Constructing compact manifolds with exceptional holonomy

Dominic Joyce
Oxford University
England
Riemannian geometry

Let $M^n$ be a manifold of dimension $n$. Let $x \in M$. Then $T_x M$ is the tangent space to $M$ at $x$. Let $g$ be a Riemannian metric on $M$. Let $\nabla$ be the Levi-Civita connection of $g$. Let $R(g)$ be the Riemann curvature of $g$. 
Holonomy groups

Fix \( x \in M \). The holonomy group \( \text{Hol}(g) \) of \( g \) is the set of isometries of \( T_xM \) given by parallel transport using \( \nabla \) about closed loops \( \gamma \) in \( M \) based at \( x \). It is a subgroup of \( O(n) \). Up to conjugation, it is independent of the base-point \( x \).
Berger’s classification

Let $M$ be simply-connected and $g$ be irreducible and nonsymmetric. Then $\text{Hol}(g)$ is one of $SO(m)$, $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m)Sp(1)$ for $m \geq 2$, or $G_2$ or $Spin(7)$. We call $G_2$ and $Spin(7)$ the exceptional holonomy groups. \(\text{Dim}(M)\) is 7 when $\text{Hol}(g)$ is $G_2$ and 8 when $\text{Hol}(g)$ is $Spin(7)$. 
Understanding Berger’s list

The four *inner product algebras* are

\[
\begin{align*}
\mathbb{R} & \quad \text{real numbers.} \\
\mathbb{C} & \quad \text{complex numbers.} \\
\mathbb{H} & \quad \text{quaternions.} \\
\mathbb{O} & \quad \text{octonions, or Cayley numbers.}
\end{align*}
\]

Here \(\mathbb{C}\) is not ordered, \(\mathbb{H}\) is not commutative, and \(\mathbb{O}\) is not associative.

Also we have \(\mathbb{C} \cong \mathbb{R}^2\), \(\mathbb{H} \cong \mathbb{R}^4\) and \(\mathbb{O} \cong \mathbb{R}^8\), with \(\text{Im} \mathbb{O} \cong \mathbb{R}^7\).
<table>
<thead>
<tr>
<th>Group</th>
<th>Acts on</th>
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<tbody>
<tr>
<td>$SO(m)$</td>
<td>$\mathbb{R}^m$</td>
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<tr>
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<td>$\mathbb{R}^m$</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>$G_2$</td>
<td>$\text{Im } \mathbb{O} \cong \mathbb{R}^7$</td>
</tr>
<tr>
<td>$Spin(7)$</td>
<td>$\mathbb{O} \cong \mathbb{R}^8$</td>
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</table>

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. 
The goal of the talk
To discuss constructions of examples of compact manifolds of holonomy $G_2$ and $Spin(7)$.

Why is this difficult?
In many problems in geometry the simplest examples are symmetric. But $G_2$- and $Spin(7)$-manifolds have no continuous symmetries.
Why is this interesting?
• Such manifolds are Ricci-flat.
• They are important to physicists working in String Theory.
• They have beautiful geometrical properties.
Geometry of $G_2$

The action of $G_2$ on $\mathbb{R}^7$ preserves the metric $g_0$ and a 3-form $\varphi_0$ on $\mathbb{R}^7$. Let $g$ be a metric and $\varphi$ a 3-form on $M^7$. We call $(\varphi, g)$ a $G_2$-structure if $(\varphi, g) \cong (\varphi_0, g_0)$ at each $x \in M$. We call $\nabla \varphi$ the torsion of $(\varphi, g)$. 
If $\nabla \varphi = 0$ then $(\varphi, g)$ is torsion-free. Also $\nabla \varphi = 0$ iff $d\varphi = d^*\varphi = 0$. If $(\varphi, g)$ is torsion-free then $\text{Hol}(g) \subseteq G_2$. Conversely, if $g$ is a metric on $M^7$ then $\text{Hol}(g) \subseteq G_2$ iff there is a $G_2$-structure $(\varphi, g)$ with $\nabla \varphi = 0$. If $M$ is compact and $\text{Hol}(g) \subseteq G_2$ then $\text{Hol}(g) = G_2$ iff $\pi_1(M)$ is finite.
The construction, 1

