Singularities of special Lagrangian submanifolds and SYZ

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and references in these
Almost Calabi-Yau $m$-folds

An almost Calabi-Yau $m$-fold $(M, J, g, \Omega)$ is a compact complex $m$-fold $(M, J)$ with a Kähler metric $g$ with Kähler form $\omega$, and a nonvanishing holomorphic $(m, 0)$-form $\Omega$, the holomorphic volume form.

It is a Calabi-Yau $m$-fold if $|\Omega|^2 \equiv 2^m$. Then $\nabla \Omega = 0$ and $g$ is Ricci-flat.
Special Lagrangian $m$-folds

Let $(M, J, g, \Omega)$ be an almost Calabi-Yau $m$-fold. Let $N$ be a real $m$-submanifold of $M$. We call $N$ special Lagrangian (SL) if $\omega|_N \equiv \text{Im} \Omega|_N \equiv 0$.

If $(M, J, g, \Omega)$ is a Calabi-Yau $m$-fold then $\text{Re} \Omega$ is a calibration on $(M, g)$, and $N$ is an SL $m$-fold iff it is calibrated with respect to $\text{Re} \Omega$. 
Singular SL $m$-folds

General singularities of SL $m$-folds may be very bad, and difficult to study. Would like a class of singular SL $m$-folds with nice, well-behaved singularities to study in depth. Would like these to occur often in real life, i.e. of finite codimension in the space of all SL $m$-folds. SL $m$-folds with isolated conical singularities (ICS) are such a class.
Let $N$ be an SL $m$-fold in $M$ whose only singular points are $x_1, \ldots, x_n$. Near $x_i$ we can identify $M$ with $\mathbb{C}^m \cong T_{x_i}M$, and $N$ near $x_i$ approximates an SL $m$-fold in $\mathbb{C}^m$ with singularity at 0. We say $N$ has isolated conical singularities if near $x_i$ it converges with order $O(r^{\mu_i})$ for $\mu_i > 1$ to an SL cone $C_i$ in $\mathbb{C}^m$ nonsingular except at 0.
SL $m$-folds with ICS have a rich theory.

• **Examples.** Many examples of SL cones $C_i$ in $\mathbb{C}^m$ have been constructed. Rudiments of classification for $m = 3$.

• **Regularity near** $x_1, \ldots, x_n$. Let $\iota: N \to M$ be the inclusion. If $\nabla^k \iota$ converges to $C_i$ near $x_i$ with order $O(r^{\mu_i-k})$ for $k = 0, 1$ then it does so for all $k \geq 0$. 
• Deformation theory. The moduli space $\mathcal{M}_N$ of deformations of $N$ is locally homeomorphic to $\Phi^{-1}(0)$, for smooth $\Phi : \mathcal{I} \rightarrow \mathcal{O}$ and fin. dim. vector spaces $\mathcal{I}, \mathcal{O}$ with $\mathcal{I}$ the image of $H^1_{cs}(N', \mathbb{R})$ in $H^1(N', \mathbb{R})$, $N' = N \setminus \{x_1, \ldots, x_n\}$, and $\dim \mathcal{O} = \sum_{i=1}^{n} s\text{-ind}(C_i)$. Here $s\text{-ind}(C_i) \in \mathbb{N}$ is the stability index, the obstructions from $C_i$. If $s\text{-ind}(C_i) = 0$ for all $i$ then $\mathcal{M}_N$ is smooth.
• **Desingularization.** Let \( C \) be an SL cone in \( \mathbb{C}^m \), nonsingular except at 0. A nonsingular SL \( m \)-fold \( L \) in \( \mathbb{C}^m \) is **Asymptotically Conical (AC)** \( C \) if \( L \) converges to \( C \) at infinity with order \( O(r^\lambda) \) for \( \lambda < 1 \). Then \( tL \) converges to \( C \) as \( t \to 0_+ \). Thus, AC SL \( m \)-folds model how families of nonsingular SL \( m \)-folds develop singularities modelled on \( C \).
If \( N \) is an SL \( m \)-fold with ICS at \( x_1, \ldots, x_n \) and cones \( C_i \), and \( L_1, \ldots, L_n \) are AC SL \( m \)-folds in \( \mathbb{C}^m \) with cones \( C_i \), then under cohomological conditions we can construct a family of compact nonsingular SL \( m \)-folds \( \tilde{N}_t \) for small \( t > 0 \) converging to \( N \) as \( t \to 0 \), by gluing \( tL_i \) into \( N \) at \( x_i \), all \( i \).
Generic codimension of singularities. Given an SL $m$-fold $N$ with ICS in $M$, we have moduli spaces $M_N$ of deformations of $N$, and $M_{\tilde{N}}$ of desingularizations $\tilde{N}$ of $N$ made by gluing in $L_1, \ldots, L_n$. Here $M_N$ is part of the boundary of $M_{\tilde{N}}$. If $M$ is a generic almost C-Y $m$-fold then $M_N$, $M_{\tilde{N}}$ are smooth with known dimension.
Call $\dim \mathcal{M}_{\tilde{N}} - \dim \mathcal{M}_N$ the index of the singularities of $N$. It is the sum over $i$ of $\text{s-ind}(C_i)$ and topological terms from $L_i$. In a dimension $k$ family $\mathcal{B}$ of $\text{SL}_m$-folds in a generic almost C-Y $m$-fold $M$, only singularities with index $\leq k$ occur. For SYZ in generic $M$ we need to know about singularities with index 1,2,3 (and 4).

