Calibrated fibrations of compact manifolds with special holonomy

Gilles Englebert

Hertford College

University of Oxford

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Abstract

We study compact manifolds of special holonomy $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ and their calibrated submanifolds, the coassociative and Cayley submanifolds. These are minimal submanifolds arising from the holonomy restriction. Calabi–Yau fourfolds appear as special examples of Spin(7)-manifolds.

Physicists expect that a Calabi–Yau threefold X admits a mirror $\hat{X}$ (where the complex geometry of X is equivalent to the symplectic geometry of $\hat{X}$ and vice-versa). A proposed geometric explanation, the SYZ conjecture [43], stipulates that both fiber over the same base B^3 with (possibly singular) calibrated torus fibres that are dual to one another.

We study the analogous existence problem of calibrated fibrations in the Spin(7) case and prove that Cayley fibrations of compact Spin(7)-manifolds, where fibres may admit certain types of conical singularities, are stable under small deformations of the Spin(7)-structure. More precisely, we require all the fibres to be unobstructed in their respective moduli spaces and the cones to have well-behaved critical rates. Furthermore, the singular locus should be of codimension at least 2 in the base and the asymptotically conical Cayleys required for the desingularisation of singular fibres should have deformations of a unique asymptotic rate. Complex fibrations of Calabi–Yau fourfolds with at worst Morse-type singularities satisfy all of these conditions.

As an application, we prove the existence of coassociative Kovalev-Lefschetz fibrations of G_2-manifolds arising as twisted connected sums. We present an explicit example of a coassociative fibration on the twisted connected sum G_2-manifold obtained from two quartic building blocks. This completes the program initiated by Kovalev to find examples of coassociative fibrations using gluing methods [24].

Along the way we revisit the deformation theory of compact Cayley submanifolds (McLean [34], Clancy [7], Moore [36]) and conically singular Cayley submanifolds (Moore [38]) and describe the deformation theory of asymptotically conical Cayley submanifolds of $\mathbb{R}^8$. We do this in the unifying framework of families of almost Cayley submanifolds (whose tangent bundles are close to a bundle of Cayley planes) in not necessarily torsion-free Spin(7)-manifolds, and define a canonical deformation operator even for submanifolds that are not Cayley. This generalises a number of results in the existing literature.

Furthermore, we study the desingularisation theory of conically singular Cayley submanifolds by attaching asymptotically conical submanifolds at the singularities and prove a general gluing theorem. As an application, we determine when immersed points of Cayley submanifolds may be smoothed by gluing in a Lawlor neck.

Contents

Introduction

The holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold (M^n, g) captures the possible parallel transport transformations of closed loops for the Levi-Civita connection ∇_g. Generically it will be isomorphic to $\mathrm{O}(n)$ (or $\mathrm{SO}(n)$ if M is oriented) in which case M only admits parallel tensors constructed from the metric. No further geometric structure compatible with the metric g can exist, as any ∇_g-parallel tensor needs to be invariant under the action of $\mathrm{Hol}(g)$ on the tangent space at every point, i.e. under the standard action of $\mathrm{SO}(n) \curvearrowright \mathbb{R}^n$.

If the holonomy group $\mathrm{Hol}(g)$ is strictly smaller than in the generic case, further parallel tensors and thus additional geometric structures appear. Take for instance Kähler manifolds, which have holonomy $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ and automatically admit a parallel integrable complex structure $J : TM \to TM$. More generally such Riemannian manifolds are known as **manifolds of special holonomy**.

Surprisingly, the list of possible holonomy groups is rather short if we restrict to simply connected, irreducible (i.e. not locally of product form) and non-symmetric metrics. Berger [3] proved that for these elementary building blocks of Riemannian manifolds $\mathrm{Hol}(g)$ must be one of the following groups:

- $\mathrm{SO}(n)$, the generic case,
- $\mathrm{U}(n)$ and $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$, from complex geometry,
- $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, from quaternionic geometry,
- $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, from octonionic geometry.

The exceptional holonomy groups G_2 and $\mathrm{Spin}(7)$ stand out among the entries of Berger's list as they are purely 7 and 8-dimensional phenomena. This already makes them mathematically interesting. However, it was unclear at first whether interesting examples of manifolds with exceptional holonomy did in fact exist. As an example, Berger's list originally included another group $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, but it was later shown that there were no non-symmetric examples. The existence question for G_2 and $\mathrm{Spin}(7)$ was resolved in 1987 by Bryant [5], who provided examples of such metrics on open balls in Euclidean space. Later in 1989 Bryant and Salamon gave complete examples in [6]. The last major contribution was made by Joyce, who in 1995-1996 constructed compact manifolds of exceptional holonomy in [11–13].

Often in mathematics one can study objects by looking at their subobjects. For example, one can study symplectic manifolds by considering the moduli spaces of pseudo-holomorphic curves. The correct subobjects to investigate in manifolds of special holonomy are the **calibrated submanifolds**. These are special examples of minimal submanifolds that solve a first-order p.d.e. which implies the second-order minimal surface p.d.e. While being easier to approach than the usual minimal surface equation, the first-order p.d.e. presupposes the existence of a particular tensor, the **calibration form**, which in our case comes from the holonomy reduction.

As a subject, calibrated geometry was first conceived by Harvey and Lawson in their foundational paper [10] as a generalisation of the geometry of complex submanifolds in Kähler manifolds. The calibrated submanifolds in the G_2 setting are the three and four-dimensional associative and coassociative submanifolds respectively. For Spin(7) the relevant submanifolds are the four-dimensional Cayley submanifolds. All of these can be seen as geometric consequences of the algebraic properties of the normed division algebra of octonions.

Both the study of manifolds with special holonomy and of calibrated submanifolds are worthwhile endeavours in differential geometry, but they are more than that. Because of their geometric structures, manifolds with special holonomy and their calibrated submanifolds have become central objects of study for theoretical physicists, especially in the context of String theory and M-theory. Here one studies the evolution of strings (embeddings of either S^1, a closed string, or $[0,1]$, an open string) on a high-dimensional Lorentzian manifold $\mathbb{R}^{3,1} \times X^6$, where X^6 is a Calabi–Yau manifold, i.e. a Riemannian manifold with holonomy SU(3), or $\mathbb{R}^{3,1} \times M^7$ with M^7 a manifold with holonomy G_2. In this way, one can associate a quantum theory to a Riemannian manifold. Manifolds of special holonomy in particular appear as inputs to these quantum theories because they admit parallel spinors, which ensure that the quantum theory is supersymmetric, a sought-after quality. Calibrated submanifolds provide boundary data for open strings to propagate along and further influence the resulting physics.

This interplay with physics becomes interesting for mathematicians once we start interpreting the physical properties of String and M-theory (which are not related to special holonomy or calibrated geometry a priori) in a mathematical light. The main impetus for this thesis is T-duality and the study of its mathematical formalisation, mirror symmetry. It states that the string theory associated with the Calabi–Yau threefold X is physically equivalent to another string theory for the Calabi–Yau threefold $\hat{X}$. One proposed way to explain mirror symmetry is the SYZ conjecture, put forward by Strominger, Yau and Zaslow in [43]. The idea is that both X and $\hat{X}$ admit fibrations by special Lagrangians (the calibrated submanifolds of interest in Calabi–Yau geometry) over the same base space and that the fibres, which are generically tori, should be dual in an appropriate sense.

In this thesis, we investigate the adjacent problem of calibrated fibrations of compact manifolds with holonomy Spin(7) and, as a result, can deduce statements about coassociative fibrations of manifolds with holonomy G_2 as well. Cayley and coassociative fibrations have been constructed before in highly symmetric situations such as the noncompact Bryant-Salamon manifolds (see the work by Karigiannis-Lotay [20] and Trinca [44]), but never before on a compact manifold without symmetry assumptions.

Kovalev [24] has outlined a construction of coassociative fibrations of G_2-manifolds obtained from Calabi–Yau ingredients, the twisted connected sum G_2-manifolds, a gluing construction also due to Kovalev [23] and later extended by Corti, Haskins, Nordström and Pacini [9]. As an application of our work on Cayley fibrations, we complete the program by Kovalev by providing the first example of a coassociative fibration on a compact G_2-manifold of full holonomy.

Main results and chapter overview

Chapter 1 reviews aspects of differential geometry and functional analysis which will be needed for the rest of the thesis. In particular, we will introduce calibrated geometry with a focus on Calabi–Yau, G_2 and Spin(7)-manifolds and their calibrated submanifolds. We note that for us G_2 and Spin(7)-manifolds do not need to be torsion free in general.

Chapter 2 focuses on moduli spaces of Cayley submanifolds in an ambient Spin(7)-manifolds, both for compact and noncompact Cayleys. The moduli space of compact Cayley submanifolds has been previously studied by McLean [34], who investigated the linearised deformation operator and the local structure of the moduli space, by Clancy [7], who investigated its global properties and proved a useful index formula, and by Moore [37], who focused on the case when the Cayley is a complex surface in a Calabi–Yau fourfold.

We reprove many of the existing results for the slightly more general setting of the family moduli space $\mathcal{M}(N, \mathcal{S})$ (where $\mathcal{S}$ is a smooth family of Spin(7)-structures, and N any smooth submanifold) of all Cayley submanifolds in (M, Φ) isotopic to N, where $\Phi \in \mathcal{S}$, and investigate the non-linear deformation operator also for non-Cayley submanifolds whose tangent planes are close to being Cayley planes, the **almost Cayley submanifolds**.

Theorem 1 (Moduli space of compact Cayley submanifolds)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let N be an immersed compact Cayley submanifold of a not necessarily torsion-free* Spin(7)*-manifold (M, Φ_{s_0}), where $\{\Phi_s\}_{s \in \mathcal{S}}$ is a smooth family of* Spin(7)*-structures parametrised by the smooth manifold $\mathcal{S}$, and $s_0 \in \mathcal{S}$. Then there is a non-linear deformation operator F which for $\epsilon > 0$ sufficiently small and $s_0 \in \mathcal{U} \subset \mathcal{S}$ an open neighbourhood is a C^∞ map:*

$$F : \mathcal{L}_\epsilon = \{v \in L^p_{k+1}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_k(E).$$

Here $\nu_\epsilon(N)$ is an ϵ-neighbourhood around the zero-section of the normal bundle $\nu(N)$ of $N \subset M$, and $E \subset \Lambda^2 T^* N$ is a certain rank four subbundle. A neighbourhood of (N, Φ_{s_0}) in the family moduli space of Cayley submanifolds $\mathcal{M}(N, \mathcal{S})$ is homeomorphic to the zero locus of F near $(0, \Phi_{s_0})$. We say that N is unobstructed if $\operatorname{Coker} \mathrm{D}F(0, \Phi_{s_0}) = \{0\}$. *In that case, near (N, Φ_{s_0}), $\mathcal{M}(N, \mathcal{S})$ is a smooth manifold of dimension*

$$\dim \operatorname{Ker} \mathrm{D}F(0, \Phi_{s_0}) = \frac{1}{2}(\sigma(N) + \chi(N)) - [N] \cdot [N] + \dim \mathcal{S}.$$

We note that the conditions $p > 4$ and $k \geqslant 1$ are there to ensure the Sobolev embedding $L^p_k \to C^0$.

The main goal of Chapter 2 is to prove the corresponding results for Cayley submanifolds which are **asymptotically conical** (AC) or **conically singular** (CS). The proof of Theorem 1 introduces all the elements required for the proof of Theorems 2 and 3, but in an analytically simpler setting. We then introduce the analytic machinery necessary to study the moduli spaces of AC and CS Cayley submanifolds, namely Lockhart and McOwen's theory [29] of weighted Sobolev spaces.

First, we prove the analogue of Theorem 1 for AC Cayleys in $\mathbb{R}^8$, which are non-compact Cayleys that have an end that is asymptotic to a cone at infinity. The moduli space $\mathcal{M}^{\lambda}_{\mathrm{AC}}(A, \mathcal{S})$ of AC Cayleys which are isotopic to A and approach the cone at infinity at least in $O(r^{\lambda-1})$ has the following structure:

Theorem 2 (Moduli space of AC Cayley submanifolds)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let A be an AC_λ Cayley submanifold of $(\mathbb{R}^8, \Phi_0)$, where Φ_0 is the standard* Spin(7)*-structure on $\mathbb{R}^8$, and let $\mathcal{S}$ be a smooth family of AC_η deformations of Φ_0 with $\eta < \lambda < 1$ (i.e. the Cayley may not have a stronger rate than the background manifold). Then there is a non-linear deformation operator F_{AC} which for $\epsilon > 0$ sufficiently small and $0 \in \mathcal{U} \subset \mathcal{S}$ an open neighbourhood is a C^∞ map:*

$$F_{\mathrm{AC}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\lambda}(\nu_\epsilon(A)), \|v\|_{L^p_{k+1,\lambda}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\lambda-1}(E).$$

A neighbourhood of (A, Φ_0) in $\mathcal{M}^{\lambda}_{\mathrm{AC}}(A, \mathcal{S})$ is homeomorphic to the zero locus of F_{AC} near $(0, \Phi_0)$. If the rate λ is not in a discrete critical set $\mathscr{D} \subset \mathbb{R}$, then F_{AC} is a Fredholm operator. In particular, if the obstruction space $\operatorname{Coker} \mathrm{D}F_{\mathrm{AC}}(0, \Phi_0)$ *vanishes, then $\mathcal{M}^{\lambda}_{\mathrm{AC}}(A, \mathcal{S})$ is a smooth manifold near (A, Φ_0), the dimension of which depends on the rate $\lambda < 1$.*

Next, we prove the analogue of Theorem 1 for conically singular Cayley submanifolds, which are compact Cayley submanifolds that admit a finite number of singular points, around which they are modelled on cones. Each singular point can be assigned a rate $1 < \mu < 2$, which is a measure of how fast the manifold approaches the cone near its vertex. When there are multiple singular points, we regroup the rates into a vector $\bar{\mu}$.

Moore [38] studied CS Cayleys on a fixed torsion-free Spin(7)-manifold. Our contribution is the extension of the result for CS Cayleys to the family moduli space $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ for varying Spin(7)-structure which may admit torsion.

Theorem 3 (Moduli space of CS Cayley submanifolds)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let N be a $\mathrm{CS}_{\bar{\mu}}$ Cayley submanifold of the not necessarily torsion-free* Spin(7)*-manifold (M, Φ_{s_0}), and suppose $\{\Phi_s\}_{s \in \mathcal{S}}$ is a smooth family of deformations of Φ_{s_0}. Let $\mathcal{F}$ be the configuration space of possible singular points and deformations of the asymptotic cones of N, where the asymptotic data of N itself is given by $f_0 \in \mathcal{F}$. This is a smooth manifold. Then there is a non-linear deformation operator F_{CS} which for $\epsilon > 0$ sufficiently small and $(s_0, f_0) \in \mathcal{U} \subset \mathcal{S} \times \mathcal{F}$ a open neighbourhood is a C^∞ map:*

$$F_{\mathrm{CS}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\bar{\mu}}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1,\bar{\mu}}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\bar{\mu}-1}(E).$$

A neighbourhood of (N, Φ_{s_0}) in $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ is homeomorphic to the zero locus of F_{CS} near $(0, s_0, f_0)$. If the rates $\bar{\mu}$ are not in a discrete critical set $\mathscr{D} \subset \mathbb{R}$, then F_{CS} is a Fredholm operator. In particular, if the obstruction space $\operatorname{Coker} \mathrm{D}F_{\mathrm{CS}}(0, \Phi_{s_0})$ *vanishes, then $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ is a smooth manifold near (N, Φ_{s_0}), the dimension of which depends on the rates $1 < \bar{\mu} < 2$.*

Chapter 3 focuses on the desingularisation of conically singular Cayley submanifolds, by gluing in matching asymptotically conical Cayleys. Here $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A)$ for $A \subset \mathbb{R}^8$ an AC

Cayley denotes the completed moduli space, which is defined to be the usual moduli space with an additional point corresponding to the asymptotic cone. We present a simplified version of the general gluing theorem 3.15, which allows for the simultaneous desingularisation of multiple singular points, as well as partial desingularisation.

Before we state the theorem, recall that when C is a Cayley cone in $\mathbb{R}^8$, with link $L = C \cap S^7$, there is a discrete subset of critical rates $\mathscr{D}_L \subset \mathbb{R}^8$ for which the deformation operators F_{AC} and F_{CS}, defined on weighted spaces, are not Fredholm. We also recall the nearly parallel G_2-structure on the round seven-sphere. Let ∂_r be the outward radial unit vector field on $S^7 \subset \mathbb{R}^8$ and r the distance to the origin in $\mathbb{R}^8$. Then we have at $p \in S^7$ that $\Phi_p = \mathrm{d}r \wedge (\varphi)_p + (\star_{S^7}\varphi)_p$. Here (S^7, φ) is a G_2-structure for which the link L of the Cayley cone is associative.

Theorem 4 (Desingularisation of CS Cayleys)**.** *Let (M, Φ) be a* $\mathrm{Spin}(7)$*-manifold and N a* CS_μ*-Cayley in (M, Φ) with unique singular point z of rate $1 < \mu < 2$, modelled on the cone $C = \mathbb{R}_+ \times L \subset \mathbb{R}^8$. Assume that N is unobstructed in $\mathcal{M}^{\mu}_{\mathrm{CS}}(N, \Phi)$, that L is an unobstructed associative in S^7, and that $\mathscr{D}_L \cap (0, \mu] = \{1\}$. Suppose that A is an unobstructed* AC_λ*-Cayley with $\lambda < 0$, such that $\mathscr{D}_L \cap [\lambda, 0) = \emptyset$. Let $\{\Phi_s\}_{s \in \mathcal{S}}$ be a smooth family* $\mathrm{Spin}(7)$*-structure, deforming $\Phi = \Phi_{s_0}$. Then there is an open neighbourhood U_{AC} of $C \in \overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A)$, an open neighbourhood $s_0 \in \mathcal{U} \subset \mathcal{S}$ and a continuous map:*

$$\Gamma : \mathcal{U} \times \mathcal{M}^{\mu}_{\mathrm{CS}}(N, \Phi) \times U_{\mathrm{AC}} \longrightarrow \mathcal{M}(N \sharp_L A, \mathcal{S}) \sqcup \mathcal{M}(N, \mathcal{S}).$$

This map is a local diffeomorphism of stratified manifolds. Thus away from the cone in $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A)$ it is a local diffeomorphism onto the nonsingular Cayley submanifolds in $\mathcal{M}(N \sharp_L A, \mathcal{S})$. It maps the points $(s, \tilde{N}, C)$ to $\tilde{N} \in \mathcal{M}^{\mu}_{\mathrm{CS}}(N, \Phi_s)$.

The gluing theorem is proven by first constructing an approximate glued Cayley and then following an iteration scheme which converges to an exact Cayley under suitable conditions. Our proof follows the outline of the analogous results for special Lagrangians by Joyce [18] and coassociative submanifolds by Lotay [30]. However, it differs in how the necessary estimates on the inverse of the linearised operator are obtained. We glue together the estimates for the pieces which, since it is adapted to the geometry, allows us in the general case of Theorem 3.15 to work with CS Cayleys that have multiple singularities of different rates and do partial desingularisation.

In particular, we can resolve negative self-intersections, as these are geometrically equivalent to a pair of special Lagrangians intersecting at a point, admitting a Lawlor neck desingularisation.

Corollary 5 (Desingularisation of immersions)**.** *Let N be an unobstructed immersed compact Cayley submanifold which admits a negative self-intersection at $p \in N$. Then there is a family of Cayley submanifolds with one fewer immersed point $\{N_t\}_{t \in (0,\epsilon)}$ such that $N_t \to N$ in the sense of currents and also in C^{∞}_{loc} away from the self-intersection as $t \to 0$.*

It is not possible to remove positive intersections, as there is no corresponding Lawlor neck in this case. In fact, in the torsion-free case, there is no non-singular Cayley homologous to an immersed Cayley with one positive self-intersection.

In Chapter 4 we prove, using our results from the two previous chapters, that Cayley fibrations of compact Spin(7)-manifolds satisfying certain conditions are stable under small perturbations of the ambient Spin(7)-structure. We proceed in two steps. First, we show stability for what we call **weak fibrations**. For $N \subset (M, \Phi)$ a compact, nonsingular Cayley submanifold we introduce the completed moduli space $\overline{\mathcal{M}}(N, \Phi)$ which adjoins to the usual moduli space the CS degenerations that can occur in $\mathcal{M}(N, \Phi)$. This is usually a partial compactification, which we assume in the following is a full compactification. Under this hypothesis we then say that $\overline{\mathcal{M}}(N, \Phi)$ **weakly fibers** (M, Φ) if every point is covered by exactly one Cayley, where we count Cayleys algebraically, i.e. with signs. Hence this is a homological notion. We introduce a regularity property of cones, **semistability** (cf. Definition 1.36), which just means that translations and deformations of the link as associatives in S^7 account for all the deformations of a Cayley cone with rate in the range $[0, 1]$. We then show the following result, which has minimal assumptions on the geometry.

Theorem 6 (Stability of weak Cayley fibrations)**.** *Let* (M, Φ) *be a* Spin(7)*-manifold that is weakly fibred by* $\overline{\mathcal{M}}(N, \Phi)$*, and suppose that* $\{\Phi_s\}_{s\in\mathcal{S}}$ *is a smooth family of* Spin(7)*-structures with* $\Phi = \Phi_{s_0}$*. Assume that all the Cayleys in* $\overline{\mathcal{M}}(N, \Phi)$ *are unobstructed and that the cones in the conically singular degenerations of* N *are semistable and unobstructed. Then there is an open set* $s_0 \in \mathcal{U} \subset \mathcal{S}$ *such that* M *is weakly fibred by* $\overline{\mathcal{M}}(N, \Phi_s)$ *for any* $s \in \mathcal{U}$*.*

Next, we build on the weak stability result to show that **strong fibrations**, i.e. fibrations in the usual sense with restrictions on the possible singularities, are also stable under certain conditions. In particular, we require the fibration to be **nondegenerate** (see Definition 4.14), which means that the initial fibres should be quantitatively separated even as one approaches the singularities. Furthermore, the fibres should all be unobstructed in their respective moduli spaces and the conically singular fibres should be **simple** (see Definition 4.13). This means that their deformation problem should have the correct index 4 just below a critical rate $\zeta < 0$ and that the linearised Cayley equation should admit solutions of at most two different rates at that rate ζ. These conditions are in particular satisfied for Cayley fibrations coming from complex fibrations of Calabi–Yau fourfolds with Morse type singularities.

Theorem 7 (Stability of strong Cayley fibrations)**.** *Let* (M, Φ_{s_0}) *be a (not necessarily torsion-free)* Spin(7)*-manifold that is strongly fibred by conically singular Cayleys which are simple, and suppose that* $\{\Phi_s\}_{s\in\mathcal{S}}$ *is a smooth family of deformations as* Spin(7)*-structures of* Φ_{s_0}*. Assume that all the Cayleys in the fibration are unobstructed and that the fibration is nondegenerate. Then there is an open set* $s_0 \in \mathcal{U} \subset \mathcal{S}$ *such that* M *can be strongly fibred for any* $s \in \mathcal{U}$*.*

The result is shown by proving a gluing theorem for the infinitesimal Cayley deformations, which are the variational vector fields associated to a family of Cayley submanifolds obtained by varying the basepoint in a Cayley fibration. This gluing result allows us to understand how the fibres of a fibration perturb near the singular points under the change of Spin(7)-structure.

Finally in Chapter 5 we construct examples of calibrated fibrations using the strong stability Theorem 7. Generally, constructing calibrated fibrations using gluing methods splits into two separate problems.

First is the issue of finding suitable fibrations on the pieces which are compatible with the gluing and fit together to give a calibrated fibration $f : M \to B$ on a small torsion manifold (M, Φ). This is already a hard problem by itself because of the difficulty of constructing calibrated submanifolds. For now, the most effective way to construct calibrated fibrations is to start with a complex fibration on a Calabi–Yau manifold, as these are abundant. Now this already excludes special Lagrangian fibrations, as they do not arise in a natural way from complex fibrations. But even though we have tools to construct Cayley and coassociative fibrations it remains challenging to construct examples which are neither trivial nor admit analytically intractable singularities.

One such singularity, which is not conical and keeps appearing in practice is:

$$f_{\mathrm{cubic}} : \mathbb{C}^4 \longrightarrow \mathbb{C}^2, \quad (x, y, z, w) \longmapsto (x^2 + y^2 + z^3, w). \tag{$\star$}$$

This singularity is expected to be of codimension 4 in the Cayley moduli space, and thus cannot simply be perturbed away if it appears in a fibration. There is currently no Fredholm deformation theory for Cayleys with this kind of singular behaviour, which is why fibrations on pre-glued manifolds may not currently include such fibres.

Next, in a gluing construction of torsion-free Spin(7)-manifolds the Cayley form Φ is deformed to a nearby torsion-free form $\tilde{\Phi}$ and the fibres of f deform accordingly to a new collection of Cayleys. Our strong stability Theorem 7 guarantees that in certain situations these new Cayleys remain fibering. At the same time, it answers the analogous question for coassociative fibrations and complex fibrations of Calabi–Yau fourfolds by surfaces.

The example we construct comes from the twisted connected sum construction of G_2-manifolds and was first proposed by Kovalev [24] with an incomplete proof of the stability theorem. It also provides an example of a fibred Spin(7)-manifold albeit with holonomy necessarily contained in G_2. In the Spin(7) case, there is a natural Cayley fibration on the pre-glued manifold (with torsion), whose local singularity model we can write down explicitly. It is the following conical Morse-type complex singularity:

$$f_0 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2, \quad (x, y, z, w) \longmapsto (x^2 + y^2 + z^2, w).$$

We then show that the fibrations persist when we perturb to the torsion-free Spin(7)-structure, which gives us the following result.

Theorem 8 (Existence of strong Kovalev-Lefschetz fibrations on compact Spin(7)-manifolds)**.** *There are compact, torsion-free* Spin(7)*-manifolds of holonomy G_2 which admit strong fibrations by Cayley manifolds.*

Finally, as our example is of product type, the fibration can be shown to split and we obtain the following corollary for the G_2 case.

Corollary 9 (Existence of coassociative fibrations on compact G_2-manifolds)**.** *There are compact, torsion-free G_2-manifolds of full holonomy which admit strong fibrations by coassociative submanifolds.*

This gives the first example of a coassociative fibration of a holonomy G_2-manifold as described by the programme of Kovalev [24].

Outlook

As explained above, even though we work in the more general framework of Spin(7) and Cayley geometry, and also prove the strong stability of fibrations in more generality than just complex fibrations, we are unable to provide example fibrations of Spin(7)-manifolds of full holonomy as of now, due to the lack of known suitable fibrations on pre-glued manifolds. Attempts to produce such holomorphic fibrations on Calabi–Yau fourfold pieces tend to include bad singularities such as ($\star$).

The twisted connected sum of G_2-manifolds presents itself as an ideal candidate in this regard, as the gluing pieces naturally admit coassociative fibrations coming from complex geometry. In this thesis we give the explicit example of a calibrated fibration on the twisted connected sum of two quartic building blocks, however, the same method should work in far greater generality, as long as one can verify the prerequisites of Theorem 7.

Finally, with the stability theorem at hand, one should keep searching for examples with full holonomy Spin(7). It seems natural to focus on the second construction of Spin(7)-manifolds [14] which starts from Calabi–Yau orbifolds. We thus may use complex geometry to help us construct candidate fibrations, so that hopefully one day Theorem 7 may unfold its true potential.

Chapter 1

Background material

1.1 Notation

In the following, we denote by C an unspecified constant, which may refer to different constants within the same derivation. To indicate the dependence of this constant on quantities $x, y, \dots$, we will write $C(x, y, \dots)$. Similarly, if an inequality holds up to an unspecified constant, we will write $A \lesssim B$ instead of $A \leqslant CB$. The application of a linear operator $D : X \to Y$ to a vector $v \in X$ is written $D[v]$ with square brackets.

1.2 Calibrated Geometry

The study of calibrated geometry starts from the following observation, already made by Harvey and Lawson in their foundational paper [10]. Let (M, g) be a Riemannian manifold and suppose that $\varphi \in \Omega^k(M)$ is a closed form, such that at each point $p \in M$ and for each oriented k-plane $\Pi \in \mathrm{Gr}(T_pM, k)$ the **calibration inequality**:

$$\varphi|_\Pi \leqslant \mathrm{dvol}_\Pi$$

is satisfied. By this, we mean that $\varphi|_\Pi = \alpha\, \mathrm{dvol}_\Pi$ with $\alpha \leqslant 1$, as both forms are top dimensional when restricted to Π. We then call φ a **calibration**. We say that an oriented k-dimensional submanifold $N \subset M$ is φ-**calibrated** if the calibration inequality becomes an equality, i.e.:

$$\varphi|_N = \mathrm{dvol}_N\, .$$

Now the key observation is that any compact calibrated N is volume minimizing in its homology class, which can be seen by an application of Stokes' theorem. Indeed, for $\tilde{N}$ homologous to N we see:

$$\mathrm{vol}(N) = \int_N \mathrm{dvol}_N = \int_N \varphi = \int_{\tilde{N}} \varphi \leqslant \int_{\tilde{N}} \mathrm{dvol}_{\tilde{N}} \leqslant \mathrm{vol}(\tilde{N}).$$

Thus in particular calibrated submanifolds are minimal submanifolds and the study of calibrated submanifolds provides a different approach to constructing minimal submanifolds, other than the more direct study of the minimal submanifold equation and variational methods.

The first known example of a calibrated geometry was Kähler geometry. If (M, J, ω, g) is a Kähler manifold of complex dimension n, with complex structure J, Kähler form ω and Riemannian metric g, then the form $\frac{\omega^k}{k!}$ is a calibration form whenever $1 \leqslant k \leqslant n$. This is also called the **Wirtinger inequality**. The calibrated submanifolds are the complex submanifolds, which are indeed minimal submanifolds of Kähler manifolds.

Kähler manifolds are moreover examples of **special holonomy manifolds**, i.e. Riemannian manifolds whose holonomy group $\mathrm{Hol}(M, g)$ is a strict subgroup of $\mathrm{SO}(n)$ (in this case $\mathrm{U}(n) \subset \mathrm{SO}(2n)$)). This is not a coincidence, as many interesting examples of calibrations exist on special holonomy manifolds. Forms φ with $\mathrm{d}\varphi = 0$ can arise from the strictly stronger condition of being parallel, i.e. $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection induced by g. At any point $p \in M$, the holonomy group acts on $\Lambda^k T_p M$ and must preserve φ_p. Then if $h \in \mathrm{Hol}(M, g)$ is any element of the holonomy group and $\Pi \in \mathrm{Gr}(T_p M, k)$ is a calibrated plane, the plane $h \cdot \Pi$ is calibrated as well. This is the reason why calibrations coming from special holonomy manifolds tend to come with a large class of calibrated planes and hence also more calibrated submanifolds.

In the following, we will review the fundamentals of three calibrated geometries, namely the Calabi-Yau, G_2 and Spin(7) geometries, which all arise this way, and study their calibrated submanifolds.

1.3 Complex Geometry

Calabi–Yau Geometry

We briefly review some aspects of Calabi–Yau and Fano manifolds which will be relevant to our discussion of Cayley fibrations of Spin(7)-manifolds. For a more in-depth introduction, we refer to [15, Ch. 6].

Definition 1.1 (Calabi–Yau manifold)**.** Let (M^{2n}, J, ω, g) be a Kähler manifold of complex dimension n which admits a nowhere vanishing holomorphic $(n, 0)$-form Ω. The line bundle of $(n, 0)$-forms is called the **canonical bundle**, so equivalently we may require (M, J) to have a holomorphically trivial canonical bundle. If we furthermore have the following normalisation condition which links the complex and symplectic geometry of M:

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}, \tag{1.1}$$

then we call $(M^{2n}, J, \omega, g, \Omega)$ a **Calabi–Yau manifold**. The form Ω is called the **holomorphic volume form**.

The holonomy of any Calabi–Yau manifold is contained in $\mathrm{SU}(n)$ and the metric g is necessarily Ricci-flat. At any point an $\mathrm{SU}(n)$-structure is isomorphic to the following standard model on $\mathbb{C}^n$ with complex coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$:

$$\begin{aligned}\omega_0 &= \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots \mathrm{d}x_n \wedge \mathrm{d}y_n,\\ g_0 &= \mathrm{d}x_1^2 + \mathrm{d}y_1^2 + \cdots + \mathrm{d}x_n^2 + \mathrm{d}y_n^2,\\ \Omega_0 &= \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.\end{aligned}$$

We have the following theorem due to Yau [47] (proving a conjecture due to Calabi) which reduces the existence of a Calabi–Yau structure to a question of complex geometry on (M, J).

Theorem 1.2 (Theorem 1 in [47]). *Let (M, J, ω, g) be a compact Kähler manifold with trivial canonical bundle. Then there is a unique Kähler form $\tilde{\omega}$ in the de Rham cohomology class of ω (with corresponding metric $\tilde{g}$) and a holomorphic volume form Ω such that $(M, J, \tilde{\omega}, \tilde{g}, \Omega)$ is a Calabi–Yau manifold.*

On Calabi–Yau manifolds there are two calibrations of interest. First we have the real part of the holomorphic volume form $\operatorname{Re}\Omega \in \Omega^n(M)$, whose calibrated submanifolds are the so-called **special Lagrangians**. These are difficult to construct, and we will not go further into discussing them here. Secondly, we have the complex submanifolds in any dimension $1 \leqslant k \leqslant n$, which are calibrated by the form $\frac{\omega^k}{k!}$, as we already pointed out above.

In two complex dimensions, Calabi–Yau manifolds are particularly well understood. Their underlying complex surfaces must either be tori T^4 or so-called **K3 surfaces**, which are the only two deformation types of complex surfaces with trivial canonical bundles. We discuss K3 surfaces in more detail now, see [15, Section 7.3.3] for a more in-depth discussion. By a result of Kodaira all complex analytic K3 surfaces S belong to a single diffeomorphism type, namely that of a quartic $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{C}P^3$. In particular, they are simply connected and all have isomorphic cohomology groups, the only non-trivial one being $H^2(S, \mathbb{Z})$. Since S is a compact closed four-manifold its second cohomology admits a nondegenerate intersection pairing, and this lattice we denote by Λ. Next, we recall that analytic K3 surfaces form a 20-dimensional moduli space. To see this explicitly we define a **marked K3 surface** to be a K3 surface S together with a choice of lattice isomorphism $h : H^2(S, \mathbb{Z}) \to \Lambda$. The complex structure of the K3 surface is then determined locally by its Hodge structure (i.e. how $H^2(S, \mathbb{C}) = H^{0,2}(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^{2,0}(S, \mathbb{C})$ splits with respect to the marking h). More precisely we define the so-called **period domain**:

$$\begin{aligned}D_{K3} &= \{u \in P(\Lambda \otimes \mathbb{C}) : u^2 = 0,\ u \cdot \bar{u} > 0\}\\ &\simeq \{\Pi \subset \Lambda \otimes \mathbb{R} : \langle \cdot, \cdot \rangle|_{\Pi} > 0\} \subset \mathrm{Gr}_+(2, \Lambda \otimes \mathbb{R}).\end{aligned} \tag{1.2}$$

This is the space of all possible complex lines $h(H^{2,0}(S, \mathbb{C}))$ in $\Lambda \otimes \mathbb{C}$. The map sending (S, h) to $[h(H^{2,0}(S, \mathbb{C}))] \in D_{K3}$ is called the **period map**. It is a local but not a global diffeomorphism, in particular, because the moduli space of marked K3 surfaces is not Hausdorff (while D_{K3} is). The isomorphism to $\mathrm{Gr}_+(2, \Lambda \otimes \mathbb{R})$ follows from identifying $H^{2,0} \oplus H^{0,2} = \Pi \otimes \mathbb{C}$ for a real two-plane Π.

Next, for our discussion, we need K3 surfaces with additional structure, so-called **lattice polarised K3 surfaces** [2]. For this, we look at the **Picard group** $\mathrm{Pic}(S, J)$, which is the abelian group of holomorphic line bundles under the tensor product. As K3 surfaces are simply connected, we can think of the Picard group as being embedded in $H^2(S,\mathbb{Z})$ via the first Chern class $c_1 : \mathrm{Pic}(S,J) \to H^{1,1}(S,\mathbb{Z})$. Thus, while the intersection form on $H^2(S,\mathbb{Z})$ is a topological invariant, we can restrict it to the Picard group to get an invariant of the complex structure, the **Picard lattice**. This is a lattice of signature $(1,\rho-1)$ where $0 \leqslant \rho \leqslant 20$ is the rank of the Picard lattice.

Assume now that we are given a sublattice $N \subset \Lambda$ of signature $(1, r-1)$ and an element $A \in N$ with $A \cdot A = 2g-2 > 0$. We say that a marked K3 surface (S,J,h) is (N,A)-**polarised** if $h^{-1}(N) \subset \mathrm{Pic}(S,J)$, this embedding is primitive, meaning that $\mathrm{Pic}(S,J)/h^{-1}(N)$ is torsion-free, and $h^{-1}(A) \in \mathrm{Pic}(S,J)$ is ample. The number g is then called the **genus** of the polarised K3 surface S. Similar to the period domain of marked K3 surfaces (1.2) one can describe a similar period domain for marked polarised K3 surfaces. For this note that as $h^{-1}(N) \subset \mathrm{Pic}(S,J) \subset H^{1,1}(S)$ the complex line $H^{2,0}(S)$ must be orthogonal to $h^{-1}(N)$. This motivates the definition of the following domain:

$$\begin{aligned} D_N &= \{u \in P(N^\perp \otimes \mathbb{C}) : u^2 = 0,\ u \cdot \bar{u} > 0\} \\ &\simeq \{\Pi \subset N^\perp \otimes \mathbb{R} : \langle \cdot,\cdot \rangle|_\Pi > 0\} \subset \mathrm{Gr}_+(2, N^\perp \otimes \mathbb{R}). \end{aligned} \tag{1.3}$$

The corresponding Torelli theorem states that the period map from above maps the moduli space of marked (N,A)-polarised K3 surfaces $\mathcal{K}^{N,A}$ to D_N by a local diffeomorphism. Hence this moduli space has dimension $20-r$.

The Kähler geometry of K3 surfaces is also rather explicit. Suppose that the (non-polarised) marked K3 surface (S,J,ω,g,h) has period point $\Pi \in D_{K3}$. We then define the **root system** corresponding to Π as:

$$\Delta_\Pi = \{\lambda \in \Lambda : \lambda \cdot \lambda = -2,\ \lambda \cdot p = 0 \ \ \forall p \in \Pi\}.$$

Then the set of **Kähler chambers** of the K3 surface is given by:

$$\{\omega \in \Lambda \otimes \mathbb{R} : \omega \cdot \omega > 0,\ \omega \cdot p = 0 \text{ for } p \in \Pi,\ \omega \cdot \lambda \neq 0 \ \ \forall \lambda \in \Delta_\Pi\}. \tag{1.4}$$

Now the Kähler cone is always a connected component of the set of Kähler chambers, and thus in particular an open subset of $H^{1,1}(S)$.