First we choose a compact 7-manifold $M$. We write down an explicit $G_2$-structure $(\varphi, g)$ on $M$ with small torsion. Then we use analysis to deform to a nearby $G_2$-structure $(\tilde{\varphi}, \tilde{g})$ with zero torsion. If $\pi_1(M)$ is finite then $\text{Hol}(\tilde{g}) = G_2$ as we want.
The construction, 2

It is not easy to find $G_2$-structures with small torsion! Here is one way to do it, in 4 steps.

**Step 1.** Choose a finite group $\Gamma$ of isometries of the 7-torus $T^7$, and a flat, $\Gamma$-invariant $G_2$-structure $(\varphi_0, g_0)$ on $T^7$. Then $T^7/\Gamma$ is compact, with a torsion-free $G_2$-structure $(\varphi_0, g_0)$. 
Step 2. However, $T^7/\Gamma$ is an orbifold. We repair its singularities to get a compact 7-manifold $M$. We can resolve complex orbifolds using algebraic geometry.

If the singularities of $T^7/\Gamma$ locally resemble $S^1 \times \mathbb{C}^3/G$ for $G \subset SU(3)$, then we model $M$ on a crepant resolution $X$ of $\mathbb{C}^3/G$. 
Step 3. $M$ is made by gluing patches $S^1 \times X$ into $T^7/\Gamma$. Now $X$ carries ALE metrics of holonomy $SU(3)$. As $SU(3) \subset G_2$, these give torsion-free $G_2$-structures on $S^1 \times X$. We join them to $(\varphi_0, g_0)$ on $T^7/\Gamma$ to get a family \{$(\varphi_t, g_t) : t \in (0, \epsilon)$\} of $G_2$-structures on $M$. 
Step 4. This \((\varphi_t, g_t)\) has \(\nabla \varphi_t = O(t^4)\). So \(\nabla \varphi_t\) is small for small \(t\). But \(R(g_t) = O(t^{-2})\) and the injectivity radius \(\delta(g_t) = O(t)\), since \(g_t\) becomes singular as \(t \to 0\).

For small \(t\) we deform \((\varphi_t, g_t)\) to \((\tilde{\varphi}_t, \tilde{g}_t)\) with \(\nabla \tilde{\varphi}_t = 0\), using analysis. Then \(\text{Hol}(\tilde{g}_t) = G_2\) if \(\pi_1(M)\) is finite.
Steps in the analysis proof:

• Arrange that $d\varphi_t = 0$ and $d^*\varphi_t = d^*\psi_t$, where $\psi_t = O(t^4)$.

• Set $\tilde{\varphi}_t = \varphi_t + d\eta_t$, where $d^*\eta_t = 0$.

• Then $(\tilde{\varphi}_t, \tilde{g}_t)$ is torsion-free iff

$$(d^*d + dd^*)(\eta_t) = d^*\psi_t + dF(d\eta_t),$$

where $F$ is nonlinear with $F(\chi) = O(|\chi|^2)$. 
• Integrating by parts gives $\|d\eta_t\|_{L^2} \leq 2\|\psi_t\|_{L^2}$ when $\|d\eta_t\|_{C^0}$ is small.

• Solve by contraction method in $L^{14}_2(\Lambda^2 T^* M)$, using elliptic regularity of $d^*d + dd^*$, balls of radius $t$ and Sobolev embedding.
The construction, 3

Using different groups $\Gamma$ acting on $T^7$ or $T^8$, and resolving $T^k/\Gamma$ in more than one way, we get many compact manifolds with holonomy $G_2$ and $Spin(7)$. We can generalize the construction by replacing $T^7$ or $T^8$ with another space made from a Calabi-Yau manifold.
Geometry of $Spin(7)$