**Problem:** classify singularities with small index.
Mirror Symmetry

String theorists believe that each Calabi–Yau 3-fold $X$ has a quantization, a SCFT. Calabi–Yau 3-folds $X, \hat{X}$ are a mirror pair if their SCFT’s are related by a certain involution of SCFT structure. Then invariants of $X, \hat{X}$ are related in surprising ways. For instance,

$$H^{1,1}(X) \cong H^{2,1}(\hat{X}) \text{ and } H^{2,1}(X) \cong H^{1,1}(\hat{X}).$$
Using physics, Strominger, Yau and Zaslow proposed:

**The SYZ Conjecture.** Let $X, \hat{X}$ be mirror Calabi–Yau 3-folds. There is a compact 3-manifold $B$ and continuous, surjective $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that

(i) For $b$ in a dense $B_0 \subset B$, the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are dual SL 3-tori $T^3$ in $X, \hat{X}$.

(ii) For $b \notin B_0$, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are singular SL 3-folds in $X, \hat{X}$. 
We call $f, \hat{f}$ special Lagrangian fibrations, and $\Delta = B \setminus B_0$ the discriminant. In (i), the nonsingular fibres $T, \hat{T}$ of $f, \hat{f}$ are supposed to be dual tori. Topologically, this means an isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. But the metrics on $T, \hat{T}$ should really be dual as well. This only makes sense in the ‘large complex structure limit’, when the fibres are small and nearly flat.
U(1)-invariant SL 3-folds

Let U(1) act on $\mathbb{C}^3$ by
$$(z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3).$$

Let $N$ be a U(1)-invariant SL 3-fold. Then locally we can write $N$ in the form
$$\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a, \quad z_1z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad x, y \in \mathbb{R}\},$$
where $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy
$$u_x = v_y \quad \text{and} \quad v_x = -2(v^2 + y^2 + a^2)^{1/2} u_y. \quad (\ast)$$
Since $u_x = v_y$, there exists a potential function $f$ with $u = f_y$ and $v = f_x$. The 2nd equation of (*) becomes

$$f_{xx} + 2(f_x^2 + y^2 + a^2)^{1/2} f_{yy} = 0.$$  

This is a second-order quasi-linear equation. When $a \neq 0$ it is locally uniformly elliptic. When $a = 0$ it is non-uniformly elliptic, except at singular points $f_x = y = 0$. 

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Theorem A. Let $S$ be a compact domain in $\mathbb{R}^2$ satisfying some convexity conditions. Let $\phi \in C^{3,\alpha}(\partial S)$. If $a \neq 0$ there exists a unique $f \in C^{3,\alpha}(S)$ satisfying $(\dagger)$ with $f|_{\partial S} = \phi$. If $a = 0$ there exists a unique $f \in C^1(S)$ satisfying $(\dagger)$ with weak second derivatives, with $f|_{\partial S} = \phi$. Also $f$ depends continuously in $C^1(S)$ on $a, \phi$. 

