Plan of talk:

1. Introduction

2. D-manifolds without boundary

3. Differential geometry of d-manifolds
1. Introduction

I will tell you about new classes of geometric objects I call \textit{d-manifolds} and \textit{d-orbifolds} — ‘derived’ smooth manifolds, in the sense of Derived Algebraic Geometry. Some properties:

- D-manifolds form a \textit{strict 2-category} \textit{dMan}. That is, we have objects \(X\), the d-manifolds, 1-morphisms \(f, g : X \rightarrow Y\), the smooth maps, and also 2-morphisms \(\eta : f \Rightarrow g\).
- Smooth manifolds \textit{Man} embed into d-manifolds as a full (2)-subcategory. So, d-manifolds generalize manifolds.
- There are also 2-categories \textit{dMan}^b, \textit{dMan}^c of d-manifolds \textit{with boundary} and \textit{with corners}, and orbifold versions \textit{dOrb}, \textit{dOrb}^b, \textit{dOrb}^c of these, \textit{d-orbifolds}.
- Much of differential geometry extends nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, \ldots.

Almost any moduli space used in any enumerative invariant problem over \(\mathbb{R}\) or \(\mathbb{C}\) has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used — Kuranishi spaces, polyfolds, \textit{C}-schemes or Deligne–Mumford \textit{C}-stacks with obstruction theories. Combining these truncation functors with known results gives d-manifold/d-orbifold structures on many moduli spaces.

Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting for their own sake.
D-manifolds are based on ideas from derived algebraic geometry. First consider an algebro-geometric version of what we want to do. A good algebraic analogue of smooth manifolds are complex algebraic manifolds, that is, separated smooth $\mathbb{C}$-schemes $S$ of pure dimension. These form a full subcategory $\text{AlgMan}_\mathbb{C}$ in the category $\text{Sch}_\mathbb{C}$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects $S$ in $\text{Sch}_\mathbb{C}$ whose cotangent complex $L_S$ is a vector bundle (i.e. perfect in the interval $[0,0]$).

To make a derived version of this, we first define an $\infty$-category $\text{DerSch}_\mathbb{C}$ of derived $\mathbb{C}$-schemes, and then define the $\infty$-category $\text{DerAlgMan}_\mathbb{C}$ of derived complex algebraic manifolds to be the full $\infty$-subcategory of objects $S$ in $\text{DerSch}_\mathbb{C}$ which are quasi-smooth (have cotangent complex $L_S$ perfect in the interval $[-1,0]$), and satisfy some other niceness conditions (separated, etc.).
2. D-manifolds without boundary

I will concentrate today on d-manifolds without boundary. We begin by discussing $C^\infty$-algebraic geometry, $C^\infty$-rings, and $C^\infty$-schemes. Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes. In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces. There is a little-known theory of schemes in differential geometry, $C^\infty$-schemes, going back to Lawvere, Dubuc, Moerdijk and Reyes, . . . in synthetic differential geometry in the 1960s-1980s. $C^\infty$-schemes are essentially algebraic objects, on which smooth real functions and calculus make sense.

2.1. $C^\infty$-rings

Let $X$ be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-algebra: we can add smooth functions $(c, d) \mapsto c + d$, and multiply them $(c, d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.

But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f(c_1(x), \ldots, c_n(x))$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x, y) \mapsto x + y$, multiplication from $(x, y) \mapsto xy$, etc. This huge collection of algebraic operations $\Phi_f$ make $C^\infty(X)$ into an algebraic object called a $C^\infty$-ring.
Definition

A \( \mathcal{C}^{\infty} \)-ring is a set \( \mathcal{C} \) together with \( n \)-fold operations \( \Phi_f : \mathcal{C}^n \to \mathcal{C} \) for all smooth maps \( f : \mathbb{R}^n \to \mathbb{R} \), \( n \geq 0 \), satisfying:

