COMPACT RIEMANNIAN 7-MANIFOLDS WITH HOLONOMY G_2. I

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1. Introduction

The list of possible holonomy groups of Riemannian manifolds given by Berger [3] includes three intriguing special cases, the holonomy groups G_2, Spin(7) and Spin(9) in dimensions 7, 8 and 16 respectively. Subsequently [1] it was shown that Spin(9) does not occur as a non-symmetric holonomy group, but Bryant [5] showed that both G_2 and Spin(7) do occur as non-symmetric holonomy groups. Bryant's proof is a local one, in that it proves the existence of many metrics of holonomy G_2 and Spin(7) on small balls in $\mathbb{R}^7$ and $\mathbb{R}^8$ respectively. He also gives some explicit examples of such metrics. In a subsequent paper [6], Bryant and Salamon construct *complete* metrics of holonomy G_2.

This is the first of two papers in which we shall construct examples of *compact* Riemannian 7-manifolds with holonomy G_2. These are, to our knowledge, the first such examples known. We believe that they are the first nontrivial examples of odd-dimensional, compact, Ricci-flat Riemannian manifolds. The author has also used similar methods to construct compact 8-manifolds with holonomy Spin(7), and is preparing a paper [12] giving the proof.

The goal of this first paper is to study a single example in depth. We shall describe a certain simply-connected, compact 7-manifold M, and construct a family of metrics on it with holonomy G_2. The 7-manifold M was chosen because it is the simplest example that we know of. The content of the paper is mostly introductory material, and proofs using a lot of analysis. The second paper will describe many different compact 7-manifolds admitting metrics of holonomy G_2, and will have a more topological emphasis. It will also contain much more discussion of the results, and some interesting questions.

The method we shall use to construct Riemannian 7-manifolds with holonomy G_2 is modelled on the Kummer construction for $K3$ surfaces (described in §1.3), which is a way of approximating some of the family of metrics of holonomy $SU(2)$ on the $K3$ surface, that exist by Yau's proof of the Calabi conjecture [19]. We begin with a flat Riemannian 7-torus T^7, divide by the action of a finite group Γ, and then resolve the resulting singularities in a certain special way to get a compact, nonsingular 7-manifold M. The finite group preserves a flat G_2- structure on T^7, and a family of metrics are defined on M modelled on the flat G_2- structure on T^7/Γ, that are close to having holonomy G_2 in a suitable sense. It is then shown using analysis that these metrics can be deformed to metrics that do have holonomy G_2.

The paper is divided into three chapters. This first chapter contains introductory sections §1.1 on the geometry of the holonomy group G_2, §1.2 on the analytic tools and results that will be needed later, and §1.3 on hyperkähler 4-manifolds and $K3$ surfaces. Chapter 2 defines a compact 7-manifold M and a family of G_2- structures φ_t on M depending on a parameter t. The main results of the paper are then stated in §2.3. They are divided into three theorems, Theorems A-C. From Theorems A-C we deduce that M has a smooth family of metrics of holonomy G_2, and this is the goal of the paper. The third chapter is then devoted wholly to the proofs of Theorems A-C of §2.3.

Now a G_2- structure on a 7-manifold M defines a metric g and hence a Levi-Civita connection ∇. There is a geometric invariant of the G_2- structure called the *torsion*, which measures how far the G_2- structure is from being preserved by ∇. The G_2- structure comes from a metric g with holonomy contained in G_2 if and only if the torsion is zero, and the G_2- structure is then called *torsion-free*. Theorem A of §2.3 shows that under certain conditions, a G_2- structure φ with small torsion on a compact 7-manifold M may be deformed to a torsion-free G_2- structure $\tilde{\varphi}$. Theorem B shows that the conditions of Theorem A apply to the G_2- structures φ_t on M of §2.2, for small enough t. Theorem C shows that a torsion-free G_2-structure on a compact 7-manifold is part of a smooth family of diffeomorphism classes of torsion-free G_2- structures, with dimension $b^3(M)$.

The most important issue behind the proofs of Theorems A and B is the following. The G_2- structures of Chapter 2 are defined by smoothing off a singular metric (from the singular manifold T^7/Γ) to make it nonsingular, and they depend on a parameter t, which we may think of as the degree of smoothing. The torsion of the G_2- structures is the error introduced by the smoothing process, so that the torsion is small when t is small. But when t is small, the metric is quite close to being singular in some sense, and this means that the curvature is large and the injectivity radius is small.

Our aim is to prove that a small deformation to a torsion-free G_2- structure exists. One would hope to prove a theorem stating that if the torsion is smaller than some a priori bound, then such a deformation exists. However, one would expect this a priori bound to depend on geometric information such as the curvature and the injectivity radius. Therefore there is a problem in proving that the G_2- structures of Chapter 2 can be deformed to zero torsion, because although one can get the torsion of the initial structure very small by choosing t very small, the a priori bound that the torsion must satisfy is also small when t is small.

Thus it is not clear that the a priori bound on the torsion will be satisfied for any t. The way this is solved is by writing inequalities on the torsion, curvature, and so on explicitly in terms of functions of t, and then proving that the powers of t come out in such a way that when t is small, the necessary conditions hold and the deformation to a torsion-free G_2- structure exists. Heuristically speaking, Theorems A and B show that if the torsion is $O(t^4)$, the curvature is $O(t^{-2})$, and the injectivity radius is at least $O(t)$, then for t sufficiently small one may deform to a G_2- structure with zero torsion.

1.1. The holonomy group G_2. We begin with some necessary facts about the structure group G_2, which can be found in [16], Chapter 11. Let $\mathbb{R}^7$ be equipped with an orientation and its standard metric g, and let $y_1, \ldots, y_7$ be an oriented orthonormal basis of $(\mathbb{R}^7)^*$. Define a 3-form φ on $\mathbb{R}^7$ by

$$
\begin{aligned}
\varphi =& y_1 \wedge y_2 \wedge y_7 + y_1 \wedge y_3 \wedge y_6 + y_1 \wedge y_4 \wedge y_5 + y_2 \wedge y_3 \wedge y_5 \\
&- y_2 \wedge y_4 \wedge y_6 + y_3 \wedge y_4 \wedge y_7 + y_5 \wedge y_6 \wedge y_7.
\end{aligned} \tag{1}
$$

Let $GL_+(7,\mathbb{R})$ be the subgroup of $GL(7,\mathbb{R})$ preserving the orientation of $\mathbb{R}^7$. The subgroup of $GL_+(7,\mathbb{R})$ preserving φ is the exceptional Lie group G_2, which is a compact, semisimple, 14-dimensional Lie group. It is a subgroup of $SO(7)$, so that g can be reconstructed from φ. Applying the Hodge star $*$ of g we get the 4-form

$$
\begin{aligned}
*\varphi =& y_1 \wedge y_2 \wedge y_3 \wedge y_4 + y_1 \wedge y_2 \wedge y_5 \wedge y_6 - y_1 \wedge y_3 \wedge y_5 \wedge y_7 + y_1 \wedge y_4 \wedge y_6 \wedge y_7 \\
&+ y_2 \wedge y_3 \wedge y_6 \wedge y_7 + y_2 \wedge y_4 \wedge y_5 \wedge y_7 + y_3 \wedge y_4 \wedge y_5 \wedge y_6.
\end{aligned} \tag{2}
$$

The subgroup of $GL_+(7,\mathbb{R})$ preserving $*\varphi$ is also G_2.

Let M be an oriented 7-manifold, and define $\Lambda^3_+ M, \Lambda^4_+ M$ to be the subsets of $\Lambda^3 T^*M$ and $\Lambda^4 T^*M$ respectively of forms admitting oriented isomorphisms with the forms φ and $*\varphi$ defined by (1) and (2). Then $\Lambda^3_+ M$ and $\Lambda^4_+ M$ are both canonically isomorphic to the bundle of oriented G_2- structures on M, and so have fibre $GL_+(7,\mathbb{R})/G_2$. Let $\Theta : \Lambda^3_+ M \to \Lambda^4_+ M$ be the (nonlinear) natural identification. A dimension count reveals that $\Lambda^3_+ M$ and $\Lambda^4_+ M$ are open subbundles of $\Lambda^3 T^*M$ and $\Lambda^4 T^*M$ respectively.

Let φ be a smooth section of $\Lambda^3_+ M$. Then φ is a smooth 3-form on M, and defines a G_2- structure on M. By abuse of notation, we will usually identify a G_2-structure on M with its 3-form φ. A G_2- structure φ induces a metric g on M from the inclusion $G_2 \subset SO(7)$. With the Hodge star $*$ of g we may define the 4-form $*\varphi$, which by definition of Θ is equal to $\Theta(\varphi)$. Now the most basic invariant of a G- structure on a manifold is called the torsion of the G- structure, and is the obstruction to finding a torsion-free connection ∇ on M preserving the G- structure. The condition for φ to be the G_2- structure of a metric with holonomy contained in G_2 is that the torsion of φ should be zero. Let ∇ be the Levi-Civita connection of g. Then the condition for φ to have zero torsion is that $\nabla\varphi = 0$. By [16, Lemma 11.5], this is equivalent to the condition $d\varphi = d * \varphi = 0$.

When a metric g on M has holonomy group H, the Levi-Civita connection ∇ of g must preserve an H- structure on M, and this in turn implies that the Riemann curvature R of g lies in a bundle with fibre $S^2\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. Therefore a holonomy reduction imposes a linear restriction on the Riemann curvature. For the case of holonomy G_2, [16, Lemma 11.8] shows that metrics with holonomy contained in G_2 are Ricci-flat, which is one reason to study them.

The action of G_2 on $\mathbb{R}^7$ gives an action of G_2 on $\Lambda^k(\mathbb{R}^7)^*$, which splits $\Lambda^k(\mathbb{R}^7)^*$ into an orthogonal direct sum of irreducible representations of G_2. Suppose that M is an oriented 7-manifold with a G_2- structure, so that M has a 3-form φ and a metric g as above. Then in the same way, $\Lambda^k T^*M$ splits into an orthogonal direct sum of subbundles with irreducible representations of G_2 as fibres. In this section we shall describe these splittings, and some results associated with them. We shall use the notation Λ^k_l for an irreducible representation of dimension l lying in $\Lambda^k T^*M$.

Proposition 1.1.1. *Let M be an oriented 7-manifold with G_2- structure, giving a 3-form φ and a metric g on M. Then $\Lambda^k T^*M$ splits orthogonally into components as follows, where Λ^k_l is an irreducible representation of G_2 of dimension l:*

(i)	$\Lambda^1 T^*M = \Lambda^1_7$,	(ii)	$\Lambda^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{14}$,
(iii)	$\Lambda^3 T^*M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$,	(iv)	$\Lambda^4 T^*M = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$,
(v)	$\Lambda^5 T^*M = \Lambda^5_7 \oplus \Lambda^5_{14}$, *and*	(vi)	$\Lambda^6 T^*M = \Lambda^6_7$.

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{7-k}_l. The spaces Λ^k_l can be described as follows:

(a) Λ^2_7 *is the contraction of φ with TM,*

(b) Λ^2_{14} *is the kernel of the map $\xi \mapsto \xi \wedge *\varphi$. It is canonically isomorphic to $\mathfrak{g}_2$.*

(c)	$\Lambda^3_1 = \langle\varphi\rangle$,	(d)	$\Lambda^4_1 = \langle *\varphi\rangle$,
(e)	$\Lambda^4_7 = \varphi \wedge T^*M$,	(f)	$\Lambda^5_7 = *\varphi \wedge T^*M$.

Proof. Part (i) holds as G_2 acts irreducibly on $\mathbb{R}^7$, and parts (ii) and (iii) are given in [16, Lemma 11.4]. Applying the Hodge star we deduce the splittings $(iv), (v)$ and (vi). Parts $(a) - (f)$ are then elementary. $\square$

Let the orthogonal projection from $\Lambda^k T^*M$ to Λ^k_l be denoted π_l. Then, for instance, if $\xi \in C^\infty(\Lambda^2 T^*M)$, $\xi = \pi_7(\xi) + \pi_{14}(\xi)$. This notation will be used throughout the paper.

Lemma 1.1.2. *There exists a 1-form μ on M such that $\pi_7(d\varphi) = 3\mu \wedge \varphi$ and $\pi_7(d * \varphi) = 4\mu \wedge *\varphi$. In particular, if $d\varphi = 0$ then $\pi_7(d * \varphi) = 0$.*

Proof. This can be calculated from the identity $(*d\varphi) \wedge \varphi + (*d * \varphi) \wedge *\varphi = 0$ of Bryant [5, p. 553], using parts (e) and (f) of Proposition 1.1.1. $\square$

Lemma 1.1.3. *Suppose M is a compact, simply-connected 7-manifold, and φ a torsion-free G_2- structure on M. Let g be the metric associated to φ. Then the holonomy of g is G_2.*

Proof. This follows immediately from [5, Lemma 1, p. 563]. It is true because if the holonomy group of g is not G_2 then it must be contained in $SU(3)$, but this forces $b^1(M) = 1$, contradicting the assumption that M is simply-connected. $\square$

1.2. Hölder spaces and elliptic regularity. Let M be a Riemannian manifold with metric g, and V a vector bundle on M with metrics on the fibres and a connection ∇ preserving these metrics. In problems in analysis it is often useful to consider infinite-dimensional vector spaces of sections of V over M, and to equip these vector spaces with norms, making them into Banach spaces. In this paper we will meet three different types of Banach spaces of this sort, written $L^2(V)$, $C^k(V)$ and $C^{k,\alpha}(V)$, and they are defined below.

Define the Lesbesgue space $L^2(V)$ to be the set of locally integrable sections v of V for which the norm $\|v\|_2 = \left(\int_M |v|^2 d\mu\right)^{1/2}$ is finite. Here $d\mu$ is the volume form of the metric g. In fact $L^2(V)$ is a Hilbert space with inner product $\langle v_1, v_2\rangle = \int_M (v_1, v_2) d\mu$, where $(,)$ is the inner product in V. When M is compact, this L^2-inner product has the useful property of integration by parts, so that for instance we have $\langle d\chi, \xi\rangle = \langle \chi, d^*\xi\rangle$ when χ is a k- form and ξ a $k+1$- form on M. For integers $k \geq 0$, define the space $C^k(V)$ to be the space of continuous, bounded sections v of V that have k continuous, bounded derivatives, and define the norm $\|v\|_{C^k}$ by $\|v\|_{C^k} = \Sigma^k_{i=0} \sup_M |\nabla^i v|$.