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Theorem A shows that the Dirichlet problem for \((+)\) is uniquely solvable in certain convex domains. The induced solutions \(u, v \in C^0(S)\) of \((*)\) yield U(1)-invariant SL 3-folds in \(\mathbb{C}^3\) satisfying certain boundary conditions over \(\partial S\). When \(a \neq 0\) these SL 3-folds are nonsingular, when \(a = 0\) they are singular when \(v = y = 0\).
Theorem B.
Let \( \phi, \phi' \in C^{3,\alpha}(\partial S) \), let \( a \in \mathbb{R} \) and let \( f, f' \in C^{3,\alpha}(S) \) or \( C^1(S) \) be the solutions of (\(+\)) from Theorem A with \( f|_{\partial S} = \phi, \ f'|_{\partial S} = \phi' \). Let \( u = f_y, \ v = f_x, \ u' = f'_y, \ v' = f'_x \). Suppose \( \phi - \phi' \) has \( k+1 \) local maxima and \( k+1 \) local minima on \( \partial S \). Then \((u, v) - (u', v')\) has no more than \( k \) zeroes in \( S^o \), counted with multiplicity.
Theorem C.
Let \( u, v \in C^0(S) \) be a singular solution of (\( \ast \)) with \( a = 0 \), e.g. from Theorem A. Then 

- **either** \( u(x, y) \equiv u(x, -y) \) and \( v(x, y) \equiv -v(x, -y) \), so that \( u, v \) is singular on the \( x \)-axis,

- **or** the singularities \( (x, 0) \) of \( u, v \) in \( S^\circ \) are isolated, with a multiplicity \( n > 0 \). Multiplicity \( n \) singularities occur in codimension \( n \) of boundary data.

All multiplicities occur.
Theorem D.
Let \( U \subset \mathbb{R}^3 \) be open, \( S \) as above, and \( \Phi : U \to C^{3,\alpha}(\partial S) \) continuous such that if \((a, b, c) \neq (a, b', c') \in U\) then \( \Phi(a, b, c) - \Phi(a, b', c') \) has 1 local maximum and 1 local minimum.
For \( \alpha = (a, b, c) \in U \), let \( f_{\alpha} \in C^1(S) \) be the solution of \((+)\) from Theorem A with \( f_{\alpha}|_{\partial S} = \Phi(\alpha) \).
Set $u_\alpha = (f_\alpha)_y$ and $v_\alpha = (f_\alpha)_x$. Let $N_\alpha$ be the SL 3-fold 
\[
\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a, \\
z_1z_2 = v_\alpha(x, y) + iy, \\
z_3 = x + iu_\alpha(x, y), (x, y) \in S^\circ\}.
\]
Then there exists an open $V \subset \mathbb{C}^3$ and a continuous map $F : V \to U$ with $F^{-1}(\alpha) = N_\alpha$.
This is a $U(1)$-invariant special Lagrangian fibration. It can include singular fibres, of every multiplicity $n > 0$. 
Example. Define $f : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by $f(z_1, z_2, z_3) = (a, b)$, where $2a = |z_1|^2 - |z_2|^2$ and

$$b = \begin{cases} 
    z_3, & z_1 = z_2 = 0, \\
    z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geq 0, \ z_1 \neq 0, \\
    z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0.
\end{cases}$$

Then $f$ is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$. The fibres $f^{-1}(a, b)$ are $T^2$-cones when $a = 0$, and nonsingular $S^1 \times \mathbb{R}^2$ when $a \neq 0$. 

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Conclusions
Using these SL fibrations as local models, if $X$ is a generic ACY 3-fold and $f : X \to B$ an SL fibration, I predict:

- $f$ is only piecewise smooth.
- All fibres have finitely many singular points.
- $\Delta$ is codim 1 in $B$. Generic singularities are modelled on the example above.
- Some codim 2 singularities are also locally $U(1)$-invariant.
• Codim 3 singularities are not locally U(1)-invariant.
• If $f : X \to B$, $\hat{f} : \hat{X} \to B$ are dual SL fibrations of mirror C-Y 3-folds, the discriminants $\Delta, \hat{\Delta}$ have different topology near codim 3 singular fibres, so $\Delta \neq \hat{\Delta}$.

This contradicts some statements of the SYZ Conjecture. I regard SYZ as primarily a limiting statement about the ‘large complex structure limit’.