The action of $Spin(7)$ on $\mathbb{R}^8$ preserves the metric $g_0$ and a 4-form $\Omega_0$ on $\mathbb{R}^8$. Let $g$ be a metric and $\Omega$ a 4-form on $M^8$. We call $(\Omega, g)$ a $Spin(7)$-structure if $(\Omega, g) \cong (\Omega_0, g_0)$ at each $x \in M$. We call $\nabla \Omega$ the torsion of $(\Omega, g)$. 
If $\nabla \Omega = 0$ then $(\Omega, g)$ is torsion-free. Also $\nabla \Omega = 0$ iff $d\Omega = 0$. If $\nabla \Omega = 0$ then $\text{Hol}(g) \subseteq \text{Spin}(7)$. If $g$ is a metric on $M^8$ then $\text{Hol}(g) \subseteq \text{Spin}(7)$ iff there is a $\text{Spin}(7)$-structure $(\Omega, g)$ with $\nabla \Omega = 0$. If $M$ is compact and $\text{Hol}(g) \subseteq \text{Spin}(7)$ then $g$ has holonomy $\text{Spin}(7)$ iff $\pi_1(M) = \{1\}$, $\hat{A}(M) = 1$. 20
Compact examples

The first examples of compact 8-manifolds with holonomy $Spin(7)$ were constructed by me in 1995. Here is how. Let $T^8$ be a torus with flat $Spin(7)$-structure $(\Omega_0, g_0)$, and let $\Gamma$ be a finite group acting on $T^8$ preserving $(\Omega_0, g_0)$. Then $T^8/\Gamma$ is an orbifold.
We choose $\Gamma$ so that the singularities of $T^8/\Gamma$ are locally modelled on $\mathbb{C}^4/G$, for $G \subset SU(4)$. Then we use complex algebraic geometry to resolve $T^8/\Gamma$, giving a compact 8-manifold $M$. Finally we use analysis to construct metrics on $M$ with holonomy $Spin(7)$.
A new construction

We shall describe a new way of making compact 8-manifolds with holonomy $Spin(7)$, where we start not with a torus $T^8$ but with a Calabi-Yau 4-orbifold $Y$ with isolated singular points $p_1, \ldots, p_k$. 
Instead of a group $\Gamma$ we use an antiholomorphic, isometric involution $\sigma$ on $Y$ fixing only the $p_j$. Then $Z = Y/\langle \sigma \rangle$ is a real 8-orbifold with singular points $p_1, \ldots, p_k$. We resolve the $p_j$ to get a compact 8-manifold $M$, and construct holonomy $\text{Spin}(7)$ metrics on $M$. 
Calabi-Yau orbifolds

A Calabi-Yau orbifold is a compact complex orbifold with a Kähler metric of holonomy $SU(m)$. One can find many examples using algebraic geometry and Yau’s proof of the Calabi conjecture.
The construction

Let $Y$ be a Calabi-Yau 4-orbifold with only isolated singular points $p_1, \ldots, p_k$, each modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator of $\mathbb{Z}_4$ acts by

$$(z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4).$$

We call this a singular point of type $\frac{1}{4}(1, 1, 1, 1)$. 
Pick an antiholomorphic, isometric involution $\sigma$ on $Y$, fixing only $p_1, \ldots, p_k$, and let $Z = Y/\langle \sigma \rangle$. As $SU(4) \subset Spin(7)$ and $Y$ has holonomy $SU(4)$, there is a torsion-free $Spin(7)$-structure $(\Omega, g)$ on $Y$. We can choose $(\Omega, g)$ to be $\sigma$-invariant, so $(\Omega, g)$ pushes down to $Z$. Thus $Z$ is a $Spin(7)$-orbifold.
All the singularities $p_j$ of $\mathbb{Z}$ are modelled on $\mathbb{R}^8/G$, where $G = \langle \alpha, \sigma \rangle$ is a non-abelian group of order 8, and $\alpha, \sigma$ act on $\mathbb{R}^8 = \mathbb{C}^4$ by

$\alpha : (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4),$

$\sigma : (z_1, \ldots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3).$

There are two different ways to resolve $\mathbb{R}^8/G$ within holonomy $\text{Spin}(7)$.  