The third class of vector spaces are the Hölder spaces $C^{k,\alpha}(V)$ for $k \geq 0$ an integer and $\alpha \in (0,1)$. We begin by defining $C^{0,\alpha}(\mathbb{R})$, where $\mathbb{R}$ is regarded as a trivial vector bundle over M. Suppose M is connected, and define the distance $d(x, y)$ between $x, y \in M$ to be the infimum of the lengths of paths γ connecting x and y. Let $\alpha \in (0,1)$. Then a function f on M is said to be *Hölder continuous with exponent α* if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{\bigl|f(x) - f(y)\bigr|}{d(x,y)^\alpha} \tag{3}$$

is finite. Any Hölder continuous function f is continuous. The vector space $C^{0,\alpha}(\mathbb{R})$ is the set of continuous, bounded functions on M which are Hölder continuous with exponent α, and the norm on $C^{0,\alpha}(\mathbb{R})$ is $\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_\alpha$.

In the same way, we would like to define Hölder norms on spaces of sections v of a vector bundle V over M. The trouble with doing this is that in (3) the term $\bigl|f(x) - f(y)\bigr|$ should be replaced by $\bigl|v(x) - v(y)\bigr|$, but $v(x)$ and $v(y)$ lie in different vector spaces, the fibres of V over x and y. To get round this, we shall identify the fibres of V over x and y by parallel translation using ∇ along a path between x and y. Thus we arrive at the following definition of $[v]_\alpha$, by analogy with (3). Define

$$G = \bigl\{\text{smooth maps } \gamma : [0,1] \to M \text{ such that } \mathrm{Im}(\gamma) \text{ is a geodesic in } M\bigr\}, \tag{4}$$

and for each $\gamma \in G$ define $l(\gamma)$ to be the length of the geodesic $\mathrm{Im}(\gamma)$. Let v be a section of V over M, and define $[v]_\alpha$ by

$$[v]_\alpha = \sup_{\gamma \in G} \frac{\bigl|v(\gamma(0)) - v(\gamma(1))\bigr|}{l(\gamma)^\alpha}, \tag{5}$$

whenever the supremum exists. Here the term $\bigl|v(\gamma(0)) - v(\gamma(1))\bigr|$ is defined by identifying the fibres of V over $\gamma(0)$ and $\gamma(1)$ by parallel translation along γ using ∇. Since ∇ preserves the metrics in the fibres, the metric on this identified vector space is well-defined.

Define $C^{k,\alpha}(V)$ to be the set of v in $C^k(V)$ for which the supremum $[\nabla^k v]_\alpha$ defined by (5) exists, working in the vector bundle $\otimes^k T^*M \otimes V$ with its natural metric and connection. The Hölder norm on $C^{k,\alpha}(V)$ is $\|v\|_{C^{k,\alpha}} = \|v\|_{C^k} + [\nabla^k v]_\alpha$. With this norm, $C^{k,\alpha}(V)$ is a Banach space. The condition of Hölder continuity is analogous to a sort of fractional differentiability. To see this, observe that if $v \in C^1(V)$, then by the mean value theorem $[v]_\alpha$ exists, and

$$[v]_\alpha \leq \bigl(2\|v\|_{C^0}\bigr)^{1-\alpha} \|\nabla v\|_{C^0}^\alpha. \tag{6}$$

Thus $[v]_\alpha$ is a sort of interpolation between the C^0- and C^1- norms of v. It can be helpful to think of $C^{k,\alpha}(V)$ as the space of sections of V that are $k+\alpha$- times differentiable.

Now Hölder spaces are useful tools for problems involving elliptic partial differential operators, because they have a property known as elliptic regularity. Suppose that V and W are vector bundles of the same dimension over M, and that $P : C^\infty(V) \to C^\infty(W)$ is a linear elliptic operator of order l. If $P(v) = w$, where $v \in C^l(V)$ and $w \in C^{k,\alpha}(W)$, then it is in general true that $v \in C^{k+l,\alpha}(V)$. In other words, v has the maximum number of (fractional) derivatives that the problem allows. However, it is not in general true that v must be in $C^{k+l}(M)$ if $w \in C^k(M)$. This is why we work with Hölder spaces rather than the simpler spaces $C^k(V)$. Here are two elliptic regularity results for elliptic operators on Hölder spaces. The first is deduced from [4, Theorems 27, 31, p. 463-4].

Proposition 1.2.1. *Suppose M is a compact Riemannian manifold, V, W are vector bundles over M of the same dimension, and $P : C^\infty(V) \to C^\infty(W)$ is a smooth, linear, elliptic differential operator of order l. Let $\alpha \in (0,1)$ and $k \geq 0$ be an integer. Then P extends to $P : C^{k+l,\alpha}(V) \to C^{k,\alpha}(W)$, and in each of these spaces* $\operatorname{Ker} P$ *is a finite-dimensional subspace of $C^\infty(V)$.*

Suppose that $P(v) = w$ holds weakly, with $v \in L^2(V)$ and $w \in L^2(W)$. If $w \in C^\infty(W)$, then $v \in C^\infty(V)$. If $w \in C^{k,\alpha}(W)$, then $v \in C^{k+l,\alpha}(V)$, and

$$\|v\|_{C^{k+l,\alpha}} \leq C\big(\|w\|_{C^{k,\alpha}} + \|v\|_2\big) \tag{7}$$

for some constant C independent of v, w. Moreover, if v is L^2- orthogonal to $\operatorname{Ker} P$ *then the term $C\|v\|_2$ may be omitted from (7) by increasing the constant C.*

Here is a similar statement for P Hölder continuous rather than smooth, that is deduced from [2, Theorem 3.55].

Proposition 1.2.2. *Let M be a compact Riemannian manifold, and V, W vector bundles over M of the same dimension. Let $\alpha \in (0,1)$, $k \geq 0$ be an integer, and $P : C^2(V) \to C^0(W)$ be a linear, elliptic differential operator of order 2 with coefficients in $C^{k,\alpha}$. Suppose that $P(v) = w$ holds almost everywhere, where $v \in C^2(V)$ and $w \in C^{k,\alpha}(W)$. Then $v \in C^{k+2,\alpha}(V)$.*

1.3. Hyperkähler 4-manifolds and $K3$ surfaces. A metric on an oriented 4-manifold with holonomy contained in $SU(2)$ is called a hyperkähler structure [16, p. 114]. A hyperkähler 4-manifold is Ricci-flat and self-dual, and its metric is Kähler w.r.t. each of three anticommuting complex structures. Equivalently, we may define a hyperkähler structure on an oriented 4-manifold X to be a triple $(\omega_1, \omega_2, \omega_3)$ of smooth, closed 2-forms on X, that can at each point x be written as

$$\omega_1 = y_1 \wedge y_2 + y_3 \wedge y_4, \qquad \omega_2 = y_1 \wedge y_3 - y_2 \wedge y_4, \qquad \omega_3 = y_1 \wedge y_4 + y_2 \wedge y_3, \tag{8}$$

where $(y_1, \ldots, y_4)$ is an oriented basis of $T_x^* X$. Hyperkähler 4-manifolds have received a lot of attention, and much is known about different compact and noncompact examples.

Perhaps the simplest nontrivial example of a hyperkähler 4-manifold is the Eguchi-Hanson space [9], which is a complete hyperkähler metric on the noncompact 4-manifold $T^*\mathbb{CP}^1$. We will give this metric explicitly in coordinates. Consider $\mathbb{C}^2$ with complex coordinates (z_1, z_2), acted upon by the involution $-1 : (z_1, z_2) \mapsto (-z_1, -z_2)$. Let X be the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at the singular point. Then X is biholomorphic to $T^*\mathbb{CP}^1$, and has $\pi_1(X) = \{1\}$ and $H^2(X, \mathbb{R}) = \mathbb{R}$. The closed, holomorphic 2-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ descends to $\mathbb{C}^2/\{\pm 1\}$, and thus lifts to X. Define closed 2-forms ω_2, ω_3 on X by $\omega_2 + i\omega_3 = dz_1 \wedge dz_2$. The function $u = |z_1|^2 + |z_2|^2$ on $\mathbb{C}^2$ descends to $\mathbb{C}^2/\{\pm 1\}$ and so lifts to X. Let $t > 0$ be a positive constant, and define a function f on X by

$$f = \sqrt{u^2 + t^4} + t^2 \log u - t^2 \log\left(\sqrt{u^2 + t^4} + t^2\right). \tag{9}$$

This is the Kähler potential for ω_1, and is taken from [13, p. 593]. Define the 2-form ω_1 on X by $\omega_1 = \frac{1}{2} i \partial\bar{\partial} f$. Then ω_1 is a closed 2-form, and it can be shown that the triple $(\omega_1, \omega_2, \omega_3)$ may be written in the form (8), and thus form a hyperkähler structure on X.

The transformation $(z_1, z_2) \mapsto (cz_1, cz_2)$ for some positive constant c induces an endomorphism of X, which takes $\omega_i(t)$ to $c^{-2}\omega_i(ct)$ for $i = 1, 2, 3$. Thus the hyperkähler metrics on X induced by different positive values of t are all equivalent modulo diffeomorphisms and homotheties. Putting $t = 0$ gives $f = u$ in (9), and then ω_1 is the Kähler form of the Euclidean metric on $\mathbb{C}^2/\{\pm 1\}$, so that in the limiting case $t = 0$ the metric becomes degenerate along the exceptional curve, and away from this is equal to the flat hyperkähler metric on $\mathbb{C}^2/\{\pm 1\}$. This indicates that the Eguchi-Hanson metric is asymptotic to the flat metric on $\mathbb{C}^2/\{\pm 1\}$ at infinity.

The only compact hyperkähler 4-manifolds are flat tori and $K3$ surfaces. $K3$ surfaces are compact complex surfaces with $b_1 = c_1 = 0$. They are all diffeomorphic, and are very interesting from a number of different points of view. By Yau's proof of the Calabi conjecture [19], the $K3$ surface possesses a 58-parameter family of metrics of holonomy $SU(2)$. An approximate description of some of these hyperkähler metrics was given by Page [15], which employs an idea known as the Kummer construction. Let T^4 be the 4-torus with a flat Riemannian metric, and let $\sigma : T^4 \to T^4$ be an isometric involution that reverses directions on T^4. Then σ has 16 fixed points, so T^4/σ has 16 singular points modelled on the origin in $\mathbb{R}^4/\{\pm 1\}$.

Page observed that gluing 16 small copies of the Eguchi-Hanson space in place of small neighbourhoods of the singular points yields a metric on $K3$ that is close to being hyperkähler, in the sense that the errors introduced by the gluing are small when the Eguchi-Hanson metrics are small. Later, Topiwala [17] and LeBrun and Singer [14] gave proofs that the $K3$ surface admits hyperkähler metrics using Page's idea. These proofs use ideas from twistor theory and the deformation theory of singular complex manifolds.

2. A 'Kummer construction' for a compact 7-manifold

Perhaps the best known and most studied nontrivial example of a compact manifold admitting metrics of special holonomy is the $K3$ surface of §1.3, which is a compact 4-manifold possessing a family of metrics with holonomy $SU(2)$. This moduli space of metrics is a manifold parametrized by the cohomology classes of 3 constant 2-forms on the $K3$ surface. The metrics are not known explicitly, and not very much is known about what the general $K3$ metric 'looks like'. However, orbifolds of the torus T^4 appear naturally as limits at the boundary of the moduli space, and therefore one can get a fairly good grasp of what the metrics in the moduli space close to these orbifolds are like, as they arise from desingularizing the flat, singular manifold T^4/Γ in a certain way, where Γ is a finite group. This construction was described in §1.3, and is known as the Kummer construction for the $K3$ surface.

The situation for metrics on holonomy G_2 on the 7-manifold M we shall describe shares many features with the $K3$ surface. There is a family of metrics of special holonomy which we cannot write down explicitly, but which is parametrized (at least locally) by the cohomology classes of constant 3- and 4- forms $\varphi, *\varphi$. Orbifolds of the torus T^7 appear in a certain way as limits at the boundary of the moduli space, and we can give a good approximate description of the metrics in the moduli space close to these limits. Thus $K3$ surfaces furnish a good analogy, and are useful as a mental picture and in deciding what to aim for – since, for instance, noone has yet succeeded in writing down a $K3$ metric explicitly, we are unlikely to be able to give an explicit metric of holonomy G_2 on M in the forseeable future, as this is surely a more difficult problem.

However, the $K3$ picture also has features which the G_2 picture probably does not share. For underlying the metric of holonomy $SU(2)$ is a complex structure – a family of complex structures, in fact – and so the rhythms of complex and even algebraic geometry run through the study of metrics on $K3$. Complex geometry, with its local triviality and lack of metrics, and algebraic geometry, with the possibility of complete description of the objects of study and their moduli, seem very different to Riemannian geometry. Since we know of no underlying structure in the G_2 case comparable to complex geometry, it seems likely that the extraordinarily good behaviour of the moduli space of $K3$ metrics may not extend to the G_2 case. For instance, metrics in the moduli space may develop singularities in a disorderly way, and the set of cohomology classes $[\varphi]$ realized by torsion-free G_2- structures on M may well contain ragged holes rather than being defined by pleasing linear constraints. But this is only speculation.

In §2.1 a finite group $\Gamma \cong \mathbb{Z}_2^3$ of automorphisms of T^7 is defined. This group preserves a flat G_2- structure on T^7. The singularities of the quotient T^7/Γ are determined and described. Now Γ has been chosen very carefully so that the singular set of T^7/Γ is particularly simple and well-behaved. It consists of 12 disjoint copies of T^3, and each component T^3 of the singular set has a neighbourhood in T^7/Γ of the form $T^3 \times \big(B^4/\{\pm 1\}\big)$, where B^4 is the open unit ball in $\mathbb{R}^4$.

Therefore, to desingularize T^7/Γ it is enough to be able to desingularize $B^4/\{\pm 1\}$. In §2.2 a compact 7-manifold M is defined by desingularizing T^7/Γ, and the method used to desingularize $B^4/\{\pm 1\}$ is exactly that used in the Kummer construction of the $K3$ surface. Also in §2.2 a family of G_2- structures φ_t on M depending on a parameter t is defined. These satisfy $d\varphi_t = 0$ and $d * \varphi_t = O(t^4)$. Then in §2.3 the main results of the paper are given, which lead to the existence of a family of metrics of holonomy G_2 on M, stated in Theorem 2.3.1. The proofs of the results are deferred to Chapter 3.

