Plan of talk:

1. PTVV’s shifted symplectic geometry
2. A Darboux theorem for shifted symplectic schemes
3. D-critical loci
4. Categorification using perverse sheaves
5. Motivic Milnor fibres
1. PTVV’s shifted symplectic geometry

Let $K$ be an algebraically closed field of characteristic zero, e.g. $K = \mathbb{C}$. Work in the context of Toën and Vezzosi’s theory of derived algebraic geometry. This gives $\infty$-categories of derived $K$-schemes $\text{dSch}_K$ and derived stacks $\text{dSt}_K$. For this talk we are interested in derived schemes, though we are working on extensions to derived Artin stacks. Think of a derived $K$-scheme $X$ as a geometric space which can be covered by Zariski open sets $Y \subseteq X$ with $Y \cong \text{Spec} \ A$ for $A = (A, d)$ a commutative differential graded algebra (cdga) over $K$.

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of $k$-shifted symplectic structure on a derived $K$-scheme or derived $K$-stack $X$, for $k \in \mathbb{Z}$. This is complicated, but here is the basic idea. The cotangent complex $\mathbb{L}_X$ of $X$ is an element of a derived category $L_{\text{qcoh}}(X)$ of quasicoherent sheaves on $X$. It has exterior powers $\Lambda^p \mathbb{L}_X$ for $p = 0, 1, \ldots$. The de Rham differential $d_{dR}: \Lambda^p \mathbb{L}_X \to \Lambda^{p+1} \mathbb{L}_X$ is a morphism of complexes, though not of $\mathcal{O}_X$-modules. Each $\Lambda^p \mathbb{L}_X$ is a complex, so has an internal differential $d: (\Lambda^p \mathbb{L}_X)^k \to (\Lambda^p \mathbb{L}_X)^{k+1}$. We have

$$d^2 = d_{dR}^2 = d \circ d_{dR} + d_{dR} \circ d = 0.$$
A p-form of degree k on X for \( k \in \mathbb{Z} \) is an element \([\omega^0]\) of \( H^k(\wedge^p \mathbb{L}_X, d) \). A closed p-form of degree k on X is an element

\[
[(\omega^0, \omega^1, \ldots)] \in H^k(\bigoplus_{i=0}^{\infty} \wedge^{p+i} \mathbb{L}_X[i], d + d_{dR}).
\]

There is a projection \( \pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0] \) from closed p-forms \([(\omega^0, \omega^1, \ldots)]\) of degree k to p-forms \([\omega^0]\) of degree k.

Note that a closed p-form is not a special example of a p-form, but a p-form with an extra structure. The map \( \pi \) from closed p-forms to p-forms can be neither injective nor surjective.
Pantev et al. prove that if $Y$ is a Calabi–Yau $m$-fold over $\mathbb{K}$ and $\mathcal{M}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on $Y$, then $\mathcal{M}$ has a natural $(2-m)$-shifted symplectic structure $\omega$. So Calabi–Yau 3-folds give $-1$-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $\omega^0$ in terms of *Serre duality*. At a point $[E] \in \mathcal{M}$, we have $h^i(\mathcal{T}_E) |_{[E]} \cong \text{Ext}^{i-1}(E, E)$ and $h^i(\mathcal{L}_E) |_{[E]} \cong \text{Ext}^{1-i}(E, E)^\ast$. The Calabi–Yau condition gives $\text{Ext}^i(E, E) \cong \text{Ext}^{m-i}(E, E)^\ast$, which corresponds to $h^i(\mathcal{T}_E) |_{[E]} \cong h^i(\mathcal{L}_E[2-m]) |_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathcal{T}_E \to \mathcal{L}_E[2-m]$.

Let $(X, \omega)$ be a $k$-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $L$ in $(X, \omega)$, which is a morphism $i : L \to X$ of derived schemes or stacks together with a homotopy $i^!(\omega) \sim 0$ satisfying a nondegeneracy condition, implying that $\mathcal{T}_L \cong \mathcal{L}_{L/X}[k-1]$.

If $L, M$ are Lagrangians in $(X, \omega)$, then the fibre product $L \times_X M$ has a natural $(k-1)$-shifted symplectic structure.