After we discussed the complex and Kähler geometry of a K3 surface, consider now a K3 surface $(S,\omega_I,I,g,\Omega_I)$ with a chosen Calabi–Yau structure. By Yau's Theorem 1.2 we see that ω_I, g and Ω_I are determined by the complex structure I and the cohomology class $[\omega] \in H^2(S)$. We can then write $\Omega_I = \omega_J + i\omega_K$. As suggested by the notation S is also Kähler with respect to the forms ω_J and ω_K for new complex structures J and K (meaning that $g(\cdot,\cdot) = \omega_J(\cdot, J\cdot) = \omega_K(\cdot, K\cdot)$). The three complex structures satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. In fact, for $(a,b,c) \in S^2 \subset \mathbb{R}^3$ any linear combination $aI + bJ + cK$ determines a further complex structure for which (S,g) is Kähler for a suitably chosen Kähler form. Riemannian manifolds that are Kähler in three compatible ways like above are called **hyperkähler** manifolds. In the K3 case, we can describe the K3 moduli space explicitly:

Proposition 1.3. *The moduli space $\mathcal{M}^{\mathrm{hk}}$ of hyperkähler K3 surfaces admits a period* *map, which is a global diffeomorphism:*

$$\mathcal{P}^{\text{hk}} : \mathcal{M}^{\text{hk}} \longrightarrow D^{\text{hk}}_{K3}. \tag{1.5}$$

Here D^{hk}_{K3} is defined as:

$$\begin{aligned} D^{\text{hk}}_{K3} = \big\{ &(\alpha_1, \alpha_2, \alpha_3) : \alpha_i \in \Lambda \otimes \mathbb{R},\ \alpha_i \cdot \alpha_j = a\delta_{ij} \textit{ with } a > 0, \\ &\textit{for each } \lambda \in \Lambda \textit{ with } \lambda \cdot \lambda = -2 \textit{ there is } i = 1, 2 \textit{ or } 3 \textit{ such that } \alpha_i \cdot \lambda \neq 0 \big\}. \end{aligned} \tag{1.6}$$

Hyperkähler manifolds admit isometries of a special kind which interchange the complex structures, called **hyperkähler rotations**. More formally, for us a hyperkähler rotation is an isometry $\varphi : S_1 \to S_2$ between K3 surfaces S_1 and S_2 with complex structures I_1, J_1, K_1 and I_2, J_2, K_2 respectively, so that

$$\varphi^* I_2 = J_1, \quad \varphi^* J_2 = I_1, \text{ and } \varphi^* K_2 = -K_1. \tag{1.7}$$

Alternatively, we can define hyperkähler rotations by their actions on the Kähler forms. Indeed the hyperkähler rotation φ from above induces the following action on the Kähler forms $(\omega_+, \omega_-, \omega_0)$ corresponding to the distinguished complex structures (I, J, K) of a K3 surface S:

$$(\omega_+, \omega_-, \omega_0) \longmapsto (\omega_-, \omega_+, -\omega_0). \tag{1.8}$$

These special isometries will be important in Section 5.3 when we discuss the construction of G_2-manifolds from Calabi–Yau pieces. We will glue asymptotically cylindrical G_2-manifolds which have ends modelled on $\mathbb{R} \times S^1 \times S^1 \times S$, where S is a K3 surface. For topological reasons explained after Equation (5.2), we need to identify the two K3 surfaces on either end by a hyperkähler rotation.

Fano Geometry

We now review some aspects of the geometry of Fano threefolds. More details can be found in the book by Kollár [22] and the survey paper by Beauville [2].

Definition 1.4. A **Fano manifold** is a compact, complex manifold M with ample anticanonical bundle, meaning that a basis of $H^0(M, (-K_M)^{\otimes k})$ gives a well-defined embedding into $\mathbb{C}P^N$ for some $k \geqslant 1$.

Being Fano is quite a restrictive condition. In each dimension $n \geqslant 1$ there are only finitely many deformation types of Fano n-folds. We are mostly concerned with Fano threefolds, of which there are 105 deformation types. In fact, our entire discussion can be adapted to what Corti, Haskins, Nordström and Pacini [8] call **semi-Fano** manifolds, however, their definition is somewhat involved and we do not present it here. Morally speaking, semi-Fanos are desingularisations of mildly singular Fanos.

We now recall some properties of (semi)-Fano manifolds that are relevant to our discussion of the twisted connected sum construction of G_2-manifolds, mainly following [8]. To begin, assume that M is a Fano three-fold. We can then define the following pairing on $H^2(M,\mathbb{Z})$:

$$\langle\cdot,\cdot\rangle_M : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow H^6(M,\mathbb{Z}) \simeq \mathbb{Z},$$
$$(a,b) \longmapsto a \cdot b \cdot c_1(-K_M).$$

This endows $H^2(M,\mathbb{Z})$ with a nondegenerate lattice structure. We can write

$$\langle -K_M, -K_M\rangle = 2g-2 > 0,$$

where g is the degree of the Fano three-fold. Next, let $S \subset M$ be an anticanonical divisor. It is known that generically this is a smooth K3 surface [42]. From now on assume that it is. The restriction map $H^2(M,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ is a primitive embedding of lattices (see the proof of [8, Prop. 5.7]), where we consider $H^2(S,\mathbb{Z})$ with the usual intersection pairing. Thus S is a $(H^2(M,\mathbb{Z}), -K_M)$-polarised K3 surface. From this it is natural to discuss the moduli space of pairs (M,S) where M is a (semi)-Fano threefold and $S \subset M$ is a smooth, anticanonical K3 divisor, together with an isomorphism $h : N \simeq H^2(M,\mathbb{Z})$, where N is a fixed lattice and $A \in N$ satisfies $h(A)^2 = -K_M^3$. Write this moduli space as $\mathcal{F}^{N,A}$. This is again a (potentially singular) complex manifold. Of course, we have a forgetful morphism:

$$s^{N,A} : \mathcal{F}^{N,A} \longrightarrow \mathcal{K}^{N,A}, \quad (M,S) \longmapsto S.$$

It has the following important property.

Proposition 1.5 (Thm. 6.8 in [8])**.** *The image of each connected component of $\mathcal{F}^{N,A}$ is an open dense subset of $\mathcal{K}^{N,A}$, and for smooth points $(M,S) \in \mathcal{F}^{N,A}, S \in \mathcal{K}^{N,A}$ we have that $s^{N,A}$ is locally a submersion.*

1.4 G_2 and coassociative Geometry

Consider $\mathbb{C}^3$ with the standard Calabi-Yau structure $(\mathbb{C}^3, J_0, \omega_0, g_0, \Omega_0)$. We can define the following three-form, called the **associative form** on $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$:

$$\varphi_0 = \mathrm{dt} \wedge \omega_0 + \operatorname{Re}\Omega_0.$$

Here t denotes the coordinate on $\mathbb{R}$. The stabiliser of this form in GL(7) is the 14-dimensional simple Lie group $G_2 \subset \mathrm{SO}(7)$. A 7-manifold M together with a three-form $\varphi \in \Omega^3(M)$ such that at each point (T_pM, φ_p) is isomorphic to the standard model $(\mathbb{R}^7, \varphi_0)$ is called a G_2-**manifold**. The **associative form** φ induces a metric g_φ on M via the pull-back of the standard metric on $\mathbb{R}^7$. If now φ is both closed and co-closed, i.e. $\mathrm{d}\varphi = 0$ and $\mathrm{d}^\star_\varphi \varphi = 0$ (the **torsion-free** case), then both φ and $\star_\varphi \varphi$ are calibrations. Their calibrated submanifolds are called **associatives** and **coassociatives** respectively. We then also have that the holonomy of (M, g_φ) is contained in G_2. Note that we take the unusual approach of not requiring the G_2-structure to be torsion-free. In our setting manifolds with holonomy G_2 are particular (torsion-free) examples of G_2-manifolds.

Example 1.6. Let $(X^6, J, \omega, g, \Omega)$ be a Calabi-Yau threefold. Consider $M^7 = X \times S^1$ with the coassociative form $\varphi = \operatorname{Re}\Omega + \mathrm{d}s \wedge \omega$, where s is the coordinate on S^1. This

G_2-structure is torsion-free, and a special Lagrangian $L \subset X$ gives rise to an associative manifold $L \times \{p\}$ for any $p \in S^1$, whereas a complex surfaces $S^4 \subset X$ gives rise to a coassociative submanifold $S \times \{p\}$.

1.5 Spin(7) and Cayley Geometry

The group Spin(7) is the double cover of SO(7), and thus a 21-dimensional connected, simply-connected and compact Lie group. Its real spinor representation $\delta_7 : \mathrm{Spin}(7) \to \mathrm{GL}(8, \mathbb{R})$ gives an embedding into SO(8), after choosing an invariant metric. Alternatively, this subgroup of SO(8) can be seen as the stabiliser of the **standard Cayley form** in $\mathbb{R}^8$. If $\mathbb{R}^8$ has coordinates $(x_1, \ldots, x_8)$ then this form is given by:

$$\begin{aligned}\Phi_0 = \mathrm{d}x_{1234} - \mathrm{d}x_{1256} - \mathrm{d}x_{1278} - \mathrm{d}x_{1357} + \mathrm{d}x_{1368} - \mathrm{d}x_{1458} - \mathrm{d}x_{1467} \\ -\mathrm{d}x_{2358} - \mathrm{d}x_{2367} + \mathrm{d}x_{2457} - \mathrm{d}x_{2468} - \mathrm{d}x_{3456} - \mathrm{d}x_{3478} + \mathrm{d}x_{5678},\end{aligned} \tag{1.9}$$

where $\mathrm{d}x_{ijkl} = \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l$.

More generally, we say that a 4-form Φ on an 8-dimensional vector space V is a **Cayley form** if V admits an isomorphism with $\mathbb{R}^8$ taking Φ to Φ_0. We call the pair (V, Φ) a Spin(7)**-vector space**. Any such form then determines a Spin(7)-subgroup $\mathrm{Spin}_\Phi(7) \subset \mathrm{GL}(V)$. Let (V, Φ) be a Spin(7)-vector space. Then Φ induces a Riemannian metric g_Φ on V obtained as the pullback of the standard metric $g_0 = \sum_{i=1}^{8} \mathrm{d}x_i^2$ via the isomorphism $V \simeq \mathbb{R}^8$. Note that the isomorphism is not unique, but since $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ the pullback metric is independent of the choice of identification with $\mathbb{R}^8$. Pulling back the standard orientation on $\mathbb{R}^8$ induces a well-defined orientation on V in the same manner. Thus a Cayley form induces a metric and an orientation. In fact, the unoriented vector space V admits two classes of Cayley forms, determined by the orientation they induce. This is reflected in the fact that SO(8) admits exactly two conjugacy classes of Spin(7)-subgroups, which are conjugated inside O(8) [45, Thm. 1.3]. Consequently, when we consider a vector space which already admits an orientation, we only consider Cayley forms which induce the given orientation. If V does not have an orientation, we allow the Cayley form to induce the orientation. In particular the Cayley form Φ then induces a Hodge star operator $\star : \Lambda^k V^* \to \Lambda^{8-k} V^*$ and musical isomorphisms $\flat : V \to V^*$ and $\sharp : V^* \to V$. The Cayley form is self-dual with respect to the Hodge star it induces. Next, the action of $\mathrm{Spin}_\Phi(7)$ on V induces representations on the tensor and exterior bundles, which decompose into irreducible representations of $\mathrm{Spin}_\Phi(7)$. We are mostly interested in the action on 2-forms, which decomposes as follows as explained in [5, p. 546]:

Proposition 1.7. *There is an orthogonal splitting:*

$$\Lambda^2 V^* = \Lambda^2_7 V^* \oplus \Lambda^2_{21} V^*, \tag{1.10}$$

where Λ^2_i is an i-dimensional irreducible representation. Explicitly they are given by:

$$\begin{aligned} \Lambda^2_7 V^* &= \{\eta \in \Lambda^2 V : \star(\Phi \wedge \eta) = 3\eta\} \\ &= \{u^\flat \wedge v^\flat - \iota(u)\iota(v)\Phi : u, v \in V\}, \qquad (1.11) \\ \Lambda^2_{21} V^* &= \{\eta \in \Lambda^2 V : \star(\Phi \wedge \eta) = -\eta\}. \qquad (1.12) \end{aligned}$$

Using the Cayley form we now define various product structures on a Spin(7)-vector space (V, Φ). First we define the **cross product** as the bilinear map $V \times V \to \Lambda^2_7 V^*$:

$$u \times v = \pi_7(u^\flat \wedge v^\flat) = \frac{1}{4}\left(u^\flat \wedge v^\flat - \iota(u)\iota(v)\Phi\right). \qquad (1.13)$$

Here $\pi_7(\eta) = \frac{1}{4}(\eta - \star(\eta \wedge \Phi))$ for $\eta \in \Lambda^2 V^*$ is the orthogonal projection onto the Λ^2_7-summand. The **triple product** is a trilinear map $V \times V \times V \to V$ defined by:

$$u \times v \times w = (\iota(u)\iota(v)\iota(w)\Phi)^\sharp, \qquad (1.14)$$

Finally, the **quadruple product** is a $\Lambda^2_7 V^*$-valued four-form:

$$\begin{aligned} \tau(u, v, w, x) = u \times (v \times w \times x) - g_\varphi(u, v)(w \times x) \\ -g_\varphi(u, w)(v \times x) + g_\varphi(u, x)(v \times w). \qquad (1.15) \end{aligned}$$

On $(\mathbb{R}^8, \Phi_0)$, this form has the following coordinate expression:

$$\begin{aligned} \tau = \frac{1}{4} \sum_{1 \leqslant i < j \leqslant 8} (e^j \wedge (\iota(e_i)\Phi) - e^i \wedge (\iota(e_j)\Phi)) \otimes (e^i \times e^j) \\ = (\mathrm{d}x_{1358} + \mathrm{d}x_{1367} - \mathrm{d}x_{1457} + \mathrm{d}x_{1468} \\ - \mathrm{d}x_{2357} + \mathrm{d}x_{2368} - \mathrm{d}x_{2458} - \mathrm{d}x_{2467}) \otimes (e_1 \times e_2) \\ + (-\mathrm{d}x_{1258} - \mathrm{d}x_{1267} + \mathrm{d}x_{1456} + \mathrm{d}x_{1478} \\ + \mathrm{d}x_{2356} + \mathrm{d}x_{2378} - \mathrm{d}x_{3458} - \mathrm{d}x_{3467}) \otimes (e_1 \times e_3) \\ + (\mathrm{d}x_{1257} - \mathrm{d}x_{1268} - \mathrm{d}x_{1356} - \mathrm{d}x_{1378} \\ + \mathrm{d}x_{2456} + \mathrm{d}x_{2478} + \mathrm{d}x_{3457} - \mathrm{d}x_{3468}) \otimes (e_1 \times e_4) \\ + (\mathrm{d}x_{1238} - \mathrm{d}x_{1247} + \mathrm{d}x_{1346} - \mathrm{d}x_{1678} \\ - \mathrm{d}x_{2345} + \mathrm{d}x_{2578} - \mathrm{d}x_{3568} + \mathrm{d}x_{4567}) \otimes (e_1 \times e_5) \\ + (\mathrm{d}x_{1237} + \mathrm{d}x_{1248} - \mathrm{d}x_{1345} + \mathrm{d}x_{1578} \\ - \mathrm{d}x_{2346} + \mathrm{d}x_{2678} - \mathrm{d}x_{3568} - \mathrm{d}x_{4568}) \otimes (e_1 \times e_6) \\ + (-\mathrm{d}x_{1236} + \mathrm{d}x_{1245} + \mathrm{d}x_{1348} - \mathrm{d}x_{1568} \\ - \mathrm{d}x_{2347} + \mathrm{d}x_{2567} + \mathrm{d}x_{3678} - \mathrm{d}x_{1568}) \otimes (e_1 \times e_7) \\ + (-\mathrm{d}x_{1235} - \mathrm{d}x_{1246} - \mathrm{d}x_{1347} + \mathrm{d}x_{1568} \\ - \mathrm{d}x_{2348} + \mathrm{d}x_{2568} + \mathrm{d}x_{3578} + \mathrm{d}x_{4678}) \otimes (e_1 \times e_8). \qquad (1.16) \end{aligned}$$

We now introduce Spin(7)-manifolds by applying these linear algebraic constructions to the tangent bundle of smooth 8-manifolds. To be precise, we take a Spin(7)**-manifold** to be a smooth 8-dimensional manifold M together with a choice of Spin(7)-structure, i.e. a choice of Spin(7)-subbundle $\mathrm{Fr}_{\mathrm{Spin}(7)}$ of the frame bundle $\mathrm{Fr}(M)$. This data is equivalent to the choice of a smooth differential 4-form Φ on M which is a Cayley form at every point. In other words, Φ is a smooth section of a bundle $\mathcal{A}(M)$ whose fibre over the point p is the set of all Cayley forms of T_pM. With a choice of Spin(7)-structure T_pM is a Spin(7)-vector space at every point $p \in M$, which gives M the structure of an oriented Riemannian manifold. Note that this is a non-standard definition as one usually makes the additional assumption that Φ be torsion-free (so that the metric has holonomy contained in Spin(7)), which we do not assume here. Next, if we are given an orientation of M, we require the Spin(7)-structure to be compatible pointwise. The form Φ will also be called a Spin(7)-structure. A Spin(7)-manifold M is a **torsion-free** if its intrinsic torsion vanishes, meaning that Fr admits a torsion-free connection compatible with the reduction to $\mathrm{Fr}_{\mathrm{Spin}(7)}$. In this case, the holonomy of (M, g_Φ) is a subgroup of Spin(7), as we will see later.

The question of when an 8-manifold is Spin(7) is topological, and can be answered via obstruction theory. Concretely we have Theorem 10.7 from [27] which states:

Proposition 1.8. *A connected oriented 8-manifold M admits a* Spin(7)*-structure inducing its orientation if and only if it is spin and its positive real spinor bundle has trivial Euler class. This last condition is satisfied exactly when:*

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0. \tag{1.17}$$

A Spin(7)*-structure induces a unique spin structure.*

Notice that $\mathrm{Spin}(7)_{\Phi_0} \subset \mathrm{SO}(8)$ can be uniquely factored as

$$\mathrm{Spin}(7)_{\Phi_0} \longrightarrow \mathrm{Spin}(8) \longrightarrow \mathrm{SO}(8),$$

as $\mathrm{Spin}(7)_{\Phi_0}$ is simply connected. The last arrow is the double cover. With regards to this embedding, $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ is the stabiliser of a non-zero positive spinor [27, Prop. 10.4]. This can be used to prove that Spin(7)-structures compatible with a fixed Riemannian metric, orientation and spin structure are equivalent to non-vanishing sections of the positive spinor bundle. Such a non-vanishing section in turn exists exactly when the Euler class vanishes. The spin structure on a manifold with Spin(7)-structure is obtained by lifting the transition maps for the Spin(7)-frame bundle via the embedding into Spin(8).

The question of when a given Spin(7)-structure admits a torsion-free compatible connection can also be answered fully. The intrinsic torsion of the structure vanishes exactly when $\nabla_{g_\Phi}\Phi = 0$, which is equivalent to $\mathrm{d}\Phi = 0$ by [5, Thm. 3]. This is entirely analogous to how the integrability of an almost complex structure can be determined from the vanishing of the Nijenhuis tensor. Note that, if $\mathrm{d}\Phi = 0$, then since Φ is self-dual, Φ will be harmonic. However, determining whether or not a Spin(7)-manifold admits a torsion-free Spin(7)-structure is highly non-trivial. Indeed it is comparable in difficulty to determining if an manifold that admits almost complex structure admits a holomorphic atlas. If a torsion-free Spin(7)-structure exists on a closed manifold M, then the moduli space of all torsion-free Spin(7)-structures is a non-empty smooth manifold of dimension $\hat{A}(M) + b^1(M) + b^4_-(M)$ [19, Thm. 11.5.9]. This dimension is determined by the topology of M. If M admits a torsion-free Spin(7)-structure, then the metric induced from this Spin(7)-structure has holonomy contained in Spin(7). Topological conditions on M can then determine when the holonomy is exactly Spin(7) and when it is a proper subgroup (see [5, Thm. 11.5.1]).

Example 1.9. There are examples of Spin(7)-manifolds which come from dimensional reductions.

- G_2 **geometry**: We can write the Cayley form in (1.9) as $\Phi_0 = \mathrm{d}x_1 \wedge \varphi_0 + \star_7\varphi_0$ for an associative form $\varphi_0 \in \Lambda^3\mathbb{R}^7$ as in Section 1.4, and where $\star_7$ is the Hodge star on $\{0\} \times \mathbb{R}^7$. More generally, if we are given a G_2-manifold, then we can define a Spin(7)-structure on $\mathbb{R} \times M$ with Cayley form $\Phi = \mathrm{d}t \wedge \varphi + \star_M\varphi$.

- **Calabi–Yau geometry**: Let $(M^n, g, \omega, J, \Omega)$ be a complex four-dimensional Calabi-Yau manifold. Such a manifold is modelled at each point on $(\mathbb{C}^n, g_0, \omega_0, J_0, \Omega_0)$, where g_0 and J_0 are the standard Riemannian metric and complex structure respectively and:

$$\omega_0 = \frac{i}{2}\sum_{i=l}^{n} \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_l,$$
$$\Omega_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

  In the complex four-dimensional case, i.e. on $\mathbb{C}^4$, we have that the Cayley form can be written as $\Phi_0 = \operatorname{Re}\Omega_0 + \frac{1}{2}\omega_0 \wedge \omega_0$. Thus in particular any almost Calabi-Yau fourfold is also a Spin(7)-manifold.

Let (M, Φ) be a Spin(7)-manifold with Spin(7)-bundle $\mathrm{Fr}_{\mathrm{Spin}(7)}$. Then the tensor and exterior bundles of M are associated to $\mathrm{Fr}_{\mathrm{Spin}(7)}$ via representations induced from the embedding $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Thus the fibres of these bundles can be seen as representations of Spin(7), and as such decompose into bundles of irreducible representations. For two-forms, Proposition 1.7 implies that there is an orthogonal splitting:

$$\Lambda^2T^*M = \Lambda^2_{21} \oplus \Lambda^2_7, \tag{1.18}$$

where the fibres of Λ^2_{21} and Λ^2_7 are given by (1.12) and (1.11) respectively. On a Spin(7)-manifold (M, Φ) we can define the cross, triple and quadruple product of tangent vectors using the differential form Φ, and these extend to bundle homomorphisms.

Cayley submanifolds

Let (V, Φ) be a Spin(7)-vector space. A fundamental property of the Cayley form Φ is that when restricted to any four-plane $\xi = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with $\{e_1, e_2, e_3, e_4\}$ a positively oriented, g_Φ-orthonormal basis, the **Cayley inequality** holds [10, Th. 1.24, Ch. IV]:

$$\Phi(e_1, e_2, e_3, e_4) \leqslant 1. \tag{1.19}$$

The oriented four-planes which satisfy $\Phi(e_1, e_2, e_3, e_4) = 1$ are called **Cayley planes** and are said to be **calibrated** by Φ. Note that if ξ is Cayley, its orthogonal complement will be

Cayley as well. If $u, v, w \in V$ are three independent vectors, then there is a unique Cayley plane which contains them, namely $\xi = \text{span}\{u, v, w, u \times v \times w\}$. Moreover, a four-plane is Cayley for one of its orientations exactly when the quadruple product τ vanishes on it.

Given a Cayley plane ξ in a Spin(7)-vector space (V, Φ), the cross product on V decomposes with regards to the splitting $V = \xi \oplus \xi^{\perp}$ (where we assumes that $\xi = \text{span}\{\partial_1, \ldots, \partial_4\}$, $\xi^{\perp} = \text{span}\{\partial_5, \ldots, \partial_8\}$, and ∂_i has dual one-form dx_i), which we will now explain. Define:

$$E_\xi = \{\omega \in \Lambda^2_7 V^* : \omega|_\xi = 0\}, \tag{1.20}$$

which is a rank four subspace of $\Lambda^2_7 V^*$ (with an orthonormal basis given by $\pi_7(\mathrm{d}x_1 \wedge \mathrm{d}x_i)$ for $i \in \{5, 6, 7, 8\}$). Also note that any $\omega \in \Lambda^2_- \xi$ can be extended by 0 on $\xi^{\perp}$ to a two-form on V, and their projections under π_7 form a rank three subspace of $\Lambda^2_7 V^*$ that we will also denote by $\Lambda^2_- \xi$. It has an orthonormal basis given by $\pi_7(\mathrm{d}x_1 \wedge \mathrm{d}x_i)$ for $i \in \{2, 3, 4\}$. Denote the orthogonal projection map to E_ξ by $\pi_E : \Lambda^2_7 \to E_\xi$. From the above we see that there is an orthogonal splitting: $\Lambda^2_7 V^* = E_\xi \oplus \Lambda^2_- \xi$. The cross-product then restricts as follows:

$$\begin{aligned} \xi \times \xi &\longrightarrow \Lambda^2_- \xi, \\ \xi^{\perp} \times \xi^{\perp} &\longrightarrow \Lambda^2_- \xi, \\ \xi \times \xi^{\perp} &\longrightarrow E_\xi. \end{aligned} \tag{1.21}$$

Let now (M, Φ) be a Spin(7)-manifold. We call a four-dimensional submanifold $N \subset M$ all of whose tangent planes are Cayley planes a **Cayley submanifold**. In this situation, the cross-product splits into:

$$\begin{aligned} TN \times TN &\longrightarrow \Lambda^2_- TN, \\ \nu(N) \times \nu(N) &\longrightarrow \Lambda^2_- TN, \\ TN \times \nu(N) &\longrightarrow E. \end{aligned} \tag{1.22}$$

Here E is the globalisation of E_ξ from Equation (1.20), where for $p \in M$ we define $E_p = E_{T_pN} \subset \Lambda^2_7 T^*_p M$. Note that we can carry out the same construction whenever we are given a rank 4 subbundle of $TM|_N$ whose fibres are Cayley planes, irrespective of whether N is Cayley.

Example 1.10. The Spin(7)-manifolds coming from reductions of the structure group to G_2 and SU(4) (see Example 1.9) admit their own classes of calibrated submanifolds, which give examples of Cayley submanifolds.

- G_2 **geometry**: The three-form φ_0 and the four-form $\star_7\varphi$ satisfy a calibration inequality which is analogous to the Cayley inequality (1.12). The calibrated hyperplanes are called **associative** 3-planes and **coassociative** 4-planes respectively. If we have an associative submanifold A^3 in (M^7, φ), then $\mathbb{R} \times A$ is a Cayley in the Spin(7)-manifold $\mathbb{R} \times M$ with the Cayley form $\Phi = \mathrm{d}t \wedge \varphi + \star_7\varphi$. Similarly, if C^4 is coassociative in (M^7, φ), then $\{t\} \times C$ is a Cayley in $\mathbb{R} \times M$ for any $t \in \mathbb{R}$.

- **Calabi-Yau geometry**: An almost Calabi-Yau fourfold $(M^4, J, \omega, g, \Omega)$ admits two kinds of calibrated four-dimensional submanifolds. First, we have the **complex surfaces**, which are calibrated by $\frac{1}{2}\omega \wedge \omega$. Second, we have the **special Lagrangian manifolds**, calibrated by $\operatorname{Re}\Omega$. As the Cayley form on M is $\Phi = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\Omega$, which is the sum of both the previous calibrations, both complex surfaces and special Lagrangian submanifolds are Cayley in the induced Spin(7)-manifold.

The Dirac bundle associated to a Cayley

We will see later that the linearised deformation operator associated to the deformation problem of a spin Cayley is a twisted Dirac operator. On a non-spin Cayley, the situation is more complicated, as neither the spinor bundles nor the bundle by which they are twisted is well-defined on their own, however, one can still make sense of their product, in the form of a **Dirac bundle** (as defined in [27, Ch. II.5]). The Cayley deformation operator will then linearise to the Dirac operator associated to this Dirac bundle, which we will define for any Cayley submanifold N (be it spin or not) in a Spin(7)-manifold (M, Φ). We have previously introduced the Spin(7)-frame bundle associated to Φ, which can be described as:

$$(\mathrm{Fr}_{\mathrm{Spin}(7)})_x = \{e : T_xM \xrightarrow{\simeq} \mathbb{R}^8 : e^*(\Phi_0) = \Phi_x\}. \tag{1.23}$$

Using the splitting $TM|_N = TN \oplus \nu(N)$, where in the Cayley case both summands are bundles of Cayley planes, we can define the **adapted** Spin(7)**-frame bundle** $\mathrm{Fr}_{\mathrm{Spin}(7),N} \subset \mathrm{Fr}_{\mathrm{Spin}(7)}|_N$ as:

$$\begin{aligned}(\mathrm{Fr}_{\mathrm{Spin}(7),N})_x = \{e : T_xM \xrightarrow{\simeq} \mathbb{R}^8 : e^*(\Phi_0) = \Phi_x, e(T_xN) = \mathbb{R}^4 \times 0,\\ e(\nu(N)) = 0 \times \mathbb{R}^4\}.\end{aligned} \tag{1.24}$$

The structure group of this bundle is isomorphic to the stabiliser of a given Cayley plane (since it automatically preserves the orthogonal complement). It is given by

$$H = (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))/(\pm(1, 1, 1)),$$

as shown in [10, Thm. IV.1.8]. Here $H \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ via the following action on $\mathbb{R}^8 \simeq \mathbb{H} \oplus \mathbb{H}$. For $[p, q, r] \in H$ and $(u, v) \in \mathbb{H} \oplus \mathbb{H}$ we have:

$$[p_1, p_2, q] \cdot (u, v) = (p_1 u \bar{q}, p_2 v \bar{q}). \tag{1.25}$$

Using the embedding $H \subset \mathrm{SO}(8)$, a number of bundles over N can be represented as associated bundles to $\mathrm{Fr}_{\mathrm{Spin}(7),N}$. Here $u, v \in \mathbb{H}$ and $w \in \operatorname{im}\mathbb{H}$.

- TN is associated via $\rho_{TN}([p_1, p_2, q]) \cdot u = (p_1 u \bar{q})$, since the projection $[p_1, p_2, q] \mapsto [p_1, q]$ maps H surjectively onto $\mathrm{SO}(\mathbb{R}^4 \times 0)$.
- $\nu(N)$ is associated via $\rho_{\nu(N)}([p_1, p_2, q]) \cdot u = (p_2 u \bar{q})$, as H also surjects onto $\mathrm{SO}(0 \times \mathbb{R}^4)$
- If N is spin, then the adapted Spin(7)-frame bundle admits a double cover by a $G = \mathrm{Sp}(1)^3$-bundle, which we will denote by $\widetilde{\mathrm{Fr}}_{\mathrm{Spin}(7),N}$. This can be seen as follows:

as N is spin, we can lift a co-cycle for the tangent bundle to $\mathrm{Spin}(4) \simeq \mathrm{Sp}(1)^2$. Similarly, since M admits a spin structure induced by the Spin(7)-structure (as Spin(7) is a simply connected subgroup of SO(8)), the normal bundle $\nu(N)$ will also be canonically spin [27, Prop. II.1.15], thus a describing co-cycle can be lifted to Spin(4) as well. Using these two lifts one can then write down a lift to G for a co-cycle of $\mathrm{Fr}_{\mathrm{Spin}(7),N}$. The tangent and normal bundle will then be associated to this double cover via the lift of the representations ρ_{TN} and $\rho_{\nu(N)}$ respectively. Furthermore, the spinor bundles of N are associated bundles to this double cover as follows:

$$\displaystyle{\not}S_\pm = \widetilde{\mathrm{Fr}}_{\mathrm{Spin}(7),N} \times_{\delta_\pm} \mathbb{H},$$

where $\delta_+ \oplus \delta_-$ acts on $\mathbb{H} \oplus \mathbb{H}$ via $(p_1, p_2, q)(u, v) = (u\bar{p_1}, v\bar{q})$. Similarly the spinor bundles of $\nu(N)$ are associated via the representation $(p_1, p_2, q)(u, v) = (u\bar{p_2}, v\bar{q})$.

- The irreducible representation Λ^2_7 of Spin(7) restricted to H can be described as follows. Let $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4 \simeq \operatorname{im}\mathbb{H} \oplus \mathbb{H}$. Then we have the following (see [34]):

$$\rho_7([p_1, p_2, q])(w, u) = (\bar{q}wq, p_2u\bar{p_1}).$$

It turns out that in this splitting, the bundle associated via

$$[p_1, p_2, q]w = \bar{q}wq$$

is exactly the bundle $\Lambda^2_- N$ of anti-self-dual two-forms, and the bundle associated via $[p_1, p_2, q]w = p_2u\bar{p_1}$ is E.

From this discussion, we see that the suggestively named bundle

$$\displaystyle{\not}S = E \oplus \nu(N)$$

arises from the representation:

$$\rho : [p_1, p_2, q] \cdot (u, v) = (p_2u\bar{p_1}, p_2v\bar{q}). \tag{1.26}$$

If N is spin, then consider the quaternionic line bundle L associated to $\widetilde{\mathrm{Fr}}_{\mathrm{Spin}(7),N}$ via the representation $\rho_L : (p_1, p_2, q)u = p_2u$. We then see from the representations, that as quaternionic bundles, $E \simeq \displaystyle{\not}S_+ \otimes_{\mathbb{H}} L$ and similarly $\nu(N) \simeq \displaystyle{\not}S_- \otimes_{\mathbb{H}} L$, which allows to represent the bundle $E \oplus \nu(N)$ as a twisted spinor bundle, if N is spin.

To complete the construction of the Dirac bundle, we need to define a Clifford module structure of $Cl(\mathbb{R}^4) \simeq Cl(\mathbb{H})$ acting on TM, and a compatible metric and connection. This is done in the following proposition:

Proposition 1.11 (Dirac bundle). *There is a Clifford multiplication map $c : TN \times \displaystyle{\not}S \to \displaystyle{\not}S$, a metric h and a connection ∇ on $\displaystyle{\not}S$ such that $(\displaystyle{\not}S, c, h, \nabla)$ is a Dirac bundle. In an adapted* Spin(7)*-frame $\{e_i\}_{i=1,\dots,8}$ the negative Dirac operator acts on $v \in C^\infty(\nu(N))$ as:*

$$\displaystyle{\not}D v = \sum_{i=1}^{4} e_i \times \nabla^\perp_{e_i} v \in C^\infty(E), \tag{1.27}$$

where $\nabla^\perp$ is induced from the Levi-Civita connection on M.

Proof. Consider the Clifford algebra $Cl(\mathbb{H}) \simeq M^2(\mathbb{H})$ (here $\mathbb{H}$ is equipped with the standard metric, and $M^2(\mathbb{H})$ is the algebra of 2×2 matrices over $\mathbb{H}$). A $Cl(\mathbb{H})$-module structure on $\mathbb{H} \oplus \mathbb{H} \simeq T_pM$ is determined by the action of vectors satisfying $h \cdot (h \cdot (v_1, v_2)) = -|h|^2(v_1, v_2)$, for $h, v_1, v_2 \in \mathbb{H}$. One natural action is given by:

$$\begin{aligned} c : \mathbb{H} \times (\mathbb{H} \oplus \mathbb{H}) &\longrightarrow \mathbb{H} \oplus \mathbb{H} \\ (h, (v_1, v_2)) &\longmapsto (v_2\bar{h}, -v_1 h). \end{aligned} \tag{1.28}$$

We use here that $h\bar{h} = \bar{h}h = |h|^2$. This action commutes with the representation determining $\not{S}$, and TN, in the sense that:

$$c \circ (\rho_{TN}, \rho_E \oplus \rho_{\nu(N)}) = (\rho_E \oplus \rho_{\nu(N)}) \circ c.$$

Thus we can extend c to a map $c : TN \times \not{S} \to \not{S}$ as required. For an adapted Spin(7)-frame $\{e_a\}_{i=a,\dots,8}$, we identify $(1,0)$, $(i,0)$, $(j,0)$ and $(k,0)$ with the basis elements $e_1 \times e_a$ for $(5 \leqslant a \leqslant 8)$ of E and we identify $(0,1)$, $(0,i)$, $(0,j)$ and $(0,k)$ with the basis elements e_a $(5 \leqslant a \leqslant 8)$ of $\nu(N)$. Using this identification we see that the Clifford multiplication $c : TN \times \nu(N) \to E$ is exactly given by the cross-product. Since the e_a are orthonormal with respect to the metric g_Φ, as are $e_1 \times e_a$ for $(5 \leqslant a \leqslant 8)$ with respect to the metric g_E induced from g_Φ on the bundle of forms, we see that $c(v)$ is an isometry of $(\not{S}, h = g_\Phi \oplus g_E)$, whenever v is a unit vector. Finally, we choose as our connection ∇ on $\nu(N)$ the connection $\nabla^\perp$. On E we choose the unique connection such that $c(e_1)v$ is a parallel section (along a curve), whenever v is a parallel section along a curve in $\nu(N)$. From these definitions, it follows readily that $(\not{S}, c, h, \nabla)$ is a Dirac bundle. The Dirac operator restricted to $\nu(N)$ is then of the required form. □

Example 1.12. When C is a Cayley cone in $\mathbb{R}^8$, with link $L = C \cap S^7$, the Dirac operator $\not{D}$ can be rewritten as an evolution equation of vector fields $u \in \nu_{S^7}(L)$. For this, we introduce the nearly parallel G_2-structures on round seven-spheres. Let ∂_r be the outward radial unit vector field on $\mathbb{R}^8 \setminus 0$. Then at the point $(r,p) \in \mathbb{R}_+ \times S^7 \simeq \mathbb{R}^8 \setminus 0$ we have:

$$\Phi_{r,p} = \mathrm{d}r \wedge (\varphi^r)_p + (\star_{rS^7}\varphi^r)_p.$$

Here φ^r is the associative form (as in Section 1.4) corresponding to the nearly parallel G_2-structure on the round sphere of radius r. A submanifold $L^3 \subset (S^7, \varphi^r)$ is an associative submanifold exactly when the associated cone $C \subset (\mathbb{R}^8, \Phi_0)$ is Cayley.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame around a point $p \in L$ with dual coframe

$\{e^1, e^2, e^3\}$. Then we can rewrite the Dirac operator (1.27) as follows for $v \in C^\infty(\nu(C))$:

$$
\begin{aligned}
\not{D}v &= \partial_r \times \nabla^\perp_{\partial_r} v + \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} v|_{rL} \\
&= \mathrm{d}r \wedge (\nabla^\perp_{\partial_r} v)^\flat - \iota(\nabla^\perp_{\partial_r} v)\varphi^r + \sum_{i=1}^{3} e^i \wedge (\nabla^\perp_{e_i} v|_{rL})^\flat - \iota(e_i)\iota(\nabla^\perp_{e_i} v) \star_{rS^7} \varphi^r \\
&= A_p(\nabla^\perp_{\partial_r} v) + B_r(v|_{rL}).
\end{aligned}
$$

Here $A_p : \nu_{r,p}(C) \to E_{r,p}$ is a linear map that is independent of the radius, and

$$
B_r : C^\infty(\nu_{rS^7}(rL)) \longrightarrow C^\infty(E|_{rL}) \tag{1.29}
$$

are a family of first-order partial differential operators on the links rL. We can furthermore identify $E|_{rL} \simeq \nu_{rS^7}(rL)$ via the map $\omega \mapsto (\iota(\partial_r)\omega)^\sharp$, and identify sections $C^\infty(\nu_{S^7}(L)) \simeq C^\infty(\nu_{rS^7}(rL))$ via rescaling, at which point the operator has the following shape:

$$
\begin{aligned}
\not{D} : C^\infty(\mathbb{R}_+, C^\infty(\nu_{S^7}(L))) &\longrightarrow C^\infty(\mathbb{R}_+, C^\infty(\nu_{S^7}(L))) \\
v &\longmapsto \frac{\mathrm{d}}{\mathrm{d}r} v + D_L v(r).
\end{aligned} \tag{1.30}
$$

Here:

$$
\begin{aligned}
D_L : C^\infty(\nu_{S^7}(L)) &\longrightarrow C^\infty(\nu_{S^7}(L)) \\
u &\longmapsto B_1(u) = \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} u,
\end{aligned} \tag{1.31}
$$

where $\times$ is the vector product associated with the associative manifold $L \subset (S^7, \varphi^1)$. It is determined by the identity $g(u \times v, w) = \varphi^1(u, v, w)$.