2.1. A finite group action on T^7. Let $(x_1, \dots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Define a section $\hat\varphi$ of $\Lambda^3_+ T^7$ by equation (1) of §1.1, where y_i is replaced by dx_i. Let α, β and γ be the involutions of T^7 defined by

$$\alpha\big((x_1, \dots, x_7)\big) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7), \tag{10}$$

$$\beta\big((x_1, \dots, x_7)\big) = (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7), \tag{11}$$

$$\gamma\big((x_1, \dots, x_7)\big) = (\tfrac{1}{2} - x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, -x_7), \tag{12}$$

By inspection, α, β and γ preserve $\hat\varphi$, because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Thus they generate a group $\langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of T^7 preserving the flat G_2-structure $\hat\varphi$.

In the next two Lemmas, we shall describe the singular set S of T^7/Γ.

Lemma 2.1.1. *The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and $\alpha\beta\gamma$ of Γ have no fixed points on T^7. The fixed points of α in T^7 are 16 copies of T^3, and the group $\langle \beta, \gamma \rangle$ acts freely on the set of 16 3-tori fixed by α. Similarly, the fixed points of β, γ in T^7*

are each 16 copies of T^3, and the groups $\langle\alpha,\gamma\rangle$ and $\langle\alpha,\beta\rangle$ act freely on the sets of 16 3-tori fixed by β,γ respectively.

Proof. The element $\beta\gamma$ acts on the coordinate x_1 by $x_1 \mapsto x_1 + \frac{1}{2}$. Therefore $\beta\gamma$ can have no fixed point x, because the x_1- coordinates of x and $\beta\gamma(x)$ are different. Similarly, $\gamma\alpha$ changes coordinates x_1 and x_3, $\alpha\beta$ changes coordinate x_2, and $\alpha\beta\gamma$ changes coordinate x_3. Thus none of these elements have fixed points.

By inspection, the fixed points of α are $x_1, x_2, x_3, x_4 \in \{0, \frac{1}{2}\}$, which clearly divide into 16 disjoint copies of T^3. The action of β on these 16 copies of T^3 fixes x_1, x_3 and x_4, and takes x_2 to $x_2 + \frac{1}{2}$. The action of γ on the 16 copies of T^3 fixes x_2 and x_4 and takes x_1 to $x_1 + \frac{1}{2}$ and x_3 to $x_3 + \frac{1}{2}$. Therefore the group $\langle\beta,\gamma\rangle$ does act freely on the set of 16 fixed 3-tori of α. The rest of the Lemma uses the same argument, and is left to the reader. $\square$

Lemma 2.1.2. *The singular set S of T^7/Γ is a disjoint union of 12 copies of T^3. There is an open subset T of T^7/Γ containing S, such that each of the 12 connected components of T is isometric to $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$, where B^4_ζ is the open ball of radius ζ in $\mathbb{R}^4$ for some positive constant ζ ($\zeta = 1/9$ will do).*

Proof. The singular set S is exactly the image in T^7/Γ of the set S' of points in T^7 that are fixed by some nonidentity element of Γ. By Lemma 2.1.1, the only nonidentity elements of Γ with fixed points are α, β and γ, and for each of these the fixed point set in T^7 is 16 copies of T^3. This gives 48 copies of T^3 in T^7. Now these 48 3-tori are all disjoint. It is clear that distinct 3-tori fixed by the same element α, β or γ are disjoint. Suppose that two 3-tori fixed by different elements, say α and β, intersect. Then the intersection point is fixed by both α and β, so it is a fixed point of $\alpha\beta$, which contradicts Lemma 2.1.1.

Thus S' is the disjoint union of 48 copies of T^3 in T^7, and $S = S'/\Gamma$. Now each T^3- component of S' is defined by setting four of the coordinates $x_1, \ldots, x_7$ to some values from $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. Therefore the distance between distinct T^3- components of S' is at least $1/4$. Let T' be the set of points in T^7 a distance less than $\zeta = 1/9$ from S'. Because $1/9 + 1/9 < 1/4$, T' splits into 48 components, each isometric to $T^3 \times B^4_\zeta$. Define $T \subset T^7/\Gamma$ by $T = T'/\Gamma$. Then T is a 'tubular neighbourhood' of S.

Since the group $\langle\beta,\gamma\rangle$ acts freely on the 16 3-tori fixed by α, it follows that the 3-tori fixed by α contribute 4 copies of T^3 to S. Similarly, the fixed 3-tori of β and γ each contribute 4 3-tori to S. Therefore S consists of 12 disjoint copies of T^3, so that T has 12 components. Examining the action of α, β and γ near their fixed 3-tori, it is clear that the component of T containing each T^3- component of S is isometric to $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$. This completes the Lemma. $\square$

2.2. A compact 7-manifold, and a family of G_2- structures. Let Γ, S and T be as in §2.1. We shall define a compact 7-manifold M using T^7/Γ, and a family of closed G_2- structures $\left\{\varphi_t : t \in (0, \theta]\right\}$ on M. Using the decomposition $T \cong S \times \left(B^4_\zeta/\{\pm 1\}\right)$, the flat G_2- structure $\hat\varphi$ on T^7/Γ and its dual $*\hat\varphi$ may be written as

$$\hat\varphi = \hat\omega_1 \wedge \delta_1 + \hat\omega_2 \wedge \delta_2 + \hat\omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3, \tag{13}$$

$$*\hat\varphi = \hat\omega_1 \wedge \delta_2 \wedge \delta_3 + \hat\omega_2 \wedge \delta_3 \wedge \delta_1 + \hat\omega_3 \wedge \delta_1 \wedge \delta_2 + \tfrac{1}{2}\hat\omega_1 \wedge \hat\omega_1, \tag{14}$$

where δ_1, δ_2 and δ_3 are constant orthonormal 1-forms on S, and $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3$ are constant 2-forms on $B^4_\zeta/\{\pm 1\}$ that can be written in the form (8) given in §1.3.

In order to desingularize T^7/Γ, we shall replace each factor $B^4_\zeta/\{\pm 1\}$ in the decomposition of T by a nonsingular 4-manifold U that agrees with $B^4_\zeta/\{\pm 1\}$ in a neighbourhood of its boundary. To define a G_2- structure φ_t upon the resulting 7-manifold M, U will be given a triple $\omega_1(t), \omega_2(t), \omega_3(t)$ of 2-forms that agree with $\hat{\omega}_j$ near the boundary. Our construction of U will follow that of the Eguchi-Hanson space in §1.3. Regard B^4_ζ as a subset of $\mathbb{C}^2$ with complex coordinates (z_1, z_2), and let U be the blow-up of $B^4_\zeta/\{\pm 1\}$ at its singular point. Define real, closed 2-forms $\omega_2(t), \omega_3(t)$ on U as in §1.3 by $\omega_2(t) + i\omega_3(t) = dz_1 \wedge dz_2$. Let u be the function $|z_1|^2 + |z_2|^2$ on U.

Let $\tau : [0, \zeta^2] \to [0, 1]$ be a fixed, smooth, nonincreasing function with $\tau(r) = 1$ for $r \le \zeta^2/4$ and $\tau(r) = 0$ for $r \ge \zeta^2/2$. For $t > 0$ define a function f_t on U by

$$f_t = \sqrt{u^2 + \tau^2(u)t^4} + \tau(u)t^2 \log u - \tau(u)t^2 \log\left(\sqrt{u^2 + \tau^2(u)t^4} + \tau(u)t^2\right), \quad (15)$$

as in (9). Define the 2-form $\omega_1(t)$ on U by $\omega_1(t) = \frac{1}{2} i\partial\overline{\partial} f_t$. Then $\omega_1(t)$ is closed. Where $u \le \zeta^2/4$ this triple $\{\omega_j(t)\}$ is the hyperkähler structure of the Eguchi-Hanson space of §1.3, and where $u \ge \zeta^2/2$ it is the flat triple $\{\hat{\omega}_j\}$. Thus the triple $\{\omega_j(t)\}$ interpolates between the hyperkähler structure of the Eguchi-Hanson space and the Euclidean structure on $B^4_\zeta/\{\pm 1\}$.

Define M to be the compact, nonsingular 7-manifold without boundary that is obtained by replacing each factor $B^4_\zeta/\{\pm 1\}$ by U in the decomposition of T, in the obvious way. Define a 3-form φ_t and a 4-form υ_t on M by $\varphi_t = \hat{\varphi}$ and $\upsilon_t = *\hat{\varphi}$ on $(T^7/\Gamma) \setminus T$, and

$$\varphi_t = \omega_1(t) \wedge \delta_1 + \omega_2(t) \wedge \delta_2 + \omega_3(t) \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3 \quad (16)$$

$$\upsilon_t = \omega_1(t) \wedge \delta_2 \wedge \delta_3 + \omega_2(t) \wedge \delta_3 \wedge \delta_1 + \omega_3(t) \wedge \delta_1 \wedge \delta_2 + \tfrac{1}{2}\omega_1(t) \wedge \omega_1(t), \quad (17)$$

on T. Since by definition the 2-forms $\omega_j(t)$ agree with $\hat{\omega}_j$ near the boundary of U, φ_t and υ_t are smooth, and as $\omega_j(t)$, δ_j are closed, φ_t and υ_t are closed.

Now the 2-forms ω_j of the Eguchi-Hanson space of §1.3 are in the form (8), so comparing the definitions (9), (15) we see that the 2-forms $\omega_j(t)$ are in the form (8) on U except where $\tau(u)$ has nonzero first or second derivatives, which is on the open annulus $u \in (\zeta^2/4, \zeta^2/2)$. Let $A \subset M$ be the subset where $u \in (\zeta^2/4, \zeta^2/2)$. Then A is an open subset of T, the product of S and an annulus in B^4_ζ. On $M \setminus A$, φ_t is a section of Λ^3_+ and $\upsilon_t = \Theta(\varphi_t)$. But on A the 2-forms $\omega_j(t)$ are not in the form (8), so φ_t need not even be a section of Λ^3_+. However, when t is small, the 2-forms $\omega_j(t)$ are close to satisfying (8), and thus φ_t is a section of Λ^3_+ because Λ^3_+ is an open subbundle of $\Lambda^3 T^*M$. For the same reason, when t is small, υ_t is close to $\Theta(\varphi_t)$ on A.

The terms in $\omega_1(t)$ due to the derivatives of $\tau(u)$ may be seen to be $O(t^4)$. Since these are the terms that cause the triple $\omega_j(t)$ to deviate from the form (8), it follows that $\upsilon_t - \Theta(\varphi_t)$ and all its derivatives are $O(t^4)$ for small t. Therefore there exist positive constants θ, D_1 such that when $0 < t \le \theta$, φ_t is a section of Λ^3_+, and the 3-form ψ_t on M defined by $*\psi_t = \Theta(\varphi_t) - \upsilon_t$ is smooth and satisfies $\|\psi_t\|_2 \le D_1 t^4$ and $\|\psi_t\|_{C^2} \le D_1 t^4$, where the metrics and Hodge star are those induced by the G_2- structure φ_t. This 3-form ψ_t satisfies $d * \psi_t = d\Theta(\varphi_t)$, as $d\upsilon_t = 0$, and this is equivalent to $d^*\psi_t = d^*\varphi_t$, since $\Theta(\varphi_t) = *\varphi_t$.

What the definitions above mean is as follows. The compact 7-manifold M is divided into three regions. The first region, the subset of T on which $u \le \zeta^2/4$, is the product of S, and a closed subset U of the Eguchi-Hanson space X of §1.3. On this region the G_2- structure φ_t is torsion-free, because it is the product of the flat structure on S and the torsion-free $SU(2)$- structure on the Eguchi-Hanson space. The Eguchi-Hanson space metric has parameter t, which is proportional to the diameter of the exceptional curve in X. Therefore t measures the diameter of the exceptional set introduced to desingularize T^7/Γ. The second region is a subset of T^7/Γ, and on it φ_t is equal to the flat G_2- structure $\hat{\varphi}$ of T^7/Γ.

The third region, A, is a collar between the first two regions. On A the G_2-structure φ_t has to interpolate smoothly between its values on the first and second regions, and it achieves this using a partitition of unity function $\tau(u)$. On A, φ_t is not torsion-free because the derivatives of $\tau(u)$ introduce torsion terms. This means that although $d\varphi_t = 0$, $d\,\Theta(\varphi_t)$ is not identically zero on A, but is $O(t^4)$. For later convenience we introduce a 3-form ψ_t which is also $O(t^4)$, such that $d{*}\psi_t = d\,\Theta(\varphi_t)$. It is important that the region A is independent of t, and does not become small when t is small, because this way the 'error term' $d\,\Theta(\varphi_t)$ is spread thinly over a large volume instead of being concentrated in a small one. Thus we make the error term as small as possible, which will be crucial to the proof.

Now let us consider the topology of M. It is made by putting patches of the form $T^3 \times U$ on the singularities of T^7/Γ. It is easy to show that both both U and T^7/Γ are simply-connected, so it follows that M is simply-connected. Thus the first Betti number $b^1(M)$ is zero. The other Betti numbers may be calculated too. It can be shown that $b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 7$, and there are 12 patches of the form $T^3 \times U$ each of which add 1 to b^2 and 3 to b^3. Therefore $b^2(M) = 12$ and $b^3(M) = 43$.

2.3. The main results. We are now ready to state the results of the paper. They hinge upon the following three theorems, which will be proved in Chapter 3.

Theorem A. *Let $E_1, \ldots, E_5$ be positive constants. Then there exist positive constants κ, K depending only on $E_1, \ldots, E_5$, such that whenever $0 < t \le \kappa$, the following is true.*

Let M be a compact 7-manifold, and φ a smooth, closed section of $\Lambda^3_+ M$ on M. Suppose that ψ is a smooth 3-form on M with $d^*\psi = d^*\varphi$, and that these four conditions hold:

(i) $\|\psi\|_2 \le E_1 t^4$ *and* $\|\psi\|_{C^{1,1/2}} \le E_1 t^4$,

(ii) *if* $\chi \in C^{1,1/2}(\Lambda^3 T^* M)$ *and* $d\chi = 0$ *then* $\|\chi\|_{C^0} \le E_2\big(t\|\nabla\chi\|_{C^0} + t^{-7/2}\|\chi\|_2\big)$ *and* $\|\nabla\chi\|_{C^0} + t^{1/2}[\nabla\chi]_{1/2} \le E_3\big(\|d^*\chi\|_{C^0} + t^{1/2}[d^*\chi]_{1/2} + t^{-9/2}\|\chi\|_2\big)$,

(iii) $1 \le E_4 \,\mathrm{vol}(M)$, *and*

(iv) *if f is a smooth, real function and* $\int_M f d\mu = 0$ *then* $\|f\|_2 \le E_5 \|df\|_2$.