Let \( m, n \geq 0 \), and \( f_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, \ldots, m \) and \( g : \mathbb{R}^m \to \mathbb{R} \) be smooth functions. Define \( h : \mathbb{R}^n \to \mathbb{R} \) by

\[
h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)),
\]

for \((x_1, \ldots, x_n) \in \mathbb{R}^n \). Then for all \( c_1, \ldots, c_n \) in \( \mathcal{C} \) we have

\[
\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).
\]

Also defining \( \pi_j : (x_1, \ldots, x_n) \mapsto x_j \) for \( j = 1, \ldots, n \) we have

\[
\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j.
\]

A morphism of \( \mathcal{C}^{\infty} \)-rings is \( \phi : \mathcal{C} \to \mathcal{D} \) with

\[
\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathcal{C}^n \to \mathcal{D}
\]

for all smooth \( f : \mathbb{R}^n \to \mathbb{R} \). Write \( \mathcal{C}^{\infty}\text{Rings} \) for the category of \( \mathcal{C}^{\infty} \)-rings.

Examples of \( \mathcal{C}^{\infty} \)-rings

Then \( \mathcal{C}^{\infty}(X) \) is a \( \mathcal{C}^{\infty} \)-ring for any manifold \( X \), and from \( \mathcal{C}^{\infty}(X) \) we can recover \( X \) up to canonical isomorphism.

If \( f : X \to Y \) is smooth then \( f^* : \mathcal{C}^{\infty}(Y) \to \mathcal{C}^{\infty}(X) \) is a morphism of \( \mathcal{C}^{\infty} \)-rings; conversely, if \( \phi : \mathcal{C}^{\infty}(Y) \to \mathcal{C}^{\infty}(X) \) is a morphism of \( \mathcal{C}^{\infty} \)-rings then \( \phi = f^* \) for some unique smooth \( f : X \to Y \). This gives a full and faithful functor \( F : \text{Man} \to \mathcal{C}^{\infty}\text{Rings}^{\text{op}} \) by

\[
F : X \mapsto \mathcal{C}^{\infty}(X), \hspace{0.5cm} F : f \mapsto f^*.
\]

Thus, we can think of manifolds as examples of \( \mathcal{C}^{\infty} \)-rings, and \( \mathcal{C}^{\infty} \)-rings as generalizations of manifolds. But there are many more \( \mathcal{C}^{\infty} \)-rings than manifolds. For example, \( \mathcal{C}^0(X) \) is a \( \mathcal{C}^{\infty} \)-ring for any topological space \( X \).

Any \( \mathcal{C}^{\infty} \)-ring \( \mathcal{C} \) has a cotangent module \( \Omega_{\mathcal{C}} \). If \( \mathcal{C} = \mathcal{C}^{\infty}(X) \) for \( X \) a manifold, then \( \Omega_{\mathcal{C}} = \mathcal{C}^{\infty}(T^*X) \).
2.2. \( C^\infty \)-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by \( C^\infty \)-rings throughout — see my arXiv:1104.4951, arXiv:1001.0023.

A \( C^\infty \)-ringed space \( \underline{X} = (X, \mathcal{O}_X) \) is a topological space \( X \) with a sheaf of \( C^\infty \)-rings \( \mathcal{O}_X \). Write \( \mathbf{C}^\infty \mathbf{R} \mathbf{S} \) for the category of \( C^\infty \)-ringed spaces.

The global sections functor \( \Gamma : \mathbf{C}^\infty \mathbf{R} \mathbf{S} \to \mathbf{C}^\infty \mathbf{R} \mathbf{ings}^{op} \) maps \( \Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X) \). It has a right adjoint, the spectrum functor \( \text{Spec} : \mathbf{C}^\infty \mathbf{R} \mathbf{ings}^{op} \to \mathbf{C}^\infty \mathbf{R} \mathbf{S} \). That is, for each \( C^\infty \)-ring \( \mathcal{C} \) we construct a \( C^\infty \)-ringed space \( \text{Spec} \mathcal{C} \). Points \( x \in \text{Spec} \mathcal{C} \) are \( \mathbb{R} \)-algebra morphisms \( x : \mathcal{C} \to \mathbb{R} \) (this implies \( x \) is a \( C^\infty \)-ring morphism). We don't use prime ideals.