Example 1.13. We noted in Example 1.10 that complex surfaces N in an (almost) CY4 manifold M are examples of Cayley submanifolds. In this case, the linearised Cayley deformation operator is a twisted Dirac operator on a Kähler surface, and thus necessarily of the form $\bar{\partial} + \bar{\partial}^*$ with twisted coefficients [39]. It has been computed in [36] and can be identified with:

$$
\bar{\partial} + \bar{\partial}^* : C^\infty(\nu^{1,0}(N) \oplus \Lambda^{0,2} N \otimes \nu^{1,0}(N)) \longrightarrow C^\infty(\Lambda^{0,1} N \otimes \nu^{1,0}(N)). \tag{1.32}
$$

For any complex surface, the kernel and cokernel of this operator are the complexifications of the kernel and cokernel respectively of $\not{D}$. Thus the real expected dimension of the Cayley moduli space is equal to the complex index of $\bar{\partial} + \bar{\partial}^*$. This can be compared to the linearised deformation operator of a complex surface deforming as a complex surface, which is just $\bar{\partial} \oplus \bar{\partial}^*$ (notice $\oplus$ instead of $+$). Hence being Cayley is a weaker condition than being complex.

We also noted in Example 1.10 that special Lagrangians in CY4 manifolds are examples of Cayley submanifolds. McLean [34] showed that the infinitesimal deformations of a special Lagrangian $N \subset M$ are given by the kernel of the operator

$$-\mathrm{d}\star \oplus \mathrm{d} : \Omega^1(N) \longrightarrow \Omega^n(N) \oplus \Omega^2(N), \tag{1.33}$$

which are the closed and co-closed one-forms. The Cayley deformation operator of a special Lagrangian is formed by a subset of these equations, reflecting the fact that the Cayley condition is a priori less restrictive than the special Lagrangian condition.

Proposition 1.14. *Let N be a special Lagrangian submanifold in a Calabi-Yau manifold $(M, J, \omega, g, \Omega)$. Then the infinitesimal Cayley deformation operator can be identified with:*

$$-\mathrm{d}\star \oplus \mathrm{d}^- : \Omega^1(N) \longrightarrow \Omega^4(N) \oplus \Omega^{2,-}(N). \tag{1.34}$$

Here $\Omega^{2,-}(N)$ is the bundle of self-dual two forms on (M, g), and $\mathrm{d}^- = \pi^- \circ \mathrm{d}$, where $\pi^-(\eta) = \frac{1}{2}(\eta - \star_N \eta)$ is the projection onto the anti-self-dual forms.

Proof. First, we show that there are canonical isomorphisms $m : T^*N \simeq \nu(N)$ and $n : E \simeq \Lambda^4 \oplus \Lambda^2_-$. We can take $m(\sigma) = J\sigma^\sharp$ to be the composition of the musical isomorphism $\sharp : T^*N \to TN$ and J. Note that J maps the tangent bundle of any Lagrangian to its normal bundle as $g(v, Jw) = \omega(v, w) = 0$ for any pair of vectors $v, w \in T_pN$ by the Lagrangian condition. As for the morphism n, we can pull back forms on T_pM via the map $\mathrm{id} \oplus J : TN \to TN \oplus \nu(N)$, which when restricted to E gives a surjection onto the anti-self-dual forms on TN. The kernel of this map is spanned by $v^\flat \wedge (Jv)^\flat$, and the projection onto these forms gives the Λ^4 summand. More concretely, recall that E_p is spanned by $e_1 \times Je_i$, where $\{e_i\}_{1\leqslant i\leqslant 4}$ is an orthonormal basis of T_pN. The morphism n then sends $e_i \times Je_i$ to the Λ^4 summand, and identifies $e_i \times e_j$ for $i \neq j$ with the anti-self-dual form $\alpha_{ij} = \mathrm{d}x_{ij} - \mathrm{d}x_{kl}$, where the $\mathrm{d}x_i$ are dual to e_i and (i, j, k, l) is a positive permutation of $(1, 2, 3, 4)$. Let now $f_i = Je_i \in \nu(N)$ complete the e_i to a frame of T_pM, and suppose that $\mathrm{d}y_i$ are the corresponding dual 1-forms. A computation shows that the vector product of $v = \sum_{i=1}^4 a_i e_i \in T_pN$ and $w = \sum_{i=1}^4 b_i f_i \in \nu(N)$ is given by:

$$\begin{aligned} v \times w = &\sum_{i=1}^{4} a_i b_i \mathrm{d}x_i \wedge \mathrm{d}y_i \\ &+ \sum_{\sigma(i,j,k,l)=1} \frac{a_i b_j}{4} \underbrace{(\mathrm{d}x_i \wedge \mathrm{d}y_j - \mathrm{d}x_j \wedge \mathrm{d}y_i - \mathrm{d}x_k \wedge \mathrm{d}y_l + \mathrm{d}x_l \wedge \mathrm{d}y_k)}_{\beta_{ij}}. \end{aligned}$$

Here $\sigma(i, j, k, l) = 1$ means that (i, j, k, l) is a positive permutation of $(1, 2, 3, 4)$. Note that k, l are uniquely determined by i and j. We now look at $n \circ \not{D} \circ m$, where $\not{D}$ is the Dirac operator from equation (1.27). For a one form $\eta = \sum_{i=1} a_i e_i \in \Omega^1(N)$, where we extended the basis $\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\}$ to a local parallel frame, we have:

$$\begin{aligned}
\not{D}[m[\eta]] &= \sum_{i,j=1}^{4} e_i \times \nabla^{\perp}_{e_i}(a_j f_j) \\
&= \sum_{i,j=1}^{4} e_i \times \frac{\partial a_j}{\partial x_i} f_i \\
&= \sum_{i=1}^{4} \frac{\partial a_i}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}y_i + \tfrac{1}{2} \sum_{i \neq j} \frac{\partial a_i}{\partial x_j} \beta_{ij}.
\end{aligned}$$

As n maps $\mathrm{d}x_i \wedge \mathrm{d}y_i$ to $-\mathrm{dvol} \in \Lambda^4$, and β_{ij} to α_{ij}, we see that

$$\begin{aligned}
n \circ \not{D}[m[\eta]] &= -\sum_{i=1}^{4} \frac{\partial a_i}{\partial x_i} \mathrm{dvol} + \sum_{i<j} \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \right) \alpha_{ij} \\
&= -\mathrm{d} \star \eta + \tfrac{1}{2} \sum_{i<j} (\mathrm{d}\eta)_{ij} \alpha_{ij} = -\mathrm{d} \star \eta + \pi^{-} \mathrm{d}\eta.
\end{aligned}$$

□

1.6 Analysis on manifolds with ends

In this section, we lay the groundwork for the analysis on Riemannian manifolds with cylindrical and conical ends. The Fredholm properties of elliptic operators on compact manifolds can be extended to these special classes of noncompact manifolds by using the theory developed by Lockhart and McOwen in [29].

Manifolds with ends

Definition 1.15. We say that a Riemannian n-manifold (M, g) is **asymptotically cylindrical** with rates $\lambda_1, \dots, \lambda_l < 0$ ($\mathrm{ACyl}_{\bar{\lambda}}$, where $\bar{\lambda} = (\lambda_1, \dots, \lambda_l)$) if the following holds. There is a compact set $K \subset M$ such that $M = K \sqcup \bigsqcup_{j=1}^{s} U_j$ with U_j connected and open. Furthermore, there are compact connected $(n-1)$-dimensional Riemannian manifolds (L_j, h_j) and diffeomorphisms $\Psi_j : (0, \infty) \times L_j \to U_j$ for $1 \leqslant j \leqslant l$, such that for $i \in \mathbb{N}$:

$$|\nabla^i (\Psi_j^*(g) - g_{j,\mathrm{cyl}})| = O(e^{\lambda_j t}) \text{ as } t \to \infty, \tag{1.35}$$

where $g_{j,\mathrm{cyl}} = \mathrm{d}t^2 + h_j$ is the cylindrical metric on $(0, \infty) \times L_j$, and ∇, $|\cdot|$ are taken with respect to these metrics.

The asymptotic convergence rates $e^{\lambda_j t}$ are chosen so that elliptic operators will be Fredholm on $\mathrm{ACyl}_{\bar{\lambda}}$ manifolds when considered between appropriately weighted spaces. Note that the condition $\lambda_j < 0$ ensures that the asymptotically cylindrical metric converges to the cylindrical metric at infinity. For the next class of noncompact manifolds, the asymptotically conical manifolds, it will be useful to have both an intrinsic as well as an extrinsic definition.

Definition 1.16. A Riemannian n-manifold (M, g) is **asymptotically conical** with rate $\eta < 1$ (AC_η) if there is a compact set $K \subset M$, a compact $(n-1)$-dimensional Riemannian manifold (L, h) and a diffeomorphism $\Psi : (r_0, \infty) \times L \to M \setminus K$ (for some $r_0 > 0$), such that for $i \in \mathbb{N}$:

$$|\nabla^i(\Psi^*(g) - g_{\mathrm{con}})| = O(r^{\eta-1-i}) \text{ as } r \to \infty, \tag{1.36}$$

where $g_{\mathrm{con}} = \mathrm{d}r^2 + r^2 h$ is the conical metric on $(r_0, \infty) \times L$, and ∇, $|\cdot|$ are taken with respect to the conical metric.

Definition 1.17. Suppose that $(\mathbb{R}^8, \Phi)$ is an AC_η manifold for some $\eta < 1$ with asymptotic cone $\mathbb{R}^8$. Let $A^m \subset \mathbb{R}^8$ be a smooth submanifold. Then A is an AC_λ **submanifold** of $\mathbb{R}^8$ ($\eta < \lambda < 1$), asymptotic to the cone $C = \mathbb{R}_+ \times L$ if there is a compact subset $K \subset A$ and a diffeomorphism $\Theta : (r_0, \infty) \times L \to A \setminus K$ such that if $\iota(r, p) = r \cdot p$ is the embedding of the cone $C \hookrightarrow \mathbb{R}^8$, then for every $i \in N$:

$$\iota(r,p) - \Psi_M^{-1} \circ \Theta(r,p) \in \nu_{(r,p)}(C) \tag{1.37}$$

$$|\nabla^i(\Psi_M^{-1} \circ \Theta(r,p) - \iota(r,p))| \in O(r^{\lambda - i}), \text{ as } r \to \infty. \tag{1.38}$$

Here the norm is computed with respect to the conical metric on $(r_0, \infty) \times L$ coming from the embedding ι, and the ∇^i are the higher covariant derivatives coming from the conical metric on C coupled to the flat connection on $C \times \mathbb{R}^8$ given by the Levi-Civita connection on $\mathbb{R}^8$. We say that L is the **link** of the AC_λ manifold A.

Remark 1.18. An AC_λ submanifold is in particular also an AC_λ manifold.

As we will not work with conically singular ambient manifolds, we will just give the extrinsic definition of the final class of noncompact manifolds that we consider.

Definition 1.19. Let (M, Φ) be a Spin(7)-manifold and consider a point $p \in M$. We say that a parametrisation $\chi : B_\eta(0) \to U$ of an open neighbourhood U of p is a **Spin(7)-parametrisation** around p if $\chi(0) = p$ and $\mathrm{D}\chi|_0^* \Phi_p = \Phi_0$, where Φ_0 is the standard Cayley form on $\mathbb{R}^8$. We say that two Spin(7)-parametrisations around p are **equivalent** if their derivatives agree at p.

Definition 1.20. Let $N^n \subset (M, g)$ be a closed subset, and suppose that there are $z_1, \ldots, z_l \in N$ such that $\hat{N} = S \setminus \{z_1, \ldots, z_l\}$ is a smooth, embedded submanifold of M. For any $1 \leqslant j \leqslant l$ let χ_j be a Spin(7)-coordinate system around z_j and let $L_j \subset S^7$ be a connected $(n-1)$-dimensional Riemannian submanifold of the round sphere in $\mathbb{R}^8$. Then N is an $\mathrm{CS}_{\bar{\mu}}$ **submanifold** of (M, g) ($\bar{\mu} = (\mu_1, \ldots, \mu_l), 1 < \mu_j < 2$), asymptotic to the cones $C_j = \mathbb{R}_+ \times L_j \subset \mathbb{R}^8$ ($1 \leqslant j \leqslant l$) if the following holds. There is a compact subset $K \subset N$ such that $N = K \sqcup \bigsqcup_{j=1}^s U_j$ with $z_j \in U_j$ open, and diffeomorphisms $\Psi_j = \chi_j \circ \Theta_j : (0, R_0) \times L_j \to U_j \setminus \{z_j\}$ for $1 \leqslant j \leqslant l$ such that if $\iota_j(p, r) = r \cdot p$ is the embedding of the cone C_j, then we have for every $i \in \mathbb{N}$:

$$\iota_j(r,p) - \Theta_j(r,p) \in \nu_{(r,p)}(C_j)$$

$$|\nabla^i(\Theta_j(r,p) - \iota_j(r,p))| \in O(r^{\mu_j - i}), \text{ as } r \to 0. \tag{1.39}$$

Here the norm is computed with respect to the conical metric on $(0, R_0) \times L_j$ coming from the embedding ι_j, and the ∇^i are the higher covariant derivatives coming from the conical metric on C_j together with the flat connection on $C_j \times \mathbb{R}^8$ given by the usual derivative on $\mathbb{R}^8$.

Remark 1.21. If we require that an embedded $\mathrm{CS}_{\bar{\mu}}$ submanifold is $\mathrm{CS}_{\bar{\mu}}$ with regards to any choice of Spin(7)-parametrisations in the equivalence classes of χ_j, we must restrict to $\mu_j < 2$. This is because the equivalence class of χ_j only determines it up to first order at the origin. The condition $\mu_j > 1$ ensures that the asymptotic cone is unique.

As before, the conditions on λ and μ_j ensure that the metrics tend towards a conical metric in the limit. For any of these three classes of manifolds, we call the connected components of $M \setminus K$ the **ends** of M, and the cross-sections the **link** of this end. In the case of ACyl and CS metrics, we write ACyl_λ and CS_μ for λ, μ real numbers when all the ends have the same decay rate. These metrics are examples of admissible metrics in the sense of [29]. Indeed this is clear for the cylindrical case. For the conical cases, note that if g_{cyl} is an ACyl_λ metric on $L \times \mathbb{R}_+$ which is asymptotic to a product metric $g_\infty = dt^2 + h$, then $e^{2t} g_{\mathrm{cyl}}$ is AC_λ and is asymptotic to the metric $g_{\mathrm{con}} = dr^2 + r^2 h$, where we introduced the new coordinate $r = e^t$ on $L \times [r_0, \infty)$. In fact, the decay rates for the AC_λ metrics were chosen so that this correspondence holds. Similarly $e^{-2t} g_{\mathrm{cyl}}$ (with $\lambda \in (-2, -1)$) is $\mathrm{CS}_{-\lambda}$ with radial coordinate $r = e^{-t}$. Turning the correspondence around, if g_{con} is either $\mathrm{CS}_{-\lambda}$ or AC_λ, then $r^{-2} g_{\mathrm{con}}$ is ACyl_λ. The admissibility allows us to use the Fredholm results of [29] in appropriate Sobolev spaces as we will see shortly. To end this section we define a generalisation of the radial coordinate on cones.

Definition 1.22. Let (M, Φ) be Spin(7)-manifold, and consider an embedded $\mathrm{CS}_{\bar{\mu}}$ submanifold $N \subset M$. A smooth function $\rho : M \to [0, R_0]$ (with $R_0 > 0$) is a **radius function** for N if near a singular point $z \in N$ it is given by the distance to z. Similarly, if $A \subset \mathbb{R}^8$ is an asymptotically conical submanifold, we say that the radial coordinate r on $\mathbb{R}^8$ is a radius function for A. This may not be smooth on all of A if $0 \in A$, but we will only consider AC radius functions at sufficiently large radii anyway.

Tubular neighbourhoods

We introduce tubular neighbourhoods of the noncompact $\mathrm{CS}_{\bar{\mu}}$ and AC_λ manifolds, which shrink or grow like the asymptotic cones. This is a straightforward extension of [38, Prop. 3.4].

Proposition 1.23. *Let C be either an AC_λ submanifold of $(\mathbb{R}^8, \Phi)$, where Φ is AC_η to Φ_0 with $\lambda < \eta < 1$, or a $\mathrm{CS}_{\bar{\mu}}$ ($1 < \mu < 2$) submanifold of (M, Φ), where M is compact. Suppose that $\rho : C \to \mathbb{R}$ is a radius function. Let $\epsilon > 0$. Define the open subset $\nu_\epsilon(C) \subset \nu(C)$ as:*

$$\nu_\epsilon(C) = \{(p, v) \in \nu(C) : |v| \leqslant \epsilon \rho(p)\}. \tag{1.40}$$

Then for sufficiently small $\epsilon > 0$ there is an open neighbourhood $N \subset U$ such that:

$$\exp : \nu_\epsilon(C) \longrightarrow U$$

is a diffeomorphism.

We note that in both cases the tubular neighbourhood scales like the radius function ρ as one approaches the singular points in a CS manifold, or infinity in the AC case.

Banach spaces

We now introduce the weighted Banach spaces that appear in the deformation theory of manifolds with ends.

Sobolev spaces

Let (M, g_{cyl}) be an asymptotically cylindrical manifold with radius function $\rho : M \to [r_0, \infty)$, and let (E, h) be a metric real vector bundle over M with a metric connection ∇^E. Let $s \in C^\infty(E)$ be a compactly supported section. We then define the (cylindrical) $L^p_{k,\delta,\text{cyl}}$ weighted Sobolev norm as

$$\|s\|_{p,k,\delta,\text{cyl}} = \left(\sum_{i=0}^{k} \int_M |(\nabla^E)^i s e^{-\delta\rho}|_h^p \mathrm{d}\mu_{\text{cyl}} \right)^{\frac{1}{p}}. \tag{1.41}$$

and the **weighted Sobolev space** $L^p_{k,\delta,\text{cyl}}(E)$ is defined to be the completion of the compactly supported sections with respect to this norm.

Now let (M, g) be an asymptotically conical or conically singular n-manifold with radius function ρ, with (E, h) a metric real vector bundle over M, together with a metric connection ∇^E. Then the (conical) $L^p_{k,\delta}$ weighted Sobolev norm of a section $s \in C^\infty(E)$ is defined to be:

$$\|s\|_{p,k,\delta} = \left(\sum_{i=0}^{k} \int_M |(\nabla^E)^i s \rho^{-\delta+i}|_h^p \rho^{-n} \mathrm{d}\mu \right)^{\frac{1}{p}}, \tag{1.42}$$

and the weighted Sobolev space $L^p_{k,\delta}(E)$ is, like in the cylindrical case, defined to be the completion of the compactly supported sections with respect to this norm. The sections in these spaces should be thought of as $L^p_{k,\text{loc}}$ sections that have decay in $o(r^\delta)$. Naturally one can extend this definition to include different weights at multiple singularities. For a vector of weights $\bar{\delta} \in \mathbb{R}^l$ the weighted Sobolev spaces will be denoted by $L^p_{k,\bar{\delta}}(E)$. In the above definition, δ must be replaced by a smooth function $w : M \to \mathbb{R}$ which interpolates between the different weights. If we assume that near a singularity w is constantly equal to the corresponding weight, then different choices of w will give rise to equivalent norms, as the norms only differ on a compact subset of M.

If E is a bundle of tensors, these spaces correspond to the spaces $W^p_{k,-\delta,-\frac{n}{p}}$ (AC case) and $W^p_{k,\delta,-\frac{n}{p}}$ (CS case) of [28, Ch.4], so we can translate their results into our setting. For instance, we have a Sobolev embedding theorem for the weighted spaces, which is an adaptation of Theorem 4.8 in [28].

Theorem 1.24. *Let (M, g) be an* CS/AC *manifold. Denote by $L^p_{k,\delta}(E)$ the corresponding weighted Sobolev space. Suppose that the following hold:*

i) $k - \tilde{k} \geqslant n \left(\frac{1}{p} - \frac{1}{\tilde{p}} \right)$ *and either:*

ii) $1 < p \leqslant \tilde{p} < \infty$ *and* $\tilde{\delta} \geqslant \delta$ (AC) *or* $\tilde{\delta} \leqslant \delta$ (CS)

ii') $1 < \tilde{p} < p < \infty$ *and* $\tilde{\delta} > \delta$ (AC) *or* $\tilde{\delta} < \delta$ (CS)

Then there is a continuous embedding:

$$L^p_{k,\delta}(E) \longrightarrow L^{\tilde{p}}_{\tilde{k},\tilde{\delta}}(E). \tag{1.43}$$

Hölder spaces

Let (M, g) be a Riemannian manifold, and consider the induced geodesic distance function $d : M \times M \to \mathbb{R}$ on M.

Definition 1.25 (Spaces of differentiable sections)**.** Let E be a metric vector bundle with a metric connection ∇^E. For a section $s \in C^k(E)$ we define the C^k**-norm** as:

$$\|s\|_{C^k} = \sum_{i=0}^{k} \sup_{p\in M} |(\nabla^E)^i s|(p). \tag{1.44}$$

If (M, g) is either an AC or CS manifold (a **conical** manifold) with a given radius function ρ, we also consider the C^k**-norm with weight** $\delta \in \mathbb{R}$ instead:

$$\|s\|_{C^k_\delta} = \sum_{i=0}^{k} \sup_{p\in M} |\rho^{i-\delta}(\nabla^E)^i s|(p). \tag{1.45}$$

Denote the set of C^k_{loc}-sections with finite C^k_δ-norm by $C^k_\delta(E)$ and set:

$$C^\infty_\delta(E) = \bigcap_{i=0}^{\infty} C^i_\delta(E). \tag{1.46}$$

Then $C^k_\delta(E)$ are Banach spaces and $C^\infty_\delta(E)$ is a Fréchet space. If multiple conical ends are present, the spaces $C^k_{\bar{\delta}}(E)$ and $C^\infty_{\bar{\delta}}(E)$ are defined analogously.

For any point $p \in M$ there is an open neighbourhood $p \in U_p \subset M$ such that for any $q \in U_p$, there is a unique shortest geodesic of length $d(p, q)$ joining p and q. In particular, there is an open neighbourhood $V \subset M \times M$ of the diagonal such that for $(p, q) \in V$ we have $q \in U_p$. Let now E be a metric vector bundle together with a metric connection. For $(p, q) \in V$ we identify the fibres E_p and E_q via parallel transport along the unique shortest geodesic connecting p and q.

Definition 1.26 (Hölder spaces)**.** For a section $s \in C^k(E)$ and a constant $0 < \alpha \leqslant 1$ we define the $C^{0,\alpha}$**-semi-norm** as:

$$[s]_\alpha = \sup_{(x,y)\in V} \frac{|s(x) - s(y)|}{d(x, y)^\alpha}. \tag{1.47}$$

The $C^{k,\alpha}_\delta$**-Hölder norm** is then defined as:

$$\|s\|_{C^{k,\alpha}_\delta} = \|s\|_{C^k_\delta} + [\rho^{k-\delta+\alpha} \cdot (\nabla^E)^k s]_\alpha. \tag{1.48}$$

The **Hölder space** $C^{k,\alpha}_\delta(E)$ is the subset of $C^k_\delta(E)$ with finite $C^{k,\alpha}_\delta$-Hölder norm. In the case of multiple weights, we denote by $C^{k,\alpha}_{\bar\delta}$ the corresponding Hölder space.

We also have a Sobolev embedding theorem into weighted Hölder spaces.

Theorem 1.27 (cf. [16, Thm. 2.9])**.** *Let (M,g) be an* CS/AC *manifold. Let $p > 1$, $k, l \geqslant 0$, $0 < \alpha < 1$ and $\delta \in \mathbb{R}$. If $k - \frac{n}{p} \geqslant l + \alpha$ then there is a continuous embedding:*

$$L^p_{k,\delta}(E) \longrightarrow C^{l,\alpha}_\delta(E). \tag{1.49}$$

Elliptic operators and Fredholm results

Every elliptic operator on a compact manifold is Fredholm. However, this useful fact does not generally hold in the noncompact setting. Consider the noncompact manifold $\mathbb{R}$ with the elliptic operator $\frac{\mathrm{d}}{\mathrm{dt}}$ acting on functions.

Proposition 1.28. *The elliptic operator $\frac{\mathrm{d}}{\mathrm{dt}} : L^2_1(\mathbb{R}) \to L^2(\mathbb{R})$ is not Fredholm.*

Proof. We show that the image of $\frac{\mathrm{d}}{\mathrm{dt}}$ is not closed in $L^2(\mathbb{R})$. Consider the functions $f_n \in L^2_1(\mathbb{R})$ which are defined as follows:

$$f_n(t) = \begin{cases} \frac{n}{t}, & t \leqslant -n \\ -1, & -n \leqslant t \leqslant -1 \\ t, & -1 \leqslant t \leqslant 1 \\ 1, & 1 \leqslant t \leqslant n \\ \frac{n}{t}, & n \leqslant t \end{cases}$$

Then clearly $f_n \in L^2_1(\mathbb{R})$, since both $\|f_n\|_{L^2} = O(n)$ and $\|\frac{\mathrm{d}}{\mathrm{dt}} f_n\|_{L^2} = O(1)$ are finite. As a consequence, this family does not admit a limit in $L^2_1(\mathbb{R})$. However, the family of derivatives does converge in L^2 to the characteristic function $\chi_{[-1,1]} \in L^2$, which is not in the image of $\frac{\mathrm{d}}{\mathrm{dt}}$. Any preimage $f \in L^2_{loc}$ would have $\lim_{t\to\infty} f(t) - f(-t) = 2$, and can hence not be square integrable. Thus the image of $\frac{\mathrm{d}}{\mathrm{dt}}$ is not closed, which precludes it from being Fredholm. □

More generally the same non-Fredholmness appears for operators on $\mathbb{R} \times N$ which are of the form $\frac{\mathrm{d}}{\mathrm{dt}} + A(t)$, where $A(t)$ is a self-adjoint elliptic operator on the compact cross-section N which converges in a suitable sense as $t \to \pm\infty$ to limiting operators $A_\pm$ with non-trivial kernel. In the example on $\mathbb{R}$, we had $A(t) = A_\pm = 0$ over the point. The proof above can be applied to the general case, if we consider $f_n\psi$ instead, where ψ is a non-zero element of the kernel of one of $A_\pm$. In fact, if $A_\pm$ have trivial kernel, the operator $\frac{\mathrm{d}}{\mathrm{dt}} + A$ will be Fredholm. A proof of this fact can be found in Robbin and Salamon's paper on the spectral flow [41]. Thus to ensure Fredholmness we need to shift the zero eigenvalues of $A_\pm$ to a non-zero value. This can be achieved by perturbing $A(t)$ to $A(t) - \delta\,\mathrm{id}_N$. It turns out that this is equivalent to varying the Banach spaces by introducing the weight $e^{-\delta t}$ into the norms, as we did in the previous section with the cylindrical Sobolev spaces. Indeed, note that the norm $\|s\|_{\mathcal{L}^p_{k,\delta}} = \|se^{-\delta t}\|_{L^p_k}$ is equivalent to the previously introduced weighted norm $\|\cdot\|_{L^p_{k,\delta,\mathrm{cyl}}}$. The advantage of this definition is that there is an isometry

$\mathcal{L}^p_{k,\delta} \to L^p_k$ given by sending $s \mapsto se^{\delta t}$. Thus an operator $\frac{\mathrm{d}}{\mathrm{dt}} + A(t) : L^p_{k,\delta,\mathrm{cyl}} \to L^p_{k-1,\delta,\mathrm{cyl}}$ will be Fredholm exactly when $e^{\delta t}(\frac{\mathrm{d}}{\mathrm{dt}} + A(t))e^{-\delta t} : L^p_k \to L^p_{k-1}$ is. However:

$$e^{\delta t}\left(\frac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)e^{-\delta t} = \frac{\mathrm{d}}{\mathrm{dt}} + A(t) - \delta\,\mathrm{id}\,.$$

In other words, perturbing the operator can be achieved by varying the weight in the definition of the Sobolev norm. This will recover the Fredholm results from the compact case. Note however that the index of an operator might depend on the weight chosen as seen in Theorem 1.32.

Let now (M, g) be a cylindrical manifold, i.e. it admits ends which are isometric to Riemannian cylinders. Let E and F be two metric vector bundles over M. A linear r-th order partial differential operator:

$$D_\infty : C^{k+r}_{\mathrm{loc}}(E) \longrightarrow C^k_{\mathrm{loc}}(F)$$

is then **cylindrical** if for every section $f \in C^{k+r}_{\mathrm{loc}}(E)$ which is supported in an end $N = (0, \infty) \times L$ we have $(D_\infty s)(t + \cdot) = D_\infty[s(t + \cdot)]$. Here $s(t + \cdot)$ denotes the translation action of $\mathbb{R}_+$ on the end. Now suppose that $D : C^{k+r}_{\mathrm{loc}}(E) \to C^k_{\mathrm{loc}}(F)$ is another operator and write these operators as:

$$D[s] = \sum_{i=0}^{r} D^i \nabla^i s, \tag{1.50}$$

$$D_\infty[s] = \sum_{i=0}^{r} D^i_\infty \nabla^i s, \tag{1.51}$$

for bounded coefficients $D^i_{(\infty)} \in C^\infty(TM^{\otimes i} \otimes F \otimes E^*)$. Then D is **asymptotically cylindrical** if for any $j \in \mathbb{N}$:

$$|\nabla^j(D^i_\infty - D^i)| \longrightarrow 0 \text{ as } t \to \infty. \tag{1.52}$$

Note that by translation invariance, the coefficients of D_∞ are independent of t. Using this one can prove the following.

Proposition 1.29. *If D is an asymptotically cylindrical operator, then for any $\delta \in \mathbb{R}$, it extends to a well-defined map:*

$$D : L^p_{k+d,\delta,\mathrm{cyl}}(E) \longrightarrow L^p_{k,\delta,\mathrm{cyl}}(F). \tag{1.53}$$

Conical operators

Suppose now that (M, g) is AC_λ or $\mathrm{CS}_{\bar{\mu}}$, and assume that we have a radius function ρ and a conical metric g_c that g is asymptotic to as $\rho \to \infty$ and $\rho \to 0$ respectively. We can now define conical operators between bundles of exterior forms:

Definition 1.30. A linear r-th order partial differential operator between forms

$D : C^{k+r}_{\mathrm{loc}}(\Lambda^m) \to C^k_{\mathrm{loc}}(\Lambda^{m'})$ is **conical with rate** $\nu \in \mathbb{R}$ if

$$D^\nu = \rho^{-m'+\nu} D \rho^m$$

is an asymptotically cylindrical operator and ν is maximal in this regard. Here the cylindrical metric is $\rho^{-2} g_c$. This definition can be extended to bundle of forms of mixed degree as well as more general tensor bundles

We now present the fundamental result concerning these operators, which is that they are Fredholm for almost all choices of weight $\delta \in \mathbb{R}$. More concretely, we have the following:

Theorem 1.31. *Let* $D : C^{k+r}_{\mathrm{loc}}(E) \to C^k_{\mathrm{loc}}(F)$ *be a conical operator on an* (M, g) *with rate* ν*. Then for any* $\delta \in \mathbb{R}$*,* P *extends to a well-defined map:*

$$D : L^p_{k+r,\delta}(E, g) \longrightarrow L^p_{k,\delta-\nu}(F). \tag{1.54}$$

Furthermore if D *is elliptic, then this map is Fredholm for* δ *in the complement of a discrete subset* $\mathscr{D} \subset \mathbb{R}$*. This subset is determined by an eigenvalue problem on the asymptotic link.*

Proof. The operator D is bounded whenever D^ν is. Now D^ν is bounded by Proposition 1.29. It is also Fredholm whenever D^ν is. This in turn is the case for all but a countable set of weights, which are determined by the cylindrical operator D asymptotes to, as in [29, Thm. 6.1]. □

Let D be a conical operator of rate ν, and let D_∞ be the cylindrical operator that D^ν asymptotes to, as in (1.52). Then the set of exceptional weights $\mathscr{D}$ can be determined as follows. With respect to the parametrisation by $(t, p) \in (0, \infty) \times L$ of the cylindrical end, D_∞ takes the following form:

$$D_\infty = \sum_{j+k \leqslant r} a^{j,k}_\infty \partial^j_t \nabla^k_L. \tag{1.55}$$

At the start of this section, we have seen that the Fredholm property fails for the first order operator $\partial_t + A(t)$ if the limit $A_+ = \lim_{t\to\infty} A(t)$ has a zero eigenvalue. This was because the kernel gained a solution whose growth was of order $O(1)$, and thus not integrable, but could nonetheless be approximated within L^p_k. More generally the operator $\partial_t + A(t) - \delta\,\mathrm{id}$ will not be Fredholm if $A_+ - \delta\,\mathrm{id}$ admits a kernel, i.e. A_+ admits a δ-eigenvector. Thus $\partial_t + A(t)$ will not be Fredholm as a map $L^p_{k,\delta} \to L^p_{k-1,\delta}$ for those values δ where a solution to the eigenvalue problem $A_+ v = \delta v$ exists. The generalisation of this to a higher-order operator in the form (1.55) is to consider the eigenvalue problem for the operator

$$\hat{D}_{\infty,\lambda} = \sum_{j+k \leqslant r} a^{j,k}_\infty (i\lambda)^j \nabla^k_L. \tag{1.56}$$

Denote by $\mathcal{C} \subset \mathbb{C}$ the set of all the complex values for λ for which (1.56) admits a non-zero eigenvector. As Lockhart and McOwen describe in more detail in their paper [29], this is a discrete subset of $\mathbb{C}$. The subset $\mathscr{D} \subset \mathbb{R}$ of exceptional weights, which again is discrete, is then given by:

$$\mathscr{D} = \{\operatorname{im} \lambda : \lambda \in \mathcal{C}\}.$$

In this way, the exceptional weights can be related to an eigenvalue problem on the link. Similar to the model case $\frac{\mathrm{d}}{\mathrm{dt}} + A(t)$, the existence of solutions to the eigenvalue problem implies that solutions of a certain exponential decay rate δ exist. These then get added to the kernel once the rate δ is passed, which makes the index jump discontinuously. Thus the Fredholm property cannot hold at these weights. Note that the dependence on the link means that operators on different CS or AC manifolds will have the same set of exceptional weights if their links agree as Riemannian manifolds, and the two operators approach the same limiting operator over that link, in the sense that the associated cylindrical operators limit to the same operator. Consider now for $\delta \in \mathscr{D}$ the dimension $d(\delta) < \infty$ of the set of solutions to $D_\infty u = 0$, which have the form $e^{-\delta t}p$, where p is a polynomial in t whose coefficients are sections of $E|_L$ and do not depend on t. This is exactly the jump in index as a weight is passed. To be more precise, let $\delta_1 < \delta_2$ be given such that $\delta_1, \delta_2 \notin \mathscr{D}$. Then we define:

$$N(\delta_1, \delta_2) = \sum_{\delta \in (\delta_1, \delta_2) \cap \mathscr{D}} d(\delta).$$

The relation between the indices of differential operators on differently weighted Sobolev spaces is then given as follows:

Theorem 1.32 (cf. [29, Thm. 1.2])**.** *Let D be an elliptic conical operator of order $r \geqslant 0$ and rate $\nu \in \mathbb{R}$. Let $1 < p < \infty$ and $k \geqslant 0$. Denote by $i_\delta(P)$ for $\delta \in \mathbb{R} \setminus \mathscr{D}$ not a critical rate the index of the following operator:*

$$D : L^p_{k+r,\delta}(E) \longrightarrow L^p_{k,\delta-\nu}(E).$$

We then have that: $i_{\delta_2}(P) - i_{\delta_2}(P) = N(\delta_1, \delta_2)$.

Proposition 1.33 (cf. [16, Lem. 2.8])**.** *Assume that $1 < p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if $n \in \mathbb{N}$ is the dimension of the underlying manifold and $\bar{\delta} \in \mathbb{R}^l$ is a vector of weights, there is a perfect pairing $L^p_{\bar{\delta}} \times L^q_{-n-\bar{\delta}} \to \mathbb{R}$, and thus $(L^p_{\bar{\delta}})^* = L^q_{-n-\bar{\delta}}$.*

Example 1.34. The simplest example of a Cayley cone is a Cayley plane $\Pi = \mathbb{R}^4 \times 0 \subset (\mathbb{R}^8, \Phi_0)$ with a round S^3 as its link. The limiting operator as $r \to \infty$ of its associated Dirac operator $\not{D}_C$ from Equation (1.27) is D_L from Equation (1.30). Even more than that, we can think of a Cayley plane as being induced by a special Lagrangian plane for a CY4 structure on $\mathbb{R}^8$ which induces the standard Spin(7)-structure. By Proposition 1.14 we can write:

$$\not{D} = -\mathrm{d} \star \oplus \mathrm{d}^- = \frac{\mathrm{d}}{\mathrm{d}r} + r^{-1} D_{S^3}. \tag{1.57}$$

Combining the work done in [31], where coassociative cones were analysed, and slightly extending the work done in [21], where the non-coassociative Cayley deformations of cones where studied (but not other critical rates), we can give all the critical rates $\mathscr{D}$ in the range $(-4, 2)$. They are $-3, -1, 0, 1$, and the eigenspaces have dimensions:

$$d(-3) = 1 + 0, \quad d(-1) = 1 + 0, \quad d(0) = 3 + 1, \quad d(1) = 8 + 4. \tag{1.58}$$

Here the first summand corresponds to the coassociative contribution, and the second is the truly Cayley contribution.

