Lagrangian L_0 contained in a small ball in (M, J, g, Ω) must shrink to a point under LMCF in bounded time, unless it becomes singular first.

Now if (L_0, b_0) lies in $D^b\tilde{F}(M, \omega)$ with L_0 in a small ball in M, then L_0 is displaceable, so that $(L_0, b_0) \cong 0$ is a zero object in $D^b\tilde{F}(M, \omega)$.

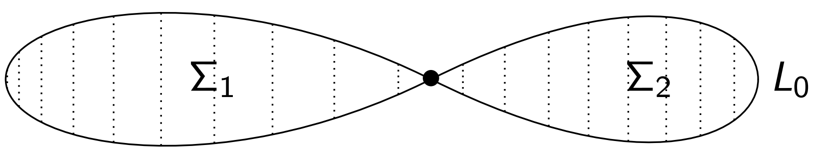
Suppose we have LMCF L_t, $t \in [0, T)$ with $L_t = L'_t \amalg L''_t$, with bounding cochains $b_t = b'_t \amalg b''_t$, where L'_t is contained in a small ball in M and shrinks to a point in M at $t = T$. Then $(L'_t, b'_t) \cong 0$, so that $(L_t, b_t) \cong (L''_t, b''_t)$ in $D^b\tilde{F}(M, \omega)$. At $t = T$ we delete (L'_t, b'_t), and continue the flow for $t > T$ by flowing L''_t. This gives an LMCF surgery which does not change the isomorphism class in $D^b\tilde{F}(M, \omega)$. Neves' 2010 examples can be explained using (a),(c).

7. What goes wrong in LMCF if HF^* is obstructed

Conjecture 3 claims long-time existence L_t, $t \in [0, \infty)$ of LMCF with surgeries starting with a Lagrangian L_0 *with HF^* unobstructed*, i.e. with an object (L_0, b_0) in $D^b\tilde{F}(M, \omega)$.
In contrast, I expect that for Lagrangians L_0 with HF^* obstructed, there may be finite time singularities at $t = T$ in LMCF such that *we cannot continue the flow for $t > T$*, even after a surgery.
In the Akaho–Joyce immersed HF^* theory, obstructions to HF^* of L_t can be caused by J-holomorphic 'teardrops' Σ_t with small area:

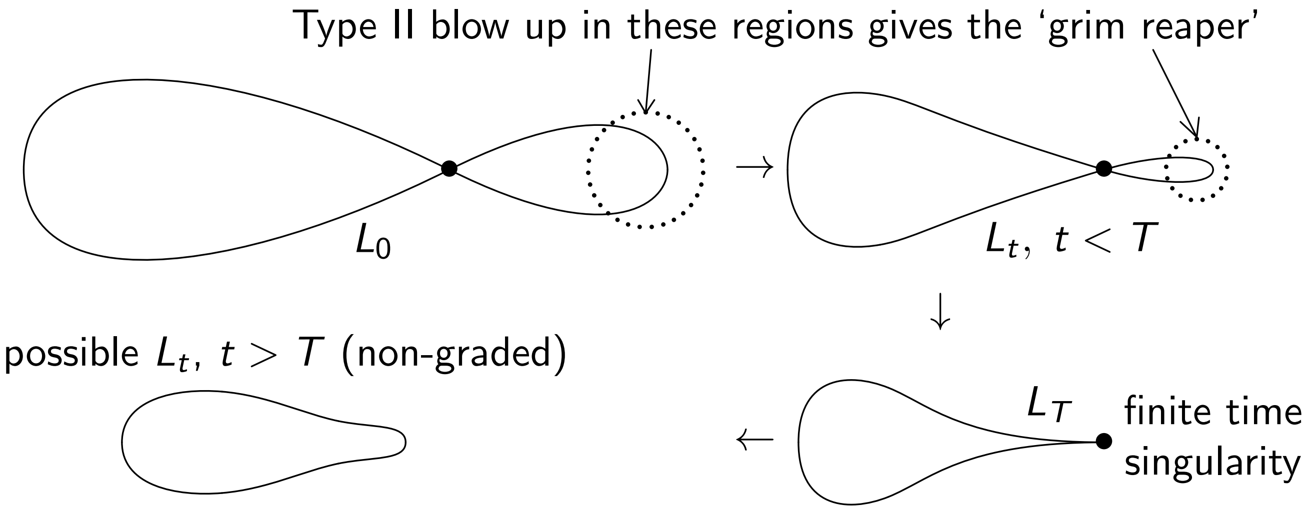


I expect that there can be Type II singularities of immersed LMCF L_t, $t \in [0, T)$ with a teardrop curve Σ_t in which $\mathrm{area}(\Sigma_t) \to 0$ as $t \to T_-$, and L_T has a singular 'cusp', after which one cannot continue the flow. For 1-dimensional LMCF in $\mathbb{C}$ we can prove this using known theorems: start with L_0 an '∞ sign' immersed graded Lagrangian in $\mathbb{C}$, with $\mathrm{area}(\Sigma_1) > \mathrm{area}(\Sigma_2)$



'∞ sign' Lagrangian L_0 in $\mathbb{C}$

Then LMCF L_t, $t \in [0, T)$ looks like this:



One can continue the flow for $t > T$, but only in *non-graded* embedded Lagrangians, which we exclude. There is no way to continue LMCF in graded Lagrangians for $t > T$.

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I also expect similar behaviour in higher dimensions. In my paper I sketch how singularities might form in m dimensions with Type II blow-up a Joyce–Lee–Tsui LMCF translator in $\mathbb{C}^m$, through shrinking J-holomorphic teardrops.

Thank you for listening!