On the subcategory of \( \text{fair} \) \( C^\infty \)-rings, \( \text{Spec} \) is full and faithful.

A \( C^\infty \)-ringed space \( \underline{X} \) is called an \emph{affine} \( C^\infty \)-scheme if \( \underline{X} \cong \text{Spec} \mathcal{C} \) for some \( C^\infty \)-ring \( \mathcal{C} \). We call \( \underline{X} \) a \( C^\infty \)-scheme if \( \underline{X} \) can be covered by open subsets \( U \) with \( (U, \mathcal{O}_X|_U) \) an affine \( C^\infty \)-scheme. Write \( \mathbf{C}^\infty \mathbf{S} \mathbf{ch} \) for the full subcategory of \( C^\infty \)-schemes in \( \mathbf{C}^\infty \mathbf{R} \mathbf{S} \).

If \( \underline{X} \) is a manifold, define a \( C^\infty \)-scheme \( \underline{X} = (X, \mathcal{O}_X) \) by \( \mathcal{O}_X(U) = C^\infty(U) \) for all open \( U \subseteq X \). Then \( \underline{X} \cong \text{Spec} C^\infty(X) \).

This defines a full and faithful embedding \( \text{Man} \hookrightarrow \mathbf{C}^\infty \mathbf{S} \mathbf{ch} \). So we can regard manifolds as examples of \( C^\infty \)-schemes.

All fibre products exist in \( \mathbf{C}^\infty \mathbf{S} \mathbf{ch} \). In manifolds \( \text{Man} \), fibre products \( X \times_{g,Z,h} Y \) need exist only if \( g : X \to Z \) and \( h : Y \to Z \) are transverse. When \( g, h \) are not transverse, the fibre product \( X \times_{g,Z,h} Y \) exists in \( \mathbf{C}^\infty \mathbf{S} \mathbf{ch} \), but may not be a manifold.

We also define \emph{vector bundles} and \emph{quasicoherent sheaves} on a \( C^\infty \)-scheme \( \underline{X} \), and write \( \text{qcoh}(X) \) for the abelian category of quasicoherent sheaves. A \( C^\infty \)-scheme \( \underline{X} \) has a well-behaved cotangent sheaf \( T^*\underline{X} \).
The topology on $\mathcal{C}^\infty$-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.

Can find smooth functions supported on (almost) any open set.

(Almost) any open cover has a subordinate partition of unity.

Our $\mathcal{C}^\infty$-rings $\mathcal{C}$ are generally not noetherian as $\mathbb{R}$-algebras. So ideals $I$ in $\mathcal{C}$ may not be finitely generated, even in $\mathcal{C}^\infty(\mathbb{R}^n)$.

We can define derived $\mathbb{C}$-schemes by replacing $\mathbb{C}$-algebras $A$ by $dg$ $\mathbb{C}$-algebras $A^\bullet$ in the definition of $\mathbb{C}$-scheme — commutative differential graded $\mathbb{C}$-algebras in degrees $\leq 0$, of the form

\[ \cdots \rightarrow A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^0, \] where $A^0$ is an ordinary $\mathbb{C}$-algebra.

The corresponding ‘classical’ $\mathbb{C}$-algebra is $H^0(A^\bullet) = A^0/d[A^{-1}]$.

There is a parallel notion of $dg$ $\mathcal{C}^\infty$-ring $\mathcal{C}^\bullet$, of the form

\[ \cdots \rightarrow \mathcal{C}^{-2} \xrightarrow{d} \mathcal{C}^{-1} \xrightarrow{d} \mathcal{C}^0, \] where $\mathcal{C}^0$ is an ordinary $\mathcal{C}^\infty$-ring, and $\mathcal{C}^{-1}, \mathcal{C}^{-2}, \ldots$ are modules over $\mathcal{C}^0$. The corresponding ‘classical’ $\mathcal{C}^\infty$-ring is $H^0(\mathcal{C}^\bullet) = \mathcal{C}^0/d[\mathcal{C}^{-1}]$.