Example 1.35. Consider the complex cone $C_q = \{x^2 + y^2 + z^2 = 0, w = 0\} \subset \mathbb{C}^4$ which has link $L \simeq \mathrm{SU}(2)/\mathbb{Z}_2$. In particular, this is also a Cayley cone. The critical rates in $(-2, 2)$ are $-1, 0, 1, -1 + \sqrt{5}$, and the eigenspaces have dimensions:

$$d(-1) = 2 + 0, \quad d(0) = 7 + 1, \quad d(1) = 16 + 6, \quad d(-1 + \sqrt{5}) = 3 + 3. \tag{1.59}$$

Here the first summand corresponds to the coassociative contribution, and the second is the truly Cayley contribution. Note that $d(0)$ corresponds to the 8-dimensional space of translations of the cone, whereas $d(1) = (21 - 1) + 2$ splits as the Spin(7)-rotations of the cone (up to a one-dimensional stabiliser) together with a two-dimensional contribution coming from a variation of the link as an associative in S^7 (which are not coming from the action of Spin(7)). We discuss this aspect more in detail in Remark 2.32.

We close out the section by discussing a regularity property of cones which simplifies our discussion of fibrations in Chapter 4.

Definition 1.36. Let $C \subset \mathbb{R}^8$ be a Cayley cone with associative link $L \subset S^7$ as in Example 1.12. If C has no homogeneous deformation with rate in $[0, 1]$ other than the translations (of weight 0), rotations of the cone and associative deformations of the link (both of weight 1) and that every such infinitesimal deformation can be integrated, we say that the cone is **semistable**.

Formulated differently, a cone C is semistable when its singular rates satisfy $\mathscr{D}_C \cap [-1, 0] = \{-1, 0\}$ with $d(-1) = 8$, $d(0) = \dim \mathrm{Spin}(7) - \dim \mathrm{Stab}(C) + \dim \mathcal{M}^{G_2}(L)$, and the cone C is unobstructed. Here $\mathcal{M}^{G_2}(L)$ is the moduli space of associatives in S^7. It is unobstructed exactly when the corresponding moduli space of Cayley cones is.

We chose the term *semistable* since a semistable cone is stable [17, Def. 3.6] if its link is rigid as an associative and thus semistability is a weaker version of stability.

Chapter 2

Deformation theory of Cayley submanifolds

In this chapter, we study the deformation theory of Cayley submanifolds in Spin(7)-manifolds, and the local structure of their moduli spaces. We will in particular revisit the deformation theory of nonsingular compact Cayleys which was first considered in the foundational paper by McLean [34], who derived a formula for the linearised deformation operator and in particular realised it as a twisted Dirac operator in the case of a Spin Cayley. Later Clancy [7] provided a formula for the expected dimension of the moduli space in terms of topological invariants of the embedding of the Cayley and studied its global properties. Finally, Moore [37] focused on the case of a compact complex surface in a Calabi–Yau fourfold. We slightly generalise the known results by considering the family moduli space, i.e. the moduli space of Cayleys for varying choices of ambient Spin(7)-structure.

To this end, we introduce almost Cayley submanifolds in Section 2.1. These are manifolds whose tangent bundle is close to being a bundle of Cayley planes, or in other words, almost Cayley manifolds are C^1-close to being Cayley. We then discuss the non-linear Cayley deformation operator in Section 2.2 which can be defined for any almost Cayley sufficiently close to being Cayley.

Then, in the three remaining sections we study compact, asymptotically conical and conically singular Cayleys respectively. We reprove the main structural result on the moduli space of compact Cayleys, Theorem 2.16, to lay the groundwork for the analytically more complicated variants that follow. We then prove the analogous result for AC Cayleys in $\mathbb{R}^8$ with a Spin(7)-structure that is AC to Φ_0, Theorem 2.23. As far as the author is aware, this has not been done previously.

To conclude, we consider the case of conically singular Cayleys. Their deformation theory has been studied before by Moore [38] in the case of a unique singular point and for a fixed torsion-free Spin(7)-structure. We generalise these results slightly by considering multiple singular points as well as families of (potentially torsion) Spin(7)-structures. We will require these generalisations as well as the result for AC Cayleys to perform the desingularisation of CS Cayleys in Chapter 3. We conclude our discussion of the deformation theory by giving formulae for the dimension of the moduli spaces of AC and

CS Cayleys.

2.1 Almost Cayley submanifolds

Denote the Grassmannian of oriented 4-planes in an 8-dimensional vector space V by $\mathrm{Gr}_+(4,V)$. If (V,Φ) is a Spin(7)-vector space we can additionally consider the **Grassmannian of Cayley planes** in (V,Φ), which we denote by $\mathrm{Cay}(V,\Phi)$. The group $\mathrm{Spin}(7)_\Phi$ acts on $\mathrm{Gr}_+(4,V)$, and acts transitively on $\mathrm{Cay}(V,\Phi)$. As the stabiliser group of any Cayley is isomorphic to $H=(\mathrm{SU}(2)^3)/\mathbb{Z}_2$, we have $\dim \mathrm{Cay}(V,\Phi)=\dim \mathrm{Spin}(7)-\dim H = 21-9=12$. As $\dim \mathrm{Gr}_+(4,V)=16$, we see that $\mathrm{Cay}(V,\Phi)$ is a codimension 4 submanifold of the Grassmannian of oriented four-planes. In other words, the Cayley condition can be given in terms of four independent equations. This is the reason why the bundle E, which will appear later as the co-domain of the deformation operator of a Cayley, is a rank 4 bundle. We can think of a fibre of E as corresponding to the normal space at a given Cayley plane of the submanifold $\mathrm{Cay}(V,\Phi)\subset \mathrm{Gr}_+(4,V)$. Since $\mathrm{Spin}(7)_\Phi$ is compact and the action is smooth, there is a metric $g_{\mathrm{Spin}(7)}$ on $\mathrm{Gr}_+(4,V)$, such that Spin(7) acts by isometries. Such a metric can be realised by embedding $\mathrm{Gr}_+(4,V)\hookrightarrow \Lambda^4 V$ via $\mathrm{span}\{e_1,e_2,e_3,e_4\}\mapsto e_1\wedge e_2\wedge e_3\wedge e_4$. The Spin(7)-invariant metric is then the restriction of the Euclidean metric on Λ^4V. The resulting distance map is uniformly equivalent to the following Spin(7)-invariant distance defined in terms of the orthogonal projections onto planes:

$$d_{\mathrm{Gr}}(E,E')=\|\pi_E-\pi_{E'}\|_{\mathrm{op}}.$$

Here $\pi_E,\pi_{E'}$ are the orthogonal projections onto E and E' respectively. Let us take a closer look at a tubular neighbourhood of the Cayley planes inside $\mathrm{Gr}_+(4,V)$.

Proposition 2.1. *Let $\epsilon_1\in[0,1),\epsilon_2\in[0,2),\epsilon_3\in[0,\infty)$ be given and consider the sets:*

$$\begin{aligned}
E_1 &= \{\xi\in \mathrm{Gr}_+(4,V) : \Phi|_\xi > (1-\epsilon_1)\,\mathrm{dvol}_\xi\},\\
E_2 &= \{\xi\in \mathrm{Gr}_+(4,V) : \|\tau(f_1,f_2,f_3,f_4)\|_{\Lambda^2_7}<\epsilon_2,\\
&\qquad \{f_i\}_{i=1,\dots,4} \text{ is an orthonormal basis of } \xi\},\\
\text{and } E_3 &= \{\xi\in \mathrm{Gr}_+(4,V) : \xi=\mathrm{span}\left\{e_1,e_2,e_3,\frac{e_4+\sqrt{\alpha}v}{\sqrt{1+\alpha}}\right\},\\
&\qquad e_4=e_1\times e_2\times e_3,\ e_i\in V \text{ orthonormal},\\
&\qquad v\perp e_i, \|v\|=1,\ 0\leqslant \alpha<\epsilon_3\}.
\end{aligned}\tag{2.1}$$

Note that $\|\tau(f_1,f_2,f_3,f_4)\|_{\Lambda^2_7}$ is independent of the choice of basis of ξ. Then for a choice of one of the ϵ_i we can determine the other two such that the three sets agree.

Proof. These three families of sets are Spin(7)-invariant. The sets E_1 and E_2 are invariant by the definition of Φ and τ respectively. The invariance of E_3 follows from the fact that there are elements of $H\subset \mathrm{Spin}(7)$ which keep ξ fixed while acting transitively on the unit sphere in $\xi^\perp$.

Now note that the orbit of a single $\xi \in E_3$ is a sphere of a given radius in the open ball E_3. Thus choosing elements for every radius, we can exhaust all of E_3. Furthermore spheres in E_1 and E_2 (i.e. replacing the inequality with equality in the definition) must be unions of this set for different values of α. What is left to show is that the value of α determines the radii of the spheres in E_i $(i = 1, 2)$ uniquely. For E_1 for instance it can be computed using the definition (1.9) of Φ_0 that $1 - r_1 = \frac{1}{\sqrt{1+\alpha}}$. For r_2 we see that $r_2 = \frac{2\sqrt{\alpha}}{\sqrt{1+\alpha}}$, using the coordinate representation (1.16) of τ. □

For $\alpha \in (0, 1)$, consider consider the set of **almost Cayley planes**:

$$\mathrm{Cay}_\alpha(V, \Phi) = \{\xi \in \mathrm{Gr}_+(4, V) : \Phi|_\xi > \alpha\, \mathrm{dvol}_\xi\}. \tag{2.2}$$

As this set of planes is Spin(7)-invariant, it admits a canonical action of Spin(7). In fact, $\mathrm{Cay}_\alpha(V, \Phi)$ is a tubular neighbourhood of $\mathrm{Cay}(V, \Phi)$ under geodesic normal coordinates for $g_{\mathrm{Spin}(7)}$ and α sufficiently close to 1. Let $0 < \alpha_0 < 1$ be such that for all $\alpha > \alpha_0$ the set $\mathrm{Cay}_\alpha(V, \Phi)$ has this tubular neighbourhood property. This is a universal constant. Let now (M, Φ) be a Spin(7)-manifold. We can then consider the associated fibre bundles:

- $\mathrm{Gr}_+(4, TM) = P_{\mathrm{Spin}(7)} \times_{\mathrm{Spin}(7)} \mathrm{Gr}_+(4, \mathbb{R}^8)$
- $\mathrm{Cay}(M) = P_{\mathrm{Spin}(7)} \times_{\mathrm{Spin}(7)} \mathrm{Cay}(\mathbb{R}^8, \Phi_0)$
- $\mathrm{Cay}_\alpha(M) = P_{\mathrm{Spin}(7)} \times_{\mathrm{Spin}(7)} \mathrm{Cay}_\alpha(\mathbb{R}^8, \Phi_0)$

We see that a submanifold $N^4 \subset M$ is Cayley exactly when TN, seen as a section of $\mathrm{Gr}_+(4, TM)$ over N, takes values in $\mathrm{Cay}(M)$. Analogously we say that a submanifold of M is α-Cayley if the section TN takes values in $\mathrm{Cay}_\alpha(M)|_N$. Now for every $p \in M$, we have $(T_pM, \Phi_p) \simeq (\mathbb{R}^8, \Phi_0)$ as Spin(7)-vector spaces, thus $\mathrm{Cay}_\alpha(T_pM, \Phi_p)$ will be a tubular neighbourhood of $\mathrm{Cay}(T_pM, \Phi_p)$ whenever $\alpha > \alpha_0$. In particular, for an α-Cayley N with $\alpha > \alpha_0$ we get a canonical section cay_N of $\mathrm{Cay}(M)|_N$ defined as the closest Cayley plane $\mathrm{cay}_N(p)$ to the given almost Cayley T_pN, as measured by the metric $d_{\mathrm{Spin}(7)}$. This Cayley plane is unique because of the tubular neighbourhood property.

Proposition 2.2 (Adapted frame for α-Cayley)**.** *There is a universal constant $1 > \alpha_1 > \alpha_0$ such that the following holds. Let N be an α'-Cayley submanifold of M, where $\alpha' > \alpha_1$, and let cay_N be the canonical Cayley section associated to N. Let $p \in N$. Write $\Phi|_N = \alpha\, \mathrm{dvol}_N$, for a smooth function $\alpha : N \to (\alpha_1, 1]$. Then we can then find a* Spin(7)*-frame $\{e_i\}_{i=1,\dots,8}$ adapted to cay_N around p such that:*

$$\begin{aligned} T_pN &= \mathrm{span}\,\{\beta_i e_i + v_i\}_{i=1,\dots,4}\,, \\ \nu_p(N) &= \mathrm{span}\,\{\beta_i e_i + v_i\}_{i=5,\dots,6}\,, \end{aligned} \tag{2.3}$$

where the basis vector fields $\beta_i e_i + v_i$ are orthonormal. Here v_i for $i = 1, \dots, 8$ are vector fields such that:

$$\begin{cases} v_1, v_2, v_3, v_4 \perp \mathrm{cay}_N, \\ v_5, v_6, v_7, v_8 \in \mathrm{cay}_N, \\ \|v_i\| \leqslant C_{\alpha_1}(1 - \alpha), \end{cases}$$

and β_i are functions such that $1 - \beta_i \geqslant C_{\alpha_1}(1-\alpha)$.

Proof. For every $p \in N$, the planes $\mathrm{cay}_N(p)$ and T_pN are sufficiently close so that $\|\pi_{\mathrm{cay}_N(p)} - \pi_{T_pN}\|_{\mathrm{op}} \lesssim 1-\alpha$, where π_V is the orthogonal projection onto $V \subset T_pM$. Thus, if we take a Spin(7)-frame $\{e_i\}_{i=1,\dots,8}$ which is adapted to cay_N, then the tangent vectors $\tilde{f}_i = \pi_{T_pN}(e_i)$ for $i = 1, \cdots, 4$ will be such that $\|e_i - \tilde{f}_i\| \lesssim 1-\alpha$. After applying the Gram-Schmidt orthogonalisation procedure to $\tilde{f}_i$ to obtain orthogonal vectors f_i, we still have that $v_i = \beta_i e_i - f_i \in \mathrm{cay}_N^\perp$ and $\|v_i\| \lesssim 1-\alpha$, for some functions β_i coming from the Gram-Schmidt algorithm. Note that the hidden constant in our $\lesssim$-notation only depends on α_1. Similarly the coefficients β_i tend to 1 as α tends to 1. An analogous argument applies to the normal vectors, using the fact that we also have $\|\pi_{\mathrm{cay}_N^\perp(p)} - \pi_{\nu_p(N)}\|_{\mathrm{op}} \lesssim 1-\alpha$. $\square$

2.2 Deformation operator

Consider a Spin(7)-manifold M. Let $\alpha_0 \in (0,1)$ be sufficiently close to 1 such that $\mathrm{Cay}_{\alpha_0}(T_pM, \Phi_p)$ is a tubular neighbourhood of $\mathrm{Cay}(T_pM, \Phi_p)$ for every $p \in M$. Let N be an α_1-Cayley (where $\alpha_1 > \alpha_0$) with a tubular neighbourhood $N \subset U \subset M$. In other words, we require that the exponential map $\exp : V \subset \nu(N) \to U$ defines a diffeomorphism onto its image, where V is some open subset of the normal bundle of N. For $v \in C^\infty(N, V)$, i.e. v is a normal vector field to N with values in V, we define $\exp_v : N \hookrightarrow M$ to be the embedding given by $\exp_v(p) = \exp(v(p))$. This is a small perturbation of N inside U, and in fact, any C^k-small perturbation of N (where $k > 0$) can be obtained as the image of $\exp_v$ of a unique C^k-small normal vector field v. We denote this image by N_v.

Our goal is to construct a Cayley submanifold of the form N_v, whenever N is close to being Cayley. As we have seen, N admits a canonical section $\mathrm{cay}_N : N \to \mathrm{Cay}(M)|_N$. This section can also be seen as a four-dimensional subbundle of $TM|_N$, with each fibre a Cayley plane. This allows us to globalise the definition of the subspace $E_\xi \subset \Lambda^2_7 V^*$ of Equation (1.20), generalising the definition of the bundle E associated to a Cayley from Equation (1.22), and define the four-dimensional vector bundle for any α-Cayley with $\alpha < 1$ sufficiently close to 1:

$$E_{\mathrm{cay}} = \{\omega \in \Lambda^2_7 : \omega|_{\mathrm{cay}_N} = 0\}. \tag{2.4}$$

Clearly, when N is Cayley, then $E_{\mathrm{cay}} = E$.

Let now $\eta \in \Omega^k(N_v, F|_{N_v})$ be a differential form with values in a bundle of tensors $F \to M$ over the submanifold N_v. The form τ from Equation (1.16) provides such an example. Ordinarily the pull-back of $\exp_v^* \eta$ is a form in $\Omega^k(N, \exp_v^* F|_{N_v})$. However, when we write $\mathrm{Exp}_v^* \eta$ in the following, we mean a form in $\Omega^k(N, F|_N)$ (i.e. we also pull back the value bundle of the form), which we define as follows. Extend the normal vector field v on N to a vector field on U, where $v(\exp_u(p))$ is defined to be the parallel transport of $v(p)$ along the geodesic which starts at p and has initial velocity $u(p)$, u being another normal vector field on N. As U is a tubular neighbourhood for geodesic normal coordinates, this gives rise to a smooth extension of v to all of U. In turn, this induces a flow $\varphi_t : U \to M$ with the property that $\varphi_1(p) = \exp_v(p)$. We now define the pullback of a decomposable form $\eta = \omega \otimes s$, $\omega \in C^\infty(\Lambda^k T * M), s \in C^\infty(F)$, by:

$$\mathrm{Exp}_v^*\eta = \varphi_1^*\omega \otimes \mathrm{D}\varphi_1^* s = \exp_v^*\omega \otimes \mathrm{D}\varphi_1^* s. \tag{2.5}$$

Here φ_1^* and $\exp_v^*$ are the usual pullback of differential forms, and $\mathrm{D}\varphi_1^* : F_{\exp_v(p)} \to F_p$ is the pullback induced from the linear isomorphism $\mathrm{D}\varphi_1 : F_p \to F_{\exp_v(p)}$ coming from φ_1, on any tensor bundle. To summarise, we extended the vector field v to have a pullback operation that also pulls back the bundle in which the differential form is valued.

We now define the **deformation operator** associated to N as follows:

$$\begin{aligned} F : C^\infty(N, V) &\longrightarrow C^\infty(E_{\mathrm{cay}}) \\ v &\longmapsto \pi_E(\star_N \mathrm{Exp}_v^*(\tau|_{N_v})). \end{aligned} \tag{2.6}$$

Here $\mathrm{Exp}_v^*\tau$ is the non-standard definition of a pullback introduced above. This addition is necessary, otherwise the resulting sections would be valued in different bundles for varying v. Moreover π_E denotes the orthogonal projection $\Lambda^2_7|_N \to E_{\mathrm{cay}}$, which depends on the Cayley section cay_N over N. We project onto E_{cay} to ensure that F is a map between bundles of the same rank, for otherwise, F cannot be elliptic. However one might ask whether such a projection loses information. Heuristically, the condition $F(v) = 0$ is given by ignoring 3 of the 7 equations obtained from the condition $\star_N \mathrm{Exp}_v^*(\tau|_{N_v}) = 0$, one for each of the basis vectors of $\Lambda^2_7|_{N_v}$. The fact that this still gives enough equations to determine the Cayleyness of a plane can be expected for a generic choice of a projection to a four-dimensional subbundle of $\Lambda^2_7|_N$, since the Grassmannian of Cayley planes is of codimension 4 in the Grassmannian of oriented four-planes, i.e. the Cayley condition on a four-plane can be described with four independent equations. Notice however that the projection is onto a fibre of E_{cay} over $p \in N$, whereas the equations we chose are situated at $\exp_v(p) \in N_v$, so this is only approximately true, as we will see in the next proposition. We will crucially rely on the fact that we choose v to be small in the C^1-norm to overcome this discrepancy and prove that we can ignore 3 equations while still retain the Cayley-detecting property of τ.

Proposition 2.3 (Detects Cayleys). *Let (M, Φ) be a* $\mathrm{Spin}(7)$*-manifold with uniformly bounded Riemann curvature tensor $|R| < C$. Let $0 < \alpha_0 < 1$ such that $\mathrm{Cay}_{\alpha_0}(T_pM, \Phi_p)$ is a tubular neighbourhood of $\mathrm{Cay}(T_pM, \Phi_p)$. Then there is $\alpha_0 < \alpha_2 < 1$ depending on Φ and a constant $C_1 = C_1(\Phi, \alpha_0, \alpha_2) < 1$ such that the following holds for any α_2-Cayley N. If $v \in C^\infty(N, V)$ is such that for all $p \in N$:*

$$\min\{|R(q)| : q \in B(p, |v(p)|)\}|v(p)| < C_1, \quad |\nabla^\perp v(p)| < C_1,$$

then N_v is Cayley exactly when $F(v) = 0$. Note that if M is flat, then v is only constrained by the requirement that it lie in the tubular neighbourhood V of N.

Proof. The submanifold N_v is Cayley exactly when $\tau|_{N_v} = 0$, which is equivalent to $\star_N \mathrm{Exp}_v^*(\tau|_{N_v}) = 0$. Thus we need to prove that $\star_N \mathrm{Exp}_v^*(\tau|_{N_v})$ vanishes identically if and only if $\pi_E(\star_N \mathrm{Exp}_v^*(\tau|_{N_v})) = 0$. Let $p \in N$ be given. For any $\alpha_0 < \tilde{\alpha} < \alpha_2$ we have that the set of $\tilde{\alpha}$-Cayley planes in a given Spin(7)-vector space is an open ϵ-neighbourhood of the set of α_1-Cayley planes, for some $\epsilon > 0$ dependent on $\tilde{\alpha}$. The same holds true globally in

$\mathrm{Gr}_+(4, TM)$, since everything is pointwise isometric to the standard model $(\mathbb{R}^8, \Phi_0)$. In particular, since $T_{\exp_v(p)}N_v$ is determined from T_pN by knowing only $v(p)$ and $\nabla v(p)$, this implies that if $|v(p)|$ is small compared to the curvature at any point within a distance $|v(p)|$ and $|\nabla v(p)|$ is sufficiently small at every point $p \in N$, say smaller than some $\tilde{C}_1$, then N_v is still $\tilde{\alpha}$-Cayley. Thus we still have a canonical Cayley section cay_{N_v} over N_v. We then apply Proposition 2.2 to choose a Spin(7)-frame $\{e_i\}_{i=1,\dots,8}$ adapted to cay_{N_v} such that an orthonormal basis of $T_{\exp_v(p)}N_v$ is given by $f_i = \beta_i e_i + v_i$, where $\beta_i : U \to [0,1]$ are smooth functions. Under $(\mathrm{D}\exp_v(p))^{-1}$ these in turn get mapped to a local frame $\{f_i'\}_{i=1,\dots,4}$ of TN, which is not necessarily orthogonal. We now have that:

$$\begin{aligned}\det(f_i') \star_N \mathrm{Exp}_v^*(\tau|_{N_v})(p) &= \mathrm{Exp}_v^*(\tau|_{N_v})_p(f_1', f_2', f_3', f_4')\\ &= \mathrm{D}\varphi_1^* \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4),\end{aligned}$$

where $\det(f_i')$ is the volume of the 4-parallelepiped spanned by the f_i'. We similarly get:

$$\begin{aligned}\det(f_i')\pi_E \star_N \mathrm{Exp}_v^*(\tau|_{N_v})(p) &= \pi_E \mathrm{D}\varphi_1^* \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4)\\ &= \mathrm{D}\varphi_1^* \pi_{\tilde{E}} \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4).\end{aligned}$$

Here π_E is the orthogonal projection onto $(E_{\mathrm{cay}})_p$ and $\pi_{\tilde{E}} = (\mathrm{D}\varphi_1^*)^{-1}\pi_E \mathrm{D}\varphi_1^*$ is a not necessarily orthogonal projection onto a subspace $\tilde{E} \subset \Lambda^2_7|_{\exp_v(p)}$. As $\mathrm{D}\varphi_1^*$ is an isomorphism, it suffices to show that:

$$\pi_{\tilde{E}} \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4) = 0 \Rightarrow \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4) = 0.$$

Let $q = \exp_v(p)$ for simplicity. Consider $\tau_q : \mathrm{Gr}_+(4, T_qM) \to (\Lambda^2_7)_q M$ as a smooth map. As τ_q vanishes on a 12-dimensional submanifold in the 16-dimensional manifold $\mathrm{Gr}^+(4, T_qM)$, its derivative at a Cayley can have rank at most 4, in other words $\mathrm{im}\, D\tau_q \subset (\Lambda^2_7)_q M$ is four-dimensional at most. Now we use the coordinate expression for τ_q in (1.16) with regards to the frame $\{e_i\}_{i=1,\dots,8}$, as well as the special form $f_i = \beta_i e_i + v_i$ with $v_i \perp \mathrm{cay}_{N_v}$ to see that $\mathrm{im}\, D\tau_q = E$. In fact $\mathrm{im}\, D\tau_q$ is spanned by all the vector of the form $\partial_t \tau_q(e_1 + tv_1, \dots, e_4 + tv_4)$. Remembering that $E_{\mathrm{cay}}^\perp$ is spanned by the vectors $e_1 \times e_2$, $e_1 \times e_3$ and $e_1 \times e_4$, we see that these components are at least quadratic in the normal contributions $\{v_i\}_{1 \leqslant i \leqslant 4}$ (which are simply linear combinations of $\{e_i\}_{5 \leqslant i \leqslant 8}$), and thus $\partial_t(\tau|_{E^\perp}) = 0$. This implies that if f_1, f_2, f_3 and f_4 span a sufficiently small perturbation of $\mathrm{cay}_{N_v}(q)$, as is the case in our setting by the $\tilde{\alpha}$-Cayleyness of N_v, then $\pi_E \tau(f_1, f_2, f_3, f_4) = 0$ can only occur if $\tau(f_1, f_2, f_3, f_4) = 0$, as E_{cay} and $E_{\mathrm{cay}}^\perp$ are transversal subspaces. In fact, if E' is any four-dimensional subspace and $\pi_{E'}$ is any projection onto E', not necessarily orthogonal, such that $\ker_{E'} \cap E_{\mathrm{cay}} = \{0\}$, then the implication:

$$\pi_{E'} \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4) = 0 \Rightarrow \tau_{\exp_v(p)}(f_1, f_2, f_3, f_4) = 0 \tag{2.7}$$

holds, given that the f_i span a sufficiently small perturbation of cay_{N_v}. In other words for a given E', if $\tilde{\alpha}$ is sufficiently close to 1, the implication above will be true. Moreover, having fixed a neighbourhood of cay_{N_v} for the f_i to vary in, satisfying the above implication is an open condition on $\pi_{E'}$.

Let now an $\tilde{\alpha}$-Cayley plane $\xi_p \subset T_pM$ be given, with associated canonical Cayley plane

cay_p. If we restrict the f_i to vary so that the planes they span are in a sufficiently small neighbourhood of $\text{cay}_p \in \text{Gr}_+(4, T_pM)$, then the implication (2.7) holds for an open subset of maps $\pi_{E'}$ which contains π_E. Note that this open subset of maps $\pi_{E'}$ depends on the neighbourhood of cay_p that we fixed. Moreover, this is true at every point $p \in M$. Denote this open subset of projections which contains π_E by Π, and let the fibre over the point p be Π_p. It thus remains to show that $\pi_{\tilde{E}} \in \Pi_q$. For this notice that from a fixed $v(p)$ and $\nabla v(p)$, we can determine $\mathrm{D}\varphi_t : T_pM \to T_qM$. This is because first-order deviations of geodesics at a later time depend solely on the first-order deviation of geodesics at an earlier time. Thus we get a family of maps $\pi^t_{\tilde{E}}$ which act on the fibre of Λ^2_7 over the point $\varphi_t(p)$. We have $\pi^0_{\tilde{E}} = \pi_E$, thus $\pi^0_{\tilde{E}} \in \Pi_p$. Next, by what we have said above it is clear that $\pi^t_{\tilde{E}}$ being contained in $\Pi_{\exp_{tv}(p)}$ is an open condition in t. Thus for a fixed $v(p)$ and $\nabla v(p)$ there is a $t_0 = t_0(v(p), \nabla v(p))$ such that for all $0 \leqslant t < t_0$ we have $\pi^t_{\tilde{E}} \in \Pi_{\exp_{tv}(p)}$. Now note that for $\lambda > 0$:

$$t_0(\lambda v(p), \lambda \nabla v(p)) = \lambda^{-1} t_0(v(p), \nabla v(p)).$$

This implies that for $|v(p)| + |\nabla v(p)|$ sufficiently small we have $\pi_{\tilde{E}} \in \Pi_q$. Now we can find a constant $C_1 < \tilde{C}_1$ such that whenever $|v(p)| + |\nabla v(p)| < C_1$, then the implication (2.7) holds at q. □

Next, we are interested in studying the linearisation of the nonlinear operator F and showing that it is elliptic at the zero section.

Proposition 2.4 (Linearisation)**.** *Let N be an $\bar{\alpha}$-Cayley with $\bar{\alpha} > \alpha_1$ with deformation operator F, such that $\Phi|_N = \alpha\, \mathrm{dvol}_N$. Let $p \in N$ and suppose that near p we have a* Spin(7)*-frame $\{e_i\}_{i=1,\dots,8}$ and a frame $\{f_j\}_{j=1,\dots,8}$ which respects the splitting $TM = TN \oplus \nu(N)$ as in Proposition 2.2. The linearisation of F at* 0 *is then given by:*

$$\begin{aligned} \not{D} : C^\infty(\nu(N)) &\longrightarrow C^\infty(E_{\text{cay}}), \\ v &\longmapsto \pi_E\big(\beta \textstyle\sum_{i=1}^4 f_i \times \nabla^\perp_{f_i} v \\ &\quad + \textstyle\sum_{i=1}^4 \sum_{j=1}^8 \beta_{ij} f_j \times \nabla^\perp_{f_i} v \\ &\quad + \nabla_v \tau(f_1, f_2, f_3, f_4)\big), \end{aligned} \tag{2.8}$$

where $\nabla^\perp$ is the Levi-Civita connection on the normal bundle $\nu(N)$, and $\nabla^\top_{f_i} v$ is the projection of $\nabla_{f_i} v$ to TN. Here β, β_{ij} are smooth functions such that $1 - \beta \geqslant C_{\alpha_1}(1-\alpha)$ and $|\beta_{ij}| \leqslant C_{\alpha_1}(1-\alpha)$. The constant C_{α_1} only depends on the choice of α_1. When N is a Cayley, then the second line is the Dirac operator associated to a Cayley from Proposition 1.11. Furthermore, the third line vanishes in this case. Finally, the fourth line vanishes if (M, Φ) is torsion-free.

Proof. We have for $v \in C^\infty(\nu(N))$:

$$\begin{aligned} \mathrm{D}F[v] &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} F(tv) \\ &= \pi_E \star_N \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \mathrm{Exp}^*_{tv}(\tau_{N_v}). \end{aligned}$$

Note that $\varphi_t = \exp_{tv}$ is the flow of a vector field on a neighbourhood of N which extends v, as we discussed before the definition of the deformation operator in equation (2.6). Thus we get from the definition of the Lie derivative, the expression (2.5) for our non-standard definition of pull-back, and the definition of the Levi-Civita covariant derivative on forms:

$$\begin{aligned}\mathrm{D}F[v] &= \pi_E \star_N \mathcal{L}_v\tau = \pi_E\mathcal{L}_v(\tau(f_1,f_2,f_3,f_4)) \\ &= \pi_E\left(\tau(\nabla_{f_1}v,f_2,f_3,f_4)+\tau(f_1,\nabla_{f_2}v,f_3,f_4)\right. \\ &\left.+\tau(f_1,f_2,\nabla_{f_3}v,f_4)+\tau(f_1,f_2,f_3,\nabla_{f_4}v)+\nabla_v\tau(f_1,f_2,f_3,f_4)\right).\end{aligned} \tag{2.9}$$

Here the last line uses the fact that ∇ is torsion-free. Now consider the torsion term $\tau(\nabla_{f_1}v,f_2,f_3,f_4)$. Since we have the orthogonal splitting $TM|_N = TN\oplus\nu(N)$, the connection ∇ on $TM|_N$ splits into $\nabla^\top+\nabla^\perp$, where $\nabla^\top = \pi_{TN}\circ\nabla$ and $\nabla^\perp = \pi_{\nu(N)}\circ\nabla$. Thus in particular:

$$\begin{aligned}\tau(\nabla_{f_1}v,f_2,f_3,f_4) &= \tau(\nabla^\perp_{f_1}v,f_2,f_3,f_4)+\tau(\nabla^\top_{f_1}v,f_2,f_3,f_4) \\ &= \nabla^\perp_{f_1}v\times(f_2\times f_3\times f_4)+\tau(\nabla^\top_{f_1}v,f_2,f_3,f_4) \\ &= \nabla^\perp_{f_1}v\times(f_2\times f_3\times f_4) \\ &= \nabla^\perp_{f_1}v\times(\beta_2e_2\times\beta_3e_3\times(\beta_4e_4+v_4)) \\ &= \nabla^\perp_{f_1}v\times(\beta_2\beta_3\beta_4e_1+\sum_{j=1}^{8}\tilde{\beta}_{1j}f_j) \\ &= \nabla^\perp_{f_1}v\times(\beta f_1+\sum_{j=1}^{8}\beta_{1j}f_j).\end{aligned}$$

For the third line we used the fact that $\nabla_w v(p)\perp T_pN$, since $v\in C^\infty(\nu(N))$. For the rest, we use the definition of τ as well as the coordinate representation of Φ_0. Here $\beta=\beta_1\beta_2\beta_3\beta_4$. The computations for the remaining three terms are similar and lead to the claimed formula.

If N is Cayley, then the second line corresponds exactly to the Dirac operator associated to a Cayley from Propositions 1.11, since in this case $\alpha\equiv 1$, which implies $\beta\equiv 1$, and the cross product of f_i and $\nabla_{f_i}v$ already lies in $E_{\text{cay}}=E$, so no further projection is required. In the same situation, we see that the third line vanishes, as all the β_{ij} vanish. Finally, τ is covariantly constant if Φ is torsion-free, and the last line vanishes in this case. To see this note we can choose a local Spin(7)-frame which is covariantly constant, from which it is clear that the cross and triple product send parallel sections to parallel sections. Since τ is defined in terms of these two products, the same must hold for τ. From this, it is immediate that τ is covariantly constant. □

The coefficients of $\not{D}$ depend on the data M, N and Φ in a very precise way.

Proposition 2.5. *Let (M,Φ) be a* Spin(7)*-manifold. Then there is an open subset $\mathcal{U}\subset\mathcal{A}(M)$ with $\Phi\in C^\infty(\mathcal{U})$ such that for α sufficiently close to* 1 *there are smooth bundle* *maps as follows:*

$$c_1 : M \times \mathrm{Cay}_\alpha(TM) \times \mathcal{U} \longrightarrow \mathrm{Hom}(T^*M \otimes TM, \Lambda^2 M),$$
$$c_0 : M \times \mathrm{Cay}_\alpha(TM) \times (T^*M \otimes \mathcal{U}) \longrightarrow \mathrm{Hom}(TM, \Lambda^2 M),$$

for which to following holds. For any immersed α-Cayley $N \subset (M, \tilde{\Phi})$ (where $\tilde{\Phi} \in C^\infty(\mathcal{U})$) the associated linearised deformation operator $\not{D}_{N,\tilde{\Phi}}$ satisfies:

$$\not{D}_{N,\tilde{\Phi}} v(p) = c_1(p, T_pN, \tilde{\Phi}(p)) \cdot \nabla v(p) + c_0(p, T_pN, \nabla\tilde{\Phi}(p)) \cdot v(p).$$

Here ∇ refers to the Levi-Civita connection for the fixed Spin(7)*-structure Φ.*

Proof. This is a consequence of (2.8), which gives a coordinate expression for $\not{D} = \not{D}_{N,\tilde{\Phi}}$ in a carefully chosen frame $\{f_j\}_{1 \leqslant j \leqslant 8}$ from Proposition 2.2 as:

$$\not{D}[v] = \pi_E(\beta \textstyle\sum_{i=1}^4 f_i \times \nabla^\perp_{f_i} v + \sum_{i=1}^4 \sum_{j=1}^8 \beta_{ij} f_j \times \nabla^\perp_{f_i} v + \nabla_v \tau(f_1, f_2, f_3, f_4)).$$

Here β, β_{ij} depend algebraically on the choice of frame (which depends on TN) and $\tilde{\Phi}(p)$, and $\nabla^\perp$ is the connection on the normal bundle induced by $\tilde{\Phi}$. The product $\times$ that appears also depends pointwise on $\tilde{\Phi}$, and the derivative of the form τ depends pointwise on $\nabla\Phi$ and TN. We remark that the Christoffel symbols of $\nabla^\perp$ also depend on $\nabla\Phi$ and that this is included in c_0. □

In particular, if two almost Cayley submanifolds are sufficiently close to one another, their deformation operators will differ in a controlled manner.