Then there exists $\eta \in C^\infty(\Lambda^2 T^* M)$ with $\|d\eta\|_{C^0} \le K t^{1/2}$, such that $\tilde{\varphi} = \varphi + d\eta$ is a smooth, torsion-free G_2- structure.

Theorem B. *Let $D_1, \ldots, D_5$ be positive constants. Then there exist positive* *constants* $E_1,\dots,E_5$ *and* λ *depending only on* $D_1,\dots,D_5$*, such that for every* $t\in(0,\lambda]$*, the following is true.*

Let M *be a compact 7-manifold, and* φ *a smooth, closed section of* Λ^3_+M *on* M*. Let* g *be the metric associated to* φ*. Suppose that* ψ *is a smooth 3-form on* M *with* $d^*\psi = d^*\varphi$*, and that these five conditions hold:*

(i) $\|\psi\|_2 \le D_1 t^4$ *and* $\|\psi\|_{C^2} \le D_1 t^4$*,*

(ii) *the injectivity radius* $\delta(g)$ *satisfies* $\delta(g) \ge D_2 t$*,*

(iii) *the Riemann curvature* $R(g)$ *of* g *satisfies* $\|R(g)\|_{C^0} \le D_3 t^{-2}$*,*

(iv) *the volume* $\mathrm{vol}(M)$ *satisfies* $\mathrm{vol}(M) \ge D_4$*, and*

(v) *the diameter* $\mathrm{diam}(M)$ *satisfies* $\mathrm{diam}(M) \le D_5$*.*

Then conditions (i)*-*(iv) *of Theorem A hold for* (M,φ)*.*

Theorem C. *Let* M *be a compact 7-manifold, let* X *be the set of torsion-free* G_2*- structures on* M*, and let* $\mathrm{Diff}_0(M)$ *be the group of diffeomorphisms of* M *isotopic to the identity. Define a map* $\Xi : X \to H^3(M,\mathbb{R})$ *by* $\Xi(\varphi) = [\varphi]$*. Then* Ξ *is invariant under the action of* $\mathrm{Diff}_0(M)$ *on* X*, and if* $\varphi \in X$ *then there exists an open subset* $Y \subset X$ *that contains* φ *and is invariant under* $\mathrm{Diff}_0(M)$*, such that* Ξ *induces an isomorphism between* $Y/\mathrm{Diff}_0(M)$ *and an open ball about* $[\varphi]$ *in* $H^3(M,\mathbb{R})$*.*

We note that the result of Theorem C was first announced by Bryant and Harvey [5, p. 561], and they have a proof of it that predates the proof given in §3.3 by a number of years, but that this proof has not yet been published. Using Theorems A-C we may prove the existence of metrics of holonomy G_2 on M, which is the main result of the paper.

Theorem 2.3.1. *The compact, simply-connected 7-manifold* M *of §2.2 admits a 43-dimensional family of metrics of holonomy* G_2*.*

Proof. We will show that for small t, the hypotheses of Theorem B hold with $\varphi = \varphi_t$ and $\psi = \psi_t$, where φ_t, ψ_t are as defined in §2.2. From §2.2 we have $\|\psi_t\|_2 \le D_1 t^4$ and $\|\psi_t\|_{C^2} \le D_1 t^4$ when $t \le \theta$, so part (i) of Theorem B holds with the constant D_1 of §2.2. Now the metric g_t induced by φ_t is defined in §2.2 by gluing ends of the form $T^3 \times U$ into T^7/Γ, where T^3 and T^7/Γ carry fixed, flat metrics, and U is a subset of the Eguchi-Hanson space shrunk by a homothety multiplying distances by t. It is therefore clear that parts (ii)-(v) of Theorem B hold for the metric g_t on M when $t \le \theta$, for some constants $D_2,\dots,D_5$ independent of t.

So parts (i)-(v) of Theorem B hold for (M,φ_t) for $t \le \theta$, and by Theorem B there exist positive constants $E_1,\dots,E_5$ and λ depending on $D_1,\dots,D_5$, such that parts (i)-(iv) of Theorem A hold for (M,φ_t) when $t \le \min(\theta,\lambda)$. By Theorem A there is a constant $\kappa > 0$, such that for $t \le \min(\theta,\lambda,\kappa)$, the G_2- structure φ_t on M may be deformed to a torsion-free G_2- structure $\tilde\varphi_t$ on M. Thus M admits torsion-free G_2- structures. By Theorem C, the family of torsion-free G_2- structures on M is a manifold locally isomorphic to $H^3(M,\mathbb{R})$. But $b^3(M) = 43$ from §2.2, so the dimension of the family is 43. Since M is simply-connected, the holonomy group of the associated metrics is G_2, by Lemma 1.1.3. Therefore there is a 43-dimensional family of metrics of holonomy G_2 on M. □

3. The existence of torsion-free G_2- structures on M

This chapter contains the proofs of Theorems A-C of §2.3. Theorem A will be proved in §3.1, Theorem B in §3.2, and Theorem C in §3.3. I have found several different proofs of the existence of metrics of holonomy G_2 on the 7-manifold M of §2.2, and the current formulation as Theorems A and B is the shortest and the one I like best. The point about the hypotheses of Theorem B is that they actually use very little information about the 7-manifold M and the G_2- structures φ. Conditions (ii)-(v) of Theorem B do not depend on the G_2- structure φ, but only on the metric g on M it induces, and they do not use global information about the manifold in any significant way.

Previous versions of the proof used geometric quantities that are global in nature, such as the first nonzero eigenvalue of a Laplacian operator, and constants estimating the norm of various Sobolev embeddings. The trouble with these proofs was that it takes rather more work to estimate a geometric quantity that does not depend on purely local information, so the proofs were lengthy, and are also not as general as the current proofs. As the price of proving a result from minimal information, the proof of Theorem A is slightly devious, and uses the special geometry of G_2 a lot, which may make it difficult to use the same method of proof for similar geometric problems.

Here are the general considerations that motivate the design of the proof of Theorem A in §3.1. Our goal is to start with a G_2- structure φ with small torsion, and deform it to a 3-form $\tilde\varphi$ with zero torsion. From §1.1, this means that $\tilde\varphi$ must satisfy the two equations $d\tilde\varphi = 0$ and $d\,\Theta(\tilde\varphi) = 0$. We may satisfy the first equation automatically by starting with $d\varphi = 0$, and choosing $\tilde\varphi = \varphi + d\eta$. Thus we seek a 2-form η satisfying the equation $d\,\Theta(\varphi + d\eta) = 0$.

Now this is not a good equation to try and solve, because its linearization in η is not elliptic. It fails to be elliptic for two reasons. The first reason is that adding a closed 2-form to η does not change $\tilde\varphi$, so that the solution space acquires an infinite-dimensional factor from the closed 2-forms. To eliminate this we shall require that $d^*\eta = 0$. The second reason is that since the problem is diffeomorphism-invariant, if $\tilde\varphi$ is a solution then so is the image of $\tilde\varphi$ under any diffeomorphism, so that there is an infinite-dimensional space of solutions.

The method we use is to construct a 2-form η and a real number ϵ satisfying a certain nonlinear equation with elliptic linearization. Having constructed these solutions, we will show using some special geometric facts about G_2 that a solution to this nonlinear equation satisfies $d\,\Theta(\varphi + d\eta) = 0$. It also turns out to satisfy $\pi_7(\eta) = 0$, and this can be interpreted as the 'gauge-fixing' condition for the diffeomorphism group: in order to pick out a unique representative in each orbit of the diffeomorphism group on $C^\infty(\Lambda^3_+)$ we expect to impose some restriction on the data η, and $\pi_7(\eta) = 0$ is the natural condition.

Having chosen the nonlinear, elliptic equation, we attempt to find a solution to it by defining a sequence $\{\eta_j\}_{j=0}^\infty$ of 2-forms that are successive approximations to a solution, and then showing that the sequence converges. One must ensure that the terms η_j remain small, and do not diverge to infinity. The art in proofs of this sort is to choose the right norms – Lesbesgue norms, Hölder norms, Sobolev norms – to control the terms η_i of the sequence. It turns out that because of an integration by parts formula, the L^2- norm of $d\eta_j$ is very well controlled; in fact for small η_j, $\|d\eta_{j+1}\|_2 \le C_1 + C_2\|d\eta_j\|_2\|d\eta_j\|_{C^0}$ for constants C_1, C_2, where $C_1 = O(t^4)$. This makes the norm $\|d\eta_j\|_2$ a good choice to work with.

However, we cannot work with this norm alone because the estimate involves $\|d\eta_j\|_{C^0}$ as well. The sequence element η_{j+1} is defined by setting $P(\eta_{j+1}) = Q(\eta_j, \nabla\eta_j, \nabla^2\eta_j)$, where P is a second-order, linear, elliptic operator, and Q is a nonlinear function. Naïve attempts to estimate some norm of η_{j+1} always seem to involve some stronger norm of η_j, just as the C^0- norm is a stronger norm than the L^2- norm. The way to break this cycle is to use elliptic regularity results, analogous to Propositions 1.2.1 and 1.2.2. If η_j is controlled in $C^{k+2,\alpha}$, then $Q(\eta_j, \nabla\eta_j, \nabla^2\eta_j)$ is controlled in $C^{k,\alpha}$, and therefore by elliptic regularity, η_{j+1} is controlled in $C^{k+2,\alpha}$.

Thus elliptic regularity allows us to estimate a Hölder norm of η_{j+1} in terms of *the same* Hölder norm of η_j. Without this property our proof would never get off the ground. In the proof of Theorem A, we shall use the norms $\|d\eta_j\|_2$ and $\|d\eta_j\|_{C^{1,1/2}}$ to control the sequence $\{\eta_j\}_{j=0}^{\infty}$, and we will be able to show that the sequence exists, is bounded in these norms, and is convergent in the associated topologies. This is the thinking behind Theorem A.

3.1. An existence result for metrics of holonomy G_2. In this section we prove the core result of the paper, Theorem A, which states that a G_2- structure φ on a compact 7-manifold M with $d\varphi = 0$ and $d*\varphi$ small in a suitable sense can, under suitable conditions, be deformed to a G_2- structure $\tilde\varphi$ with zero torsion. We begin with a result estimating the function Θ of §1.1.

Lemma 3.1.1. *There exist real, positive, universal constants $e_1, \ldots, e_4$ such that whenever M is a 7-manifold and φ is a closed section of Λ^3_+, the following is true.*

Suppose that $\chi \in C^0(\Lambda^3 T^*M)$ and $\|\chi\|_{C^0} \le e_1$. Then $\varphi + \chi \in C^0(\Lambda^3_+ M)$ and $\Theta(\varphi+\chi)$ is given by

$$\Theta(\varphi+\chi) = *\varphi + \tfrac{4}{3} * \pi_1(\chi) + *\pi_7(\chi) - *\pi_{27}(\chi) - F(\chi), \tag{18}$$

where F is a smooth function from the closed ball of radius e_1 in $\Lambda^3 T^*M$ into $\Lambda^4 T^*M$ with $F(0) = 0$. Suppose that $\chi, \xi \in C^{1,1/2}(\Lambda^3 T^*M)$ and $\|\chi\|_{C^0}, \|\xi\|_{C^0} \le e_1$. Then $F(\chi) - F(\xi)$ satisfies the inequalities

$$\bigl|F(\chi) - F(\xi)\bigr| \le e_2 |\chi - \xi| \bigl(|\chi| + |\xi|\bigr), \tag{19}$$

$$\bigl|d\bigl(F(\chi)-F(\xi)\bigr)\bigr| \le e_3 \Bigl\{ |\chi-\xi|\bigl(|\chi|+|\xi|\bigr)|d^*\varphi| + \bigl|\nabla(\chi-\xi)\bigr|\bigl(|\chi|+|\xi|\bigr) + |\chi-\xi|\bigl(|\nabla\chi|+|\nabla\xi|\bigr)\Bigr\}, \tag{20}$$

$$\begin{aligned} \bigl[d\bigl(F(\chi) - F(\xi)\bigr)\bigr]_{1/2} \le e_4 \Bigl\{ & [\chi-\xi]_{1/2}\bigl(\|\chi\|_{C^0} + \|\xi\|_{C^0}\bigr)\|d^*\varphi\|_{C^0} + \|\chi-\xi\|_{C^0}\bigl([\chi]_{1/2} + [\xi]_{1/2}\bigr)\|d^*\varphi\|_{C^0} \\ & + \|\chi-\xi\|_{C^0}\bigl(\|\chi\|_{C^0} + \|\xi\|_{C^0}\bigr)[d^*\varphi]_{1/2} \\ & + \bigl[\nabla(\chi-\xi)\bigr]_{1/2}\bigl(\|\chi\|_{C^0} + \|\xi\|_{C^0}\bigr) + \bigl\|\nabla(\chi-\xi)\bigr\|_{C^0}\bigl([\chi]_{1/2} + [\xi]_{1/2}\bigr) \\ & + [\chi-\xi]_{1/2}\bigl(\|\nabla\chi\|_{C^0} + \|\nabla\xi\|_{C^0}\bigr) + \|\chi-\xi\|_{C^0}\bigl([\nabla\chi]_{1/2} + [\nabla\xi]_{1/2}\bigr)\Bigr\}. \end{aligned} \tag{21}$$

Proof. As $\Lambda^3_+ M$ is an open subset of $\Lambda^3 T^*M$, we may choose $e_1 > 0$ independent of (M, φ), such that the closed ball of radius e_1 about φ in $\Lambda^3 T^*M$ is contained in $\Lambda^3_+ M$. Let the function F be defined by (18). Then (18) holds, and F is a smooth function of ξ. Now by calculating in coordinates (using Schur's Lemma) one can write $\Theta(\varphi+\chi)$ as an explicit function of χ up to first order in χ, and the answer is the first four terms of the r.h.s. of (18). So the function $F(\chi)$ has zero first derivative in χ.