One could use $dg$ $\mathcal{C}^\infty$-rings to define ‘derived $\mathcal{C}^\infty$-schemes’; an alternative is to use simplicial $\mathcal{C}^\infty$-rings, see Spivak arXiv:0810.5175, Borisov–Noel arXiv:1112.0033, and Borisov arXiv:1212.1153.
My d-spaces are a 2-category truncation of derived $\mathcal{C}^\infty$-schemes. To define them, I use a special class of dg $\mathcal{C}^\infty$-rings called square zero dg $\mathcal{C}^\infty$-rings, which form a 2-category $\text{SZC}^\infty\text{Rings}$.

A dg $\mathcal{C}^\infty$-ring $\mathcal{C}^\bullet$ is square zero if $\mathcal{C}^i = 0$ for $i < -1$ and $\mathcal{C}^{-1} \cdot d[\mathcal{C}^{-1}] = 0$. Then $\mathcal{C}$ is $\mathcal{C}^{-1} \xrightarrow{d} \mathcal{C}^0$, and $d[\mathcal{C}^{-1}]$ is a square zero ideal in the (ordinary) $\mathcal{C}^\infty$-ring $\mathcal{C}^0$, and $\mathcal{C}^{-1}$ is a module over the ‘classical’ $\mathcal{C}^\infty$-ring $H^0(\mathcal{C}^\bullet) = \mathcal{C}^0 / d[\mathcal{C}^{-1}]$.

A 1-morphism $\alpha^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ in $\text{SZC}^\infty\text{Rings}$ is maps $\alpha^0 : \mathcal{C}^0 \to \mathcal{D}^0$, $\alpha^{-1} : \mathcal{C}^{-1} \to \mathcal{D}^{-1}$ preserving all the structure. Then $H^0(\alpha^\bullet) : H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is a morphism of $\mathcal{C}^\infty$-rings.

For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ a 2-morphism $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathcal{C}^0 \to \mathcal{D}^{-1}$ with $\beta^0 = \alpha^0 + d \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ d$.

There is an embedding of (2-)categories $\mathcal{C}^\infty\text{Rings} \subset \text{SZC}^\infty\text{Rings}$ as the (2-)subcategory of $\mathcal{C}^\bullet$ with $\mathcal{C}^{-1} = 0$.

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of virtual cotangent modules.
Let $V$ be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we call $(V, E, s)$ a Kuranishi neighbourhood (compare Kuranishi spaces); for d-orbifolds, we take $V$ an orbifold. Associate a square zero dg $C^\infty$-ring $\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0$ to $(V, E, s)$ by

\[
\mathcal{E}^0 = C^\infty(V)/I_s^2, \quad \mathcal{E}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),
\]

\[d(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,
\]

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by $s$. The d-manifold $X$ associated to $(V, E, s)$ is $\text{Spec} \mathcal{E}^\bullet$. It only knows about functions on $V$ up to $O(s^2)$, and sections of $E$ up to $O(s)$.

### 2.4. D-spaces

A d-space $X$ is a topological space $X$ with a sheaf of square zero dg-$C^\infty$-rings $\mathcal{O}_X^\bullet = \mathcal{O}_X^{-1} \xrightarrow{d} \mathcal{O}_X^0$, such that $X = (X, H^0(\mathcal{O}_X^\bullet))$ and $(X, \mathcal{O}_X^0)$ are $C^\infty$-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $X$. We call $X$ the underlying classical $C^\infty$-scheme.

D-spaces form a strict 2-category $d\text{Spa}$, with 1-morphisms and 2-morphisms defined using sheaves of 1-morphisms and 2-morphisms in $\text{SZC}^\infty\text{Rings}$ in the obvious way.

All fibre products exist in $d\text{Spa}$. $C^\infty$-schemes include into d-spaces as those $X$ with $\mathcal{O}_X^{-1} = 0$. Thus we have inclusions of (2-)categories $\text{Man} \subset C^\infty\text{Sch} \subset d\text{Spa}$, so manifolds are examples of d-spaces.