If $(S, \omega)$ is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L, M \subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L \cap M = L \times_S M$ is a $-1$-shifted symplectic derived scheme.
2. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose \((X, \omega)\) is a \(k\)-shifted symplectic derived \(\mathbb{K}\)-scheme for \(k < 0\). If \(k \neq 2 \mod 4\), then each \(x \in X\) admits a Zariski open neighbourhood \(Y \subseteq X\) with \(Y \simeq \text{Spec} A\) for \((A, d)\) an explicit cdga over \(\mathbb{K}\) generated by graded variables \(x_j^{-i}, y_j^{k+i}\) for \(0 \leq i \leq -k/2\), and \(\omega|_Y = [(\omega^0, 0, 0, \ldots)]\) where \(x_j^l, y_j^l\) have degree \(l\), and

\[
\omega^0 = \sum_{i=0}^{[-k/2]} \sum_{j=1}^{m_i} dRy_j^{k+i} dRx_j^{-i}.
\]

Also the differential \(d\) in \((A, d)\) is given by Poisson bracket with a Hamiltonian \(H\) in \(A\) of degree \(k + 1\).

If \(k \equiv 2 \mod 4\), we have two statements, one étale local with \(\omega^0\) standard, and one Zariski local with the components of \(\omega^0\) the degree \(k/2\) variables depending on some invertible functions.

Sketch of the proof of the theorem

Suppose \((X, \omega)\) is a \(k\)-shifted symplectic derived \(\mathbb{K}\)-scheme for \(k < 0\), and \(x \in X\). Then \(\mathbb{L}_X\) lives in degrees \([k, 0]\). We first show that we can build Zariski open \(x \in Y \subseteq X\) with \(Y \simeq \text{Spec} A\), for \(A = \bigoplus_{i \leq 0} A^i\) a cdga over \(\mathbb{K}\) with \(A^0\) a smooth \(\mathbb{K}\)-algebra, and such that \(A\) is freely generated over \(A^0\) by graded variables \(x_j^{-i}, y_j^{k+i}\) in degrees \(-1, -2, \ldots, k\). We take \(\dim A^0\) and the number of \(x_j^{-i}, y_j^{k+i}\) to be minimal at \(x\).

Using theorems about periodic cyclic cohomology, we show that on \(Y \simeq \text{Spec} A\) we can write \(\omega|_Y = [(\omega^0, 0, 0, \ldots)]\), for \(\omega^0\) a 2-form of degree \(k\) with \(d\omega^0 = dR\omega^0 = 0\). Minimality at \(x\) implies \(\omega^0\) is strictly nondegenerate near \(x\), so we can change variables to write \(\omega^0 = \sum_{i,j} dRy_j^{k+i} dRx_j^{-i}\). Finally, we show \(d\) in \((A, d)\) is a symplectic vector field, which integrates to a Hamiltonian \(H\).
When $k = -1$ the Hamiltonian $H$ in the theorem has degree 0. Then the theorem reduces to:

**Corollary**

Suppose $(X, \omega)$ is a $-1$-shifted symplectic derived $K$-scheme. Then $(X, \omega)$ is Zariski locally equivalent to a derived critical locus $\text{Crit}(H : U \to \mathbb{A}^1)$, for $U$ a smooth classical $K$-scheme and $H : U \to \mathbb{A}^1$ a regular function. Hence, the underlying classical $K$-scheme $X = t_0(X)$ is Zariski locally isomorphic to a classical critical locus $\text{Crit}(H : U \to \mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

**Corollary**

Let $Y$ be a Calabi–Yau 3-fold over $K$ and $\mathcal{M}$ a classical moduli $K$-scheme of coherent sheaves, or complexes of coherent sheaves, on $Y$. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\text{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $K$-scheme.

Here we note that $\mathcal{M} = t_0(\mathcal{M})$ for $\mathcal{M}$ the corresponding derived moduli scheme, which is $-1$-shifted symplectic by PTVV. A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler. Note that the proof of the corollary is wholly algebro-geometric.
Corollary

Let \((S, \omega)\) be a classical smooth symplectic \(\mathbb{K}\)-scheme, and \(L, M \subseteq S\) be smooth algebraic Lagrangians. Then the intersection \(L \cap M\), as a \(\mathbb{K}\)-subscheme of \(S\), is Zariski locally isomorphic to the critical locus \(\text{Crit}(H : U \to \mathbb{A}^1)\) of a regular function on a smooth \(\mathbb{K}\)-scheme.