Thus the principal part of the function $F(\chi)$ is quadratic in χ. It follows that inequality (19) holds for some constant e_2, and as this is a calculation at a point, e_2 is a universal constant, independent of M and φ. By the chain rule, the derivative $dF(\chi)$ can be written $dF(\chi) = F_1(\chi, \nabla\varphi) + F_2(\chi, \nabla\chi)$, where F_1 and F_2 are linear in their second arguments, and are universal functions in the sense that their definition at $m \in M$ depends only on the value of φ at the point m.

Since $d\varphi = 0$ by assumption, $\nabla\varphi$ is determined pointwise by $d^*\varphi$. Using this and estimates on F_1, F_2 it easy to show that dF satisfies an estimate of the form (20) for some $e_3 > 0$. But because of the universal property of F_1 and F_2, this e_3 is independent of M and φ, as we want. In a similar way, we can estimate the Hölder norm $\left[d\big(F(\chi) - F(\xi)\big)\right]_{1/2}$ using 'universal functions', and it is easy to see that (21) holds for some universal constant e_4. This completes the proof. □

Now we shall prove Theorem A of §2.3.

Theorem A. *Let $E_1, \ldots, E_5$ be positive constants. Then there exist positive constants κ, K depending only on $E_1, \ldots, E_5$, such that whenever $0 < t \le \kappa$, the following is true.*

Let M be a compact 7-manifold, and φ a smooth, closed section of $\Lambda^3_+ M$ on M. Suppose that ψ is a smooth 3-form on M with $d^*\psi = d^*\varphi$, and that these four conditions hold:

(i) $\|\psi\|_2 \le E_1 t^4$ *and* $\|\psi\|_{C^{1,1/2}} \le E_1 t^4$,

(ii) if $\chi \in C^{1,1/2}(\Lambda^3 T^*M)$ *and* $d\chi = 0$ *then* $\|\chi\|_{C^0} \le E_2\big(t\|\nabla\chi\|_{C^0} + t^{-7/2}\|\chi\|_2\big)$ *and* $\|\nabla\chi\|_{C^0} + t^{1/2}[\nabla\chi]_{1/2} \le E_3\big(\|d^*\chi\|_{C^0} + t^{1/2}[d^*\chi]_{1/2} + t^{-9/2}\|\chi\|_2\big)$,

(iii) $1 \le E_4 \operatorname{vol}(M)$, *and*

(iv) if f is a smooth, real function and $\int_M f\,d\mu = 0$ *then* $\|f\|_2 \le E_5\|df\|_2$.

Then there exists $\eta \in C^\infty(\Lambda^2 T^*M)$ with $\|d\eta\|_{C^0} \le K t^{1/2}$, such that $\tilde\varphi = \varphi + d\eta$ is a smooth, torsion-free G_2- structure.

Proof. The idea of the proof is to construct solutions $\eta \in C^\infty(\Lambda^2 T^*M)$, $\epsilon \in \mathbb{R}$ to the equations $d^*\eta = 0$ and

$$\epsilon = \frac{1}{3\operatorname{vol}(M)} \int_M d\eta \wedge *\psi, \tag{22}$$

$$d\,\Theta(\varphi + d\eta) = \tfrac{7}{3} d\big(*\pi_1(d\eta)\big) + 2d\big(*\pi_7(d\eta)\big) - \epsilon d * \varphi, \tag{23}$$

and then to show that η satisfies the requirements of the Theorem. Suppose that $\eta \in C^{2,1/2}(\Lambda^2 T^*M)$ and $\|d\eta\|_{C^0} \le e_1$. Then by Lemma 3.1.1, $\varphi + d\eta \in C^0(\Lambda^3_+ M)$ and

$$d\,\Theta(\varphi + d\eta) = d * \varphi - dF(d\eta) - d(*d\eta) + \tfrac{7}{3} d\big(*\pi_1(d\eta)\big) + 2d\big(*\pi_7(d\eta)\big), \tag{24}$$

so that (23) is equivalent to the equation $d(*d\eta) = (1+\epsilon) d * \varphi - dF(d\eta)$, which gives

$$d^* d\eta = (1+\epsilon) d^*\psi + *dF(d\eta), \tag{25}$$

as $*d* = -d^*$ acting on $\Lambda^3 T^*M$, and $d^*\psi = d^*\varphi$.

The rest of the Theorem is an immediate consequence of the following three Propositions:

Proposition 3.1.2. *Suppose t is sufficiently small. Then there exist convergent sequences $\{\eta_j\}_{j=0}^{\infty}$ in $C^{2,1/2}(\Lambda^2 T^*M)$ and $\{\epsilon_j\}_{j=0}^{\infty}$ in $\mathbb{R}$ with $\eta_0 = \epsilon_0 = 0$, satisfying the equations $d^*\eta_j = 0$ and*

$$\epsilon_j = \frac{1}{3\,\mathrm{vol}(M)} \int_M d\eta_j \wedge *\psi, \tag{26}$$

$$d^* d\eta_j = (1 + \epsilon_{j-1}) d^*\psi + *dF(d\eta_{j-1}), \tag{27}$$

for each $j > 0$, and the inequalities
(a) $\|d\eta_j\|_2 \le 2E_1 t^4$, (b) $|\epsilon_j| \le E_6 t^8$, (c) $\|d\eta_j\|_{C^0} \le K t^{1/2}$,
(d) $\|\nabla d\eta_j\|_{C^0} \le E_7 t^{-1/2}$, (e) $[\nabla d\eta_j]_{1/2} \le E_7 t^{-1}$
(A) $\|d\eta_j - d\eta_{j-1}\|_2 \le 2E_1 2^{-j} t^4$, (B) $|\epsilon_j - \epsilon_{j-1}| \le E_6 2^{-j} t^8$, (C) $\|d\eta_j - d\eta_{j-1}\|_{C^0} \le K 2^{-j} t^{1/2}$,
(D) $\|\nabla(d\eta_j - d\eta_{j-1})\|_{C^0} \le E_7 2^{-j} t^{-1/2}$, (E) $[\nabla(d\eta_j - d\eta_{j-1})]_{1/2} \le E_7 2^{-j} t^{-1}$,
where E_6, E_7 and K are positive constants depending only on $E_1, \dots, E_5$. Let η be the limit of the sequence $\{\eta_j\}_{j=0}^{\infty}$ in $C^{2,1/2}(\Lambda^2 T^*M)$ and ϵ be the limit of the sequence $\{\epsilon_j\}_{j=0}^{\infty}$ in $\mathbb{R}$. Then η, ϵ satisfy $d^*\eta = 0$, (22) and (23) and the estimate $\|d\eta\|_{C^0} \le K t^{1/2} \le e_1$.

Proposition 3.1.3. *Let M, φ, η and ϵ be as in Proposition 3.1.2. Then η is smooth.*

Proposition 3.1.4. *Let M, φ, η and ϵ be as in Proposition 3.1.2, and suppose t is sufficiently small. Then $d\,\Theta(\varphi + d\eta) = 0$, so that $\tilde{\varphi} = \varphi + d\eta$ is a torsion-free G_2- structure.*

The proofs of the Propositions will be given in order. At a number of points in the proofs we shall need t to be smaller than some positive constant defined in terms of $e_1, \dots, e_4$ and $E_1, \dots, E_5$. As a shorthand we shall simply say that this holds since $t \le \kappa$, and suppose without remark that κ has been chosen such that the relevant restriction holds. Since $e_1, \dots, e_4$ are universal constants, the restriction really depends only on $E_1, \dots, E_5$. The reader may if she wishes go through the proof collecting the restrictions on the constant κ, and hence obtain an explicit expression for κ for which the Theorem holds.

Proof of Proposition 3.1.2. The operator $d^*d + dd^* : C^\infty(\Lambda^2 T^*M) \to C^\infty(\Lambda^2 T^*M)$ is elliptic and self-adjoint and has kernel $W \cong H^2(M, \mathbb{R})$, the vector space of Hodge representatives for $H^2(M, \mathbb{R})$. Let $W^\perp$ be the subspace of $L^2(\Lambda^2 T^*M)$ that is L^2-orthogonal to W. Since $d^*d + dd^*$ is self-adjoint, its image is the orthogonal complement of its kernel, so $W^\perp = \mathrm{Im}(d^*d + dd^*)$. Let $\xi \in W^\perp$. Then $\xi \in \mathrm{Im}(d^*d + dd^*)$, so there exists a unique $\chi \in W^\perp$ with $(d^*d + dd^*)\chi = \xi$. Moreover, by Proposition 1.2.1 there exists some constant $C(\varphi)$ independent of χ and ξ such that $\|\chi\|_{C^{2,1/2}} \le C(\varphi)\|\xi\|_{C^{0,1/2}}$ whenever $\xi \in C^{0,1/2}(\Lambda^2 T^*M)$. The constant is written $C(\varphi)$ to indicate that it depends on φ.

The Proposition will be proved by induction in j. The first step holds trivially because $\eta_0 = \epsilon_0 = 0$. For the inductive step, suppose that $\eta_0, \dots, \eta_k$ and $\epsilon_0, \dots, \epsilon_k$ exist and satisfy the conditions of the Proposition for $j \le k$. Then $\|d\eta_k\|_{C^0} \le K t^{1/2} \le e_1$, as $t \le \kappa$. So $F(d\eta_k)$ is well-defined. Now $(1 + \epsilon_k) d^*\psi + *dF(d\eta_k)$ lies in $W^\perp$ by integration by parts, since $dw = 0$ for each $w \in W$. Therefore from above there exists a unique $\eta_{k+1} \in W^\perp$ such that $(d^*d + dd^*)\eta_{k+1} = (1 +$

$\epsilon_k)d^*\psi + *dF(d\eta_k)$. Moreover, since $\eta_k \in C^{2,1/2}(\Lambda^2T^*M)$, $d^*\psi + *dF(d\eta_k) \in C^{0,1/2}(\Lambda^2T^*M)$, so that $\eta_{k+1} \in C^{2,1/2}(\Lambda^2T^*M)$ and $\|\eta_{k+1}\|_{C^{2,1/2}} \leq C(\varphi)\|d^*\psi + *dF(d\eta_k)\|_{C^{0,1/2}}$.

Because $\operatorname{Im} d$ and $\operatorname{Im} d^*$ are L^2- orthogonal, and $(d^*d + dd^*)\eta_{k+1} \in \operatorname{Im} d^*$, it follows that $dd^*\eta_{k+1} = 0$. Taking the inner product with η_{k+1} and integrating by parts shows that $d^*\eta_{k+1} = 0$, and so $d^*d\eta_{k+1} = (1+\epsilon_k)d^*\psi + *dF(d\eta_k)$. Therefore the element η_{k+1} we have defined satisfies $d^*\eta_{k+1} = 0$ and equation (27), as we have to prove. Define ϵ_{k+1} by (26) for $j = k+1$. To complete the induction it remains to show that parts (a)-(e) and (A)-(E) hold for $j = k+1$. The proof will be split into two cases, the case $k = 0$ and the case $k > 0$.

For the first case we must show that parts (a)-(e) and (A)-(E) hold for $j = 1$. Since $\eta_0 = 0$, parts (A)-(E) imply parts (a)-(e), so it is sufficient to prove parts (A)-(E). As $\eta_0 = \epsilon_0 = 0$ and $F(0) = 0$, (27) gives $d^*d\eta_1 = d^*\psi$. Taking the inner product with η_1 and integrating by parts gives $\|d\eta_1\|_2^2 \leq \|d\eta_1\|_2\|\psi\|_2$ by Hölder's inequality, so cancelling $\|d\eta_1\|_2$ from each side and using condition (i) of Theorem A gives $\|d\eta_1\|_2 \leq E_1t^4$, which proves part (A) for $j = 1$. By (26) and condition (iii) of Theorem A we have $|\epsilon_1| \leq \frac{1}{3}E_4\|d\eta_1\|_2\|\psi\|_2$, so condition (i) of Theorem A and part (A) for $j = 1$ give part (B) for $j = 1$, with $E_6 = 2E_4E_1^2/3$. Also, as $d^*d\eta_1 = d^*\psi$, $\|d^*d\eta_1\|_{C^{0,1/2}} \leq E_1t^4$ by condition (i) of Theorem A, and thus condition (ii) with $\chi = d\eta_1$ gives

$$\|\nabla d\eta_1\|_{C^0} + t^{1/2}[\nabla d\eta_1]_{1/2} \leq E_3\bigl(E_1t^4 + t^{-9/2}E_1t^4\bigr), \tag{28}$$

so that parts (D) and (E) hold for $j = 1$ with $E_7 = 3E_1E_3$ since $t \leq \kappa$, and again by part (ii) we have

$$\|d\eta_1\|_{C^0} \leq E_2\bigl(tE_7t^{-1/2} + t^{-7/2}E_1t^4\bigr), \tag{29}$$

so that part (C) holds for $j = 1$ with $K = 2E_2(E_7 + E_1)$. So parts (a)-(e) and (A)-(E) hold for $j = 1$, as we have to prove.

To finish the induction we must show that parts (a)-(e) and (A)-(E) hold for $j = k+1$ for $k > 0$, when the conditions of the Proposition hold for $j \leq k$. Since $\eta_0 = 0$, parts (a)-(e) for $j = k+1$ follow from parts (A)-(E) for $j = 1, \ldots, k+1$ by induction on j. Thus it suffices to prove parts (A)-(E) for $j = k+1$. The difference of (27) for $j = k, k+1$ gives

$$d^*d(\eta_{k+1} - \eta_k) = (\epsilon_k - \epsilon_{k-1})d^*\psi + *d\bigl(F(d\eta_k) - F(d\eta_{k-1})\bigr). \tag{30}$$

Taking the inner product of (30) with $\eta_{k+1} - \eta_k$ and integrating by parts, we deduce that $\|d\eta_{k+1} - d\eta_k\|_2 \leq |\epsilon_k - \epsilon_{k-1}|\|\psi\|_2 + \bigl\|F(d\eta_k) - F(d\eta_{k-1})\bigr\|_2$. Applying inequality (19) of Lemma 3.1.1 gives

$$\|d\eta_{k+1} - d\eta_k\|_2 \leq |\epsilon_k - \epsilon_{k-1}| \cdot \|\psi\|_2 + e_2\|d\eta_k - d\eta_{k-1}\|_2\bigl(\|d\eta_k\|_{C^0} + \|d\eta_{k-1}\|_{C^0}\bigr). \tag{31}$$

By parts (A), (B) for $j = k$, part (c) for $j = k-1, k$, condition (i) of Theorem A and as $t \leq \kappa$, we deduce that part (A) holds for $j = k+1$. Part (B) for $j = k+1$ then follows from part (A) for $j = k+1$, equation (26) and conditions (i), (iii) of Theorem A.