The cotangent complex $\mathbb{L}_X^\bullet$ of $X$ is the sheaf of cotangent complexes of $\mathcal{O}_X^\bullet$, a 2-term complex $\mathbb{L}_X^{-1} \xrightarrow{d_X} \mathbb{L}_X^0$ of quasicoherent sheaves on $X$. Such complexes form a 2-category $q\text{colh}^{-1,0}(X)$. 

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2.5. D-manifolds

A d-manifold X of virtual dimension \( n \in \mathbb{Z} \) is a d-space X whose topological space \( X \) is Hausdorff and second countable, and such that \( X \) is covered by open d-subspaces \( Y \subset X \) with equivalences \( Y \simeq U \times_{g, W, h} V \), where \( U, V, W \) are manifolds with \( \dim U + \dim V - \dim W = n \), and \( g : U \to W, h : V \to W \) are smooth maps, and \( U \times_{g, W, h} V \) is the fibre product in the 2-category \( \text{dSpa} \). (The 2-category structure is essential to define the fibre product here.)

Write \( \text{dMan} \) for the full 2-subcategory of d-manifolds in \( \text{dSpa} \). Alternatively, we can write the local models as \( Y \simeq V \times_{0, E, s} V \), where \( V \) is a manifold, \( E \to V \) a vector bundle, \( s : V \to E \) a smooth section, and \( n = \dim V - \text{rank} E \). Then \( (V, E, s) \) is a Kuranishi neighbourhood on \( X \).

We call such \( V \times_{0, E, s} V \) affine d-manifolds.

2.6. D-orbifolds, d-manifolds with corners

In a similar way, I define 2-categories of d-stacks \( \text{dSta} \), which are a Deligne–Mumford stack version of d-spaces locally modelled on quotients \( [X/G] \) for \( X \) a d-space and \( G \) a finite group, and d-orbifolds \( \text{dOrb} \subset \text{dSta} \). D-orbifolds \( X \) are locally modelled by Kuranishi neighbourhoods \( (V, E, s) \) with \( V \) an orbifold, \( E \to V \) a vector bundle and \( s : V \to E \) a smooth section (that is, \( X \) is locally equivalent to a fibre product \( V \times_{0, E, s} V \) in \( \text{dSta} \)).

I also define 2-categories \( \text{dSpa}^b, \text{dSpa}^c, \text{dMan}^b, \text{dMan}^c, \text{dSta}^b, \text{dSta}^c, \text{dOrb}^b, \text{dOrb}^c \) of d-spaces, d-manifolds, d-stacks and d-orbifolds with boundary, and with corners.

Many moduli spaces of J-holomorphic curves in symplectic geometry will be d-orbifolds, possibly with corners. Doing ‘things with corners’ properly, especially in the derived context, is more complicated than you would expect.
2.7. Why should $d\text{Man}$ be a 2-category?

Here is one reason why any class of ‘derived manifolds’ should be (at least) a 2-category. One property we want of $d\text{Man}$ (or of Kuranishi spaces, etc.) is that it contains manifolds $\text{Man}$ as a subcategory, and if $X, Y, Z$ are manifolds and $g : X \to Z$, $h : Y \to Z$ are smooth then a fibre product $W = X \times_{g,Z,h} Y$ should exist in $d\text{Man}$, characterized by a universal property in $d\text{Man}$, and should be a d-manifold of ‘virtual dimension’

$$\text{vdim } W = \dim X + \dim Y - \dim Z.$$  

Note that $g, h$ need not be transverse, and $\text{vdim } W$ may be negative. Consider the case $X = Y = *,$ the point, $Z = \mathbb{R}$, and $g, h : * \mapsto 0$. If $d\text{Man}$ were an ordinary category then as $*$ is a terminal object, the unique fibre product $* \times_{0,\mathbb{R},0} *$ would be $*$. But this has virtual dimension 0, not $-1$. So $d\text{Man}$ must be some kind of higher category.

Why is a 2-category enough?