In real or complex symplectic geometry, where Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

3. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let \(X\) be a classical \(\mathbb{K}\)-scheme. Then there exists a canonical sheaf \(S_X\) of \(\mathbb{K}\)-vector spaces on \(X\), such that if \(R \subseteq X\) is Zariski open and \(i : R \hookrightarrow U\) is a closed embedding of \(R\) into a smooth \(\mathbb{K}\)-scheme \(U\), and \(I_{R,U} \subseteq \mathcal{O}_U\) is the ideal vanishing on \(i(R)\), then

\[
S_X|_R \cong \ker \left( \frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{d} \frac{T^*U}{I_{R,U} \cdot T^*U} \right).
\]

Also \(S_X\) splits naturally as \(S_X = S_X^0 \oplus \mathbb{K}_X\), where \(\mathbb{K}_X\) is the sheaf of locally constant functions \(X \to \mathbb{K}\).
The meaning of the sheaves \( \mathcal{S}_X, \mathcal{S}_X^0 \)

If \( X = \text{Crit}(f : U \to \mathbb{A}^1) \) then taking \( R = X, i = \text{inclusion} \), we see that \( f + I_{X,U}^2 \) is a section of \( \mathcal{S}_X \). Also \( f|_{X_{\text{red}}} : X_{\text{red}} \to \mathbb{K} \) is locally constant, and if \( f|_{X_{\text{red}}} = 0 \) then \( f + I_{X,U}^2 \) is a section of \( \mathcal{S}_X^0 \). Note that \( f + I_{X,U} = f|_X \in \mathcal{O}_X = \mathcal{O}_U/I_{X,U} \). The theorem means that \( f + I_{X,U}^2 \) makes sense \textit{intrinsically on} \( X \), without reference to the embedding of \( X \) into \( U \).

That is, if \( X = \text{Crit}(f : U \to \mathbb{A}^1) \) then we can remember \( f \) up to second order in the ideal \( I_X \) as a piece of data on \( X \), not on \( U \).

Suppose \( X = \text{Crit}(f : U \to \mathbb{A}^1) = \text{Crit}(g : V \to \mathbb{A}^1) \) is written as a critical locus in two different ways. Then \( f + I_{X,U}^2, g + I_{X,V}^2 \) are sections of \( \mathcal{S}_X \), so we can ask whether \( f + I_{X,U}^2 = g + I_{X,V}^2 \). This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (\textit{algebraic}) \textit{d-critical locus} \((X, s)\) is a classical \( \mathbb{K} \)-scheme \( X \) and a global section \( s \in H^0(\mathcal{S}_X^0) \) such that \( X \) may be covered by Zariski open \( R \subseteq X \) with an isomorphism \( i : R \to \text{Crit}(f : U \to \mathbb{A}^1) \) identifying \( s|_R \) with \( f + I_{R,U}^2 \), for \( f \) a regular function on a smooth \( \mathbb{K} \)-scheme \( U \).

That is, a d-critical locus \((X, s)\) is a \( \mathbb{K} \)-scheme \( X \) which may Zariski locally be written as a critical locus \( \text{Crit}(f : U \to \mathbb{A}^1) \), and the section \( s \) remembers \( f \) up to second order in the ideal \( I_{X,U} \).

We also define \textit{complex analytic d-critical loci}, with \( X \) a complex analytic space locally modelled on \( \text{Crit}(f : U \to \mathbb{C}) \) for \( U \) a complex manifold and \( f \) holomorphic.
**Orientations on d-critical loci**

**Theorem (Joyce arXiv:1304.4508)**

Let \((X, s)\) be an algebraic d-critical locus and \(X^{\text{red}}\) the reduced \(\mathbb{K}\)-subscheme of \(X\). Then there is a natural line bundle \(K_{X, s}\) on \(X^{\text{red}}\) called the **canonical bundle**, such that if \((X, s)\) is locally modelled on \(\text{Crit}(f : U \to \mathbb{A}^1)\) then \(K_{X, s}\) is locally modelled on \(K_U^{\otimes 2} |_{\text{Crit}(f)^{\text{red}}}\), for \(K_U\) the usual canonical bundle of \(U\).