Applying inequality (20) of Lemma 3.1.1 to (30) we deduce that

$$\left\|d^*d(\eta_{k+1}-\eta_k)\right\|_{C^0} \le |\epsilon_k-\epsilon_{k-1}|\cdot\|\psi\|_{C^1}+e_3\Big\{\|d\eta_k-d\eta_{k-1}\|_{C^0}\big(\|d\eta_k\|_{C^0}+\|d\eta_{k-1}\|_{C^0}\big)\|d^*\psi\|_{C^0}$$
$$+\|\nabla d\eta_k-\nabla d\eta_{k-1}\|_{C^0}\big(\|d\eta_k\|_{C^0}+\|d\eta_{k-1}\|_{C^0}\big)$$
$$+\|d\eta_k-d\eta_{k-1}\|_{C^0}\big(\|\nabla d\eta_k\|_{C^0}+\|\nabla d\eta_{k-1}\|_{C^0}\big)\Big\}. \tag{32}$$

By parts (B), (C) and (D) for $j=k$, parts (c) and (d) for $j=k-1,k$ and condition (i) of Theorem A we deduce from (32) that $\left\|d^*d(\eta_{k+1}-\eta_k)\right\|_{C^0}=O(2^{-k})$, so that (D) holds for $j=k+1$, as $t\le\kappa$.

Applying inequality (21) of Lemma 3.1.1 to (30) we deduce an inequality for $\left[d^*d(\eta_{k+1}-\eta_k)\right]_{1/2}$ that is similar to (32), but which will not be given as it has many terms. The only problem terms in this inequality involve terms like $[d\eta_k]_{1/2}$. This can be estimated using $\|d\eta_k\|_{C^0}$ and $\|\nabla d\eta_k\|_{C^0}$, as $[\chi]^2_{1/2}\le 2\|\chi\|_{C^0}\|\nabla\chi\|_{C^0}$ from (6). Using this trick and parts (B)-(E) for $j=k$, parts (c)-(e) for $j=k-1,k$, condition (i) of Theorem A and the inequality for $\left[d^*d(\eta_{k+1}-\eta_k)\right]_{1/2}$ mentioned above that is similar to (32), it can be shown that $\left[d^*d(\eta_{k+1}-\eta_k)\right]_{1/2}=O(2^{-k}t^{-1/2})$, so that (E) holds for $j=k+1$, as $t\le\kappa$.

Part (C) for $j=k+1$ then follows from parts (A) and (D) for $j=k+1$ and part (ii) of Theorem A. Thus parts (A)-(E) hold for $j=k+1$. This completes the inductive step. Therefore, by induction on j, the sequences $\{\eta_j\}_{j=0}^\infty$ and $\{\epsilon_j\}_{j=0}^\infty$ exist and satisfy $d^*\eta_j=0$, equations (26) and (27) and parts (a)-(e) and (A)-(E) for all j.

By parts (D) and (E) of the Proposition, the sequence $\{d^*d\eta_j\}_{j=0}^\infty$ converges in $C^{0,1/2}(\Lambda^2T^*M)$, and as $d^*\eta_j=0$ this means that $\left\{(d^*d+dd^*)\eta_j\right\}_{j=0}^\infty$ converges in $C^{0,1/2}(\Lambda^2T^*M)$. So by the material at the beginning of the proof, the sequence $\{\eta_j\}_{j=0}^\infty$ converges in $C^{2,1/2}(\Lambda^2T^*M)$, as we have to prove. By part (B) of the Proposition the sequence $\{\epsilon_j\}_{j=0}^\infty$ converges in $\mathbb{R}$. Let η, ϵ be the limits of these sequences. Taking the limit in (26) gives (22), and taking the limit in (27) gives (25), so that η and ϵ satisfy (23). Also $d^*\eta=0$ as $d^*\eta_j=0$ for all j, and part (c) shows that $\|d\eta\|_{C^0}\le Kt^{1/2}$. This completes the proof of Proposition 3.1.2. □

Proof of Proposition 3.1.3. There is a standard method called the 'bootstrap argument' for proving smoothness of solutions to a nonlinear differential equation with elliptic principal part. However, we cannot immediately apply this argument to equation (25), since both sides involve the second derivatives of η. Let us therefore collect on the left hand side the terms in (25) involving $\nabla d\eta$ and add on the equation $dd^*\eta=0$, giving

$$(d^*d+dd^*)\eta+P(d\eta,\nabla d\eta)=G(\epsilon,d^*\varphi,d\eta). \tag{33}$$

Here P is a function that is linear in $\nabla d\eta$, smooth in $d\eta$, and is zero when $d\eta$ is zero, and G is a smooth function of its arguments.

Since d^*d+dd^* is elliptic and ellipticity is an open condition, we deduce that d^*d+dd^*+P is also elliptic whenever $d\eta$ is sufficiently small in C^0. But by Proposition 3.1.2, $\|d\eta\|_{C^0}\le Kt^{1/2}$, so as $t\le\kappa$, d^*d+dd^*+P is elliptic. Now $d\eta\in C^{1,1/2}(\Lambda^3T^*M)$ by Proposition 3.1.2. Suppose that $d\eta\in C^{k,1/2}(\Lambda^3T^*M)$. Then the coefficients of d^*d+dd^*+P and the right hand side of (33) are both in $C^{k,1/2}$.

Thus by Proposition 1.2.2, $\eta \in C^{k+2,1/2}(\Lambda^2T^*M)$, so $d\eta \in C^{k+1,1/2}(\Lambda^3T^*M)$. Therefore by induction in k, $d\eta \in C^{k,1/2}(\Lambda^3T^*M)$ and $\eta \in C^{k,1/2}(\Lambda^2T^*M)$ for all positive integers k. So η is smooth. $\square$

Proof of Proposition 3.1.4. We begin the proof with a Lemma.

Lemma 3.1.5. *Let M, φ, η and ϵ be as in Proposition 3.1.2, and suppose t is sufficiently small. Define*

$$\begin{aligned} x_7 &= \pi_7\bigl(d\,\Theta(\varphi+d\eta)\bigr), & y_7 &= \tfrac{7}{3}\pi_7\bigl(d*\pi_1(d\eta)\bigr), & z_7 &= 2\pi_7\bigl(d*\pi_7(d\eta)\bigr),\\ x_{14} &= \pi_{14}\bigl(d\,\Theta(\varphi+d\eta)\bigr), & y_{14} &= \tfrac{7}{3}\pi_{14}\bigl(d*\pi_1(d\eta)\bigr)-\epsilon d*\varphi, & z_{14} &= 2\pi_{14}\bigl(d*\pi_7(d\eta)\bigr). \end{aligned} \tag{34}$$

Then x_7, y_7, z_7 and x_{14}, y_{14}, z_{14} satisfy the equations

$$x_7 = y_7+z_7, \qquad x_{14} = y_{14}+z_{14}, \qquad \|z_7\|_2 = \sqrt{2}\|z_{14}\|_2, \qquad \langle x_7, z_7\rangle = 2\langle x_{14}, z_{14}\rangle, \tag{35}$$

and the inequalities $\|x_7\|_2 \le \|x_{14}\|_2$ and $\|y_{14}\|_2 \le \frac{1}{4}\|y_7\|_2$.

Proof. Since φ and η satisfy (23) by Proposition 3.1.2 and $d*\varphi \in C^\infty(\Lambda^5_{14})$ by Lemma 1.1.2, taking π_7 and π_{14} of (23) and using (34) gives $x_7 = y_7+z_7$ and $x_{14} = y_{14}+z_{14}$, as we have to prove. Now $d*\pi_7(d\eta) = \frac{1}{4}d(\nu\wedge\varphi) = \frac{1}{4}d\nu\wedge\varphi$ for some smooth 1-form ν. But it can be shown by calculation in coordinates that if $\xi \in \Lambda^2_7$ then $\xi\wedge\varphi = 2*\xi$, and if $\xi \in \Lambda^2_{14}$ then $\xi\wedge\varphi = -*\xi$. Therefore $z_7 = \frac{1}{2}\pi_7(d\nu)\wedge\varphi = *\pi_7(d\nu)$ and $z_{14} = \frac{1}{2}\pi_{14}(d\nu)\wedge\varphi = -\frac{1}{2}*\pi_{14}(d\nu)$, so that $d\nu = *z_7 - 2*z_{14}$.

As φ is closed, $d\nu\wedge d\nu\wedge\varphi$ is an exact 7-form, and $\int_M d\nu\wedge d\nu\wedge\varphi = 0$ by Stokes' Theorem. But $d\nu\wedge d\nu\wedge\varphi = \bigl\{2|\pi_7(d\nu)|^2 - |\pi_{14}(d\nu)|^2\bigr\}d\mu$, so integration gives $2\bigl\|\pi_7(d\nu)\bigr\|_2^2 - \bigl\|\pi_{14}(d\nu)\bigr\|_2^2 = 0$. As $d\nu = *z_7 - 2*z_{14}$ this gives $\|z_7\|_2 = \sqrt{2}\|z_{14}\|_2$, as we have to prove. By definition x_7+x_{14} is a closed 5-form, and $d\nu = *z_7-2*z_{14}$ is an exact 2-form. Therefore $(x_7+x_{14})\wedge(*z_7-2*z_{14})$ is an exact 7-form, and integrating over M shows that $\langle x_7, z_7\rangle = 2\langle x_{14}, z_{14}\rangle$, as we have to prove.

Let us write $\tilde\varphi = \varphi+d\eta$, and let $\Lambda^5T^*M = \tilde\Lambda^5_7\oplus\tilde\Lambda^5_{14}$ be the splitting defined by Proposition 1.1.1 using $\tilde\varphi$. By Lemma 1.1.2, as $d\tilde\varphi = 0$ we have $x_7+x_{14} = d\,\Theta(\tilde\varphi) \in C^\infty(\tilde\Lambda^5_{14})$. Since $\tilde\Lambda^5_{14}$ approaches Λ^5_{14} as $|d\eta|$ becomes small, if $|d\eta|$ is sufficiently small then $|\pi_7(\mu)| \le |\pi_{14}(\mu)|$ whenever $\mu \in \tilde\Lambda^5_{14}$. But this holds since $\|d\eta\|_{C^0} \le Kt^{1/2}$ by Proposition 3.1.2 and $t \le \kappa$. Therefore $\|x_7\|_2 \le \|x_{14}\|_2$, as we have to prove.

Write $\frac{7}{3}\pi_1(d\eta)-\epsilon\varphi = f\varphi$ for f a smooth real function. Then $\frac{7}{3}d*\pi_1(d\eta)-\epsilon d*\varphi = df\wedge*\varphi + fd*\varphi$. But $d*\varphi \in C^\infty(\Lambda^5_{14})$ by Lemma 1.1.2, and $df\wedge*\varphi \in C^\infty(\Lambda^5_7)$. Therefore $y_7 = df\wedge*\varphi$ and $y_{14} = fd*\varphi = fd*\psi$. A calculation in coordinates shows that $|df\wedge*\varphi| = \sqrt{3}|df|$. Therefore $\|y_7\|_2 = \sqrt{3}\|df\|_2$ and $\|y_{14}\|_2 \le \|f\|_2\|\psi\|_{C^1}$. Now

$$7\int_M f d\mu = \int_M f\varphi\wedge*\varphi = \frac{7}{3}\int_M d\eta\wedge*\varphi - \epsilon\int_M \varphi\wedge*\varphi = \frac{7}{3}\int_M d\eta\wedge*\psi - 7\epsilon\,\mathrm{vol}(M), \tag{36}$$

since $\int_M d\eta\wedge*\varphi = -\int_M \eta\wedge d*\varphi = -\int_M \eta\wedge d*\psi = \int_M d\eta\wedge*\psi$ by integration by parts, and $\int_M \varphi\wedge*\varphi = 7\,\mathrm{vol}(M)$. But by (22) the r.h.s. of (36) is zero, so that $\int_M f d\mu = 0$. Thus by part (*iv*) of Theorem A we have $\|f\|_2 \le E_5\|df\|_2$, so that $\|y_{14}\|_2 \le 3^{-1/2}E_1E_5t^4\|y_7\|_2$ from above, using part (*i*) of Theorem A. Therefore $\|y_{14}\|_2 \le \frac{1}{4}\|y_7\|_2$ as $t \le \kappa$. This concludes the proof of Lemma 3.1.5. $\square$

To finish the Proposition, we shall show that the conclusions of the Lemma force $x_7 = y_7 = z_7 = 0$ and $x_{14} = y_{14} = z_{14} = 0$. Now as $x_7 = y_7 + z_7$, we have $\|y_7\|_2 \leq \|x_7\|_2 + \|z_7\|_2$, and therefore

$$\|x_{14} - z_{14}\|_2 = \|y_{14}\|_2 \leq \frac{1}{2\sqrt{2}}\big(\|x_{14}\|_2 + \|z_{14}\|_2\big). \tag{37}$$

But $2\langle x_{14}, z_{14}\rangle = \|x_{14}\|_2^2 + \|z_{14}\|_2^2 - \|x_{14} - z_{14}\|_2^2$. Using this, (37) and the inequality $\|x_{14}\|_2^2 + \|z_{14}\|_2^2 \geq 2\|x_{14}\|_2\|z_{14}\|_2$ we can show that $\langle x_{14}, z_{14}\rangle \geq \frac{3}{4}\|x_{14}\|_2\|z_{14}\|_2$. However, since $\langle x_7, z_7\rangle = 2\langle x_{14}, z_{14}\rangle$ this implies that

$$\|x_7\|_2\|z_7\|_2 \geq \langle x_7, z_7\rangle \geq \frac{3\sqrt{2}}{4}\|x_7\|_2\|z_7\|_2, \tag{38}$$

which is a contradiction unless $\|x_7\|_2 = 0$ or $\|z_7\|_2 = 0$, as $3\sqrt{2}/4 > 1$. Thus $x_7 = 0$ or $z_7 = 0$.