Corollary 2.6. *Let $N \subset (M, \Phi)$ be a compact almost Cayley with linearised deformation operator $\not{D}_N$. Let $v \in C^\infty(\nu(N))$ be a sufficiently small normal vector field in a tubular neighbourhood of N so that N_v again admits a deformation operator. Identify the normal bundles of N and N_v via parallel transport and orthogonal projection. We can then write:*

$$\not{D}_{N_v} = \not{D}_N + \delta\not{D}_v,$$

where $\delta\not{D}_v[w] = a_1(v, N) \cdot \nabla w + a_0(v, N) \cdot w$, and:

$$|\nabla^k a_i| \lesssim |\nabla^{k+1} v|.$$

Proof. This follows from the previous Proposition 2.5 by realising that the variation in T_pN is governed by the first derivative of v, and similarly for higher derivatives. Finally, we note that the c_i from the previous proposition only depend on the ambient Spin(7)-structure and not on the submanifold. □

Proposition 2.7 (Ellipticity)**.** *There is a universal constant $\alpha_{\mathrm{ell}} > \alpha_1$, where α_1 is as in Proposition 2.2, such that if N is an α_{ell}-Cayley submanifold, then its associated linearised deformation operator $\not{D}$ is elliptic.*

Proof. From the previous proposition we see that the symbol at $p \in N$ is given in an adapted frame as follows, where $\xi \in T_p^*N$ and $\xi_i = \xi(f_i)$:

$$\sigma_\xi(\not{D}) = \pi_E\left(\beta\sum_{i=1}^{4} f_i \times \xi_i v + \sum_{i=1}^{4}\sum_{j=1}^{8} \beta_{ij} f_j \times \xi_i v\right)$$
$$= \pi_E\left(\left(\beta\xi^\sharp + \sum_{i=1}^{4}\sum_{j=1}^{8} \beta_{ij} f_j \times \xi_i\right) \times v\right). \tag{2.10}$$

Now if the f_i span a Cayley plane, then this is exactly the symbol of the Dirac operator associated with a Cayley from Proposition 1.28, and thus invertible. As we perturb the plane continuously, we see that both the product $\pi_E(\square \times \square)$ and the expression inside the bracket vary continuously. Thus invertibility of the composed expression is an open condition on the set of four-planes, which holds at Cayley planes. Since α-Cayley planes for $\alpha \in [\alpha_1, 1)$ form a neighbourhood basis of the Cayley planes in $\mathrm{Gr}_+(4, \mathbb{R}^8)$ we see that there is a universal $\alpha_{\mathrm{ell}} > \alpha_1$ such that whenever the f_i span an α_{ell}-Cayley, the symbol is invertible, and thus $\not{D}$ is elliptic. □

2.3 Compact case

We now study the deformation theory of a compact Cayley submanifold in a Spin(7)-manifold (M, Φ_{s_0}), where we also allow the Spin(7)-structure to vary in a finite-dimensional smooth family $\{\Phi_s\}_{s\in\mathcal{S}}$, with $s_0 \in \mathcal{S}$. As we are only interested in the local deformation theory, we can and will assume that M is compact by restricting to a closed tubular neighbourhood of N. The analysis will be done for almost Cayleys, which will be useful later when we desingularise Cayleys with conical singularities.

Let $N \subset (M, \Phi_{s_0})$ be α-Cayley with α strictly bigger than α_{ell} from Proposition 2.7, so that the linearised deformation operator $\not{D}_N$ is well-defined and elliptic. It will then also remain elliptic for small C^1-perturbations of N and smooth perturbations of Φ. For $\epsilon > 0$ denote by $\nu_\epsilon(N)$ the ϵ-neighbourhood of the zero section in the normal bundle $\nu(N) = TM|_N/TN$, as measured by $g_{\Phi_{s_0}}$. If $\epsilon > 0$ is sufficiently small then its image under the exponential map corresponding to Φ_s will be a tubular neighbourhood of N, for any $s \in \mathcal{S}$, after potentially restricting $\mathcal{S}$ to a neighbourhood of s_0.

Let $C^\infty(\nu_\epsilon(N)) \subset C^\infty(\nu(N))$ denote the subset of sections which take value in $\nu_\epsilon(N)$. Consider now the perturbation $N_v = \exp_v(N)$ of this compact almost Cayley. Its failure to be Cayley is measured by the deformation operator (2.6), and is a section of E_{cay}. We now mildly extend the results from the previous section for the compact case:

Proposition 2.8. *Let N be a compact α-Cayley submanifold of (M, Φ_0) with $\alpha < 1$ sufficiently close to 1. Let $\{\Phi_s\}_{s\in\mathcal{S}}$ be a smooth finite dimensional family of* Spin(7)*-structures such that $s_0 \in \mathcal{S}$. Consider the map:*

$$\begin{aligned} F : C^\infty(\nu_\epsilon(N)) \times \mathcal{S} &\longrightarrow C^\infty(E_{\mathrm{cay}}) \\ (v, s) &\longmapsto \pi_E(\star_N \mathrm{Exp}_v^*(\tau_s|_{N_v})). \end{aligned}$$

Here Exp, $\star$ *and π_E are induced from Φ_{s_0}. After shrinking $\mathcal{S}$, there is a constant $C > 0$ which depends on M, Φ_{s_0} and on the injectivity radius of N, such that if furthermore*

$\|v\|_{C^1} < C$, *then* N_v *is Cayley for* Φ_s *exactly when* $F(v, s) = 0$. *Moreover,* $F(\cdot, s)$ *is elliptic at the zero section for every* $s \in \mathcal{S}$.

Proof. The only non-trivial issue is that for $s \neq s_0$, we used the exponential map, Hodge star and π_E associated to Φ_{s_0}, not Φ_s. However by shrinking $\mathcal{S}$ we can ensure that both ellipticity and the Cayley detecting property are preserved, by the compactness of N, and the openness of these conditions in the space of all smooth Spin(7)-structures. □

This is a generalisation of previous work by Kim Moore [38, Prop. 2.3], where a more direct proof can be found for the case where N is Cayley and the Spin(7)-structure is fixed. This in turn is based on earlier work by McLean in his foundational paper [34] on the deformation theory of calibrated submanifolds. We will now consider F and $\not{D}$ as maps between Banach manifolds. For this, we need some auxiliary results.

Lemma 2.9. *Let* $N, M, \Phi_{s_0}, \{\Phi_s\}_{s\in\mathcal{S}}, F$ *be as in Proposition 2.8. The map* F *has the following form, for* $v \in C^\infty(\nu_\epsilon(N))$ *and* $p \in N$:

$$\begin{aligned} F(v, s)(p) &= \mathbf{F}(p, v(p), \nabla v(p), T_pN, s) \\ &= F(0, s)(p) + (\not{D}_s v)(p) + \mathbf{Q}(p, v(p), \nabla v(p), T_pN, s). \end{aligned}$$

Here $\not{D}_s$ *is the linearisation of* $F(\cdot, s)$ *at* 0 *and* $\mathbf{F}, \mathbf{Q}$ *are smooth fibre-preserving maps:*

$$\mathbf{F}, \mathbf{Q} : TM_\epsilon \times_N (T^*M \otimes TM)_\epsilon \times \mathrm{Cay}_\alpha(M) \times \mathcal{S} \longrightarrow \mathbf{E}_{\mathrm{cay}},$$

where $\mathbf{E}_{\mathrm{cay}} = \{(p, \pi, e) : (p, \pi) \in \mathrm{Cay}_\alpha(M), e \in E_\pi\}$ *and* α *is sufficiently large. Here we see both sides as fibre bundles over* $\mathrm{Cay}_\alpha(M) \times \mathcal{S}$. *We define the map* $Q : C^\infty(\nu_\epsilon(N)) \times \mathcal{S} \to C^\infty(E_{\mathrm{cay}})$ *as* $Q(v, s) = F(v, s) - \not{D}_s v$.

Note that Q is a map between function spaces, while $\mathbf{Q}$ is a smooth map between manifolds of finite dimension, and the same naming convention applies to F and $\mathbf{F}$. We also write $F_s(\cdot) = F(\cdot, s)$ to emphasise that F_s is a differential operator depending on the parameter $s \in \mathcal{S}$, and similar for Q and the smooth functions $\mathbf{F}$ and $\mathbf{Q}$.

Proof. The value of $\exp_v(p)$ is determined by $p \in N$ and $v(p) \in \nu_p(N)$ as a geodesic is uniquely determined by its starting point and initial velocity vector. Similarly, $\mathrm{D}\exp_v(p)$ is a smooth function of $p, v(p)$ and $\nabla v(p) \in T_p^*N \otimes \nu_p(N)$ since the first order deviation of two geodesics is determined by the first order deviation at a previous time. Finally, $\mathrm{Exp}_v^*(p)$ can be entirely determined from $p, v(p), \nabla v(p)$ and the tangent space T_pN. Thus F itself is of the form $F(v, s)(p) = \mathbf{F}(p, v(p), \nabla v(p), T_pN, s)$, as it is the pullback of a differential form (which depends on s) by $\exp_v$. Here $\mathbf{F}$ is a smooth map which is independent of N. The smoothness of $\mathbf{F}$ follows from the smoothness of $\exp_v$ in v. In particular, the same argument applies to the map $Q(v, s) = F(v, s) - \not{D}_s v$, since $(\not{D}_s v)(p)$ is a smooth map in $p, v(p), \nabla v(p)$ and T_pN only, as it is a first-order operator. □

The name $\mathbf{Q}$ is meant to suggest that the term $\mathbf{Q}_{s,\pi}(p, x, y)$ contains all the quadratic and higher terms in the variables x and y. Indeed, we clearly have $\mathbf{Q}_{s,\pi}(p, 0, 0) = 0$, so no constant term. Let us denote by ∂_x and ∂_y respectively the partial derivatives with respect to x and y. Note that this does not require choosing a connection, as $\mathbf{Q}$ is a fibre-preserving map between subsets of metric vector bundles, and x and y are exactly the fibre coordinates. Let $v \in C^\infty(\nu_\epsilon(N))$ be such that $v(p) = x_0$ and $\nabla v(p) = 0$. Then:

$$\begin{aligned}
\partial_x \mathbf{Q}_{s,\pi}(p,0,0)[x_0] &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{Q}_{s,T_pN}(p, tx_0, 0) \\
&= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_s(tv) - F_s(0) - t\not{D}_s v \right)(p) \\
&= (\not{D}_s v - \not{D}_s v)(p) = 0.
\end{aligned}$$

An analogous derivation for the variable y shows that $\mathbf{Q}_{s,\pi}$ satisfies the following for $p \in M$:

$$\mathbf{Q}_{s,\pi}(p,0,0) = 0,\ \partial_x \mathbf{Q}_{s,\pi}(p,0,0) = 0 \text{ and } \partial_y \mathbf{Q}_{s,\pi}(p,0,0) = 0. \tag{2.11}$$

From this, we will now obtain bounds on $Q_s(v) - Q_s(w)$ which are formally similar to the bounds obtained for homogeneous quadratic polynomials on $\mathbb{R}^n$. If q is such a polynomial, one can show that for a constant $C > 0$ there is an inequality of the form:

$$|q(x) - q(y)| \leqslant C|x - y|(|x| + |y|).$$

In the following, we use the notation $|v|_{C^k} = \sum_{i=0}^k |\nabla^i v|$ for a pointwise norm of the derivative, and we also think of TN as a section of $\Lambda^4 T^*M|_N \to N$, so that $|TN|_{C^k}$ is well-defined. The analogous result for Q_s is then the following:

Lemma 2.10. *There is an $\epsilon > 0$ which only depends on Φ_s for $s \in \mathcal{S}$ such that for $k \geqslant 0$ and $v, w \in C^k(\nu_\epsilon(N))$ with $\|v\|_{C^1}, \|w\|_{C^1} < \epsilon$ we have the following inequality:*

$$|Q_s(v) - Q_s(w)|_{C^{k+1}} \leqslant C \sum_{\substack{i+|J|+r \leqslant k+2 \\ 0 \leqslant r \leqslant k}} |\nabla^i (v - w)|(|\nabla^J v| + |\nabla^J w|)|\nabla^r TN|, \tag{2.12}$$

where the summation is over a multi-index I, and $\nabla^I = \nabla^{I_1} \otimes \ldots \otimes \nabla^{I_r}$. If we assume that $|v|_{C^{k+1}}, |w|_{C^{k+1}}$ are sufficiently small, then this can be simplified to yield:

$$\begin{aligned}
|Q_s(v) - Q_s(w)|_{C^{k+1}} \leqslant & C(1 + |TN|_{C^{k+1}}) \Big(|v - w|_{C^{k+1}}(|v|_{C^k} + |w|_{C^k}) \\
& + |v - w|_{C^k}(|v|_{C^{k+1}} + |w|_{C^{k+1}}) \Big).
\end{aligned} \tag{2.13}$$

Proof. Let $x \in T_pM$ and $y \in T_p^*M \otimes T_pM$ be of sufficiently small norm. Then Taylor's theorem gives us uniformly in $s \in \mathcal{S}$ and $\pi \in \mathrm{Cay}_p(M)_\alpha$:

$$\begin{aligned}
\mathbf{Q}_{s,\pi}(p,x,y) = & \mathbf{Q}_{s,\pi}(p,0,0) + \partial_x \mathbf{Q}_{s,\pi}(p,0,0)x + \partial_y \mathbf{Q}_{s,\pi}(p,0,0)y \\
& + \mathbf{R}_{1,s,\pi}(p,x,y)x \otimes x + \mathbf{R}_{2,s,\pi}(p,x,y)x \otimes y \\
& + \mathbf{R}_{3,s,\pi}(p,x,y)y \otimes y,
\end{aligned} \tag{2.14}$$

where the $\mathbf{R}_{i,s,\pi}$ are smooth, non-linear remainder terms which describe the higher order behaviour of $\mathbf{Q}_{s,\pi}$. If we now consider small $x_1, x_2 \in T_pM$ and $y_1, y_2 \in T_p^*M \otimes T_pM$ and use the properties (2.11) of $\mathbf{Q}$, we see that:

$$\begin{aligned}\mathbf{Q}(p, x_2, y_2) - \mathbf{Q}(p, x_1, y_1) = \mathbf{R}_1 x_2 \otimes x_2 + \mathbf{R}_2 x_2 \otimes y_2 + \mathbf{R}_3 y_2 \otimes y_2 \\ - \mathbf{R}_1 x_1 \otimes x_1 - \mathbf{R}_2 x_1 \otimes y_1 - \mathbf{R}_3 y_1 \otimes y_1,\end{aligned} \tag{2.15}$$

Consider for the moment only the difference

$$\mathbf{R}_1(p, x_2, y_2, s) x_2 \otimes x_2 - \mathbf{R}_1(p, x_1, y_1, s) x_1 \otimes x_1.$$

We can rearrange as follows:

$$\begin{aligned}&\mathbf{R}_1(p, x_2, y_2, s) x_2 \otimes x_2 - \mathbf{R}_1(p, x_1, y_1, s) x_1 \otimes x_1 \\ &= \mathbf{R}_1(p, x_2, y_2, s)(x_2 \otimes x_2 - x_1 \otimes x_1) + (\mathbf{R}_1(p, x_2, y_2, s) - \mathbf{R}_1(p, x_1, y_1, s)) x_1 \otimes x_1 \\ &= \mathbf{R}_1(p, x_2, y_2, s)(x_2 \otimes (x_2 - x_1) + (x_2 - x_1) \otimes x_1) + (\mathbf{R}_1(p, x_2, y_2, s) - \mathbf{R}_1(p, x_1, y_2, s) \\ &\quad + \mathbf{R}_1(p, x_1, y_2, s) - \mathbf{R}_1(p, x_1, y_1, s)) x_1 \otimes x_1.\end{aligned}$$

Using the mean value inequality we can find $\tilde{x} = t_x x_1 + (1 - t_x) x_2, \tilde{y} = t_y y_1 + (1 - t_y) y_2$ with $t_x, t_y \in [0, 1]$ such that:

$$\begin{aligned}|\mathbf{R}_1(p, x_2, y_2, s) - \mathbf{R}_1(p, x_1, y_2, s)| \leqslant |\partial_x \mathbf{R}_1(p, \tilde{x}, y_2, s)||x_1 - x_2|. \\ |\mathbf{R}_1(p, x_1, y_2, s) - \mathbf{R}_1(p, x_1, y_1, s)| \leqslant |\partial_y \mathbf{R}_1(p, x_1, \tilde{y}, s)||y_1 - y_2|.\end{aligned}$$

Now since M is compact, after shrinking $\mathcal{S}$, the subset $\mathbb{V}_c$ of $(p, x, y, \pi s) \in TM \times (T^*M \otimes TM) \times \mathrm{Cay}_\alpha(M) \times \mathcal{S}$ such that $|x|, |y|, |\tau(\pi)| \leqslant c$ for a fixed constant $c \in \mathbb{R}$ is also compact. As $\mathbf{Q}$ and the $\mathbf{R}_i$ are smooth, we can thus bound the norm of any derivative of a fixed degree over such a subset. This gives us the following point-wise estimate, provided that x_1, x_2, y_1 and y_2 all have sufficiently small norms.

$$\begin{aligned}&|\mathbf{R}_1(p, x_2, y_2, s) x_2 \otimes x_2 - \mathbf{R}_1(p, x_1, y_1, s) x_1 \otimes x_1| \\ &\quad \leqslant C(|x_1| + |x_2|)|x_2 - x_1| + (|x_2 - x_1| + |y_2 - y_1|)|x_1|^2 \\ &\quad \leqslant C(|x_2 - x_1| + |y_2 - y_1|)(|x_1| + |x_2| + |y_1| + |y_2|).\end{aligned}$$

Here the constant C is independent of $p \in M$ and $\pi \in \mathrm{Cay}_\alpha(M)$. We can bound the rest of Equation (2.15) by the same expression, using similar arguments, i.e. there is a pointwise estimate:

$$|\mathbf{Q}_{s,\pi}(p, x_2, y_2) - \mathbf{Q}_{s,\pi}(p, x_1, y_1)| \leqslant C(|x_2 - x_1| + |y_2 - y_1|)(|x_1| + |x_2| + |y_1| + |y_2|). \tag{2.16}$$

We will now adapt the above reasoning to obtain bounds on the covariant derivatives $|\nabla^k(Q_s(v) - Q_s(w))|$. For this consider $v \in C^\infty(\nu_\epsilon(N))$ and note that for a curve $\gamma : \mathbb{R} \to M$

with $\gamma(0) = p \in N$ and $\gamma'(0) = \xi \in T_pN$:

$$\begin{aligned}\nabla_\xi(Q_s(v))(p) &= \Pi_{TE\to E}\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \mathbf{Q}_s(\gamma(t), v(\gamma(t)), \nabla v(\gamma(t)), T_{\gamma(t)}N) \\ &= \partial_p\mathbf{Q}_s[w] + \partial_x\mathbf{Q}_s[\nabla_\xi v(p)] + \partial_y\mathbf{Q}_s[\nabla_\xi \nabla v(p)] + \partial_\pi\mathbf{Q}_s[\nabla_\xi TN]. \qquad (2.17)\end{aligned}$$

Here $\Pi_{TE\to E}$ is the connection map, which maps $T_{(p,T_{P_N},s)}\mathbb{E} \to \mathbb{E}_{(p,T_{P_N},s)}$ as induced from the Levi-Civita connection on N, which only depends on T_pN and not on the curvature of N. The derivative ∂_p is given as:

$$\partial_p\mathbf{Q}[w] = \mathrm{D}\mathbf{Q}(p, v, \nabla v, T_pN, s)[w, w^{h,TM}, w^{h,T^*M\otimes T^M}, w^{h,TM}, 0].$$

Here $w^{h,E}$ means the horizontal lift of the vector w to the corresponding bundle E. Finally, we consider TN as a section of Λ^4T^*M, so that ∇TN is well-defined. The conclusion is that the dependence of $\nabla(Q_s(v))(p)$ on $\nabla^2 v(p)$ and $\nabla TN(p)$ is affine, and the coefficients can be bounded on subsets of the form $\mathbb{V}_c$. The same argument also applies to the $\mathbf{R}_i$, and using Equation 2.15 we can show that:

$$\nabla^k Q_s(v) = \sum_{\substack{|I|+j\leqslant k+2\\ 0\leqslant j\leqslant k}} \mathbf{R}^{I,j}(p, v, \nabla v, TN, s)\nabla^I v \otimes \nabla^j TN, \qquad (2.18)$$

where the $\mathbf{R}^{I,j}$ are smooth maps on $\mathbb{V}_c$, for sufficiently small c, and for $I = (i_1, \ldots, i_r)$ a multi-index we set $\nabla^I v = \nabla^{i_1} v \otimes \ldots \otimes \nabla^{i_l} v$. Notice that there are no products of the form $\nabla^{k+1}v \otimes \nabla^{k+1}v$ appearing. From this, we can deduce the claimed bounds, since the $\mathbf{R}^{I,j}$ are defined on compact sets. □

Corollary 2.11. *The map $Q_s : C^\infty(\nu_\epsilon(N)) \to C^\infty(E_{\mathrm{cay}})$ is a continuous map of Fréchet manifolds. Similarly, the maps $Q_s : C^{k+1}(\nu_\epsilon(N)) \to C^k(E_{\mathrm{cay}})$ and $Q_s : C^{k+1,\alpha}(\nu_\epsilon(N)) \to C^{k,\alpha}(E_{\mathrm{cay}})$ are continuous maps of Banach manifolds in the same way.*

Proof. If $v \to w \in C^{k+1}(\nu_\epsilon(N))$, then by Lemma 2.10 and by compactness of N:

$$\|Q_s(v) - Q_s(w)\|_{C^k} \leqslant \tilde{C}\|v - w\|_{C^{k+1}}(\|v\|_{C^{k+1}} + \|w\|_{C^{k+1}}) \longrightarrow 0. \qquad (2.19)$$

The proof for the Hölder case is identical, and the statement about C^∞ is obtained by combining the statements for C^k for all finite k. □

Lemma 2.12. *Let $p > 4$ and $k \geqslant 1$. Then there is an $\epsilon > 0$ and $C > 0$ which depend on Φ_s for $s \in \mathcal{S}$ and N such that for $v, w \in L^p_{k+1}(\nu_\epsilon(N))$ with $\|v\|_{L^p_{k+1}}, \|w\|_{L^p_{k+1}} < \epsilon$ we have the following inequality:*

$$\|Q_s(v) - Q_s(w)\|_{L^p_k} \leqslant C\|v - w\|_{L^p_{k+1}}(\|v\|_{L^p_{k+1}} + \|w\|_{L^p_{k+1}}). \qquad (2.20)$$

Proof. As $k + 1 \geqslant 2$ and $p > 4$, we have that $L^p_{k+1} \hookrightarrow C^k$ continuously by the Sobolev embedding theorem. Thus by making $\epsilon > 0$ small, we can make sure that the C^k norms of v and w are arbitrarily small, say less than δ. We then prove the L^p_k estimate on Q_s

using the pointwise estimate (2.13) from Proposition 2.10 as follows:

$$\begin{aligned}
&\int_N |\nabla^k Q_s(v) - \nabla^k Q_s(w)|^p \,\mathrm{dvol} \\
&\leqslant C \int_N |v-w|^p_{C^{k+1}} (|v|_{C^k} + |w|_{C^k})^{kp} + |v-w|^p_{C^k} (|v|_{C^k} + |w|_{C^k})^{(k-1)p} (|v|_{C^{k+1}} + |w|_{C^{k+1}})^p \,\mathrm{dvol} \\
&\leqslant C\delta^{(k-1)p} (|v|_{C^k} + |w|_{C^k})^p \int_N |v-w|^p_{C^{k+1}} \,\mathrm{dvol} \; + C\delta^{(k-1)p} |v-w|^p_{C^k} \int_N (|v|^p_{C^{k+1}} + |w|^p_{C^{k+1}}) \,\mathrm{dvol} \\
&\leqslant C \|v-w\|^p_{L^p_{k+1}} (\|v\|^p_{L^p_{k+1}} + \|w\|^p_{L^p_{k+1}}).
\end{aligned}$$

Here, we used Minkowski's inequality in the second inequality and the Sobolev embedding of $L^p_{k+1} \hookrightarrow C^k$ in the third. This is also where the dependence of the constant C on N appears. Note that the key fact used in deducing this L^p_k bound was that there were no terms of the form $\nabla^{k+1} v \otimes \nabla^{k+1} v$ in our expression for $\nabla^k Q_s$. In fact, the mapping $v \to \nabla^{k+1} v \otimes \nabla^{k+1} v$ is not bounded from L^p_{k+1} to L^p, thus the presence of such a term would make it impossible to deduce a bound of the above form on Sobolev norms. ☐

One can capture the dependence on the parameter $s \in \mathcal{S}$ similarly.

Lemma 2.13. *For any $s_0 \in \mathcal{S}$ and sufficiently small $\epsilon > 0$, there is an open neighbourhood $\mathcal{U} \subset \mathcal{S}$ of s_0 and a constant $C(\mathcal{S}) > 0$, such that for all $s \in \mathcal{U}$ and $v \in C^\infty(\nu_\epsilon(N))$ with $\|v\|_{C^k} < \epsilon$ we have:*

$$|F_s(v) - F_{s_0}(v)|_{C^k} \leqslant C d(s, s_0). \tag{2.21}$$

Proof. Using Taylor's theorem we get that:

$$|F_s(v) - F_{s_0}(v)|(p) \leqslant 2 |\partial_s \mathbf{F}(p, v(p), \nabla v(p), T_p N, s_0)| d(s, s_0).$$

Thus the case $k = 0$ follows from the same argument as we had before for the v-dependence. Higher derivatives follow analogously to what we had before as well. ☐

Proposition 2.14. *Let $p > 4$ and $k \geqslant 1$. For sufficiently small ϵ, and after potentially shrinking $\mathcal{S}$ the map F from Proposition 2.8 extends to a C^∞ map between Banach manifolds:*

$$F : \mathcal{L}_\epsilon = \{v \in L^p_{k+1}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1}} < \epsilon\} \times \mathcal{S} \longrightarrow L^p_k(E_{\mathrm{cay}})$$

Its linearisation at $(0, \Phi_{s_0})$ is Fredholm.

Proof. Notice that F is a continuous or C^k map between Banach spaces exactly when Q is. This is because the constant term $F_s(0)$ is smooth in s, as is the linear term $\not{D}_s$. Both those terms are smooth in v, as they are constant and linear respectively. Continuity of Q between Sobolev spaces follows from Proposition 2.12 and 2.13 in the same way that continuity between C^k-spaces was proven in Corollary 2.11.

It remains to show differentiability. We see from an application of Taylor's theorem that for $v, w \in C^\infty(\nu(N))$, $s : \mathbb{R} \to \mathcal{S}$ a smooth curve and $t \in \mathbb{R}$ sufficiently small:

$$Q(v + tw, s(t)) - Q(v, s(0)) = \partial_x \mathbf{Q}[tw] + \partial_y \mathbf{Q}[t \nabla w] + \partial_s \mathbf{Q}[t \dot{s}(0)] + O(t^2). \tag{2.22}$$

Let now $v \in \mathcal{L}_\epsilon$ and $s \in \mathcal{S}$. Thus by the Sobolev embedding theorem we have that $v \in C^k(\nu_\epsilon(N))$ has bounded C^k-norm. Define the operator:

$$L_{v,s}(w,\xi) = \partial_x \mathbf{Q}[w] + \partial_y \mathbf{Q}[\nabla w] + \partial_s \mathbf{Q}[\xi].$$

The operator $L_{v,s}$ is first order with continuous coefficients, and as such is a bounded operator $L^p_{k+1} \times T_s\mathcal{S} \to L^p_k$. From (2.22) it is clear that $L_{v,s}$ is the Fréchet derivative of Q at the point (v,s). It remains to show that varying (v,s) continuously in $L^p_{k+1} \times \mathcal{S}$ entails a continuous variation of $L_{v,s}$ in the space of bounded operators $B(L^p_{k+1} \times T_s\mathcal{S}, L^p_k)$. By computations analogous to the ones from Proposition 2.10 on may obtain a bound of the form:

$$\|(L_{v,s} - L_{v+tu,s(t)})[w,\xi]\|_{L^p_k} \leqslant Ct(\|u\|_{L^p_{k+1}}\|w\|_{L^p_2} + |\dot{s}(0)||\xi|),$$

were t is assumed sufficiently small, $v \in \mathcal{L}_\epsilon$, s a smooth curve in $\mathcal{C}$, $u, w \in L^p_{k+1}$ and $\xi \in T_{s(0)}\mathcal{S}$ (we identify the tangent spaces $T_{s(t)}\mathcal{S}$ via a fixed trivialisation). Here we crucially use the fact that $v \in \mathcal{L}_\epsilon$ have uniformly bounded C^k-norm. From this we see that:

$$\|(L_{v,s} - L_{v+tu,s(t)})\|_{\mathrm{op}} \leqslant Ct(\|u\|_{L^p_{k+1}} + |\dot{s}(0)|), \tag{2.23}$$

which shows that the derivative $L_{v,s}$ varies continuously as (v,s) varies continuously. Higher differentiability follows analogously. Finally $L_{0,s_0} = 0 \oplus T$ by (2.11), where $T : T_{s_0}\mathcal{S} \to L^p_k(E_{\mathrm{cay}})$ so the derivative of F at $(0,s_0)$ is the sum of the elliptic operator $\not{D}_{s_0}$ on a compact manifold and a bounded linear map, and as such Fredholm. □

Solutions to the equation $F_s(v) = 0$ which are in L^p_2 will be automatically smooth by elliptic regularity.

Proposition 2.15. *Any $v \in \mathcal{L}_\epsilon$ such that $F_s(v) = 0$ for some $s \in \mathcal{S}$ is smooth.*

Proof. The operator $\not{D}_s$ is an elliptic operator with smooth coefficients. It thus admits a formal adjoint $\not{D}^*_s$. Since $F_s(v) = 0$, we of course also have that $\not{D}^*_s F(v) = 0$. From the Taylor expansion $F_s(v) = F_s(0) + \not{D}_s v + Q_s(v)$ and our expansion of $\nabla Q_s(v)$ from equation (2.18) we obtain that:

$$\begin{aligned}\not{D}^*_s F(v) &= \not{D}^*_s \not{D}_s v + \tilde{S}_s(v, \nabla v) + \tilde{R}_s(v, \nabla v)\nabla^2 v \\ &= R_s(v, \nabla v)\nabla^2 v + S_s(v, \nabla v).\end{aligned}$$

Here $S_s, \tilde{S}_s, R_s, \tilde{R}_s$ are smooth in their arguments. For fixed $v \in \mathcal{L}_\epsilon$ define the linear differential operator $K_{v,s}$ as follows:

$$\begin{aligned}K_{v,s} : L^p_{k+1}(\nu(N)) &\longrightarrow L^p_{k-1}(E_{\mathrm{cay}}) \\ w &\longmapsto R_s(v, \nabla v)\nabla^2 w.\end{aligned}$$

As $v \in L^p_{k+1}(\nu(N)) \subset C^{k,\alpha}(\nu(N))$ and R_s is smooth in its arguments, this linear differential operator has coefficients in $C^{k-1,\alpha}$. It is elliptic, as $\not{D}^*_s F$ is elliptic at 0. Thus v is a solution to the following equation:

$$K_{v,s}(v) = S_s(v),$$

which is a second order elliptic equation with $C^{k-1,\alpha}$ coefficients. However $S_s(v, \nabla v)$ is actually in $C^{k,\alpha}$. Thus we can apply Schauder regularity results such as Theorem 1.4.2 in [19], which allows us to improve the regularity of v to $C^{k+2,\alpha}$. Consequently, the coefficients of $K_{v,s}$ will have regularity $C^{k+1,\alpha}$, as will the section $S_s(v)$. It follows by bootstrapping that $v \in C^\infty(\nu(N))$. □

We can now specialise to Cayley submanifolds, and describe their family moduli spaces locally. To be precise, we consider the following moduli space for $N \subset (M, \Phi_{s_0})$ an immersed submanifold and $\{\Phi_s\}_{s\in\mathcal{S}}$ a smooth family of Spin(7)-structures.

$$\mathcal{M}(N, \mathcal{S}) = \{(\tilde{N}, \Phi_s) : \tilde{N} \text{ is an immersed Cayley submanifold of } (M, \Phi_s) \text{ with } \tilde{N} \text{ isotopic to } N\}. \tag{2.24}$$

We endow $\mathcal{M}(N, \mathcal{S})$ with the C^∞ topology. Note that if N is Cayley, then the bundles E_{cay} and E agree, as $TN = \text{cay}_N$. Now we can apply the theory of Kuranishi models to the Fredholm map F to arrive at the following structure theorem for its zero set.

Theorem 2.16 (Structure)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let N be an immersed compact Cayley submanifold of a* Spin(7)*-manifold (M, Φ_{s_0}), where $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of not necessarily torsion-free* Spin(7)*-structures parametrised by the smooth manifold $\mathcal{S}$, and $s_0 \in \mathcal{S}$. Then there is an open neighbourhood $s_0 \in \mathcal{U} \subset \mathcal{S}$ and a non-linear deformation operator F which for $\epsilon > 0$ sufficiently small is a C^∞ map:*

$$F : \mathcal{L}_\epsilon = \{v \in L^p_{k+1}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_k(E).$$

*A neighbourhood of (N, Φ_{s_0}) in $\mathcal{M}(N, \mathcal{S})$ is homeomorphic to the zero locus of F near $(0, \Phi_{s_0})$. Furthermore we can define the **deformation space** $\mathcal{I}(N, \mathcal{S}) \subset C^\infty(\nu(N)) \oplus T_{s_0}\mathcal{S}$ to be the kernel of $\not{D}_{\mathcal{S}} = \mathrm{D}F(0, s_0)$, and the **obstruction space** $\mathcal{O}(N, \mathcal{S}) \subset C^\infty(E_{\text{cay}})$ to be the cokernel of $\not{D}_{\mathcal{S}}$. Then a neighbourhood of (N, s_0) in $\mathcal{M}(N, \mathcal{S})$ is also homeomorphic to the zero locus of a Kuranishi map:*

$$\kappa : \mathcal{I}(N, \mathcal{S}) \longrightarrow \mathcal{O}(N, \mathcal{S}).$$

*In particular if $\mathcal{O}(N, \mathcal{S}) = \{0\}$ is trivial, $\mathcal{M}(N, \mathcal{S})$ admits the structure of a C^1-manifold near (N, s_0). We say that N is **unobstructed** in this case.*

We remark that $\not{D}_{\mathcal{S}} = \not{D}_N \oplus T$, where $\not{D}_N$ is the deformation operator for N in the fixed Spin(7)-structure Φ_{s_0} and $T : T_s\mathcal{S} \to L^p_k(E)$ is linear map.

To conclude, we refer to the DPhil thesis of Robert Clancy [7, Theorem 6.3.1] for a proof of the following formula for the index of $\not{D}_{\mathcal{S}}$. We add $\dim \mathcal{S}$ since our deformation problem also allows for deformations of the Spin(7)-structure.

Theorem 2.17 (Index)**.** *Suppose we are in the situation of Theorem 2.16. The index of* $\not{D}_{\mathcal{S}} = \mathrm{D}F(0, s_0)$ *for the family* $\mathcal{S}$ *is given by:*

$$\operatorname{ind} \not{D}_{\mathcal{S}} = \frac{1}{2}(\sigma(N) + \chi(N)) - [N] \cdot [N] + \dim \mathcal{S}. \tag{2.25}$$

Here $\sigma(N)$ *denotes the signature of* N *as a compact oriented four-manifold,* $\chi(N)$ *the Euler characteristic, and* $[N] \cdot [N]$ *the self-intersection number in* M*. This is the expected dimension of the moduli space* $\mathcal{M}(N, \mathcal{S})$*. In particular, if* N *is unobstructed* $\mathcal{M}(N, \mathcal{S})$ *will be a smooth manifold of this dimension.*

Example 2.18. Suppose that $N \subset M$ is a complex submanifold in a CY4 manifold $(M, \omega, g, J, \Omega)$. Then N will also be Cayley in the Spin(7)-manifold $(M, \operatorname{Re}\Omega + \frac{1}{2}\omega \wedge \omega)$. If N is compact, then Hodge theory on the Kähler manifold N allows us to conclude that $\ker \bar\partial + \bar\partial^* = \ker \bar\partial \oplus \bar\partial^*$, and so according to Example 1.10, infinitesimal Cayley deformations and infinitesimal complex deformations agree. Thus, if we furthermore assume that every infinitesimal complex deformation N integrates, then any Cayley deformation integrates as well (as complex surfaces are examples of Cayley submanifolds).

Example 2.19. If $L \subset M$ is a compact special Lagrangian submanifold in a CY4 manifold $(M, \omega, g, J, \Omega)$ then it is Cayley in the Spin(7)-manifold $(M, \operatorname{Re}\Omega + \frac{1}{2}\omega \wedge \omega)$. For Lagrangian submanifolds, the normal bundle is intrinsic, as $\nu(N) \simeq TN$. Thus the formula for the index (2.25) yields:

$$\begin{aligned} \operatorname{ind} \not{D}_N = \tfrac{1}{2}(\sigma(N) + \chi(N)) - [N] \cdot [N] &= \tfrac{1}{2}(\sigma(N) - \chi(N)) \\ &= b_1(N) - b_2^-(N) - 1. \end{aligned}$$

Compare this to the special Lagrangian deformation theory as described in [34], where it is shown that the moduli space of special Lagrangians isotopic to N has dimension $b_1(N)$. Thus the obstruction space for compact Cayleys coming from special Lagrangians never vanishes, as the obstruction space necessarily has dimension:

$$\dim \mathcal{O}(N) \geqslant b_2^-(N) + 1. \tag{2.26}$$

Looking at the explicit form for the Cayley operator in Proposition 1.14, we see that:

$$\begin{aligned} \operatorname{Ker} \not{D}_N &\simeq \mathcal{H}^1, \\ \operatorname{Coker} \not{D}_N &\simeq \mathcal{H}^0 \oplus \mathcal{H}^{2,-}. \end{aligned}$$

Here $\mathcal{H}^k$ is the space of harmonic k-forms on N, and $\mathcal{H}^{2,-}$ is the space of harmonic anti-self-dual forms. On a compact manifold, if $\mathrm{d}^-\sigma = 0$, then:

$$\begin{aligned} 0 = \int_{\partial N} \sigma \wedge \mathrm{d}\sigma = \int_N \mathrm{d}\sigma \wedge \mathrm{d}\sigma &= \int_N \mathrm{d}^+\sigma \wedge \mathrm{d}^+\sigma \\ &= \int_N \mathrm{d}^+\sigma \wedge \star \mathrm{d}^+\sigma = \|\mathrm{d}^+\sigma\|_{L^2}. \end{aligned}$$

Now we obtain the first isomorphism by Hodge theory on N. As for the second isomorphism, note that the adjoint of $-\mathrm{d}\star\oplus\mathrm{d}^-$ is exactly $-\mathrm{d}\star+\mathrm{d}^*:\Omega^n\oplus\Omega^{2,-}\to\Omega^1$. Hodge theory again leads to the desired result.

2.4 Asymptotically conical case

The deformation theory of noncompact Cayleys with conical ends is analogous to the compact case, however, to stay in the Fredholm setting we need to consider the deformation map acting between weighted function spaces, as exemplified in Proposition 1.28.

Let $(\mathbb{R}^8, \Phi_{s_0})$ be an AC_η Spin(7)-manifold asymptotic to $(\mathbb{R}^8, \Phi_0)$, and suppose that $A \subset \mathbb{R}^8$ is a an α-Cayley submanifold that is AC_λ for some $\eta < \lambda < 1$. This is the level of generality we require for the desingularisation results in the next chapter. Working with Spin(7)-structures that are AC_η to the flat Φ_0 instead of just working with Φ_0, while not strictly necessary, simplifies the proof of the gluing theorem 3.15.