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The first way is to take a crepant resolution $W_1$ of $\mathbb{C}^4/\langle \alpha \rangle$, and lift $\sigma$ to a free antiholomorphic involution of $W_1$. Then $X_1 = W_1/\langle \sigma \rangle$ is a resolution of $\mathbb{R}^8/G$. There is an ALE metric with holonomy $SU(4)$ on $W_1$ which pushes down to a metric on $W_1/\langle \sigma \rangle$ with holonomy $\mathbb{Z}_2 \times SU(4)$. 
But there is a second complex structure on $\mathbb{R}^8$, so that $\sigma$ is holomorphic and $\alpha$ anti-holomorphic. Resolve $\mathbb{C}^4/\langle \sigma \rangle$ to get $W_2$, lift $\alpha$ to $W_2$, and $X_2 = W_2/\langle \alpha \rangle$ is a resolution of $\mathbb{R}^8/G$, with ALE metrics of holonomy $\mathbb{Z}_2 \ltimes SU(4)$. Note that we have used two different inclusions of $\mathbb{Z}_2 \ltimes SU(4)$ in $Spin(7)$. 
We resolve each point $p_j$ in $Z$ using either $X_1$ or $X_2$, to get a compact 8-manifold $M$. Now $Z$, $X_1$ and $X_2$ carry torsion-free $Spin(7)$-structures. We glue these together to get a $Spin(7)$-structure $(\Omega_t, g_t)$ on $M$ for $t \in (0, \epsilon)$, with torsion $O(t^{24}/5)$. 
For small $t$ we can deform $(\Omega_t, g_t)$ to a torsion-free $Spin(7)$-structure $(\tilde{\Omega}, \tilde{g})$ on $M$. If we resolve using $X_1$ for all $p_j$ then $\pi_1(M) = \mathbb{Z}_2$ and $\text{Hol}(\tilde{g}) = \mathbb{Z}_2 \rtimes SU(4)$. If we use $X_2$ for any $p_j$ then $\pi_1(M) = \{1\}$ and $\text{Hol}(\tilde{g}) = Spin(7)$. This is what we want.
An example

Let $Y$ be the degree 12 hypersurface in the weighted projective space $\mathbb{C}P^{5}_{1,1,1,1,4,4}$ given by
\[
\left\{ [z_0, \ldots, z_5] \in \mathbb{C}P^{5}_{1,\ldots,4} : \\
z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} \\
+ z_4^3 + z_5^3 = 0 \right\}.
\]
Then $c_1(Y) = 0$, so $Y$ is a Calabi-Yau 4-orbifold. It has 3 singularities $p_1, p_2, p_3$, of type $\frac{1}{4}(1, 1, 1, 1)$. 
Define $\sigma : Y \to Y$ by

\[\sigma : [z_0, \ldots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].\]

Then $\sigma$ is an anti-holomorphic involution, fixing only $p_1, p_2, p_3$. We apply our construction to $Y$ and $\sigma$, to get a compact 8-manifold $M$ with holonomy Spin(7) and Betti numbers $b^2 = 0$, $b^3 = 0$ and $b^4 = 2446$. 

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Conclusions

Using hypersurfaces in other weighted projective spaces, and dividing by finite groups, we can find many new examples of compact 8-manifolds with holonomy $\text{Spin}(7)$. Here are some of their Betti numbers.
### Betti numbers \((b^2, b^3, b^4)\)

\[
\begin{align*}
(4, 33, 200) & \quad (3, 33, 202) \\
(2, 33, 204) & \quad (1, 33, 206) \\
(0, 33, 208) & \quad (1, 0, 908) \\
(0, 0, 910) & \quad (1, 0, 1292) \\
(0, 0, 1294) & \quad (1, 0, 2444) \\
(0, 0, 2446) & \quad (0, 6, 3730) \\
(0, 0, 4750) & \quad (0, 0, 11662)
\end{align*}
\]

Note that \(b^4\) tends to be rather large — bigger than in the first construction, where \(b^4 \approx 100-200\).