For the case $z_7 = 0$, $z_{14} = 0$ as $\|z_7\|_2 = \sqrt{2}\|z_{14}\|_2$, and therefore $x_7 = y_7$ and $x_{14} = y_{14}$. But then $\|x_7\|_2 \leq \|x_{14}\|_2 = \|y_{14}\|_2 \leq \frac{1}{4}\|y_7\|_2 = \frac{1}{4}\|x_7\|_2$, so that $\|x_7\|_2 = 0$, and thus $x_7 = x_{14} = y_7 = y_{14} = 0$. For the case $x_7 = 0$, we have $y_7 = -z_7$, and therefore $\|y_{14}\|_2 \leq \frac{1}{4}\|y_7\|_2 = \frac{1}{4}\|z_7\|_2 = \frac{\sqrt{2}}{4}\|z_{14}\|_2$. Now if $z_{14} \neq 0$ then $\|y_{14}\|_2 < \|z_{14}\|_2$ so that $\langle x_{14}, z_{14}\rangle > 0$, projecting the equation $x_{14} = y_{14} + z_{14}$ onto z_{14}. But this means that $\langle x_7, z_7\rangle > 0$, which contradicts $x_7 = 0$. Therefore $z_{14} = 0$, so $z_7 = 0$ and we have reduced to the previous case. Thus in both cases we have $x_7 = y_7 = z_7 = 0$ and $x_{14} = y_{14} = z_{14} = 0$. In particular, $d\,\Theta(\varphi + d\eta) = x_7 + x_{14} = 0$. So putting $\tilde{\varphi} = \varphi + d\eta$, we have $d\tilde{\varphi} = d\,\Theta(\tilde{\varphi}) = 0$, and $\tilde{\varphi}$ is a torsion-free G_2- structure. This completes the proofs of Proposition 3.1.4 and Theorem A. $\square$

3.2. The proofs of some inequalities. In this section we shall prove Theorem B of §2.3. We begin with three Lemmas.

Lemma 3.2.1. *Let $B_{1/2}, B_1$ be the balls about the origin in $\mathbb{R}^7$ with radii $\frac{1}{2}$ and 1 respectively, and let h be the Euclidean metric on B_1. Then there exist positive constants F_1, F_2, F_3 such that if $\tilde{h}$ is a Riemannian metric on B_1 and $\|\tilde{h} - h\|_{C^{1,1/2}} \leq F_1$, then the following is true.*

Let $\chi \in C^{1,1/2}(\Lambda^3 T^* B_1)$ and suppose $d\chi = 0$. Then χ satisfies the inequalities

$$\|\chi\|_{C^0} \leq F_2\big(\|\nabla\chi\|_{C^0} + \|\chi\|_2\big), \tag{39}$$

$$\big\|\nabla\chi|_{B_{1/2}}\big\|_{C^0} + \big[\nabla\chi|_{B_{1/2}}\big]_{1/2} \leq F_3\big(\|d^*\chi\|_{C^0} + [d^*\chi]_{1/2} + \|\chi\|_{C^0}\big). \tag{40}$$

Here the connection ∇ and are all norms are w.r.t. the metric $\tilde{h}$ on B_1.

Proof. Using the mean value theorem, it can be seen that for a real function $f \in C^1(B_1)$, where B_1 has the Euclidean metric h,

$$\sup_{B_1}|f| \leq \inf_{B_1}|f| + 2\|\nabla f\|_{C^0}. \tag{41}$$

But $\inf_{B_1}|f| \leq \mathrm{vol}(B_1)^{-1/2}\|f\|_2$. Thus for $f \in C^1(B_1)$, we have $\|f\|_{C^0} \leq \frac{1}{2}F_2\big(\|\nabla f\|_{C^0} + \|f\|_2\big)$ for some constant $F_2 > 0$ independent of f. Since ∇ depends on $\tilde{h}$ and its first derivative, the inequality $\|f\|_{C^0} \leq F_2\big(\|\nabla f\|_{C^0} + \|f\|_2\big)$ will hold for any metric

$\tilde h$ on B_1 such that $\|\tilde h-h\|_{C^1}\le F_1$, for some small $F_1>0$. Putting $f=|\chi|$ for some $\chi\in C^1(\Lambda^3T^*B_1)$ gives

$$\|\chi\|_{C^0}\le F_2\big(\big\|\nabla|\chi|\big\|_{C^0}+\|\chi\|_2\big), \tag{42}$$

where norms are with respect to $\tilde h$. So the inequality $\big|\nabla|\chi|\big|\le|\nabla\chi|$ gives (39), as we have to prove.

Now consider the operator $d+d^*:\bigoplus_{i=0}^7 C^\infty(\Lambda^iT^*B_1)\rightarrow\bigoplus_{i=0}^7 C^\infty(\Lambda^iT^*B_1)$, where d^* is w.r.t. the metric $\tilde h$ on B_1. This is a first-order, elliptic, self-adjoint operator. Suppose that $\chi\in C^{1,1/2}(\Lambda^3T^*B_1)$ and $d\chi=0$. To get (40) we shall apply an elliptic regularity result for $d+d^*$ to the equation $(d+d^*)\chi=d^*\chi$. The elliptic regularity result we need is [7, Theorem 1, p. 517], which gives interior estimates for the Hölder norms of the solutions of an elliptic equation. Using this result it is easy to show that (40) holds for some constant F_3 depending on upper bounds for the $C^{0,1/2}$ norms of the coefficients of the operator $d+d^*$, and a positive lower bound for the elliptic constant of $d+d^*$ on B_1. But if $\|\tilde h-h\|_{C^{1,1/2}}\le F_1$ and F_1 is sufficiently small, then such bounds hold independently of $\tilde h$. Therefore when F_1 is chosen sufficiently small and $\|\tilde h-h\|_{C^{1,1/2}}\le F_1$, both (39) and (40) hold for some constant $F_3>0$ independent of $\tilde h$ and χ. This completes the Lemma. $\square$

Lemma 3.2.2. *Let D_2,D_3 and t be positive constants, and suppose (M,g) is a complete Riemannian 7-manifold with injectivity radius $\delta(g)$ satisfying $\delta(g)\ge D_2t$, and Riemann curvature $R(g)$ satisfying $\big\|R(g)\big\|_{C^0}\le D_3t^{-2}$. Then there exists a constant $F_4>0$ depending only on D_2 and D_3, such that for each $r\in(0,F_4t]$ and each $m\in M$, there exists an open ball $B_r(m)$ about m in M and a diffeomorphism $\Psi_{r,m}:B_1\rightarrow B_r(m)$ where B_1 is the open unit ball in $\mathbb{R}^7$, such that $\Psi_{r,m}(0)=m$ and*

$$\big\|r^{-2}\Psi^*_{r,m}(g)-h\big\|_{C^{1,1/2}}\le F_1, \tag{43}$$

where F_1 is the constant of Lemma 3.2.1.

Proof. Consider the conformally rescaled metric $t^{-2}g$ in place of g. Then $\delta(t^{-2}g)=t^{-1}\delta(g)$, and $R(t^{-2}g)=t^2R(g)$, so that $\delta(t^{-2}g)\le D_2$ and $\big\|R(t^{-2}g)\big\|_{C^0}\le D_3$. Therefore it suffices to prove the Lemma for $t=1$. What is required is systems of coordinates on open balls in M, in which the metric g appears close to the Euclidean metric in the $C^{1,1/2}$ norm. These are provided by Jost and Karcher's theory of *harmonic coordinates* ([11], [10 p. 124]). Jost and Karcher show that if the injectivity radius is bounded below and the sectional curvature is bounded above, then there exist coordinate systems on all balls of a given radius, in which the $C^{1,\alpha}$ norm of the metric is bounded in terms of α for each $\alpha\in(0,1)$. Using this result, the Lemma quickly follows. $\square$

Lemma 3.2.3. *Let D_2,D_3 and t be positive constants, and suppose (M,g) is a compact Riemannian 7-manifold with injectivity radius $\delta(g)$ satisfying $\delta(g)\ge D_2t$, and Riemann curvature $R(g)$ satisfying $\big\|R(g)\big\|_{C^0}\le D_3t^{-2}$. Let $F_1,\dots,F_4$ be the constants of Lemmas 3.2.1 and 3.2.2. There exists a positive constant F_5 depending on D_2,D_3 such that for each $r\in(0,F_4t]$, the following is true.*

Let $\chi\in C^{1,1/2}(\Lambda^3T^*M)$ and suppose $d\chi=0$. Then χ satisfies the inequalities

$$\|\chi\|_{C^0}\le F_2\big(r\|\nabla\chi\|_{C^0}+r^{-7/2}\|\chi\|_2\big), \tag{44}$$

$$\|\nabla\chi\|_{C^0} + r^{1/2}[\nabla\chi]_{1/2} \le F_5\bigl(\|d^*\chi\|_{C^0} + r^{1/2}[d^*\chi]_{1/2} + r^{-1}\|\chi\|_{C^0}\bigr). \qquad (45)$$

Proof. By Lemma 3.2.2, there exist balls $B_r(m)$ and diffeomorphisms $\Psi_{r,m}$ for each m, such that (43) holds. Therefore, Lemma 3.2.1 holds for the metric $\tilde h = r^{-2}\Psi^*_{r,m}(g)$ on B_1. We deduce that if $\chi \in C^{1,1/2}(\Lambda^3 T^*M)$ and $d\chi = 0$, then for each $m \in M$ we have the inequalities

$$\bigl\|\chi|_{B_r(m)}\bigr\|_{C^0} \le F_2\Bigl(r\bigl\|\nabla\chi|_{B_r(m)}\bigr\|_{C^0} + r^{-7/2}\bigl\|\chi|_{B_r(m)}\bigr\|_2\Bigr), \qquad (46)$$

$$\bigl\|\nabla\chi|_{B_{r/2}(m)}\bigr\|_{C^0} + r^{1/2}\bigl[\nabla\chi|_{B_{r/2}(m)}\bigr]_{1/2} \le F_3\Bigl(\bigl\|d^*\chi|_{B_r(m)}\bigr\|_{C^0} + r^{1/2}\bigl[d^*\chi|_{B_r(m)}\bigr]_{1/2} + r^{-1}\bigl\|\chi|_{B_r(m)}\bigr\|_{C^0}\Bigr). \qquad (47)$$

Here the powers of r inserted in (46), (47) are those necessary to compensate for the change of metric from $\tilde h$ to $r^2\tilde h = \Psi^*_{r,m}(g)$, and $B_{r/2}(m)$ is $\Psi_{r,m}(B_{1/2})$.

Since (46) holds for all $m \in M$, inequality (44) holds on M by taking the supremum over all $m \in M$, as we have to prove. Similarly, taking the supremum of (47) over $m \in M$ shows that

$$\|\nabla\chi\|_{C^0} \le F_3\bigl(\|d^*\chi\|_{C^0} + r^{1/2}[d^*\chi]_{1/2} + r^{-1}\|\chi\|_{C^0}\bigr), \qquad (48)$$

$$r^{1/2}\sup_{m\in M}\bigl[\nabla\chi|_{B_{r/2}(m)}\bigr]_{1/2} \le F_3\bigl(\|d^*\chi\|_{C^0} + r^{1/2}[d^*\chi]_{1/2} + r^{-1}\|\chi\|_{C^0}\bigr). \qquad (49)$$

By making the constant F_1 of Lemma 3.2.1 smaller if necessary, we can ensure that the geodesic ball of radius $r/4$ about m is contained in $B_{r/2}(m)$ for each $m \in M$. Now every geodesic of length $r/4$ lies in a geodesic ball of radius $r/4$, and is thus contained in some $B_{r/2}(m)$. Therefore by the definition of Hölder norms in §1.2, the l.h.s. of (49) is greater or equal to the supremum of the expression $\bigl|\nabla\chi(\gamma(0)) - \nabla\chi(\gamma(1))\bigr|/l(\gamma)^{1/2}$ (interpreted in the sense of §1.2) over geodesics $\gamma : [0,1] \to M$ of length less than or equal to $r/4$. But if $l(\gamma) \ge r/4$ then

$$\frac{\bigl|\nabla\chi(\gamma(0)) - \nabla\chi(\gamma(1))\bigr|}{l(\gamma)^{1/2}} \le 4r^{-1/2}\|\nabla\chi\|_{C^0}. \qquad (50)$$

We deduce that

$$[\nabla\chi]_{1/2} \le \max\left(\sup_{m\in M}\bigl[\nabla\chi|_{B_{r/2}(m)}\bigr]_{1/2}, 4r^{-1/2}\|\nabla\chi\|_{C^0}\right). \qquad (51)$$

Therefore, combining (48), (49) and (51) gives (45), with $F_5 = 5F_3$. This completes the proof. □

Now we can prove Theorem B.

Theorem B. *Let $D_1, \ldots, D_5$ be positive constants. Then there exist positive constants $E_1, \ldots, E_5$ and λ depending only on $D_1, \ldots, D_5$, such that for every $t \in (0, \lambda]$, the following is true.*

Let M be a compact 7-manifold, and φ a smooth, closed section of $\Lambda^3_+ M$ on M. Let g be the metric associated to φ. Suppose that ψ is a smooth 3-form on M with $d^*\psi = d^*\varphi$, and that these five conditions hold:

(*i*) $\|\psi\|_2 \le D_1 t^4$ *and* $\|\psi\|_{C^2} \le D_1 t^4$,
(*ii*) *the injectivity radius* $\delta(g)$ *satisfies* $\delta(g) \ge D_2 t$,
(*iii*) *the Riemann curvature* $R(g)$ *of* g *satisfies* $\|R(g)\|_{C^0} \le D_3 t^{-2}$,
(*iv*) *the volume* $\mathrm{vol}(M)$ *satisfies* $\mathrm{vol}(M) \ge D_4$, *and*
(*v*) *the diameter* $\mathrm{diam}(M)$ *satisfies* $\mathrm{diam}(M) \le D_5$.

Then conditions (*i*)-(*iv*) *of Theorem A hold for* (M, φ).