If $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of AC_η perturbations of Φ_{s_0}, then we would like to examine the moduli space:

$$\mathcal{M}^\lambda_{\mathrm{AC}}(A,\mathcal{S}) = \{(\tilde{A},\Phi_s) : \tilde{A} \text{ is an } \mathrm{AC}_\lambda \text{ Cayley submanifold of } (\mathbb{R}^8,\Phi_s) \\ \text{isotopic to } A \text{ and asymptotic to the same cone}\}. \tag{2.27}$$

For α sufficiently close to 1, A admits a canonical deformation map, just as in the compact setting. However, we need to modify the definition from Section 2.2 to account for the AC_λ condition and to ensure Fredholmness of the linearised problem. Thus we define:

$$F_{\mathrm{AC}} : C^\infty_\lambda(\nu_\epsilon(A)) \times \mathcal{S} \longrightarrow C^\infty_{\mathrm{loc}}(E_{\mathrm{cay}}), \tag{2.28}$$

for $\epsilon > 0$ sufficiently small, and after potentially shrinking $\mathcal{S}$. Recall that $\nu_\epsilon(A)$ is a tubular neighbourhood that grows linearly in the distance from the origin, so any sufficiently small AC_λ deformation will be contained in this neighbourhood. We would like to show the following result:

Proposition 2.20. *Let $p > 4$ and $k \geqslant 1$. For sufficiently small $\epsilon > 0$ and $\eta < 0$, after potentially shrinking $\mathcal{S}$, the map F_{AC} extends to a C^∞ map of Sobolev spaces:*

$$F_{\mathrm{AC}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\lambda}(\nu_\epsilon(A)) : \|v\|_{L^p_{k+1,\lambda}} \leqslant \epsilon\} \times \mathcal{S} \longrightarrow L^p_{k,\lambda-1}(E_{\mathrm{cay}}).$$

Furthermore, any $v \in L^p_{k+1,\lambda}(\nu_\epsilon(A))$ such that $F_{\mathrm{AC}}(v) = 0$ is smooth and lies in $C^\infty_{\lambda+1}(\nu_\epsilon(A))$. The linearisation at 0 is the bounded linear map:

$$\not{D}_{\mathrm{AC},\mathcal{S}} : L^p_{k+1,\lambda}(\nu(A)) \times T_{s_0}\mathcal{S} \longrightarrow L^p_{k,\lambda-1}(E_{\mathrm{cay}}).$$

Finally, $\not{D}_{\mathrm{AC},\mathcal{S}}$ is Fredholm for λ in the complement of a discrete set $\mathscr{D}_L \subset \mathbb{R}$, which is determined by the asymptotic link $L \subset S^7$ as an associative submanifold.

The proof of this result follows the same outline as in the compact case. The crucial step is to obtain estimates on the weighted C^k_δ and $L^p_{k,\delta}$ norms of the various terms involved in the Taylor expansion:

$$F_{\mathrm{AC}}(v,s) = F_{\mathrm{AC}}(0,s) + \not{D}_{\mathrm{AC},s}[v] + Q_{\mathrm{AC}}(v,s),$$

where $\not{D}_{\mathrm{AC},s} = \mathrm{D}F(\cdot, s)$ for a fixed s. After that, we need to investigate the dependence on the parameter $s \in \mathcal{S}$. Let us first examine the constant term.

Proposition 2.21. *Let $k \in \mathbb{N}$. Suppose that A is AC_λ to a Cayley cone, α-Cayley for α sufficiently close to 1. Then, after shrinking $\mathcal{S}$ there is a constant $C > 0$, independent of $s \in \mathcal{S}$ such that $\|F_{\mathrm{AC}}(0,s)\|_{C^k_{\lambda-1}} < C$. Thus $F_{\mathrm{AC}}(0,s) \in C^\infty_{\lambda-1}(E_{\mathrm{cay}})$.*

Proof. We can think of τ at any given point $p \in \mathbb{R}^8$ as a smooth map:

$$\tau_p : \Lambda^4\mathbb{R}^8 \longrightarrow \Lambda^2_7\mathbb{R}^8.$$

As Φ_s is AC_η, this linear map approaches τ_0 (corresponding to the standard Spin(7)-structure) uniformly in $O(r^{\eta-1})$ and for all $s \in \mathcal{S}$. Moreover, all derivatives up to a finite order can be bounded by a constant independent of s. We now think of the tangent bundles of A and C as maps $C \to \Lambda^4\mathbb{R}^8$, ignoring the compact region of A. The AC_λ condition on A now gives us that for $p \in C$ there are constants $K_i > 0, i \in \mathbb{N}$:

$$|\nabla^i(T_pA - T_pC)| \leqslant K_i r^{\lambda-1-i}, \text{ as } r \to \infty.$$

Here ∇ is with respect to the cone metric on C and the flat metric on $\mathbb{R}^8$. However the same is true for the metric induced from the embedding of A in $(\mathbb{R}^8, \Phi_s)$ and the connection associated to Φ_s. This is because changing the metric to an asymptotically conical one only introduces errors which are asymptotically smaller than the right-hand side. Thus an application of Taylor's theorem leads to:

$$\begin{aligned}|\nabla^k\tau(T_pA)| &\leqslant |\nabla^k\tau(T_pC)| + |\nabla^k(\tau(T_pA) - \tau(T_pC))| \\ &\leqslant Cr^{\eta-k-1} + \sum_{i+j=k}|D^i\tau||\nabla^j(T_pA - T_pC)| \leqslant Cr^{\lambda-1-k}.\end{aligned}$$

Here we used that if Φ_s is AC_η to Φ_0 then $|\nabla^k\tau(T_pC)| < Cr^{\eta-1-k}$, as C is a Φ_0-Cayley cone. The projection π_E worsens this bound by a constant factor by an analogous argument. □

Next, let us look at the quadratic term.

Proposition 2.22. *Fix $k \in \mathbb{N}$. Then there is a constant $C_k < 0$ such that if $\eta \leqslant C_k$, the following holds. Suppose A is AC_λ to a Cayley cone with $\eta < \lambda < 1$ and α-Cayley for α sufficiently close to 1, and $\epsilon > 0$ sufficiently small. Let furthermore $u, v \in C^k_\lambda(\nu_\epsilon(A))$ satisfy $|u|_{C^1_1}, |v|_{C^1_1} \leqslant \epsilon$. Then there is a constant $C > 0$ such that:*

$$|Q_{\mathrm{AC}}(u,s) - Q_{\mathrm{AC}}(v,s)|_{C^k_{\lambda-1}} < C\big(|u-v|_{C^{k+1}_\lambda}\big(|u|_{C^k_\lambda} + |v|_{C^k_\lambda}\big) + |u-v|_{C^k_\lambda}\big(|u|_{C^{k+1}_\lambda} + |v|_{C^{k+1}_\lambda}\big)\big).$$

Proof. We first consider the flat, translation-invariant Spin(7)-structure Φ_0. For this structure $\mathbf{Q}_\pi(p, v, w)$ is independent of the point p. Furthermore, $\mathbf{Q}$ is translation invariant in the following sense:

$$\mathbf{Q}_\pi(\gamma \cdot v, w) = \mathbf{Q}_\pi(v, w), \text{ for all } \gamma > 0. \tag{2.29}$$

This is a reformulation of the fact that $Q_{\mathrm{AC}}(\gamma \cdot v) = Q_{\mathrm{AC}}(v)$ after identifying $\mathbb{R}^8 \simeq T_p\mathbb{R}^8$ for each $p \in \mathbb{R}^8$. Recall the Taylor expansion (2.14) for small v, w:

$$\mathbf{Q}_\pi(v, w) = \mathbf{R}_{1,\pi} v \otimes v + \mathbf{R}_{2,\pi} v \otimes w + \mathbf{R}_{3,\pi} w \otimes w. \tag{2.30}$$

For v outside the initial domain of definition, we can define:

$$\begin{aligned}
\mathbf{R}_{1,\pi}(\gamma \cdot v, w) &= \gamma^{-2}\mathbf{R}_{1,\pi}(v, w),\\
\mathbf{R}_{2,\pi}(\gamma \cdot v, w) &= \gamma^{-1}\mathbf{R}_{2,\pi}(v, w),\\
\mathbf{R}_{3,\pi}(\gamma \cdot v, w) &= \mathbf{R}_{3,\pi}(v, w).
\end{aligned}$$

The extended $\mathbf{R}_{i,\pi}$ then also satisfy Equation (2.30). Near infinity, the derivatives have the following scaling behaviour:

$$\partial_x^k \partial_y^l \mathbf{R}_{i,\pi}(\gamma \cdot v, w) = \gamma^{-3+i-k}\mathbf{R}_{i,\pi}(v, w). \tag{2.31}$$

In particular for $v \in \nu_\epsilon(A)$ and $|w| \leqslant \epsilon$ with ϵ sufficiently small, there are bounds $|\partial_x^k \partial_y^l \mathbf{R}_{i,\pi}(v, w)| \leqslant C_{k,l}|v|^{-3+i-k}$. From this we deduce for $u, v \in C^\infty_\lambda(\nu_\epsilon(A))$:

$$\begin{aligned}
|Q_{\mathrm{AC}}(u) - Q_{\mathrm{AC}}(v)|\rho^{-2(\lambda-1)} &\leqslant |\rho^2 R_1(u)||u - v|\rho^{-\lambda}(|u| + |v|)\rho^{-\lambda}\\
&+ |\rho^{2+\lambda}\partial_x R_1(v)||u - v|\rho^{-\lambda}(|v|\rho^{-\lambda})^2\\
&+ |\rho^{1+\lambda}\partial_y R_1(v)||\nabla u - \nabla v|\rho^{1-\lambda}(|v|\rho^{-\lambda})^2 + (\cdots)\\
&\leqslant C(1 + \rho^{\lambda+1})|u - v|_{C^1_\lambda}(|u|_{C^0_\lambda} + |v|_{C^0_\lambda}) + (\cdots)\\
&\leqslant C|u - v|_{C^1_\lambda}(|u|_{C^0_\lambda} + |v|_{C^0_\lambda}) + (\cdots).
\end{aligned}$$

Here we used the fact that $\rho^{\lambda-1} \to 0$ as $\rho \to +\infty$. The terms containing R_2 and R_3 have been omitted as they admit analogous scaling behaviour. We ultimately obtain $|Q_{\mathrm{AC}}(u) - Q_{\mathrm{AC}}(v)|_{C^0_{2(\lambda-1)}} \leqslant C|u-v|_{C^1_\lambda}(|u|_{C^1_\lambda} + |v|_{C^1_\lambda})$. For higher derivatives, note that the translation invariance of $\mathbf{Q}$ gives us the following analogue of Equation (2.17):

$$\nabla_\xi(R_i(v)) = \partial_x\mathbf{Q}[\nabla_\xi v] + \partial_y\mathbf{Q}[\nabla_\xi \nabla v] + \partial_\pi\mathbf{Q}[\nabla_\xi TA].$$

Now again from the Taylor expansion (2.14) we see that:

$$\partial_x\mathbf{Q}(v, \nabla v) = \partial_x\mathbf{R}_1 v \otimes v + (\mathbf{R}_1 + \partial\mathbf{R}_2)v \otimes \nabla v + (\mathbf{R}_2 + \partial\mathbf{R}_3)\nabla v \otimes \nabla v.$$

All the terms have the same scaling behaviour so that $\partial_x\mathbf{Q}(\gamma \cdot v, \nabla v) = \gamma^{-1}\partial_x\mathbf{Q}(v, \nabla v)$. The terms $\partial_y\mathbf{Q}$ and $\partial_\pi\mathbf{Q}$ can be treated similarly. The upshot is that one can express $\nabla^k Q_{\mathrm{AC}}(u) - \nabla^k Q_{\mathrm{AC}}(v)$ as a sum of terms which are products of $\partial_x^k \partial_y^l \partial_\pi^m \mathbf{R}_i$, $\nabla^i u - \nabla^i v$,

$\nabla^j u + \nabla^j v$ and $\nabla^r TA$. Then manipulations as above allow us to conclude that:

$$|\nabla^k Q_{\mathrm{AC}}(u) - \nabla^k Q_{\mathrm{AC}}(v)|\rho^{k-2(\lambda-1)} \leqslant C|u-v|_{C^{k+1}_\lambda}\left(|u|_{C^k_\lambda} + |v|_{C^k_\lambda}\right) + \\ |u-v|_{C^k_\lambda}\left(|u|_{C^{k+1}_\lambda} + |v|_{C^{k+1}_\lambda}\right),$$

from which the claim of the proposition follows, in the flat case.

Now, if Φ is an AC_η perturbation of Φ_0, one has for $k, l \geqslant 0$ and $|v| \leqslant \epsilon r$, $|w| \leqslant \epsilon$:

$$|\partial_x^k \partial_y^l (\mathbf{F}_\Phi(p,v,w) - \mathbf{F}_{\Phi_0}(p,v,w))| = O(\|\Phi - \Phi_0\|_{C^{k+1}_\eta} |p|^{\eta+1-k-l}). \tag{2.32}$$

This can be seen by first observing that $|\nabla^k(\exp_\Phi - \exp_{\Phi_0})| = O(r^{\eta-k})$. This, in turn, can be obtained by analysing the geodesic equation for the curve $x(t)$:

$$\ddot{x}_k = \Gamma^k_{ij}\dot{x}_i\dot{x}_j, \tag{2.33}$$

where Γ^k_{ij} are the Christoffel symbols for the usual coordinates on $\mathbb{R}^8$. The AC_η condition implies that $|\Gamma^k_{ij}| \lesssim r^{\eta-2}$. Now, as $\dot{x}$ is a vector uniformly bounded with respect to both g_Φ and the flat metric, we can deduce that

$$\begin{aligned}|\exp_{p,\Phi}(v) - \exp_{p,\Phi_0}(v)| = |x(t) - x(0) - t\dot{x}(0)| &\leqslant \int_0^t \int_0^t |\Gamma^k_{ij}| \mathrm{d}s \mathrm{d}s' \\ &= O(t^2 \|\Phi - \Phi_0\|_{C^1_\eta} r^{\eta-2}) = O(\epsilon^2 \|\Phi - \Phi_0\|_{C^1_{\eta-1}} r^\eta).\end{aligned}$$

One can write down similar ODEs for the variation of exp with regards to the initial condition and perform an analogous analysis to bound the quantities $|\nabla^k(\exp_\Phi - \exp_{\Phi_0})|$ for $k \geqslant 1$. As τ_p is obtained from Φ_p by a smooth mapping, it too is in C^∞_η with the same norm, up to a universal multiplicative constant. Thus we see that:

$$\begin{aligned}|\mathbf{F}_\Phi(p,v,w) - \mathbf{F}_{\Phi_0}(p,v,w))| &\lesssim |\exp_\Phi - \exp_{\Phi_0}| + |\nabla(\exp_\Phi - \exp_{\Phi_0})| + |\tau - \tau_0| \\ &= O(\|\Phi - \Phi_0\|_{C^1_\eta} |p|^\eta).\end{aligned}$$

The proof for higher derivatives works in a similar way. We can now use this to get decay estimates similar to Equation (2.31). For any AC_η Spin(7)-structure Φ we can define

$$\begin{aligned}\mathbf{R}_1(p,v,w) &= \int_0^1 (1-t)\partial_x^2 \mathbf{F}(p, t(v,w))\mathrm{d}t, \\ \mathbf{R}_2(p,v,w) &= \int_0^1 (1-t)\partial_{x,y}^2 \mathbf{F}(p, t(v,w))\mathrm{d}t, \\ \mathbf{R}_3(p,v,w) &= \int_0^1 (1-t)\partial_y^2 \mathbf{F}(p, t(v,w))\mathrm{d}t.\end{aligned}$$

From the bounds (2.32) we see that for $\gamma \geqslant 1$ and $\eta \leqslant 0$:

$$\begin{aligned}|\mathbf{R}_{1,\Phi}(p,\gamma\cdot v,w)| &\leqslant |\mathbf{R}_{1,\Phi_0}(p,\gamma\cdot v,w)| + (\gamma|p|)^{\eta-2}\\ &\leqslant \gamma^{-2}|\mathbf{R}_{1,\Phi_0}(p,v,w)| + (\gamma|p|)^{\eta-2}\\ &\leqslant \gamma^{-2}(|\mathbf{R}_{1,\Phi}(p,v,w)| + \|\Phi-\Phi_0\|_{C^\infty_\eta}|p|^{\eta-2}) + \|\Phi-\Phi_0\|(\gamma|p|)^{\eta-2}\\ &\leqslant \gamma^{-2}|\mathbf{R}_{1,\Phi}(p,v,w)| + \|\Phi-\Phi_0\|_{C^\infty_\eta}\gamma^{-2}|p|^{\eta-2}(1+\gamma^\eta)\\ &\lesssim \gamma^{-2}(|\mathbf{R}_{1,\Phi}(p,v,w)| + \|\Phi-\Phi_0\|_{C^\infty_\eta}).\end{aligned}$$

Thus formally similar estimates to the flat case hold if we replace the equality by an inequality and introduce an error term which depends on the size of the perturbation. Note also that the constant introduced in the last step only depends on $\|\Phi-\Phi_0\|_{C^\infty_\eta}$, thus can be bounded uniformly in $\mathcal{S}$. In fact the estimates on $\mathbf{R}_{2,\Phi}$, $\mathbf{R}_{3,\Phi}$ and all the derivatives follow in a similar way, given that η is sufficiently negative. We can now conclude the proof like in the flat case. □

Proof of Theorem 2.20. To prove that $F_{\mathrm{AC}}(\cdot,s) : L^p_{k+1,\lambda} \to L^p_{k,\lambda-1}$ is C^∞ for a fixed Spin(7)-structure Φ_s, we can repeat the proof for the compact case, using our estimates from Propositions 2.21 and 2.22 as well as the fact that $\not{D}_{\mathrm{AC},s}$ is an asymptotically conical operator, and therefore bounded between the Sobolev spaces in question. Indeed, it can be seen from the presentation (2.8) of $\not{D}_{\mathrm{AC},s}$ that its coefficients and all derivatives approach the values for the conical operator on C. It now also follows from the Lockhart and McOwen theory that there is a discrete set $\mathscr{D}_L \subset \mathbb{R}$ such that $\not{D}_{\mathrm{AC},s}$ is Fredholm for λ in the complement of $\mathscr{D}_L$.

For $v \in L^p_{2,\lambda}$ such that $F_{\mathrm{AC}}(v) = 0$, elliptic bootstrapping applies locally like in Proposition 2.15, so that such v are immediately in C^∞_{loc}. Now we can invoke the Sobolev embedding theorem 1.27 to get that $v \in C^\infty_\lambda$.

What remains to show is that F_{AC} is also smooth with respect to the parameter $s \in \mathcal{S}$. Certainly, the derivatives $\partial^k_s F_{\mathrm{AC}}(v,s)$ exist as smooth functions. The key issue is that they might not be in $L^p_{\lambda-1}$ a priori. Note however that the perturbations in the Spin(7)-structure induced by a change in s lie in $C^\infty_\eta \subset L^p_{k,\lambda}$ for any k, as $\eta < \lambda$ by assumption. From this, it can easily seen that $\partial_s F_{\mathrm{AC}}(v,s)$ will be in $L^p_{k,\lambda-1}$ as well, and the argument applies equally to higher derivatives. □

We can now prove the analogue of Theorem 2.16 for the asymptotically conical case.

Theorem 2.23 (Structure for AC Cayleys)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let A be an AC_λ Cayley submanifold of $(\mathbb{R}^8, \Phi_0)$, where Φ_0 is the standard Spin(7)-structure on $\mathbb{R}^8$, and let $\mathcal{S}$ be a smooth family of AC_η deformations of Φ_0 with $\eta < \lambda < 1$. Then there is an open neighbourhood $0 \in \mathcal{U} \subset \mathcal{S}$ and a non-linear deformation operator F_{AC} which for $\epsilon > 0$ sufficiently small is a C^∞ map:*

$$F_{\mathrm{AC}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\lambda}(\nu_\epsilon(A)), \|v\|_{L^p_{k+1,\lambda}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\lambda-1}(E).$$

*A neighbourhood of (A,Φ_0) in $\mathcal{M}^\lambda_{\mathrm{AC}}(A,\mathcal{S})$ is homeomorphic to the zero locus of F_{AC} near $(0,\Phi_0)$. Assuming that $\lambda \notin \mathscr{D}_L$ we define the **deformation space** $\mathcal{I}^\lambda_{\mathrm{AC}}(A) \subset C^\infty_\lambda(\nu(A)) \times T_0\mathcal{S}$ to be the the kernel of $\not{D}_{\mathrm{AC},\mathcal{S}} = \mathrm{D}F_{\mathrm{AC}}(0,\Phi_0)$, and the **obstruction space** $\mathcal{O}^\lambda_{\mathrm{AC}}(A) \subset$*

$C^\infty_{-4-\lambda}(E)$ *to be the cokernel of* $\not{D}_{\mathrm{AC},\mathcal{S}}$. *Then a neighbourhood of* A *in* $\mathcal{M}^\lambda_{\mathrm{AC}}(A,\mathcal{S})$ *is also homeomorphic to the zero locus of a Kuranishi map:*

$$\kappa^\lambda_{\mathrm{AC}} : \mathcal{I}^\lambda_{\mathrm{AC}}(A) \longrightarrow \mathcal{O}^\lambda_{\mathrm{AC}}(A).$$

In particular if $\mathcal{O}^\lambda_{\mathrm{AC}}(A) = \{0\}$ *is trivial,* $\mathcal{M}^\lambda_{\mathrm{AC}}(A,\mathcal{S})$ *admits the structure of a smooth manifold near* A. *We say that* A *is* ***unobstructed*** *as an* AC_λ *Cayley in this case.*

The expected dimension of this moduli space can be expressed like in the compact case in Theorem 2.17, but it requires the more general Atiyah-Patodi-Singer theorem [1], which computes the index not only in terms of topological data but also from analytical data that depends on the cone.

First, a few definitions are in order. For N a possibly noncompact $2n$-manifold, we define its **signature** $\sigma(N)$ to be the signature of the nondegenerate pairing $H^n(N) \times H^n_{\mathrm{cs}}(N) \to \mathbb{R}$, where $H^n_{\mathrm{cs}}(N)$ denotes cohomology with compact support. This of course agrees with the usual definition of the signature for compact manifolds. For $A \subset \mathbb{R}^8$ an AC manifold asymptotic to the cone $C = \mathbb{R}_+ \times L$, we will consider the following version of the intersection number. Pick a section $u \in C^\infty(L, S\nu(A)|_L)$ in the sphere bundle of normal vector fields (to A) on L. For any normal vector field $v \in C^\infty(A, \nu(A))$ that converges to u at infinity, the algebraic count of its zeros will only depend on the homotopy class of u in the sphere bundle. We denote this number by $[A] \cdot_{[u]} [A]$.

Proposition 2.24 (Index)**.** *Let* $A \subset \mathbb{R}^8$ *be* AC_λ *with* $\lambda < 1$, *asymptotic to a Cayley cone* $C = \mathbb{R}_+ \times L$, *and* α*-Cayley for* α *sufficiently close to* 1. *Assume moreover that* $(\lambda, 1) \cap \mathscr{D}_L = \emptyset$. *Pick a homotopy class* $[u] \in [L, S\nu(A)|_L]$ *(where* $S\nu(A)$ *is the sphere bundle of* $\nu(A)$*) of a section* $u : L \to S\nu(A)|_L$. *Then the following holds:*

$$\operatorname{ind} \not{D}_{\mathrm{AC},\mathcal{S}} = \frac{1}{2}(\sigma(A) + \chi(A)) - [A] \cdot_{[u]} [A] + \eta(L) + T([u]) + \dim \mathcal{S}. \tag{2.34}$$

Here $\eta(L)$ *is a real number that depends on* $L \subset S^7$ *as an associative submanifold, and* $T([u]) \in \mathbb{R}$ *is a topological term depending on the homotopy class of* u.

Here $\mathcal{M}^{G_2}(L)$ is the moduli space of associative submanifolds of S^7 isotopic to L. It can be defined similarly to the Cayley moduli space as the zero locus of a non-linear deformation operator. However, as we noted in Example 1.12, $L \subset S^7$ is associative exactly when the cone $C = L \times \mathbb{R}_+$ is a Cayley cone. Thus we simply define $\dim \mathcal{M}^{G_2}(L)$ to be the dimension of the Cayley cone deformations of C, and say that L is unobstructed exactly when all the infinitesimal deformations of C integrate to full deformations.

Proof. This is a consequence of the Atiyah-Singer theorem and additivity of the index of the Cayley operator under gluing. We have yet to formally introduce how to glue an AC Cayley onto a singularity in a CS Cayley $\hat{N}$ with a matching cone. This will be the subject of the next chapter. We will in particular see (during the proof of Proposition 3.14) that the index is additive in the following sense, where we assume $\mathcal{S} = \{\Phi\}$ is a fixed Spin(7) structure:

$$\operatorname{ind} \not{D}_{\mathrm{AC}} + \operatorname{ind} \not{D}_{\mathrm{CS}} = \operatorname{ind} \not{D}_{N \sharp_L A}.$$

Here A is our AC Cayley, and $\hat{N}$ is a CS Cayley with one singular point modelled on the cone C. The $\operatorname{ind} \not{D}_{\mathrm{CS}}$ is considered at rate $\lambda < 1$, which is different from the standard CS Cayley operator and the reason why the formula (2.34) has an additional term compared to formula (2.46). The compact, nonsingular Cayley $N = \hat{N} \sharp_L A$ has the topological type of a connected sum along the link L of C. Now by the Atiyah-Singer index theorem as used in [7, Theorem 6.3.1] we have

$$\operatorname{ind} \not{D}_{\hat{N}\sharp_L A} = \int_{\hat{N}\sharp_L A} \mathrm{td}$$

for a certain characteristic class td computed from the curvatures of the bundles ν and E. By the AC and CS conditions it is well-defined to write:

$$\operatorname{ind} \not{D}_{\mathrm{AC}} = \int_A \mathrm{td} + \eta_{\mathrm{AC}}(\not{D}), \quad \operatorname{ind} \not{D}_{\mathrm{CS}} = \int_{\hat{N}} \mathrm{td} + \eta_{\mathrm{CS}}(\not{D}).$$

Here $\eta_{\mathrm{AC/CS}}$ is a term that a priori depends on the entire Cayley operator on the submanifold. However we can perturb both operators to exactly conical operators (with cone C) without affecting the index. Thus we can write:

$$\begin{aligned}\int_{\hat{N}\sharp_L A} \mathrm{td} = \operatorname{ind} \not{D}_{\hat{N}\sharp_L A} = \operatorname{ind} \not{D}_{\mathrm{AC}} + \operatorname{ind} \not{D}_{\mathrm{CS}} &= \int_{\hat{N}} \mathrm{td} + \int_A \mathrm{td} + \eta_{\mathrm{AC}}(\not{D}) + \eta_{\mathrm{CS}}(\not{D}) \\ &= \int_{\hat{N}\sharp_L A} \mathrm{td} + \eta_{\mathrm{AC}}(\not{D}) + \eta_{\mathrm{CS}}(\not{D}).\end{aligned}$$

From this we see that $\eta_{\mathrm{AC}}(\not{D}) = -\eta_{\mathrm{CS}}(\not{D})$ does in fact not depend on the whole operator $\not{D}$ on either submanifold, but the information that they share, i.e. the asymptotic conical operator restricted to the link L. Thus we can write:

$$\operatorname{ind} \not{D}_{\mathrm{AC}} = \int_A \mathrm{td} + \eta(L), \quad \operatorname{ind} \not{D}_{\mathrm{CS}} = \int_N \mathrm{td} - \eta(L).$$

Here η is not necessarily equal to the η-invariant of a partial differential operator, but simply a term depending analytically on the link as an associative in S^7. I

Next, we remark that on A we have $\mathrm{td} = \frac{1}{2}(\frac{1}{3}p_1(TA) + e(TA)) - e(\nu(A))$, where p_1 and e are the Chern-Weil representatives of the first Pontryagin class of a bundle and the Euler class respectively. Now we can apply the Atiyah-Patodi-Singer index theorem to conclude that the integral of $p_1(TA)$ is exactly given by the signature $\sigma(A)$ (up to a term depending on the boundary, which may be absorbed into η).

The integral of the Euler class of bundles on A can be computed in terms of zeros of a generic section, as is the case for bundles on compact manifolds. However in the noncompact case the number of zeros of a section may not be constant, and we require the Mathai-Quillen current defined in [32, Section 7]. For a metric bundle $\pi : (F^4, g_F) \to A$ with compatible connection ∇^F it is given as a form $\psi(F, \nabla^F) \in \Omega^3(F)$. If now δ_A is the Dirac-δ current on F representing the zero section $A \subset F$. By [4, Thm. 3.7] we have the following identity as currents:

$$\mathrm{d}\psi(F,\nabla^F)=\pi^*e(F,\nabla^F)-\delta_A.$$

Thus in particular if $s:A\to F$ is a transversal section with $z(s)$ zeros we have:

$$\int_A e(F,\nabla^F)=\int s^*\psi(F,\nabla^F)+s^*\delta_A=z(s)+\int s^*\psi(F,\nabla^F).$$

Thus we can express $\int_A e(TA)$ and $\int_A e(\nu)$ as counts of zeros if we fix a homotopy class $[u]$. Here u is the limiting section of s restricted to the link at large radii. For the tangent bundle, we can canonically fix a section that is everywhere outward pointing, but for ν there is no canonical choice, hence the dependence of $T([u])=\int s^*\psi(\nu,\nabla^\nu)$. We redefine $\eta(L)$ so as to absorb the $\int s^*\psi(F,\nabla^F)$ terms. Finally, if Φ varies in a family $\mathcal{S}$, the index will increase by $\dim\mathcal{S}$. □

Remark 2.25. Consider the complex fibration:

$$\begin{aligned} f_0:\mathbb{C}^4&\longrightarrow\mathbb{C}^2\\ (x,y,z,u)&\longmapsto(x^2+y^2+z^2,u).\end{aligned}$$

Its singular fibres are cones of the form $C_q=f_0^{-1}(0,0)=\{x^2+y^2+z^2=0,u=0\}\subset\mathbb{C}^4$. For each $\epsilon\in\mathbb{C}\setminus\{0\}$ we get a complex surface $A_\epsilon=f_0(\epsilon,0)$. We can write a compact subset of A_ϵ as the image of a normal section as follows:

$$\begin{aligned} C_q&\longrightarrow A_\epsilon\\ (p,u)&\longmapsto\left(p+\frac{\epsilon\bar{p}}{2|p|^2},u\right).\end{aligned}$$

Here $p=(x,y,z)$ and $\nu_{(p,0)}(A_\epsilon)=\mathrm{span}_{\mathbb{C}}\{\bar{p}\}$. From this we see that A_ϵ is AC_{-1} to the cone C_q. It can be shown that for $\delta>0$ small:

$$\begin{aligned} \mathcal{M}^{-1+\delta}_{\mathrm{AC}}(A_\epsilon)&\simeq\mathbb{C}\setminus\{0\}\\ [A_\epsilon]&\longmapsto\epsilon,\end{aligned}$$

and that all $A\in\mathcal{M}^{-1+\delta}_{\mathrm{AC}}(A_\epsilon)$ are unobstructed as we will see in Proposition 5.2. In particular:

$$\operatorname{ind}\not{D}_{\mathrm{AC}}=\dim\mathcal{M}^{-1+\delta}_{\mathrm{AC}}(A_\epsilon)=2.$$

The critical rates $\mathscr{D}_L\cap[-1,1]$ for this cone were determined in Example 1.35 to be $\{-1,0,1\}$. As $d(-1)=2$, this means that there are no deformations for rates below -2. The next critical rate above -1 is 0, which corresponds to translations. Thus:

$$\dim\mathcal{M}^{\delta}_{\mathrm{AC}}(A_\epsilon)=2+8.$$

Finally, the remaining critical rate is 1, which corresponds to rotations and deformations of the link as an associative, but which our theory does not take into account so far, as it would correspond to $\lambda = 1$.

Corollary 2.26. *Suppose that $A \subset \mathbb{C}^4$ is a special Lagrangian AC_λ submanifold for $\lambda < 1$. With the notation from Proposition 2.24, we have that the Cayley deformation operator of A has index:*

$$\operatorname{ind} \not{D}_{\mathrm{AC},\mathcal{S}} = \tfrac{1}{2}(\sigma(A) - \chi(A)) + \tilde{\eta}(L) - \dim \mathcal{M}^{G_2}(L). \tag{2.35}$$

Here $\tilde{\eta}(L)$ is a canonically defined invariant of any special Lagrangian cone, not necessarily equal to $\eta(L)$ from Proposition 2.24.

Proof. There is a distinguished section of the normal bundle of a special Lagrangian cone in $(\mathbb{C}^4, \omega, \Omega, J)$, owing to the isomorphism

$$\begin{aligned} \sharp : TA &\longrightarrow \nu(A) \\ v &\longmapsto J(v). \end{aligned}$$

The outward pointing radial vector field ∂_r is tangent to the cone, and so $J(\partial_r)$ is normal. The Poincaré-Hopf theorem then allows us to equate $[A] \cdot_{[J(\partial_r)|_L]} [A] = \chi(A)$, from which the formula follows by setting $\tilde{\eta}(L) = \eta(L) + T([J(\partial_r)])$. □

Example 2.27 (Cayley plane)**.** Consider the case of a Cayley plane $\Pi \subset \mathbb{R}^8$ as an AC_λ manifold of rate $\lambda < 1$. It can also be seen as a special Lagrangian plane, by choosing an appropriate Calabi-Yau structure on $\mathbb{R}^8$. Up to translations, a Cayley plane is rigid and unobstructed. This can be seen by solving the infinitesimal deformation equation from Proposition 1.14 explicitly. It is given by:

$$\not{D}_{\mathrm{AC}} = -\mathrm{d}\star \oplus \mathrm{d}^- : \Omega^1_\lambda \longrightarrow \Omega^4_{\lambda-1} \oplus \Omega^{2,-}_{\lambda-1}.$$

If $\sigma \in \operatorname{Ker} \not{D}_{\mathrm{AC}}$, we get that $\mathrm{d}\sigma \in \Omega^{2,+}$, and so we can deduce:

$$\mathrm{d}^*\mathrm{d}\sigma = \star\mathrm{d}\star\mathrm{d}\sigma = \star\mathrm{d}\mathrm{d}\sigma = 0.$$

Together with $\mathrm{d}\star\sigma = 0$, we get $\delta\sigma = 0$. As we are in flat $\mathbb{R}^4$, we see that $\sigma = \sum_{i=1}^4 f_i \mathrm{d}x_i$ with f_i harmonic functions that decay like r^λ. Thus each of the f_i must be a constant and $\dim \operatorname{Ker} \not{D}_{\mathrm{AC}} = 4$. Now for the obstruction space $\mathcal{O}^\lambda_{\mathrm{AC}}$, we can equivalently look at the kernel of the adjoint map:

$$(-\mathrm{d}\star \oplus \mathrm{d}^-)^* = -\mathrm{d}\star + \mathrm{d}^* : \Omega^n_{-4-\lambda} \oplus \Omega^{2,-}_{-4-\lambda} \longrightarrow \Omega^1_{-5-\lambda}. \tag{2.36}$$

If $f \in \Omega^n_{-4-\lambda}$ and $\eta \in \Omega^{2,-}_{-4-\lambda}$ satisfy $\mathrm{d}^*\eta = \mathrm{d}\star f$, then $\mathrm{d}^*\mathrm{d}\star f = 0$, i.e. $\star f$ is a harmonic function on Π. Similarly, using the anti-self-duality of η, we see that

$$\begin{aligned} \mathrm{d}\mathrm{d}^*\eta &= \mathrm{d}\mathrm{d}\star f = 0, \\ \mathrm{d}^*\mathrm{d}\eta &= -\star\mathrm{d}\mathrm{d}^*\eta = 0. \end{aligned}$$

Thus η is an anti-self-dual harmonic two form. Now for $\lambda > 0$, we have that both f and η are in L^2. Thus by [28, Example 0.15] we see that both must vanish, as $H^0_{\mathrm{cs}}(\Pi) =$

$H^2_{\mathrm{cs}}(\Pi) = 0$. Hence for a round sphere $S^3 \subset S^7$ we find from Corollary 2.26 that:

$$4 = \dim \mathcal{M}^\lambda_{\mathrm{AC}}(\Pi) = \tfrac{1}{2}(\sigma(\Pi) - \chi(\Pi)) + \tilde{\eta}(S^3) - \dim \mathcal{M}^{G_2}(S^3).$$

This implies that $\tilde{\eta}(S^3) - \dim \mathcal{M}^{G_2}(S^3) = 4\frac{1}{2}$. Thus, as $\dim \mathcal{M}^{G_2}(S^3) = d(1) = 12$ by Example 1.34, we see that $\tilde{\eta}(S^3) = 16\frac{1}{2}$.

Example 2.28 (Lawlor neck)**.** Consider two distinct Lagrangian subspaces Π_1 and Π_2 in $\mathbb{C}^4$ that intersect transversely. They are given as:

$$\Pi_1 = \mathrm{span}\{v_1, v_2, v_3, v_4\} \text{ and } \Pi_2 = \mathrm{span}\{e^{i\theta_1}v_1, e^{i\theta_2}v_2, e^{i\theta_3}v_3, e^{i\theta_4}v_4\},$$

with characteristic angles $\theta_1 + \theta_2 + \theta_3 + \theta_4 = k\pi$, $0 \leqslant \theta_i \leqslant \pi$ and $k = 1, 2, 3$. Lawlor showed in [26] that if $k = 1, 3$, then there is a one parameter family of AC_{-3} manifolds asymptotic to the cone $\Pi_1 \cup \Pi_2$, the **Lawlor necks** L_t. They are all diffeomorphic to $S^3 \times \mathbb{R}$.

We can now apply Corollary 2.26 again to determine the expected dimension of their Cayley moduli space:

$$\begin{aligned} \dim \mathcal{M}^\lambda_{\mathrm{AC}}(L_t) &= \tfrac{1}{2}(\sigma(L_t) - \chi(L_t)) + 2\tilde{\eta}(S^3) - 2\dim \mathcal{M}^{G_2}(S^3) \\ &= \tfrac{1}{2}(0 - 0) + 2 \cdot (4\tfrac{1}{2}) = 9. \end{aligned} \tag{2.37}$$

This corresponds to the translations in $\mathbb{R}^8$ and the rescaling action. Thus there are no additional infinitesimal strictly Cayley deformations of the Lawlor necks. Note that if $k = 2$, then there are no minimal desingularisations of $\Pi_1 \cup \Pi_2$ [26], so, in particular, no Cayley desingularisations. Next, notice that the same argument that allowed us to show unobstructedness of the Cayley plane in the previous Example 2.27 also gives us unobstructedness for the Lawlor necks for small $\lambda < 1$, since $H^0_{\mathrm{cs}}(L_t) = H^2_{\mathrm{cs}}(L_t) = 0$.