**Definition**

Let \((X, s)\) be a d-critical locus. An **orientation** on \((X, s)\) is a choice of square root line bundle \(K_{X, s}^{1/2}\) for \(K_{X, s}\) on \(X^{\text{red}}\).

This is related to **orientation data** in Kontsevich–Soibelman 2008.

---

**A truncation functor from \(-1\)-symplectic derived schemes**

**Theorem (Brav, Bussi and Joyce arXiv:1305.6302)**

Let \((X, \omega)\) be a \(-1\)-shifted symplectic derived \(\mathbb{K}\)-scheme. Then the classical \(\mathbb{K}\)-scheme \(X = t_0(X)\) extends naturally to an algebraic d-critical locus \((X, s)\). The canonical bundle of \((X, s)\) satisfies \(K_{X, s} \cong \det \mathbb{L}_X |_{X^{\text{red}}}\).

That is, we define a **truncation functor** from \(-1\)-shifted symplectic derived \(\mathbb{K}\)-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as **classical truncations** of \(-1\)-shifted symplectic derived \(\mathbb{K}\)-schemes.

An alternative semi-classical truncation, used in D–T theory, is **schemes with symmetric obstruction theory**. D-critical loci appear to be better, for both categorified and motivic D–T theory.
The corollaries in §2 imply:

**Corollary**

Let \( Y \) be a Calabi–Yau 3-fold over \( K \) and \( \mathcal{M} \) a classical moduli \( K \)-scheme of coherent sheaves, or complexes of coherent sheaves, on \( Y \). Then \( \mathcal{M} \) extends naturally to a \( d \)-critical locus \((\mathcal{M}, s)\). The canonical bundle satisfies \( K_{\mathcal{M}, s} \cong \det(\mathcal{E}^*)|_{\mathcal{M}^{\text{red}}} \), where \( \phi : \mathcal{E}^* \to \mathbb{I}_{\mathcal{M}} \) is the (symmetric) obstruction theory on \( \mathcal{M} \) defined by Thomas or Huybrechts and Thomas.

**Corollary**

Let \((S, \omega)\) be a classical smooth symplectic \( K \)-scheme, and \( L, M \subseteq S \) be smooth algebraic Lagrangians. Then \( X = L \cap M \) extends to naturally to a \( d \)-critical locus \((X, s)\). The canonical bundle satisfies \( K_{X, s} \cong K_L|_{X^{\text{red}}} \otimes K_M|_{X^{\text{red}}} \). Hence, choices of square roots \( K_L^{1/2} \) and \( K_M^{1/2} \) give an orientation for \((X, s)\).

---

Dominic Joyce, Oxford University  
**Categorification of D–T theory**

---

4. Categorification using perverse sheaves

**Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)**

Let \((X, s)\) be an algebraic \( d \)-critical locus over \( K \), with an orientation \( K_{X,s}^{1/2} \). Then we can construct a canonical perverse sheaf \( P_{X,s}^* \) on \( X \), such that if \((X, s)\) is locally modelled on \( \text{Crit}(f : U \to \mathbb{A}^1) \), then \( P_{X,s}^* \) is locally modelled on the perverse sheaf of vanishing cycles \( P_{U,f}^* \) of \((U, f)\).  
Similarly, we can construct a natural \( \mathcal{D} \)-module \( D_{X,s}^* \) on \( X \), and when \( K = \mathbb{C} \) a natural mixed Hodge module \( M_{X,s}^* \) on \( X \).
Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover \( \{ R_i : i \in I \} \) of \( X \) with \( R_i \cong \text{Crit}(f_i : U_i \to \mathbb{A}^1) \), and showing that \( \mathcal{PV}_{U_i,f_i} \) and \( \mathcal{PV}_{U_j,f_j} \) are canonically isomorphic on \( R_i \cap R_j \), so we can glue the \( \mathcal{PV}_{U_i,f_i} \) to get a global perverse sheaf \( P_{X,s}^\bullet \) on \( X \).