Usually in derived algebraic geometry, one considers an $\infty$-category of objects (derived stacks, etc.). But we work in a 2-category, a truncation of Spivak’s $\infty$-category of derived manifolds. Here are two reasons why this truncation does not lose important information. Firstly, d-manifolds correspond to quasi-smooth derived schemes $X$, whose cotangent complexes $L_X$ lie in degrees $[-1, 0]$. So $L_X$ lies in a 2-category of complexes, not an $\infty$-category. Note that $f : X \to Y$ is étale in $d\text{Man}$ iff $\Omega f : f^*(L_Y) \to L_X$ is an equivalence.

Secondly, the existence of partitions of unity in differential geometry means that our structure sheaves $\mathcal{O}_X$ are ‘fine’ or ‘soft’, which simplifies behaviour. Partitions of unity are also essential in gluing by equivalences in $d\text{Man}$. Our ‘2-category style derived geometry’ would not work well in a conventional algebro-geometric context, rather than a differential-geometric one.
3. Differential geometry of d-manifolds

3.1. Cotangent complexes of d-manifolds

If $X$ is a d-manifold, its cotangent complex $L^\bullet_X$ is perfect, that is, $L^\bullet_X$ is equivalent locally on $X$ in the 2-category $\text{qcoh}[-1,0](X)$ of 2-term complexes of quasicoherent sheaves on $X$ to a complex of vector bundles $E^{-1} \to E^0$, and $\text{rank } E^0 - \text{rank } E^{-1} = \text{vdim } X$.

For $x \in X$, define the cotangent space $T^*_x X = H^0(L^\bullet_X|_x)$ and the obstruction space $O_x X = H^{-1}(L^\bullet_X|_x)$, with $\dim T^*_x X = \dim O_x X = \text{vdim } X$. A 1-morphism of d-manifolds $f : X \to Y$ induces a 1-morphism $df : f^*(L^\bullet_Y) \to L^\bullet_X$ in $\text{qcoh}[-1,0](X)$.

**Theorem**

A 1-morphism $f : X \to Y$ in $\text{dMan}$ is étale if and only if $df : f^*(L^\bullet_Y) \to L^\bullet_X$ is an equivalence in $\text{qcoh}[-1,0](X)$, if and only if $H^0(df|_x) : T^{*}_{f(x)} Y \to T^*_x X$ and $H^{-1}(df|_x) : O^*_{f(x)} Y \to O^*_x X$ are isomorphisms for all $x \in X$.

3.2. D-transversality and fibre products

Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds. Then $g$, $h$ are transverse if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in $Z$, the map $dg|_x \oplus dh|_y : T^*_Z Z \to T^*_Z X \oplus T^*_Z Y$ is injective. If $g$, $h$ are transverse then a fibre product $X \times^g, Z, h Y$ exists in $\text{Man}$.

Similarly, we call 1-morphisms of d-manifolds $g : X \to Z$, $h : Y \to Z$ d-transverse if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in $Z$, the map $H^{-1}(dg|_x) \oplus H^{-1}(dh|_y) : O^*_Z Z \to O^*_Z X \oplus O^*_Z Y$ is injective. Note that d-transversality is much weaker than transversality of manifolds, and often holds automatically.
Let $g : X \to Z$ and $h : Y \to Z$ be $d$-transverse 1-morphisms in $\text{dMan}$. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\text{dMan}$, with $\text{vdim } W = \text{vdim } X + \text{vdim } Y - \text{vdim } Z$.

If $Z$ is a manifold, $O^*_Z = 0$ and $d$-transversality is trivial, giving:

**Corollary**

All fibre products of the form $X \times_Z Y$ with $X, Y$ $d$-manifolds and $Z$ a manifold exist in the 2-category $\text{dMan}$.

The same holds in $\text{dOrb}$. This is a very useful property of $d$-manifolds and $d$-orbifolds.

### 3.3. Gluing by equivalences

A 1-morphism $f : X \to Y$ in $\text{dMan}$ is an equivalence if there exist $g : Y \to X$ and 2-morphisms $\eta : g \circ f \Rightarrow \text{id}_X$ and $\zeta : f \circ g \Rightarrow \text{id}_Y$.