Note that the Lawlor necks can a priori only desingularise the union of two special Lagrangian planes. It turns out however that any pair of transversely intersecting Cayley planes can be realised as a pair of special Lagrangian planes for a suitably chosen Calabi–Yau structure.

Proposition 2.29. *Let $\Pi_1, \Pi_2 \in \mathrm{Cay}(\mathbb{R}^8, \Phi_0)$ be two transversely intersecting Cayley subspaces. Then there is an $\mathrm{SU}(4)$-structure (J, g, ω, Ω) on $\mathbb{R}^8$, such that both the Π_i are special Lagrangian with respect to it.*

Proof. Recall from [10, Thm. IV.1.8] that Spin(7) acts transitively on Cayley planes with stabilizer $H \simeq (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))/\pm \mathrm{id}$. We now show that generically the action of Spin(7) on a pair of Cayley planes is free. First note that SO(8) acts transitively on pairs of four-planes with fixed characterizing angles $0 < \beta_1 \leqslant \beta_2 \leqslant \beta_3 \leqslant \beta_4 \leqslant \frac{\pi}{2}$, as explained in [26, Section 4]. Now $\dim \mathrm{SO}(8) = 2\dim \mathrm{Gr}(4, 8) - 4$, and so this action can at most admit discrete stabilisers for generic choices of the β_i. More precisely, this is the case when the β_i are pairwise different and not equal to $\frac{\pi}{2}$, in which case the stabilisers are trivial. In these cases, the action on Cayley plane pairs is free as well. Now note that for Cayley planes, the angle β_4 can be derived from the other three, by using the triple product. Thus generically a family of Cayley plane pairs with fixed angles is $2\dim \mathrm{Cay} - 3 = 21 = \dim \mathrm{Spin}(7)$ dimensional. In particular, since both Spin(7) and Cayley plane pairs of given angles are connected the action of Spin(7) is transitive when restricted to pairs with fixed angles. Now for a fixed SU(4)-structure, the set of special Lagrangian plane pairs contains examples for all possible characteristic angles that can appear for Cayleys. Thus we can always Spin(7) rotate a pair of generic Cayley planes to a pair of special Lagrangian two planes. To conclude, notice that the continuous action of Spin(7) on transversely intersecting special Lagrangian plane pairs sweeps out a closed and dense subset of the transversely intersecting Cayley plane pairs. Hence it must reach them all, and this proves the proposition. □

Lemma 2.30. *Suppose that $\Pi_1, \Pi_2 \in \mathrm{Cay}(\mathbb{R}^8)$ are two Cayley planes that intersect negatively in a single point. Then there is a one-dimensional family of unobstructed AC_{-3} Cayley submanifolds, the* ***Cayley-Lawlor necks*** *which are asymptotic to the cone $\Pi_1 \cup \Pi_2$.*

Proof. Two transversely intersecting special Lagrangian planes $\Pi_1, \Pi_2 \subset \mathbb{C}^4$, where $\mathbb{C}^4$ has coordinates $z_1 = x_1 + iy_1, \dots, z_4 = x_4 + iy_4$, can be SU(4)-rotated to be of the form

$$\Pi_1 = \mathrm{span}\{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}\} \text{ and } \Pi_2 = \mathrm{span}\{e^{i\theta_1}\partial_{x_1}, e^{i\theta_2}\partial_{x_2}, e^{i\theta_3}\partial_{x_3}, e^{i\theta_4}\partial_{x_4}\}$$

respectively, where $\theta_i \in (0, \pi)$,$\theta_1 + \theta_2 + \theta_3 + \theta_4 = k\pi$ and $k = 1, 2$ or 3. Now by the previous proposition, the same is true for Cayley planes. Recall that the Cayley form can be written as follows on $\mathbb{C}^4$ by example 1.10:

$$\Phi = \tfrac{1}{2}\omega \wedge \omega + \mathrm{Re}\,\Omega.$$

Now, since both Π_1 and Π_2 are Lagrangian, $\omega|_{\Pi_i} = 0$ and they are Cayley exactly when $\mathrm{Re}\,\Omega|_{\Pi_i} = \mathrm{dvol}_{\Pi_i}$. The holomorphic volume $\mathrm{Re}\,\Omega$ form is given by the expression $\mathrm{Re}\,\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4$, so Π_i is clearly Cayley with the orientation coming from the given basis. However for Π_2 we see that:

$$\mathrm{Re}\,\Omega[e^{i\theta_1}\partial_{x_1}, e^{i\theta_2}\partial_{x_2}, e^{i\theta_3}\partial_{x_3}, e^{i\theta_4}\partial_{x_4}] = \prod_{j=1}^{4} \mathrm{d}z_j[e^{i\theta_j}\partial_{x_j}] = e^{i(\theta_1+\theta_2+\theta_3+\theta_4)} = (-1)^k.$$

Thus Π_2 with the given orientation is Cayley for $k = 2$, and otherwise $-\Pi_2$ is Cayley. In particular Π_1 and Π_2 intersect negatively exactly when $k = 1, 3$, which are exactly the cases where Lawlor showed the existence of Lawlor necks. □

Suppose that $0 < \lambda < 1$ is such that $(\lambda, 1) \cap \mathscr{D}_L = \emptyset$. For any $\lambda < \tilde{\lambda} < 1$ we will then have an isomorphism $\mathcal{M}^{\lambda}_{\mathrm{AC}} \simeq \mathcal{M}^{\tilde{\lambda}}_{\mathrm{AC}}$, as no additional deformations appear for these rates. At $\tilde{\lambda} = 1$, which our theory does not cover at the moment, the deformations of the Cayley cone as a cone with a fixed vertex appear. As in Example 1.12 these can be understood as associative deformations of the link in a moduli space $\mathcal{M}^{G_2}(L)$. We assume that the family of deformations of the cone is smooth, and that the cone is unobstructed.

There then exists a smooth family $\{A_{\tilde{L}}\}_{\tilde{L} \in \mathcal{U}}$ of \$α-Cayley AC_{λ}-manifolds, such that $A_{\tilde{L}}$ has link $\tilde{L} \in \mathcal{U} \subset \mathcal{M}^{G_2}(L)$. Such a family can be obtained by finding ambient isotopies that perturb the cones in the desired fashion (which we do in more detail in Proposition 2.38). We obtain a smooth family of maps $\exp_{\tilde{L},v} : \nu_{\epsilon}(A) \to \mathbb{R}^8$ which form tubular neighbourhoods of the family $A_{\tilde{L}}$, all parametrised by the normal bundle of our initial A. We would like to study the moduli space:

$$\mathcal{M}^1_{\mathrm{AC}}(A,\mathcal{S}) = \bigsqcup_{\tilde{L}\in\mathcal{M}^{G_2}(L)} \mathcal{M}^\lambda_{\mathrm{AC}}(A_{\tilde{L}},\mathcal{S}).$$

This is independent of our choice of λ as long as $(\lambda,1)\cap\mathscr{D}_{\tilde{L}}=\emptyset$ for $\tilde{L}\in\mathcal{M}^{G_2}(L)$. We define the following deformation operator:

$$\begin{aligned} F_{\mathrm{AC},1} : C^\infty_\lambda(\nu_\epsilon(N))\times\mathcal{S}\times\mathcal{M}^{G_2}(L) &\longrightarrow C^\infty_{\mathrm{loc}}(E_{\mathrm{cay}}), \\ (v,s,\tilde{L}) &\longmapsto \pi_E \star_4 \operatorname{Exp}^*_{\tilde{L}}(\tau_{\Phi_s}|_{\exp_{\tilde{L},v}(N)}). \end{aligned} \tag{2.38}$$

We can now give this operator the same treatment as F_{AC}, with some mild modifications to make sure that we get a smooth map of Banach manifolds which is also smooth in the parameter $\tilde{L}\in\mathcal{M}^{G_2}(L)$. This is identical to the conically singular case, which we will treat in great detail in the next section. The upshot is the following theorem:

Theorem 2.31 (Structure)**.** *Suppose $p>4$ and $k\geqslant 1$. Let A be an AC_λ Cayley submanifold of $(\mathbb{R}^8,\Phi_0)$ with unobstructed link L, where Φ_0 is the standard $\mathrm{Spin}(7)$-structure on $\mathbb{R}^8$, and let $\mathcal{S}$ be a smooth family of AC_η deformations of Φ_0 with $\eta<\lambda<1$. Then there is an open neighbourhood $(0,L)\in\mathcal{U}\subset\mathcal{S}\times\mathcal{M}^{G_2}(L)$ and a non-linear deformation operator $F_{\mathrm{AC},1}$ which for $\epsilon>0$ sufficiently small is a C^∞ map:*

$$F_{\mathrm{AC},1} : \mathcal{L}_\epsilon = \{v\in L^p_{k+1,\lambda}(\nu_\epsilon(A)), \|v\|_{L^p_{k+1,\lambda}}<\epsilon\}\times\mathcal{U}\longrightarrow L^p_{k,\lambda-1}(E).$$

A neighbourhood of A in $\mathcal{M}^1_{\mathrm{AC}}(A,\mathcal{S})$ is homeomorphic to the zero locus of $F_{\mathrm{AC},1}$ near $(0,\Phi_0,L)$. Assuming that $\lambda\notin\mathscr{D}_L$ we define the ***deformation space*** *$\mathcal{I}^\lambda_{\mathrm{AC}}(A)\subset C^\infty_\lambda(\nu(A))\oplus T_0\mathcal{S}\oplus T_L\mathcal{M}^{G_2}(L)$ to be the the kernel of $\not{D}_{\mathrm{AC},1}=\mathrm{D}F_{\mathrm{AC},1}(0,\Phi_0,L)$, and the* ***obstruction space*** *$\mathcal{O}^\lambda_{\mathrm{AC}}(A)\subset C^\infty_{-4-\lambda}(E)$ to be the cokernel of $\not{D}_{\mathrm{AC},1}$. Then a neighbourhood of A in $\mathcal{M}^1_{\mathrm{AC}}(A,\mathcal{S})$ is also homeomorphic to the zero locus of a Kuranishi map:*

$$\kappa^\lambda_{\mathrm{AC},1} : \mathcal{I}^\lambda_{\mathrm{AC}}(A)\longrightarrow\mathcal{O}^\lambda_{\mathrm{AC}}(A).$$

In particular if $\mathcal{O}^\lambda_{\mathrm{AC}}(A)=\{0\}$ is trivial, $\mathcal{M}^1_{\mathrm{AC}}(A)$ admits the structure of a smooth manifold near A. We say that A is ***unobstructed*** *as an AC_1 Cayley in this case. The index of $\not{D}_{\mathrm{AC},1}$ is given as:*

$$\operatorname{ind}\not{D}_{\mathrm{AC},1} = \operatorname{ind}\not{D}_{\mathrm{AC}} + \dim\mathcal{M}^{G_2}(L).$$

Finally, the map $\mathcal{M}^1_{\mathrm{AC}}(A)\to\mathcal{M}^{G_2}(L)$ sending a manifold to its asymptotic link is a smooth fibre bundle.

Remark 2.32. Looking again at the fibration from Remark 2.25, we note that the cone $C_q=L_q\times\mathbb{R}_+$ is in fact part of a two-dimensional family of Cayley cones, up to the action of $\mathrm{Spin}(7)$:

$$\mathcal{C} = \{C_{\bar{a}} : \{a_1x^2+a_2y^2+a_3z^2=0, w=0\}|\ a_1+a_2+a_3=1, a_i\in\mathbb{R}_+\}.$$

A generic cone in this family has stabiliser $\{(e^{it}, e^{it}, e^{it}, e^{-3it}), t \in \mathbb{R}\} \subset \mathrm{Spin}(7)$, and so we have:

$$\dim \mathcal{M}^{G_2}(L_q) = 21 + 2 - 1 = 22. \tag{2.39}$$

This corresponds exactly to the expected dimension $d(1)$, and so the link L_q is unobstructed. Thus $\mathcal{M}^1_{\mathrm{AC}}(A_\epsilon)$ is a smooth manifold of dimension 32 near A_ϵ.

There is a natural completion of the moduli space $\mathcal{M}^\lambda_{\mathrm{AC}}(A)$ by adjoining the cone.

Definition 2.33. The **completed moduli space** $\overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A)$ is the topological space $\mathcal{M}^\lambda_{\mathrm{AC}}(A) \cup \{C\}$ such that the rescaling map $A \mapsto t \cdot A$, where $0 \cdot A = C$, is continuous, and $\mathcal{M}^\lambda_{\mathrm{AC}}(A)$ embeds homeomorphically into $\overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A)$.

We have for example that $\overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A_\epsilon)$ from Remark 2.25 is homeomorphic to $\mathbb{C}$. Note that the scaling action $A \mapsto t \cdot A$ acts as $\epsilon \mapsto t^2\epsilon$ in this picture. There is a notion of scale implicit in this description of the moduli space, which we now make precise.

Suppose for this that every $A \in \mathcal{M}^\lambda_{\mathrm{AC}}(\tilde{A})$ is unobstructed and that we have chosen a smooth cross-section $S \subset \mathcal{M}^\lambda_{\mathrm{AC}}(\tilde{A})$ of the scaling action.

Definition 2.34. The **scale** of $A \in \overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(\tilde{A})$ with respect to the cross-section S is $t(A, S)$, such that $t(A, S) \cdot A \in S$. Note that the scale functions corresponding to different cross-sections are all uniformly equivalent.

Later we will need a bound on the inverse of $\not{D}_{\mathrm{AC}}$ on the complement of its kernel when A is an α-Cayley.

Proposition 2.35. *Suppose that $A \subset (\mathbb{R}^8, \Phi_0)$ is AC_λ to a Cayley cone $C = L \times \mathbb{R}_+$ with $\lambda < 1$ and α-Cayley for α sufficiently close to 1. Let $\delta \in \mathbb{R}$ with $\delta \notin \mathscr{D}_L$ and suppose $p > 4$, $k \geqslant 1$, $\epsilon > 0$ small. Then there is a subspace $\kappa_{\mathrm{AC}} \subset C^\infty_c(\nu(A))$ such that for any $v \in \ker \not{D}_{\mathrm{AC}} \subset L^p_{k+1,\delta-\epsilon}(\nu_\epsilon(A))$ we have that if v is $L^2_{\delta-\epsilon}$-orthogonal to κ_{AC}, then v must vanish. This subspace called a **pseudo-kernel**, can be chosen of the same dimension as $\ker \not{D}_{\mathrm{AC}}$. If we identify the normal bundles of A for small AC_η perturbations of the $\mathrm{Spin}(7)$-structure via orthogonal projection, then κ_{AC} is also a pseudo-kernel for small perturbations.*

Proof. As the operator $\not{D}_{\mathrm{AC}}$ is Fredholm by assumption, we know that $\ker \not{D}_{\mathrm{AC}}$ is finite-dimensional. Now by [28, Cor. 4.5] we can approximate a given basis $\{a_i\}_{i=1}^l$ of $\ker \not{D}_{\mathrm{AC}}$ arbitrarily well in $L^p_{k+1,\delta}$ by C^∞_c sections. By the Sobolev embedding $L^p_{k+1,\delta} \hookrightarrow L^2_{0,\delta-\epsilon}$ the same is true for $L^2_{\delta-\epsilon}$. These approximations give us the desired subspace κ_{AC}. For nearby $\mathrm{Spin}(7)$-structures this result remains true, as the kernel is perturbed continuously in $L^p_{k+1,\delta-\epsilon}$ by AC_η perturbations of the ambient $\mathrm{Spin}(7)$-structure. □

Proposition 2.36. *In the situation of Proposition 2.35 there is a constant C_{AC} such that the following holds. If $v \in L^p_{k+1,\delta}(\nu(A))$ is $L^2_{\delta-\epsilon}$-orthogonal to κ_{AC} then:*

$$\|v\|_{L^p_{k+1,\delta}} \leqslant C_{\mathrm{AC}} \|\not{D}_{\mathrm{AC}} v\|_{L^p_{k,\delta-1}}. \tag{2.40}$$

The same inequality holds for small AC_η perturbations of the $\mathrm{Spin}(7)$-structure.

Proof. The map $\not{D}_{\mathrm{AC}} : L^p_{k+1,\delta} \to L^p_{k,\delta-1}$ is continuous by Proposition 2.23 and has finite-dimensional co-kernel by the assumption on the weight δ. We claim that $= \not{D}_{\mathrm{AC}}|_{\kappa^\perp_{\mathrm{AC}}}$ is an isomorphism onto its image, where the orthogonal complement is taken with respect to the $L^2_{\delta-\epsilon}$ inner product. Indeed it is injective by the construction of κ_{AC}. Moreover if $w \in \operatorname{im} \tilde{\not{D}}$, then we can find a pre-image $v \in L^p_{k+1,\delta}(\nu(A))$ of w as follows. Since $L^p_{k+1,\delta}(\nu(A)) = \kappa^\perp_{\mathrm{AC}} \oplus \ker \not{D}_{\mathrm{AC}}$, we can consider the $\kappa^\perp_{\mathrm{AC}}$-component v' of v, and note that $\not{D}_{\mathrm{AC}} v' = w$, thus proving surjectivity. Since $\tilde{\not{D}}$ is bijective and continuous, it admits a bounded inverse by the open mapping theorem for Banach spaces. $\square$

We know how $\not{D}$ varies if an almost Cayley is perturbed by a vector field (see Corollary 2.6), hence we can determine precisely what the convergence rate of $\not{D}$ to the conical model is depending on the rate of the AC manifold.

Lemma 2.37. *Let $A \subset (\mathbb{R}^8, \Phi_0)$ be an almost Cayley submanifold which can be seen as a perturbation of a Cayley cone $C \subset \mathbb{R}^8$ by a normal vector field $v \in C^\infty(\nu(C))$ with $\|v\|_{C^{k+1}_\gamma} = 1$ for $\gamma \in \mathbb{R}$. We identify the tensor bundles on A and C so that the Cayley operator $\not{D}$ of A and the Cayley operator of the cone $\not{D}_{\mathrm{con}}$ are both defined on C. For any rate $\zeta \in \mathbb{R}$ we then have the pointwise estimate:*

$$|(\not{D} - \not{D}_{\mathrm{con}})s|_{C^k_{\zeta-1}} \lesssim r^{\gamma-1}|s|_{C^{k+1}_\zeta}.$$

2.5 Conically singular case

Let $N \subset (M, \Phi_{s_0})$ be a $\mathrm{CS}_{\bar{\mu}}$ Cayley submanifold with conical singularities at $\{z_1, \ldots, z_l\}$ and rates $\bar{\mu} = (\mu_1, \ldots, \mu_l)$, where the $\mu_i \in (1, 2)$ for $1 \leqslant i \leqslant l$. Fix a Spin(7)-parametrisation χ_i around the singular point z_i. With respect to the parametrisation χ_i, let N be asymptotic to the cone $C_i \subset \mathbb{R}^8$. Thus N decays to the cone C_i in $O(r^{\mu_i})$ as the distance r to the singular point goes to 0. We denote the link of the cone C_i by $L_i \subset S^7$. In the deformations of N that we will consider, we will allow the singular points and the asymptotic cones to move via translations, rotations and associative deformations of the link. Furthermore we will allow the Spin(7)-structure to vary in a family $\{\Phi_s\}_{s \in \mathcal{S}}$ where $s_0 \in \mathcal{S}$. Thus we will study the following moduli space:

$$\begin{aligned}\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S}) = \{(\tilde{N}, \Phi_s) : \; & \tilde{N} \subset (M, \Phi_s) \text{ is a } \mathrm{CS}_{\bar{\mu}}\text{-Cayley with} \\ & \text{singularities } \tilde{z}_1, \ldots, \tilde{z}_l \text{ and cones } \tilde{C}_1, \ldots, \tilde{C}_l, \text{. Here } \tilde{N} \text{ is} \\ & \text{isotopic to } N, \text{ where the isotopy takes } z_i \text{ to } \tilde{z}_i, \text{ and} \\ & \tilde{C}_i \text{ is a deformation of } C_i\}.\end{aligned}$$

Locally around the fixed Cayley N this moduli space will be given as a zero set of a nonlinear operator between suitable Banach manifolds. This is an extension of the work done in [36], where the deformations are required to fix the cones. To define the nonlinear operator, we first define the configuration space of small deformations of the tuple $(C_1, \ldots, C_l)$. Let U_i be an open neighbourhood of $z_i \in M$ and let $G_i \subset \mathrm{Spin}(7)$ be the stabiliser of the cone C_i, which we also assume is the stabiliser of any deformation of

C_i. The configuration space is then given by:

$$\mathcal{F} = \prod_{i=1}^{l} \{(\tilde{L}_i, e_i, s) \mid e_i : \mathbb{R}^8 \to T_{\tilde{z}_i} M : e_i \text{ Spin(7)-frame for } \Phi_s,\ \tilde{z}_i \in U_i,\ \tilde{L}_i \in \mathcal{M}^{G_2}(L_i)\}/G_i.$$

It is a $H = \prod_{i=1}^{l} \mathrm{Spin}(7)/G_i$-bundle over the spaces $\mathcal{V}$ of possible vertex locations and cones for every member of $\{\Phi_s\}_{s\in\mathcal{S}}$, i.e. $\mathcal{V} = \mathcal{S} \times \prod_{i=1}^{l}(U_i \times \mathcal{M}(C_i))$. Each element $(\bar{x}, \bar{L}, \bar{e}, s) \in \mathcal{F}$ corresponds bijectively to a unique configuration of cones, since we took quotients by the stabilisers G_i. The asymptotic data for N is given by:

$$f_0 = (z_1, \ldots, z_l, L_1, \ldots, L_l, \mathrm{D}\chi_1(0), \ldots, \mathrm{D}\chi_l(0), s_0).$$

We now fix such a reference $\mathrm{CS}_{\bar{\mu}}$-fourfold for each point in a small neighbourhood of f_0.

Proposition 2.38. *There is a smooth family N_f of $\mathrm{CS}_{\bar{\mu}}$-manifolds which is parametrised by $f \in \mathcal{U} \subset \mathcal{F}$, where $\mathcal{U}$ is an open neighbourhood of f_0, such that N_f has asymptotic data f. We can choose $N_{f_0} = N$.*

Proof. Without loss of generality, we can restrict to the case of a single vertex, while only perturbing N in an arbitrarily small neighbourhood of the vertex to obtain the desired family. Let $z_0 \in N$ be singular with cone $C = \mathbb{R}_+ \times L$ with regards to the Spin(7)-parametrisation χ_0. Consider diffeomorphisms of the unit ball in $\mathbb{R}^8$ by the action of $(A, v) \in \mathrm{GL}(8) \rtimes \mathbb{R}^8$, denoted by $\varphi_{A,v}$. They are isotopic to the identity and in fact can be extended to a smooth family (also denoted by $\varphi_{A,v}$) of self-diffeomorphisms of $\mathbb{R}^8$ which leave everything outside of the ball with radius 2 unchanged (see for instance the Homogeneity Lemma in Chapter 4 of [35]). This family, scaled down sufficiently, can be applied in the chart given by χ_0 to apply any desired small translation and rotation to the asymptotic cone, while only perturbing an arbitrarily small neighbourhood of the vertex. Finally, since any $\tilde{L} \in \mathcal{M}^{G_2}(L)$ is smoothly isotopic to L, we can perturb any CS_μ manifold asymptotic to C to be asymptotic to $\mathbb{R}_+ \times \tilde{L}$ instead, with the same rate. □

If we restrict the previous family to a sufficiently small neighbourhood $\mathcal{U}$ of f_0, its members will be α-Cayley for any desired $\alpha < 1$. Let now ρ be a radius function for N. By Proposition 1.6 we know that $\nu_\epsilon(N)$ maps onto a tubular neighbourhood U_{f_0} of N inside M, for $\epsilon > 0$ sufficiently small. Composing this open embedding with the ambient isotopy from 2.38 taking N_{f_0} to N_f gives a tubular neighbourhood U_f of N_f, for f sufficiently close to f_0. We denote these maps by:

$$\exp_f : \nu_\epsilon(N) \to U_f.$$

Furthermore, given a normal vector field $v \in C^\infty(\nu_\epsilon(N))$ we define the embedding $\exp_{f,v} : N \to M$ as the composition $\exp_{f,v} = \exp_f \circ v$. Thus varying f will perturb the asymptotic cones while changing v alters the shape of the $\mathrm{CS}_{\bar{\mu}}$-manifold, keeping the cones fixed. The moduli space $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N)$ is given as the zero set of the following non-linear differential operator:

$$\begin{aligned} F_{\mathrm{CS}} : C^{\infty}_{\bar{\mu}}(\nu_\epsilon(N)) \times \mathcal{U} &\longrightarrow C^{\infty}_{\mathrm{loc}}(E_{\mathrm{cay}}) \\ (v, f = (\bar{x}, \bar{L}, \bar{e}, s)) &\longmapsto \pi_E \star_4 \mathrm{Exp}_f^*(\tau_{\Phi_s}|_{\exp_{f,v}(N)}). \end{aligned} \tag{2.41}$$

We will now address the necessary modifications to the proofs for compact Cayleys so that they extend to the conically singular setting. First, let us define the correct Banach manifolds. Let, for $\epsilon > 0$:

$$\mathcal{L}_\epsilon = \{v \in L^p_{k+1,\bar{\mu}}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1,\bar{\mu}}} < \epsilon\}. \tag{2.42}$$

In fact F_{CS} is mapping $\mathcal{L}_\epsilon \times \mathcal{U} \to L^p_{k,\bar{\mu}-1}(E_{\mathrm{cay}})$ for sufficiently small ϵ. We will again prove boundedness separately for the constant, linear and quadratic and higher terms in the expansion:

$$F_{\mathrm{CS}}(v, f) = F_{\mathrm{CS}}(0, f) + \not{D}_{\mathrm{CS},f} v + Q_{\mathrm{CS}}(v, f). \tag{2.43}$$

First, note that the closeness of N_f to a Cayley cone gives a bound on $F_{\mathrm{CS}}(0, f)$, which measures the failure of N_f to be Cayley.

Proposition 2.39. *There is a constant $C_k > 0$ such that for any $f \in \mathcal{U}$ in an open neighbourhood $f_0 \in U_k$, we have $\|F_{\mathrm{CS}}(0, f)\|_{C^k_{\bar{\mu}-1}} \leqslant C_k$.*

Proof. Let $C_f \subset \mathbb{R}^8$ be the asymptotic cone of N_f near a fixed singular point $z \in M$, where N_f has decay rate μ, and consider everything in a small ball $B_\eta(0) \subset \mathbb{R}^8$ via the parametrisation χ. Let ι_f be the embedding of the abstract cone C as C_f and let Θ_f be a parametrisation of the end of N_f by C. For both of these, we implicitly choose some identification of the potentially different links for varying f. Note that in this formulation the Spin(7)-structure on $B_\eta(0)$ only needs to agree with Φ_0 at the origin. The assignment $(r, p, f) \to \Theta_f(r, p)$ is smooth, and thus we have from the CS_μ-condition:

$$|\nabla^i(\Theta_f(r, p) - \iota_f(r, p))| \leqslant K_{i,f} r^{\mu - i}, \tag{2.44}$$

where the constant $K_{i,f}$ is continuous in f. In particular, after shrinking $\mathcal{O}$, we can replace $K_{i,f}$ by a single constant K_i. Consider τ now as a vector bundle morphism:

$$\tau : \Lambda^4 B_\eta(0) \to E_{\mathrm{cay}}.$$

The (higher) covariant derivatives of τ can then be considered as maps:

$$\nabla^i \tau : \Lambda^4 B_\eta(0) \otimes (TM)^{\otimes^i} \to E_{\mathrm{cay}}.$$

By the compactness of the base, any finite number of derivatives can be bounded by a constant. We can also consider TN_f and TC_f as maps $C \to \Lambda^4\mathbb{R}^8$, by the embedding $\mathrm{Gr}(4, M) \to \Lambda^4 M$. The CS_μ condition (2.44) then translates to:

$$|\nabla^i(T_p N_f - T_p C_f)| \leqslant K_i r^{\mu - 1 - i}.$$

By Taylor's theorem, τ has the following decay behaviour near $z \in N_f$:

$$|\nabla^i \tau_{r,p}(T_{r,p}C_f)| = |\nabla^i(\tau_{r,p}(T_{r,p}C_f) - \tau_0(T_{r,p}C_f))| \leqslant 2r|\nabla^{i+1}\tau_0| \lesssim r$$

Now we see that:

$$\begin{aligned}|\nabla^i \tau_{r,p}(T_{r,p}N_f)| &\lesssim r + |\nabla^i(\tau_{r,p}(T_{r,p}N_f) - \tau_{r,p}(T_{r,p}C_f))| \\ &\lesssim r + \sum_{a+b=i} |\nabla^a \tau_0||\nabla^b(T_pN_f - T_pC_f)| \lesssim r^{\mu-1-i}.\end{aligned}$$

Thus the result holds for a single singular point. The generalisation to multiple singular points is straightforward. □

Next, we turn our attention to the quadratic estimates:

Proposition 2.40. *Suppose that N is $\mathrm{CS}_{\bar{\mu}}$ and α-Cayley for α sufficiently close to 1. Fix $k \in \mathbb{N}$ and let $u, v \in C^k_{\bar{\mu}}(\nu_\epsilon(N))$ with $\epsilon > 0$ sufficiently small and $|u|_{C^1}, |v|_{C^1} \leqslant \epsilon$. Then there is an open neighbourhood $f \in \tilde{\mathcal{U}} \subset \mathcal{U}$ such that:*

$$\begin{aligned}|Q_{\mathrm{CS}}(u,f) - Q_{\mathrm{CS}}(v,f)|_{C^k_{\bar{\mu}-1}} \lesssim |u-v|_{C^{k+1}_{\bar{\mu}}}\left(|u|_{C^k_{\bar{\mu}}} + |v|_{C^k_{\bar{\mu}}}\right) &+ \\ |u-v|_{C^k_{\bar{\mu}}}\left(|u|_{C^{k+1}_{\bar{\mu}}} + |v|_{C^{k+1}_{\bar{\mu}}}\right)&.\end{aligned}$$

Here the constant hidden in $\lesssim$ is independent of f.

Proof. Without loss of generality, consider the case where N has just one singular point z with rate μ. We then define the smooth function $\mathbf{Q}(p, v, \nabla v, T_pN, s)$ as we did for the compact case in Lemma 2.9. Now, even though N is not compact, there are still bounds on all derivatives of $\mathbf{Q}$ as in the compact case. From our assumptions on u and v, we can ensure that $(u, \nabla u)$ vary in a compact set for all the sections in question and any point in $N \times \mathcal{S}$. Thus we can prove the bound (2.12) for a constant independent of $(p,s) \in N \times \mathcal{S}$ or the section in question. Thus we obtain:

$$\begin{aligned}&|\nabla^k(Q_{\mathrm{CS}}(u) - Q_{\mathrm{CS}}(v))|\rho^{-(k+2)(\mu-1)+k} \\ &\lesssim \sum_{\substack{i+|J|+r\leqslant k+2 \\ 0\leqslant r\leqslant k}} |\nabla^i(u-v)|\rho^{i-\mu}(|\nabla^J u| + |\nabla^J v|)\rho^{|J|-\sharp J\cdot\mu}|\nabla^r TN|\rho^{r-\mu+1} \\ &\lesssim \sum_{\substack{i+|J|+r\leqslant k+2 \\ 0\leqslant r\leqslant k}} |u-v|_{C^i_{\mu+1}}(|u|_{C^J_\mu} + |v|_{C^J_\mu})|TN|_{C^r_{\mu-1}}\end{aligned}$$

Here $\sharp J$ denotes the number of entries in the multi-index J and C^J is the product of the norms $\prod_{s=1}^l |v|_{C^{i_s}}$. We used the fact that $\rho^{i+2-\mu}$ is bounded on N to remove extraneous factors of ρ. For this, it was crucial to assume $\mu < 2$. Now simply note that $\|TN\|_{C^r_{\mu-1}} < \infty$ by the CS_μ condition. The result now follows, since $C^k_{(k+2)(\mu-1)} \hookrightarrow C^k_{\mu-1}$ is a continuous embedding. □

The deformation map F_{CS} then extends to a map between Sobolev spaces as follows:

Proposition 2.41. *Let $p > 4$ and $k \geqslant 1$. For sufficiently small $\epsilon > 0$ the map F_{CS} extends to a C^∞ map of Sobolev spaces:*

$$F_{\mathrm{CS}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\bar{\mu}}(\nu_\epsilon(N)) : \|v\| \leqslant \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\bar{\mu}-1}(E_{\mathrm{cay}}).$$

Furthermore, any $v \in L^p_{k+1,\delta}(\nu_\epsilon(N))$ such that $F_{\mathrm{CS}}(v) = 0$ is smooth and lies in $C^\infty_{\bar{\mu}}$. The linearisation at 0 is the bounded linear map:

$$\not{D}_{\mathrm{CS}} : L^p_{k+1,\bar{\mu}}(\nu(N)) \oplus T_f\mathcal{U} \longrightarrow L^p_{k,\bar{\mu}-1}(E_{\mathrm{cay}}).$$

Finally, $\not{D}_{\mathrm{CS}}$ is Fredholm if all the rates in $\bar{\mu}$ are in the complement of a discrete set $\mathscr{D} \subset \mathbb{R}$, which is determined by asymptotic cones $C_i \subset \mathbb{R}^8$, seen as a Cayley.

Proof. The proof is identical to the one for the AC case 2.20, except that one needs to check that the dependence in $f \in \mathcal{F}$ respects the weighted space, i.e. that the derivatives $\partial^k_f \mathcal{F}(v, f)$, which are a priori maps $C^\infty_{\bar{\mu}} \times (T_f\mathcal{F})^k \to C^\infty_{loc}$ can be extended to maps $L^p_{k+1,\bar{\mu}} \times (T_f\mathcal{F})^k \to L^p_{k,\bar{\mu}-1}$. For this, consider a smooth deformation $f(t) \in \mathcal{F}$ of a manifold N with a unique singular point at the origin of $\mathbb{R}^8$ and rate μ. Up to first-order this is equivalent to deforming the Spin(7)-structure in C^∞_1 (which also takes care of the translations, since we always compare to the model CS manifold N_f), and perturbing the manifold near the cone by a vector field $u \in C^\infty_1(\nu(N))$, while keeping the singular point fixed, as well as the Spin(7)-structure at the origin. Let φ_t be the flow associated to u. We then have for $v \in C^\infty_\mu(\nu_\epsilon(N))$ and $p \in N$:

$$\begin{aligned}\partial_f F_{v,f(0)}[\dot{f}(0)](p) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{F}(\varphi_t(p), (\varphi_t)_* v(p), (\varphi_t)_*\nabla v(p), (\varphi_t)_* T_p N, \Phi(t)) \\ &= \mathrm{D}\mathbf{F}\left[u, -\mathcal{L}_u v, -\mathcal{L}_u \nabla v, -\mathcal{L}_u TN, \dot{\Phi}(0)\right].\end{aligned}$$

Now we see that all the arguments in square brackets are in $O(r^{\mu-1})$, either by definition (like u an $\dot{\Phi}(0)$), or as a consequence thereof. The norm of the term $\mathcal{L}_u\nabla v$ for instance can be bounded by $|u||\nabla^2 v| + |\nabla u||\nabla v|$, which is in $O(r^{\mu-1})$ by assumption. As $\mathrm{D}\mathbf{F}$ can be bounded by a constant independent of the chosen CS manifold, we find that:

$$|\partial_f F_{v,f(0)}| \lesssim |v|_{C^2_1} + |TN|_{C^1_0} + 1.$$

The argument we presented also applies to higher derivatives and so we see that $\partial_f F$ maps $L^p_{k+1,\bar{\mu}} \times (T_f\mathcal{F})^k \to L^p_{k,\bar{\mu}-1}$, as required. □

Using this we can now prove the following result about the local structure of the family moduli space $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$, where we now also include the deformation of the Spin(7)-structure, just as in the AC case.

Theorem 2.42 (Structure)**.** *Suppose $p > 4$ and $k \geqslant 1$. Let N be an $\mathrm{CS}_{\bar{\mu}}$ Cayley submanifold of (M, Φ_{s_0}), and suppose $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of deformations of Φ_{s_0}. Let $\mathcal{F}$ be the configuration space of possible singularities and deformations of the asymptotic cones* *of N, where the asymptotic data of N itself is given by $f_0 \in \mathcal{F}$. Then there is an open neighbourhood $(s_0, f_0) \in \mathcal{U} \subset \mathcal{S} \times \mathcal{F}$ and a non-linear deformation operator F_{CS} which for $\epsilon > 0$ sufficiently small is a C^∞ map:*

$$F_{\mathrm{CS}} : \mathcal{L}_\epsilon = \{v \in L^p_{k+1,\bar{\mu}}(\nu_\epsilon(N)), \|v\|_{L^p_{k+1,\bar{\mu}}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\bar{\mu}-1}(E).$$

A neighbourhood of (N, Φ_{s_0}) in $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ is homeomorphic to the zero locus of F_{CS} near $(0, \Phi_{s_0}, f_0)$. Furthermore we can define the ***deformation space*** *$\mathcal{I}^{\bar{\mu}}_{\mathrm{CS}}(N) \subset C^\infty_{\bar{\mu}}(\nu(N)) \oplus T_{s_0}\mathcal{S} \oplus T_{f_0}\mathcal{F}$ to be the the kernel of $\not{D}_{\mathrm{CS},\mathcal{S}} = \mathrm{D}F_{\mathrm{CS}}(0, \Phi_0, f_0)$, and the* ***obstruction space*** *$\mathcal{O}^{\bar{\mu}}_{\mathrm{CS}}(N) \subset C^\infty_{-4-\bar{\mu}}(E)$ to be the cokernel of $\not{D}_{\mathrm{CS},\mathcal{S}}$. Then a neighbourhood of N in $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ is also homeomorphic to the zero locus of a Kuranishi map:*

$$\kappa^{\bar{\mu}}_{\mathrm{CS}} : \mathcal{I}^{\bar{\mu}}_{\mathrm{CS}}(N) \longrightarrow \mathcal{O}^{\bar{\mu}}_{\mathrm{CS}}(N).$$

In particular if $\mathcal{O}^{\bar{\mu}}_{\mathrm{CS}}(N) = \{0\}$ is trivial, $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ admits the structure of a smooth manifold near N. We say that N is ***unobstructed*** *in this case.*

We can now define a notion of pseudo-kernel as in 2.35. This is entirely analogous, except that the Sobolev embedding $L^2_{\delta+\epsilon} \to L^p_{k+1,\delta}$ requires us to slightly increase the rate of the L^2 sections.