In fact things are more complicated: the (local) isomorphisms \( \mathcal{PV}_{U_i,f_i} \cong \mathcal{PV}_{U_j,f_j} \) are only canonical up to sign. To make them canonical, we use the orientation \( K^{1/2}_{X,s} \) to define natural principal \( \mathbb{Z}_2 \)-bundles \( Q_i \) on \( R_i \), such that \( \mathcal{PV}_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j \) is canonical, and then we glue the \( \mathcal{PV}_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \) to get \( P_{X,s}^\bullet \).

The first corollary in §2 implies:

**Corollary**

Let \( Y \) be a Calabi–Yau 3-fold over \( \mathbb{K} \) and \( \mathcal{M} \) a classical moduli \( \mathbb{K} \)-scheme of coherent sheaves, or complexes of coherent sheaves, on \( Y \), with (symmetric) obstruction theory \( \phi : \mathcal{E}^\bullet \to \mathbb{L}_\mathcal{M} \). Suppose we are given a square root \( \det(\mathcal{E}^\bullet)^{1/2} \) for \( \det(\mathcal{E}^\bullet) \) (i.e. orientation data, K–S). Then we have a natural perverse sheaf \( P_{\mathcal{M},s}^\bullet \) on \( \mathcal{M} \).

The hypercohomology \( \mathbb{H}^*(P_{\mathcal{M},s}^\bullet) \) is a finite-dimensional graded vector space. The pointwise Euler characteristic \( \chi(P_{\mathcal{M},s}^\bullet) \) is the Behrend function \( \nu_{\mathcal{M}} \) of \( \mathcal{M} \). Thus

\[
\sum_{i \in \mathbb{Z}} (-1)^i \dim \mathbb{H}^i(P_{\mathcal{M},s}^\bullet) = \chi(\mathcal{M}, \nu_{\mathcal{M}}).
\]

Now by Behrend 2005, the Donaldson–Thomas invariant of \( \mathcal{M} \) is \( DT(\mathcal{M}) = \chi(\mathcal{M}, \nu_{\mathcal{M}}) \). So, \( \mathbb{H}^*(P_{\mathcal{M},s}^\bullet) \) is a graded vector space with dimension \( DT(\mathcal{M}) \), that is, a categorification of \( DT(\mathcal{M}) \).
The second corollary in §2 implies:

**Corollary**

Let \((S, \omega)\) be a classical smooth symplectic \(\mathbb{K}\)-scheme of dimension \(2n\), and \(L, M \subseteq S\) be smooth algebraic Lagrangians, with square roots \(K_L^{1/2}, K_M^{1/2}\) of their canonical bundles. Then we have a natural perverse sheaf \(P_{L,M}^\bullet\) on \(X = L \cap M\).

This is related to Behrend and Fantechi 2009, and Kai’s talk. We think of the hypercohomology \(H^*(P_{L,M}^\bullet)\) as being morally related to the Lagrangian Floer cohomology \(HF^*(L, M)\) by

\[
H^i(P_{L,M}^\bullet) \approx HF^{i+n}(L, M).
\]

We are working on defining ‘Fukaya categories’ for algebraic/complex symplectic manifolds using these ideas.

---

**5. Motivic Milnor fibres**

By similar arguments to those used to construct the perverse sheaves \(P_{X,s}^\bullet\) in §4, we prove:

**Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)**

Let \((X, s)\) be an algebraic \(d\)-critical locus over \(\mathbb{K}\), with an orientation \(K_X^{1/2}\). Then we can construct a natural motive \(MF_{X,s}\) in a certain ring of \(\hat{\mu}\)-equivariant motives \(\overline{M}_X^{\hat{\mu}}\) on \(X\), such that if \((X, s)\) is locally modelled on \(\text{Crit}(f : U \to \mathbb{A}^1)\), then \(MF_{X,s}\) is locally modelled on \(L^{-\dim U/2}([X] - MF_{U,f}^{\text{mot}})\), where \(MF_{U,f}^{\text{mot}}\) is the motivic Milnor fibre of \(f\).

Vittoria Bussi’s talk will give more details.
Relation to motivic D–T invariants

The first corollary in §2 implies:

**Corollary**

Let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on $Y$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_\mathcal{M}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. orientation data, K–S). Then we have a natural motive $MF_{\mathcal{M},s}^\bullet$ on $\mathcal{M}$.

This motive $MF_{\mathcal{M},s}^\bullet$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman 2008. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.