**Theorem**

Let $X, Y$ be $d$-manifolds, $\emptyset \neq U \subseteq X$, $\emptyset \neq V \subseteq Y$ open $d$-submanifolds, and $f : U \to V$ an equivalence. Suppose the topological space $Z = X \cup_{U=V} Y$ made by gluing $X, Y$ using $f$ is Hausdorff. Then there exists a $d$-manifold $\hat{Z}$, unique up to equivalence, open $\hat{X}, \hat{Y}$, with $Z = \hat{X} \cup \hat{Y}$, equivalences $g : X \to \hat{X}$ and $h : Y \to \hat{Y}$, and a 2-morphism $\eta : g|_U \Rightarrow h \circ f$.  

Equivalence is the natural notion of when two objects in $\text{dMan}$ are ‘the same’. In the theorem $Z$ is a pushout $X \amalg_{\text{id}_{U},u} \amalg f Y$ in $\text{dMan}$. The theorem generalizes to gluing families of $d$-manifolds $X_{i} : i \in I$ by equivalences on double overlaps $X_{i} \cap X_{j}$, with (weak) conditions on triple overlaps $X_{i} \cap X_{j} \cap X_{k}$.

We can take the $X_{i}$ to be ‘standard model’ $d$-manifolds $S_{V_{i},E_{i},s_{i}}$, and the equivalences on overlaps $X_{i} \cap X_{j}$ to be 1-morphisms $S_{e_{ij},\hat{e}_{ij}}$. This is very useful for proving existence of $d$-manifold or $d$-orbifold structures on moduli spaces. Essentially, from a ‘good coordinate system’ on a topological space $X$ we can build a $d$-manifold or $d$-orbifold $X$ with the same topological space $X$.

Let $Y$ be a manifold. Define the bordism group $B_{k}(Y)$ to have elements $\sim$-equivalence classes $[X, f]$ of pairs $(X, f)$, where $X$ is a compact oriented $k$-manifold and $f : X \to Y$ is smooth, and $(X, f) \sim (X', f')$ if there exists a $(k + 1)$-manifold with boundary $W$ and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with $[X, f] + [X', f'] = [X \amalg X, f \amalg f']$. 

3.4. D-manifold bordism
Similarly, define the *derived bordism group* $dB_k(Y)$ with elements $\approx$-equivalence classes $[X, f]$ of pairs $(X, f)$, where $X$ is a compact oriented d-manifold with $\text{vdim } X = k$ and $f : X \to Y = F^{\text{dMan}}_{\text{Man}}(Y)$ is a 1-morphism in $\text{dMan}$, and $(X, f) \approx (X', f')$ if there exists a d-manifold with boundary $W$ with $\text{vdim } W = k + 1$ and a 1-morphism $e : W \to Y$ in $\text{dMan}^b$ with $\partial W \simeq X \amalg -X'$ and $e|_{\partial W} \simeq f \amalg f'$. It is an abelian group, with $[X, f] + [X', f'] = [X \amalg X, f \amalg f']$.

There is a morphism $\Pi^{\text{dbo}}_{\text{bo}} : B_k(Y) \to dB_k(Y)$ mapping $[X, f] \mapsto [F^{\text{dMan}}_{\text{Man}}(X), F^{\text{dMan}}_{\text{Man}}(f)]$.

**Theorem**

$\Pi^{\text{dbo}}_{\text{bo}} : B_k(Y) \to dB_k(Y)$ is an isomorphism, with $dB_k(Y) = 0$ for $k < 0$.

This holds because every d-manifold can be perturbed to a manifold. Composing $(\Pi^{\text{dbo}}_{\text{bo}})^{-1}$ with the projection $B_k(Y) \to H_k(Y, \mathbb{Z})$ gives a morphism $\Pi^{\text{hom}}_{\text{dbo}} : dB_k(Y) \to H_k(Y, \mathbb{Z})$. We can interpret this as a *virtual class map* for compact oriented d-manifolds. Virtual classes (in homology over $\mathbb{Q}$) also exist for compact oriented d-orbifolds.