Proof. Since $[\nabla\psi]_{1/2}^2 \le 2\|\nabla\psi\|_{C^0}\|\nabla^2\psi\|_{C^0}$ by (6), we deduce that $\|\nabla\psi\|_{C^{1,1/2}} \le 3D_1 t^4$ by part (*i*) of Theorem B. Therefore part (*i*) of Theorem A holds with constant $E_1 = 3D_1$, as we have to prove. To prove part (*ii*) of Theorem A, suppose that $\chi \in C^{1,1/2}(\Lambda^3 T^*M)$ and $d\chi = 0$. Putting $r = F_4 t$ in Lemma 3.2.3, inequality (44) shows that $\|\chi\|_{C^0} \le E_2\bigl(t\|\nabla\chi\|_{C^0} + t^{-7/2}\|\chi\|_2\bigr)$ for some constant $E_2 > 0$ depending on F_2 and F_4, which in turn depend on D_2 and D_3. This is the first part of condition (*ii*) of Theorem A.

Similarly, (45) shows that for some $F_6 > 0$ depending on F_2 and F_5, we have

$$\|\nabla\chi\|_{C^0} + t^{1/2}[\nabla\chi]_{1/2} \le F_6\bigl(\|d^*\chi\|_{C^0} + t^{1/2}[d^*\chi]_{1/2} + t^{-1}\|\chi\|_{C^0}\bigr). \tag{52}$$

Using (44) to substitute for $\|\chi\|_{C^0}$ in (52) shows that

$$(1-F_2F_6rt^{-1})\|\nabla\chi\|_{C^0}+t^{1/2}[\nabla\chi]_{1/2} \le F_6\bigl(\|d^*\chi\|_{C^0}+t^{1/2}[d^*\chi]_{1/2}+F_2t^{-1}r^{-7/2}\|\chi\|_2\bigr), \tag{53}$$

for any r with $0 < r \le F_4 t$. Choosing $r = \min\bigl(F_4, (2F_2F_6)^{-1}\bigr)t$ gives $1-F_2F_6rt^{-1} \ge \frac{1}{2}$, and so (53) shows that

$$\|\nabla\chi\|_{C^0} + t^{1/2}[\nabla\chi]_{1/2} \le E_3\bigl(\|d^*\chi\|_{C^0} + t^{1/2}[d^*\chi]_{1/2} + t^{-9/2}\|\chi\|_2\bigr), \tag{54}$$

for some constant $E_3 > 0$ depending on F_2, F_4 and F_6, which in turn depend only on D_2 and D_3. This is the second part of condition (*ii*) of Theorem A. Condition (*iii*) of Theorem A follows trivially from condition (*iv*) of Theorem B.

It remains to prove condition (*iv*) of Theorem A. Consider the Laplacian $\Delta = d^*d$ acting on real functions on M. This is a self-adjoint, elliptic operator with nonnegative eigenvalues and kernel the constant functions. Yau [18, Theorem 7, p. 504] has given an explicit, positive lower bound for the smallest positive eigenvalue of Δ on a compact Riemannian manifold of dimension n without boundary. His lower bound depends only on the dimension n, an upper bound for the diameter $\mathrm{diam}(M)$, a positive lower bound for the volume $\mathrm{vol}(M)$, (which both follow from parts (*iv*), (*v*) of Theorem B), and a bound for the Ricci curvature $\mathrm{Ric}(g)$ of the manifold.

Now a torsion-free G_2- structure yields a Ricci-flat metric, and the Ricci curvature of the metric depends on the torsion of the G_2- structure. Since $\nabla\varphi$ is linear in $d\varphi$ and $d^*\varphi$, and $d\varphi = 0$ and $d^*\varphi = d^*\psi$, it follows that $\nabla\varphi$ and $\nabla^2\varphi$ depend on $\nabla\psi$ and $\nabla^2\psi$ respectively. Therefore, by part (*i*) of Theorem B we deduce that the Ricci curvature of g is $O(t^4)$ for small t. Thus the Ricci curvature of g is bounded for small t.

So by Yau's result, the smallest positive eigenvalue of Δ on (M, g_t) is bounded below, for small t, by a positive constant, E_5^{-2} say, depending only on $D_1, \dots, D_5$. Therefore when t is small, say when $t \le \lambda$ for some λ depending on $D_1, \dots, D_5$, if f is a smooth function on M and $\int_M f d\mu = 0$, we have $\langle f, \Delta f\rangle \ge E_5^{-2}\|f\|_2^2$. But $\langle f, \Delta f\rangle = \|df\|_2^2$, so $\|f\|_2 \le E_5\|df\|_2$, which is part (*iv*) of Theorem A. Thus parts

(i)-(iv) of Theorem A hold for (M,φ) when $t \in (0,\lambda]$, and the proof of Theorem B is complete. $\square$

3.3. Deformations of metrics with holonomy contained in G_2. In this section we prove Theorem C of §2.3. This result was first announced by Bryant and Harvey [5, p. 561], but their proof has not yet been published.

Theorem C. *Let M be a compact 7-manifold, let X be the set of torsion-free G_2- structures on M, and let $\mathrm{Diff}_0(M)$ be the group of diffeomorphisms of M isotopic to the identity. Define a map $\Xi : X \to H^3(M,\mathbb{R})$ by $\Xi(\varphi) = [\varphi]$. Then Ξ is invariant under the action of $\mathrm{Diff}_0(M)$ on X, and if $\varphi \in X$ then there exists an open subset $Y \subset X$ that contains φ and is invariant under $\mathrm{Diff}_0(M)$, such that Ξ induces an isomorphism between $Y/\mathrm{Diff}_0(M)$ and an open ball about $[\varphi]$ in $H^3(M,\mathbb{R})$.*

Proof. The standard method for proving a result of this sort is to find a 'slice' for the action of $\mathrm{Diff}_0(M)$ on X, which is some equation imposed on $x \in X$ that locally has a unique solution in each orbit of $\mathrm{Diff}_0(M)$. We shall give such a slice. Let φ be some fixed element of X, and let $\tilde\varphi$ be a nearby element of X. Then $\tilde\varphi$ may be written *uniquely* as $\tilde\varphi = \varphi + w + d\eta$, where w is a smooth 3-form satisfying $dw = d^*w = 0$, and η is a d^*- exact 2-form. Our 'slice' for the action of $\mathrm{Diff}_0(M)$ on X is the equation $\pi_7(\eta) = 0$, and it can be shown that close to φ this equation is transverse to the orbits of $\mathrm{Diff}_0(M)$.

The proof using this method would then show that for each sufficiently small w with $dw = d^*w = 0$, there is a unique small η that is d^*- exact and satisfies $\pi_7(\eta) = 0$ and $d\,\Theta(\varphi + w + d\eta) = 0$. (This part of the proof follows the proof of Theorem A closely.) Thus the slice in X is locally isomorphic to the vector space of 3-forms w with $dw = d^*w = 0$, which is isomorphic to $H^3(M,\mathbb{R})$ by Hodge theory. Therefore the slice in X is locally isomorphic to $H^3(M,\mathbb{R})$, and thus $X/\mathrm{Diff}_0(M)$ is locally isomorphic to $H^3(M,\mathbb{R})$, and the proof is complete.

For an example of a 'slice theorem' similar to the one required, the reader may consult [8], which gives a slice for the space of Riemannian metrics on a manifold. Now the proof we have sketched above can be fleshed out into a valid proof of Theorem C in a straightforward way. However, the author finds it rather dull and unenlightening. Therefore we will give a more motivated proof, that treats infinite-dimensional spaces somewhat informally. The reader objecting to this lack of rigour may follow the approach above, filling in the formal details on the local equivalence of $X/\mathrm{Diff}_0(M)$ with the 'slice' $\pi_7(\eta) = 0$ using [8].

Our proof relies on the following two Propositions.

Proposition 3.3.1. *Suppose M is a compact 7-manifold and φ a torsion-free G_2- structure on M. Then for each sufficiently small $\sigma \in H^3(M,\mathbb{R})$, there exists a torsion-free G_2- structure $\tilde\varphi$ close to φ, with $[\tilde\varphi] = [\varphi] + \sigma$.*

Proof. The idea of the proof is to choose a closed 3-form χ representing σ, and to look for a smooth 2-form η satisfying $d^*\eta = 0$ and $d\,\Theta(\varphi + \chi + d\eta) = 0$. Using Theorems A and B, it is easy to show that when $\|\chi\|_{C^2}$ is sufficiently small then η exists, and $\|d\eta\|_{C^0}$ is bounded in terms of $\|\chi\|_{C^2}$. Putting $\tilde\varphi = \varphi + \chi + d\eta$, the proof is complete. $\square$

Proposition 3.3.2. *Suppose M is a compact 7-manifold and $\{\varphi_t : t \in (-\epsilon,\epsilon)\}$ is a smooth family of torsion-free G_2- structures on M with $[\varphi_t] = [\varphi_0]$ in $H^3(M,\mathbb{R})$ for each $t \in (-\epsilon,\epsilon)$. Let X be the set of torsion-free G_2- structures on M, and let*

$\mathrm{Diff}_0(M)$ *be the group of diffeomorphisms of* M *isotopic to the identity. Then all* φ_t *lie in the same orbit of* $\mathrm{Diff}_0(M)$ *in* X.

Proof. Since the cohomology class $[\varphi_t]$ is constant in $H^3(M,\mathbb{R})$, the derivative $\partial\varphi_t/\partial t$ is an exact 3-form. Let us choose χ such that $\partial\varphi_t/\partial t = d\chi$ and $\|\pi_{14}(\chi)\|_2$ is minimized, where the splitting $\Lambda^2T^*M = \Lambda^2_7\oplus\Lambda^2_{14}$ and the metric are those induced by φ_t. It can easily be shown that there exists a smooth 2-form χ satisfying these conditions. Now if ν is a 1-form then $\chi' = \chi + d\nu$ also satisfies $\partial\varphi_t/\partial t = d\chi'$, so we must have $\|\pi_{14}(\chi)\|_2 \le \|\pi_{14}(\chi')\|_2$. We deduce that $\langle\pi_{14}(\chi), d\nu\rangle = 0$, and as this holds for all 1-forms ν, by integration by parts we have $d^*\pi_{14}(\chi) = 0$. Thus we have shown that $\partial\varphi_t/\partial t = d\chi$ for some χ with $d^*\pi_{14}(\chi) = 0$.

Because $\{\varphi_t : t \in (-\epsilon,\epsilon)\}$ is a family of torsion-free G_2- structures, it follows from Lemma 3.1.1 that if $\partial\varphi_t/\partial t = \xi$ then $\frac{4}{3} * \pi_1(\xi) + *\pi_7(\xi) - *\pi_{27}(\xi)$ is a closed 4-form. Putting $\xi = d\chi$, we deduce that

$$d\big(\tfrac{4}{3} * \pi_1(d\chi) + *\pi_7(d\chi) - *\pi_{27}(d\chi)\big) = 0. \tag{55}$$

If $\chi \in C^\infty(\Lambda^2_7)$, then $\chi = v\cdot\varphi_t$ for some vector field v, and $d\chi = \mathcal{L}_v\varphi_t$. But then (55) holds trivially, since if φ_t changes by Lie translation then $\Theta(\varphi_t)$ remains closed. Thus (55) holds with $\pi_7(\chi)$ in place of χ, and so by subtraction it holds with $\pi_{14}(\chi)$ in place of χ. Therefore

$$d^*\Big\{\tfrac{4}{3}\pi_1\big(d\pi_{14}(\chi)\big) + \pi_7\big(d\pi_{14}(\chi)\big) - \pi_{27}\big(d\pi_{14}(\chi)\big)\Big\} = 0. \tag{56}$$

Now $\pi_{14}(\chi)\wedge*\varphi_t = 0$, so $d\pi_{14}(\chi)\wedge*\varphi_t = 0$ as $d*\varphi_t = 0$, and thus $\pi_1\big(d\pi_{14}(\chi)\big) = 0$. Also, $*\pi_{14}(\chi) = -\pi_{14}(\chi)\wedge\varphi_t$, so $d^*\pi_{14}(\chi)$ is proportional to $d\pi_{14}(\chi)\wedge\varphi_t$, which is proportional to $\pi_7\big(d\pi_{14}(\chi)\big)$. But $d^*\pi_{14}(\chi) = 0$ from above, and thus $\pi_7\big(d\pi_{14}(\chi)\big) = 0$. Therefore from (56) we deduce that $d^*d\pi_{14}(\chi) = 0$, so $d\pi_{14}(\chi) = 0$ by integration by parts. However, χ was chosen such that $\|\pi_{14}(\chi)\|_2$ is minimum modulo addition of closed 2-forms, so since $\pi_{14}(\chi)$ is closed, it must be zero.

Therefore $\chi \in C^\infty(\Lambda^2_7)$, and so $\partial\varphi_t/\partial t = \mathcal{L}_v\varphi_t$ for some vector field v, as above. But Lie translation by vector fields generates the action of $\mathrm{Diff}_0(M)$ on X, and so $\mathcal{L}_v\varphi_t$ is a tangent vector to the orbit of $\mathrm{Diff}_0(M)$ through φ_t. Thus $\partial\varphi_t/\partial t$ is always tangent to the orbit of $\mathrm{Diff}_0(M)$, and all φ_t lie in the same orbit. □

Now we can prove Theorem C. By Proposition 3.3.1, we can deduce that the map $\Xi : X \to H^3(M,\mathbb{R})$ is locally surjective, and further, that the first derivative of Ξ is also surjective. (This is because the construction of $\tilde\varphi$ in the Proposition actually yields $\tilde\varphi$ with $\|\tilde\varphi - \varphi\| \le C\|\sigma\|$ in some suitable norms.) Here is the informal step in the reasoning: we deduce that in some neighbourhood of φ in X, the submanifolds Ξ = constant are path-connected by piecewise-smooth paths. This can probably be made rigorous using some sort of Implicit Function Theorem.

By Proposition 3.3.2, piecewise-smooth paths in the submanifolds Ξ = constant remain within a single orbit of $\mathrm{Diff}_0(M)$, so these submanifolds are locally contained in a single orbit of $\mathrm{Diff}_0(M)$. Now cohomology classes in $H^3(M,\mathbb{R})$ are homotopy invariants, so they are invariant under diffeomorphisms isotopic to the identity, and thus φ and $\tilde\varphi$ must be in distinct orbits of $\mathrm{Diff}_0(M)$ if $\Xi(\varphi) \ne \Xi(\tilde\varphi)$. Therefore Ξ induces a local isomorphism between $H^3(M,\mathbb{R})$ and $X/\mathrm{Diff}_0(M)$, and Theorem C is proved. □

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