Proposition 2.43. *Suppose that N is $\mathrm{CS}_{\bar{\mu}}$ to Cayley cones and α-Cayley for α sufficiently close to 1. Let $\delta \in \mathbb{R}$ with $\delta \notin \mathscr{D}_{L_i}$ not critical for any of the links of N and suppose $p > 4$, $k \geqslant 1$ and $\epsilon > 0$ small. Then there is a subspace $\kappa_{\mathrm{CS}} \subset C^\infty_c(\nu(N))$ such that for any $v \in \ker \not{D}_{\mathrm{CS}} \subset L^p_{k+1,\delta}(\nu(N))$ we have that if v is $L^2_{\delta+\epsilon}$-orthogonal to κ_{CS}, then v must vanish. This subspace, called a* ***pseudo-kernel*** *can be chosen of dimension dim $\ker \not{D}_{\mathrm{CS}}$.*

Proposition 2.44. *In the situation of Proposition 2.43 there is a constant C_{CS} such that the following holds. If $v \in L^p_{k+1,\bar{\delta}}(\nu(N))$ is $L^2_{\bar{\delta}+\epsilon}$-orthogonal to κ_{CS} then:*

$$\|v\|_{L^p_{k+1,\bar{\delta}}} \leqslant C_{\mathrm{CS}} \|\not{D}_{\mathrm{CS}} v\|_{L^p_{k,\bar{\delta}-1}}. \tag{2.45}$$

The same inequality is true for perturbations of N with $\bar{\mu} \geqslant \bar{\delta}$.

Remark 2.45. The operator F_{CS} allows for the points of the singular cones to move. We could also fix the points while still allowing the links of the cones to deform, giving us an operator $F_{\mathrm{CS,cones}}$. We can give this operator the same treatment and reprove all the theorems in this section. Similarly one can consider an operator $F_{\mathrm{CS,fix}}$, where neither the points nor the cones are allowed to deform and again all the same statements are true for this operator. We denote the associated families of points and cones by $\mathcal{U}_{\mathrm{cones}}$ and $\mathcal{U}_{\mathrm{fix}}$ respectively. These will be submanifolds of $\mathcal{U}$, where all movements of the points and cones are allowed.

We again have a formula for the index, where we define $\sigma(N)$ and $[N] \cdot_{[u_1],\ldots,[u_l]} [N]$ in the same way as for the AC case.

Proposition 2.46 (Index)**.** *Let N be an $\mathrm{CS}_{\bar{\mu}}$ Cayley submanifold of (M, Φ) with cones $C_i = \mathbb{R}_i \times L_i (1 \leqslant i \leqslant l)$, and suppose $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of small perturbations* *of Φ. Assume that $(1,\mu_i] \cap \mathscr{D}_{L_i} = \emptyset$. Pick homotopy classes $[u_i] \in [L_i, S\nu(N)|_{L_i}]$ (where $S\nu$ is the sphere bundle). Then the following holds:*

$$\operatorname{ind} \not{D}_{\mathrm{CS},\mathcal{S}} = \frac{1}{2}(\sigma(N)+\chi(N)) - [N]\cdot_{[u_1],\dots,[u_l]}[N] - \sum_{i=1}^{l}(\eta(L_i)+T([u_i])) + \dim\mathcal{F}. \quad (2.46)$$

Here $\eta(L)$ and $T([u_i])$ are the quantities from Proposition 2.24.

Remark 2.47. We used the additivity of the index for $\not{D}$ in the proof of the index formulae for both the AC and the CS case, hence it will not be surprising that the formulae also satisfy the same additivity. Let us look at a concrete example in more detail. Suppose hence that $\hat{N}$ is a $\mathrm{CS}_{\bar{\mu}}$ Cayley in (M,Φ) with a unique singular point, with an unobstructed cone $C = \mathbb{R}_+ \times L$. We consider the deformations of $\hat{N}$ for a fixed Spin(7)-structure and point, but moving cone. Let furthermore $A \subset \mathbb{R}^8$ be an asymptotically conical Cayley of rate $\lambda < 1$, with the same cone.

We now look at an almost Cayley manifold $N = \hat{N}\sharp_L A$, obtained as a connected sum of $\hat{N}$ with A over their common end $\mathbb{R}_+ \times L$. The nonsingular manifold N admits a deformation operator $\not{D}_N$. Pick an arbitrary class $[u] \in [L, S\nu(N)|_L] \simeq [L, S\nu(A)|_L]$, and assume that $[\lambda,1)\cap\mathscr{D}_L = (1,\mu]\cap\mathscr{D}_L = \emptyset$. Then we have the following, where we consider the deformation problem with fixed points and cones on the conically singular side:

$$\begin{aligned}
\operatorname{ind}_\mu \not{D}_{\mathrm{AC}} + \operatorname{ind}_\mu \not{D}_{\mathrm{CS,fix}} &= \frac{1}{2}(\sigma(A)+\chi(A)) - [A]\cdot_{[u]}[A] \\
&\quad + \frac{1}{2}(\sigma(\hat{N})+\chi(\hat{N})) - [\hat{N}]\cdot_{[u]}[\hat{N}] \\
&\quad + \eta(L) + T([u]) - \eta(L) - T([u]) \\
&= \frac{1}{2}(\sigma(A)+\sigma(\hat{N})+\chi(A)+\chi(\hat{N})) \\
&\quad - ([A]\cdot_{[u]}[A] + [\hat{N}]\cdot_{[u]}[\hat{N}]) \\
&= \frac{1}{2}(\sigma(N)+\chi(N)) - [N]\cdot[N] \\
&= \operatorname{ind} \not{D}_N.
\end{aligned} \quad (2.47)$$

Note that we only proved our AC index formula for rates < 1. However, using the unobstructedness of the cone (meaning $d(1) = \dim \mathcal{M}^{G_2}(L)$) and Theorem 1.32 we see that $\operatorname{ind}_\mu \not{D}_{\mathrm{AC}} = \operatorname{ind}_\lambda \not{D}_{\mathrm{AC}} + \dim \mathcal{M}^{G_2}(L)$. We note that by construction:

$$\operatorname{ind} \not{D}_{\mathrm{CS}} - \operatorname{ind} \not{D}_{\mathrm{CS,fix}} = \dim\mathcal{F} - \dim\mathcal{S} = \dim\mathcal{F},$$

as $\dim\mathcal{S} = 0$ by assumption. For the third inequality, notice that the link L is a compact three-manifold, and thus has Euler characteristic $\chi(L) = 0$. Thus $\chi(A) + \chi(\hat{N}) = \chi(N)$. Similarly, the signature is also additive [1, Theorem 4.14]. Finally, the intersection numbers with fixed boundary behaviour are also additive, as they can be obtained by simply counting self-intersection points. Thus the indices of the conical operators add up to the index of the glued manifold, which is to be expected, as perturbations of the glued manifold should correspond bijectively to perturbations in either piece. Note that we equally well have:

$$\operatorname{ind} \not{D}_N = \operatorname{ind}_\lambda \not{D}_{\mathrm{AC}} + \operatorname{ind}_\mu \not{D}_{\mathrm{CS,cones}}. \tag{2.48}$$

Whereas before all perturbations of rates $\leqslant 1$ were considered part of the asymptotically conical piece, now the perturbations of rate exactly 1 are considered perturbations of the conically singular piece. If the cone satisfies has no critical rates between 0 and 1, we can even go one step further and pick some $\lambda' < 0$ with $[\lambda', 0) \cap \mathscr{D}_L = \emptyset$. We then have:

$$\operatorname{ind} \not{D}_N = \operatorname{ind}_{\lambda'} \not{D}_{\mathrm{AC}} + \operatorname{ind}_\mu \not{D}_{\mathrm{CS}}. \tag{2.49}$$

Chapter 3

Desingularisation of conically singular Cayley submanifolds

In this chapter, we discuss the desingularisation of conically singular Cayley submanifolds. We first describe a gluing construction which, in its simplest form, takes a CS Cayley N and an AC Cayley A with identical asymptotic cones and produces an approximate Cayley desingularisation by gluing a rescaled version of A onto the singularity of N. Next, we describe an iteration scheme that allows us to perturb the approximate Cayley to a nearby exact Cayley. We modify the construction from Lotay's work [30] on coassociative submanifolds (which in turn builds on previous work by Joyce [18] for special Lagrangians) to work in families and rework some analytic aspects to remove the requirements on the rate λ of the asymptotically conical piece. This leads us to the main theorem of this section, Theorem 3.15. It includes the desingularisation of multiple singular points at different rates as well as partial desingularisation.

We then conclude the chapter by considering in more detail the desingularisation of a particular kind of conical singularity, namely the transverse intersections of immersed Cayleys. We will see that negative intersections may be resolved by gluing in a Lawlor neck, while positive intersections cannot be resolved while at the same time preserving the Cayley condition.

3.1 Approximate Cayley submanifolds

Let (M, Φ) be a Spin(7)-manifold and let $\{\Phi_s\}_{s\in\mathcal{S}}$ be a smooth family of deformations of $\Phi = \Phi_{s_0}$. Suppose N is an unobstructed $\mathrm{CS}_{\bar{\mu}}$-Cayley in (M, Φ) with singular points $\{z_i\}_{i=1,\dots,l}$. We now consider the family moduli space $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \mathcal{S})$ of deformations of N. Note that if the locus of singular points for a fixed Spin(7)-structure (which is the image of the smooth map sending a Cayley to one of its singular points, by unobstructedness) were to move by an ambient isotopy I_s, we can choose a new family $\{I_s^*\Phi_s\}_{s\in\mathcal{S}}$ that leaves the singular locus invariant. Furthermore, we can also assume that $\Phi_s(z_i) = \Phi_{s_0}(z_i)$.

For $B_\eta(0)$ the ball of radius $\eta > 0$ in $\mathbb{R}^8$, let $\chi_i : B_\eta(0) \to M$ be a Spin(7)-coordinate system centred around z_i. Recall that this means that χ_i is a parametrisation of a neighbourhood of z_i, such that $\chi_i(0) = z_i$ and $\mathrm{D}\chi_i|_0^*\Phi_{z_i} = \Phi_0$. After identifying $T_{z_i}M$ with $\mathbb{R}^8$

via the Spin(7)-isomorphism $\mathrm{D}\chi_i|_0$, we let $(L_i, h_i) \subset (S^7, g_{\mathrm{round}})$ be the link on which the conical singularity is modelled. Assume it comes in a smooth, finite-dimensional moduli space $\mathcal{M}^{G_2}(L_i)$, and that $\bar{\mu}$ be such that $(1, \mu_i] \cap \mathscr{D}_{L_i} = \emptyset$.

Fix now $1 \leqslant k \leqslant l$ and $\lambda < 1$. For each $1 \leqslant i \leqslant k$ let A_i be an AC_λ Cayley in $\mathbb{R}^8$ for the standard Spin(7)-structure, with $\mathscr{D}_{L_i} \cap (\lambda, 1) = \emptyset$. Let the link of A_i be (L_i, h_i) (the same as the link of the i-th singularity of N), and choose a scale function $t_i : \mathcal{M}^{\lambda}_{\mathrm{AC}}(A_i) \to \mathbb{R}$.

We will now describe a procedure which allows us to glue elements of sufficiently small scale in $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A_i)$ onto the first k singular points of $N \in \mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N_0, \Phi_s)$, to produce Cayleys in (M, Φ_s) that are close to being singular. Here we need to make sure to glue compatible cones, as both moduli spaces allow for deformations of the cone.

In the gluing construction, the scale t_i determines both the scaling of the AC piece A_i as well as the inner radius of the annuli joining A_i to N, which is comparable to $L_i \times (t_i r_0, R_0)$, where $r_0, R_0 > 0$ are constants. In particular, when $t_i = 0$ (which corresponds to the cone in $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A_i)$) we do not glue anything into the singularity at z_i.

Recall from the definition of a conically singular submanifold that there is a compact set $K_N \subset N$ and decomposition $N = K_N \bigsqcup_{i=1}^{l} U_i$ such that we have diffeomorphisms $\Psi^i_{\mathrm{CS}} : L_i \times (0, R_0) \to U_i$. Choose η and R_0 in such a way that the image of Ψ^i_{CS} is contained in $\chi_i(B_\eta(0))$. We can then factor $\Psi^i_{\mathrm{CS}} = \chi_i \circ \Theta^i_{\mathrm{CS}}$, where Θ^i_{CS} is a smooth map $\Theta^i_{\mathrm{CS}} : L_i \times (0, R_0) \to B_\eta(0)$. For $1 \leqslant i \leqslant k$ there is a similar diffeomorphism $\Theta^i_{\mathrm{AC}} : L_i \times (r_0, \infty) \to A \setminus K_{A_i} \subset \mathbb{R}^8$, where K_{A_i} is a compact subset of A_i, which can be chosen such that $\|\Theta^i_{\mathrm{AC}}(p) - \iota_i(p)\|_{\mathbb{R}^8} = O(|p|^\lambda)$ as $p \to \infty$. After reducing the scale of the A_i, we can assume that $r_0 < R_0$ and $A_i \setminus \Theta^i_{\mathrm{AC}}(L_i \times (R_0, \infty))$ is contained in $B_\eta(0)$. In particular we can then also consider the map $\Psi^i_{\mathrm{AC}}|_{A_i \setminus L_i \times (R_0, \infty)} = \chi_i \circ \Theta^i_{\mathrm{AC}}$. Now fix a smooth cut-off function $\varphi_{\mathrm{cut}} : \mathbb{R} \to [0, 1]$ with the property that:

$$\varphi_{\mathrm{cut}}|_{(-\infty, \frac{1}{4}]} = 0, \quad \varphi_{\mathrm{cut}}|_{[\frac{3}{4}, +\infty)} = 1. \tag{3.1}$$

Let a constant $0 < \nu < 1$ be given and suppose $t > 0$ is sufficiently small so that we have the inequalities $0 < r_0 t < \frac{1}{2} t^\nu < t^\nu < R_0 < 1$. Suppose that $\bar{A} = (A_1, \ldots, A_k)$ is a collection of AC_λ manifolds (or cones) as above such that $t_i = t_i(A_i) \leqslant t$. If $t \geqslant 0$ is minimal with this property we call it the **global scale** of $\bar{A}$. We then define the subsets $N^{\bar{A}}$ of M as follows:

$$\begin{aligned} N^{\bar{A}} = &\left(N \setminus \bigsqcup_{i=1}^{l} \Psi^i_{\mathrm{CS}}(L_i \times (0, t_i^\nu)) \right) \sqcup \bigsqcup_{i=1}^{k} \Psi^i_{\bar{A}}(L_i \times (r_0 t_i, t_i^\nu)) \\ &\sqcup \bigsqcup_{i=1}^{k} \chi_i(A_i \setminus \Theta^i_{\mathrm{AC}}(r_0 t_i, \infty)) \sqcup \bigsqcup_{i=k+1}^{l} \Psi^i_{\mathrm{CS}}(L_i \times (0, t_i^\nu)). \end{aligned} \tag{3.2}$$

Here $\Theta^i_{\bar{A}}$ is defined as the following interpolation between the A_i and U_i pieces:

$$\begin{aligned} &\Theta^i_{\bar{A}} : L_i \times (r_0 t_i, R_0) \longrightarrow \mathbb{R}^8 \\ &(p, s) \longmapsto (1 - \varphi_{\mathrm{cut}}) \left(\frac{2s}{t_i^\nu} - 1 \right) \Theta^i_{\mathrm{AC}}(p, s) + \varphi_{\mathrm{cut}} \left(\frac{2s}{t_i^\nu} - 1 \right) \Theta^i_{\mathrm{CS}}(p, s). \end{aligned} \tag{3.3}$$

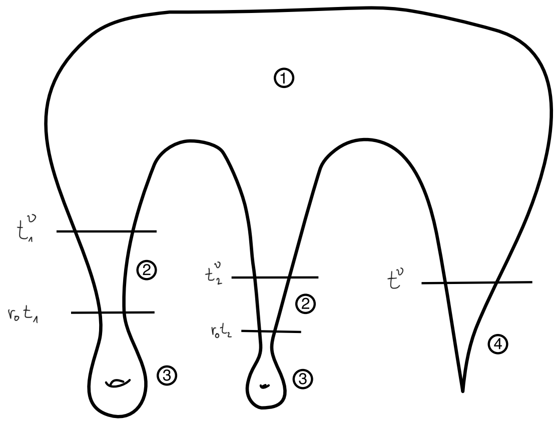

Figure 3.1: Glued manifold

If we reduce the scale of a subset of the asymptotically conical pieces, the resulting family are desingularisations of N where some tips collapse back to conically singular points. In particular, if $t_i = 0$, we should interpret the above definition as $\Theta^i_{\bar{A}} = \Theta^i_{\mathrm{CS}}$, thus the corresponding singularity is left as is, without gluing. As before, we also have maps $\Psi^i_{\bar{A}} = \chi_i \circ \Theta^i_{\bar{A}}$, so we can work in local coordinates around a singularity. Notice that as φ is locally constant in neighbourhoods of 0 and 1, $N^{\bar{A}}$ is a smooth submanifold. For analytic purposes we usually consider $N^{\bar{A}}$ as a union of four parts:

① $N^{\bar{A}}_u = \left(N \setminus \bigsqcup_{i=1}^k \Psi^i_{\mathrm{CS}}\left(L_i \times (0, t_i^\nu)\right)\right).$

② $N^{\bar{A}}_m = \bigsqcup_{i=1}^k \Psi^i_{\bar{A}}(L_i \times (r_0 t_i, t_i^\nu)) = \bigsqcup_{i=1}^k N^{A_i}_m.$

③ $N^{\bar{A}}_l = \bigsqcup_{i=1}^k \chi_i \left(A_i \setminus \Theta^i_{\mathrm{AC}}\left(r_0 t_i, \infty\right)\right) = \bigsqcup_{i=1}^k N^{A_i}_l.$

④ $N^{\bar{A}}_p = \bigsqcup_{i=k+1}^l \bigsqcup_{i=1}^l \Psi^i_{\mathrm{CS}}(L_i \times (0, t^\nu)).$

Notice that since we chose our family $\mathcal{S}$ in such a way as to leave the singular locus as well as the Spin(7)-structure at the singular points unchanged, we can use the same Spin(7)-parametrisations χ_i for all deformations of the Spin(7)-structure, and all nearby gluing data with matching cones.

The reason for making the lower part shrink sub-linearly while the tip shrinks linearly, is that $N^{\bar{A}}_l$ will stretch out and approximate any compact subset of the A_i arbitrarily well as the scale is reduced.

We now show that indeed this construction results in an approximation that is C^1-close to a Cayley in the following sense:

Proposition 3.1. *Let $\alpha \in (-1, 1)$ be given. Then if the global scale t is sufficiently small, $N^{\bar{A}}$ is α-Cayley.*

Proof. It is clear that $N_u^{\bar{A}}$ and $N_p^{\bar{A}}$ are always α-Cayley, since they are subsets of N and thus Cayley for the Spin(7)-structure Φ_s. Now for $N_m^{\bar{A}}$ and $N_l^{\bar{A}}$, we note that for $x \in \mathbb{R}^8$ near 0 we have $(\mathrm{D}\chi_i)_x^*(\Phi_{\chi_i(x)}) = \Phi_0 + O(\|x\|)$. Thus for t sufficiently small, we have for any $p \in N_m^{\bar{A}} \cup N_l^{\bar{A}}$ that $(\mathrm{D}\chi_i)^*_{\chi_i^{-1}(p)}(\Phi_p) = \Phi_0 + O(t^\nu)$. As A_i is already Cayley for Φ_0, it will also be α-Cayley for $(\mathrm{D}\chi_i)^*_{\chi_i^{-1}(p)}(\Phi_p)$ for sufficiently small values of t.

It remains to show that $N_m^{\bar{A}}$ is α-Cayley for t sufficiently small. Now by assumption on N and the A_i, $\Theta^i_{\mathrm{CS}}(p,s)$ and $t_i\Theta^i_{\mathrm{AC}}(p,s)$ approach the same Cayley cone as long as $s \in (\frac{1}{2}t^\nu, t^\nu)$ and $t_i \to 0$, and thus the respective tangent planes become arbitrarily close to the same Cayley plane, in particular they will be α'-Cayley for t small enough and any $\alpha' > \alpha$. Now for every α there is an $\alpha' > \alpha$ such that if ξ_1, ξ_2 are α'-Cayley graphs over a Cayley ξ, any linear interpolation of the between the maps having image ξ_1 and ξ_2 will have image an α-Cayley. Thus $N_m^{\bar{A}}$ will also be α-Cayley for t small enough. □

Our goal is to construct Cayley submanifolds close to the almost Cayley submanifolds $N^{\bar{A}}$. To simplify the analytic details, we will introduce Banach spaces tailored to this particular desingularisation, which were first defined by Lotay in [30]. Before that, we extend our notion of a radius function to the $N^{\bar{A}}$, combining the definitions of radius functions on CS- and AC-manifolds.

Definition 3.2. A collection of **radius functions** on $N^{\bar{A}}$ for all $\bar{A}$ with global scale bounded by $t > 0$ is a smooth function $\rho : N^{\bar{A}} \to [0,1]$ such that:

$$\rho(x) = \begin{cases} \Theta(R_0), & x \in K_N \\ \Theta(r_0 t_i), & 1 \leqslant i \leqslant k, x \in \chi_i(A_i \setminus L_i \times (r_0 t_i, \infty)) \\ \Theta(s), & 1 \leqslant i \leqslant k, x = \Psi^i_t(s,p) \text{ for } p \in L_i \text{ and } s \in (r_0 t_i, R_0) \\ \Theta(s), & k < i \leqslant l, x = \Psi^i_{\mathrm{CS}}(s,p) \text{ for } p \in L_i \text{ and } s \in (0, R_0) \end{cases} \tag{3.4}$$

Here we mean by $\Theta(f)$ a quantity that is bounded on both sides by f, up to constants that are independent of the choice of $\bar{A}$. Furthermore over $\Psi^i_{\bar{A}}((r_0 t_i, R_0) \times L_i)$ we require ρ to be an increasing function of the radial component $s \in (r_0 t_i, R_0)$.

Choose ρ to be the distance in M to the closest singular point of N and modified away from the singular points such that the functions are bounded by 1. This will be an example of a family of radius functions. From this, we also see that we can choose the family to be smooth and have uniformly bounded derivative. We can now define alternative Sobolev-norms on L^p_k-spaces on $N^{\bar{A}}$ that take into account the scale of the glued pieces. Suppose E is a metric vector bundle over $N^{\bar{A}}$ with a connection ∇. Let $\bar{\delta} \in \mathbb{R}^l$ be a vector of arbitrary weights. We then define the $L^p_{k,\bar{\delta},\bar{A}}$-norm of a section $s \in C^\infty(E)$ as:

$$\|s\|_{L^p_{k,\bar{\delta},\bar{A}}} = \left(\sum_{i=0}^{k} \int_{N^{\bar{A}}} |\rho^{-w+i} \nabla^i s|^p \rho^{-4} \,\mathrm{dvol} \right)^{\frac{1}{p}}. \tag{3.5}$$

Here $w : N^{\bar{A}} \to \mathbb{R}$ is a smooth weight function that interpolates between the chosen weights near each singularity. If all singularities are removed, so that $N^{\bar{A}}$ is nonsingular and compact, these norms are all uniformly equivalent for different values of $\bar{\delta}$, but they are not uniformly equivalent in $\bar{A}$, in the sense that the comparison constant will be unbounded. As we reduce the global scale, these norms reduce over the glued pieces to the norms for conical manifolds we introduced above. This will allow us to transplant results for the conical parts A_i and N onto the glued $N^{\bar{A}}$. Near the singularities that we did not remove, this norm is exactly the weighted Sobolev norm for conically singular manifolds. We can define Hölder spaces that vary with $\bar{A}$, the $C^{k,\alpha}_{\delta,\bar{A}}$-spaces, in a similar manner. We note that the Sobolev constants for different values of $\bar{A}$ will all be uniformly comparable.

3.2 Estimates

Consider the approximate Cayleys $N^{\bar{A}}$ that we have defined above, together with a family of radius functions ρ. For sufficiently small global scale t we have by Proposition 3.1 that $N^{\bar{A}}$ are α-Cayley for any fixed $\alpha < 1$. Thus in particular $N^{\bar{A}}$ admits a canonical deformation operator as in (2.6). Similar to (2.41) it can be augmented to include CS deformations of the unglued conical singularities as well as deformations of the Spin(7)-structure:

$$F_{\bar{A}} : C^{\infty}_{\bar{\delta}}(\nu_\epsilon(N^{\bar{A}})) \times \mathcal{U} \longrightarrow C^{\infty}_{\bar{\delta}-1}(E_{\mathrm{cay}}). \tag{3.6}$$

Here $\mathcal{U} \subset \mathcal{S} \times \mathcal{F}$ is an open neighbourhood of the point which corresponds to the initial Spin(7)-structure and the initial vertices and cones of $N^{\bar{A}}$. Moreover we define $\nu_\epsilon(N^{\bar{A}}) = \{(v,p) \in \nu(N^{\bar{A}}), \|v\| < \epsilon\rho(p)\}$, similar to the CS and AC cases. The weights $\bar{\delta} \in \mathbb{R}^l$ are chosen such that for $1 \leqslant i \leqslant k$ (corresponding to the singular points that are desingularised) we have $\lambda < \delta_i < \mu_i$ and for $k+1 \leqslant i \leqslant l$ (i.e. the singular points that are kept) we set $\delta_i = \mu_i$. We will see later that this condition arises naturally.

In the following we will write $\mu = \min_i(\mu_i)$ and $\delta = \min_i(\delta_i)$. This may seem like a restriction, however thanks to the assumption $(1, \mu_i] \cap \mathscr{D}_{L_i} = \emptyset$, we do not lose anything by doing this. Any CS_μ manifold can be improved to be CS_{μ_i} by [16, Thm. 5.5] as long as no critical weights are present in the range (μ, μ_i). We will also write $\delta = \min_i(\delta_i)$.

We denote the linearisation of $F_{\bar{A}}$ at 0 by $\not{D}_{\bar{A}}$. We can now establish bounds on the glued deformations operators, using our results for the CS and AC cases. In particular, we will take into account the dependence of various constants on the parameter $\bar{A}$. This will be important later when we deform all the $N^{\bar{A}}$ simultaneously to become Cayleys.

In this regard, the most important property of the deformation operator is its dependence on N, v and Φ. In particular, we have pointwise dependence only on $p, v(p), \nabla v(p)$ and T_pN as in the following proposition, adapted from Proposition 2.9.

Proposition 3.3. *The deformation operator on $N^{\bar{A}}$ for the varying* Spin(7)*-structures and cone configuration can be written as follows, for $v \in C^{\infty}_{loc}(\nu(N^{\bar{A}}))$, $s \in \mathcal{U}$ and $p \in N^{\bar{A}}$:*

$$\begin{aligned} F_{\bar{A}}(v,s)(p) &= \mathbf{F}(p, v(p), \nabla v(p), T_pN^{\bar{A}}, s) \\ &= F_{\bar{A}}(0,s)(p) + \not{D}_{\bar{A},s}v(p) + \mathbf{Q}(p, v(p), \nabla v(p), T_pN^{\bar{A}}, s). \end{aligned} \tag{3.7}$$

Here $\not{D}_{\bar{A},s}$ is the linearisation of $F_{\bar{A}}(\cdot, s)$ at 0 and $\mathbf{F}, \mathbf{Q}$ are smooth fibre-preserving maps:

$$\mathbf{F}, \mathbf{Q} : TM_\epsilon \times_N (T^*M \otimes TM)_\epsilon \times \mathrm{Cay}_\alpha(M) \times \mathcal{U} \longrightarrow \mathbf{E}_{\mathrm{cay}},$$

where $\mathbf{E}_{\mathrm{cay}} = \{(p, \pi, e) : (p, \pi) \in \mathrm{Cay}_\alpha(M), e \in E_\pi\}$ and α is sufficiently large. Here we see both sides as fibre bundles over $\mathrm{Cay}_\alpha(M) \times \mathcal{U}$. We define the map $Q_{\bar{A}} : C^\infty(\nu_\epsilon(N^{\bar{A}})) \times \mathcal{U} \to C^\infty(E_{\mathrm{cay}})$ as $Q_{\bar{A}}(v, s) = F_{\bar{A}}(v, s) - \not{D}_{\bar{A},s} v$.

We stress that the smooth maps $\mathbf{F}$ and $\mathbf{Q}$ only depend on the family of Spin(7)-structures Φ_s, and not on the Cayley submanifold. The term $Q_{\bar{A}}$ contains the contributions of v and ∇v which are quadratic and higher. Since $N^{\bar{A}}$ is both conically singular and has nonsingular regions of high curvature as the global scale decreases, we need to apply both the compact and the conically singular theory to prove the following:

Proposition 3.4. *Let $p > 4$, $k \geqslant 1$ and $\lambda < \delta < \mu$. Then the deformation map $F_{\bar{A}}$ is well-defined, Fredholm, and C^∞ as a map between Banach manifolds:*

$$F_{\bar{A}} : \mathcal{M}_{\bar{A}} = \{v \in L^p_{k+1,\delta,\bar{A}}(\nu_\epsilon(N^{\bar{A}})) : \|v\|_{L^p_{k+1,\delta,\bar{A}}} < \epsilon\} \times \mathcal{U} \longrightarrow L^p_{k,\delta-1,\bar{A}}(E_{\mathrm{cay}}), \tag{3.8}$$

whenever $\epsilon > 0$ is sufficiently small and can be chosen the same for all $\bar{A}$. Any $v \in L^p_{k+1,\delta,\bar{A}}(\nu_\epsilon(N^{\bar{A}}))$ such that $F_{\bar{A}}(v) = 0$ is smooth.

The proof of the smoothness of $F_{\bar{A}}$ is essentially the same as for Theorem 2.42, with all the norms replaced by their appropriate counterparts. As in the usual deformation theory, it relies on separate estimates of the first few terms of the Taylor expansion of $F_{\bar{A}}$, where we will now need to take into account the dependence on $\bar{A}$. Next, as the Hölder space $C^{k,\alpha}_{\delta,\bar{A}}$ for a fixed $\bar{A}$ can be seen as $C^{k,\alpha}_\delta$ for a conically singular manifold, usual elliptic regularity arguments apply and show smoothness, such as in the proof of Theorem 2.42. Let us now in turn take a look at the constant, linear and quadratic estimates of $F_{\bar{A}}$ and pay close attention to the constants involved.

Estimates for τ

We first investigate how well $N^{\bar{A}}$ approximates a Cayley as a function of the global scale t. Our main result will be that a priori $N^{\bar{A}}$ should converge to an ideal Cayley in $C^\infty_{\delta,\bar{A}}$ for $\lambda < \delta < \mu$, uniformly in $\bar{A}$.

Proposition 3.5 (Pointwise estimates). *Denote by $g^{\bar{A}}$ the Riemannian metric on $N^{\bar{A}}$ coming from the embedding into M. For t sufficiently small and for $s \in \mathcal{S}$ sufficiently close to our initial* Spin(7)*-structure, we have the following estimates on the derivative*

$\nabla^k \tau|_{N^{\bar{A}}}$ *for* $k \geqslant 0$*:*

$$|\chi_i^* \tau|_{N^{\bar{A}}}|(x) \lesssim |x|, \tag{3.9}$$

$$|\nabla^k \chi_i^* \tau|_{N^{A_i}}|(x) \lesssim t_i^{-k+1}, \quad k \geqslant 1 \tag{3.10}$$

$$|\nabla^k \tau|_{N_m^{A_i}}| \lesssim t_i^{-\lambda} \rho^{\lambda-k-1} + \rho^{\mu-j-1}, \quad \rho \in (r_0 t_i, \frac{1}{4} t_i^{\nu}) \tag{3.11}$$

$$|\nabla^k \tau|_{N_m^{A_i}}| \lesssim t_i^{-k\nu}(t_i^{(\nu-1)(\lambda-1)} + t_i^{\nu(\mu-1)}), \tag{3.12}$$

$$|\nabla^k \tau|_{N_u^{\bar{A}}}| \lesssim d(s, s_0). \tag{3.13}$$

Here ∇ *and* $|\cdot|$ *are computed with respect to* $\chi^* g^{\bar{A}}$ *in the first line, and* $g^{\bar{A}}$ *in the last two lines. Furthermore, the constants hidden in the* $\lesssim$*-notation are independent of* $\bar{A}$*.*

Proof. We adapt the method of proof from Proposition 8.1 in [30]. Note first that $N_u^{\bar{A}}$ are Cayley by construction for our initial Spin(7)-structure, and therefore τ_{s_0} and all its derivatives vanish on them. As $N_u^{\bar{A}}$ is compact, we can easily get the bound (3.13).

Consider next $N_l^{\bar{A}}$. In what follows we can think of the conically singular points of $N^{\bar{A}}$ as being obtained by gluing in a Cayley cone, and thus they can be treated no differently from the desingularised regions. We have by Taylor's theorem that:

$$|\chi_i^* \tau|(x) = |\chi_i^* \tau|(0) + O(|x|).$$

We have chosen χ_i to be a Spin(7)-coordinate system, so that $\chi_i^* \Phi(0) = \Phi_0$, where Φ_0 is the standard Cayley form on $\mathbb{R}^8$. We therefore also have $\chi_i^* \tau(0) = \tau_0$, where τ_0 is the standard quadruple product on $\mathbb{R}^8$. Now since $\chi_i^{-1}(N_l^{\bar{A}})$ is Cayley with respect to Φ_0, we get that:

$$|\chi_i^* \tau|_{N_l^{\bar{A}}}|(x) = |\tau_0|_{N_l^{\bar{A}}}|(0) + O(|x|) = O(|x|).$$

Thus we get (3.9). Now for $k \geqslant 1$, we have $|\nabla^k \tau_0|_{N_l^{\bar{A}}}| = 0$, as the A_i are Cayley for Φ_0. So we would like to bound:

$$|\nabla^k (\chi_i^* \tau - \tau_0)|_{N_l^{\bar{A}}}|.$$

For $t > 0$, think of tA_i as a map $f_t : A_i \to \mathbb{R}^8 \times \Lambda^4$ which maps $p \in A_i \mapsto (tp, T_p A_i)$, and of $\chi_i^* \tau - \tau_0$ as a map $\tilde{\tau} : \mathbb{R}^8 \times \Lambda^4 \mapsto \Lambda^2_7$ with the property that $\tilde{\tau}(0, \omega) = 0$. We therefore have a Taylor expansion for small $v \in \mathbb{R}^8$:

$$\tilde{\tau}(v, \omega) = L_\omega[v] + R_{\omega,v}[v \otimes v].$$

Here L_ω is a linear map depending smoothly on ω and $R_{\omega,v}$ is a bilinear maps that depends smoothly on ω and v which encodes second and higher order behaviour in v. Thus, we see that:

$$\tilde{\tau} \circ f_t(p) = t L_{T_p A}[p] + t^2 R_{T_p A, p}[p \otimes p],$$

From this we can deduce that:

$$\begin{aligned}\nabla_\xi(\tilde{\tau}\circ f_t)(p) &= t(L_{T_pA}[\xi] + \mathrm{D}L_{T_pA}[p, \nabla_\xi T_pA]) \\ &\quad + t^2(2R_{T_pA,p}[\xi\otimes p] + \mathrm{D}R_{T_pA,p}[p\otimes p, \nabla_\xi T_pA, \xi]).\end{aligned}$$

The linear maps and their derivatives can be bound uniformly, as both $p \in B_\eta(0)$ and T_pA vary in compact sets. Thus we see that:

$$|\nabla(\tilde{\tau}\circ f_t)(p)| \leqslant C(A,\tau)(t + t^2(|p| + |p|^2|\nabla T_pA|)) \leqslant C(A,\tau)t.$$

Here we used the fact that $|\nabla T_pA| \in O(|p|^{-1})$ and $|p| \in O(1)$. Thus going back to our original situation, after rescaling by t_i to account for the fact that the metric on tA_i scales as well, we obtain:

$$|\nabla\chi_i^*\tau|_{N_l^{\bar{A}}}| = |\nabla(\chi_i^*\tau - \tau_0)|_{N_l^{\bar{A}}}| \lesssim 1.$$

The higher derivatives can be deduced the same. The key point is that naively rescaling will lead to a factor t_i^{-k}, but since the A_i are Cayley, we can improve it by one factor of t_i via the above Taylor expansion argument.

Finally, we consider $N_m^{\bar{A}}$, where the interpolation happens and where we also expect the biggest error to appear. We will consider $(\Psi^i_{\bar{A}})^*\tau|_{N_m^{\bar{A}}}$, which is a form on the cone portion $C = (r_0t_i, t_i^\nu)\times L$, and we will prove the analogue of (3.11) and (3.12) with respect to the cone metric. Now as $t\to 0$, the pullback metric $(\Psi^i_{\bar{A}})^*g^{\bar{A}}$ will converge uniformly in t to the conical metric. In particular, the conical metric and the pullback metrics for small t are all uniformly equivalent with proportionality factors independent of the global scale. Thus all quantities of the form $|\nabla^k s|$, computed with regards to any of these metrics, will be in the same asymptotic class. Denote by $\iota : C\to\mathbb{R}^8$ the embedding of the cone. We then have that:

$$\begin{aligned}|\nabla^k(\Psi^i_{\bar{A}})^*\tau|_{N_m^{\bar{A}}}| &= |\nabla^k(\Theta^i_{\bar{A}})^*\chi_i^*\tau|_{N_m^{\bar{A}}}| \\ &\leqslant |\nabla^k(\Theta^i_{\bar{A}} - \iota)^*\chi_i^*\tau| + |\nabla^k\iota^*\chi_i^*\tau|\end{aligned}$$

Upper bounds for the second term can be given in an analogous way to what we have done for $N_l^{\bar{A}}$, as the cone is Cayley and scaling invariant. We are interested in the region with radius in (r_0t_i, t_i^ν), thus we can run the above argument again while only rescaling by t_i^ν, and thus only get an error $t_i^{-k\nu}$. This is always the asymptotically better term. For the remaining term, notice that $\chi_i^*\tau$ is a fixed quantity, and the only dependence on $\bar{A}$ is within $\Theta^i_{\bar{A}} - \iota$. So let us more generally bound:

$$|\nabla^k f^*\omega|,$$

for $f : C\to\mathbb{R}^8$ a smooth function, and $\omega\in\Omega^k$ a smooth form. From the definition of pullback we see that there are smooth maps E_k, independent of f such that:

$$\nabla^k f^*\omega(p) = E_k(f(p), \nabla f(p), \dots, \nabla^{k+1}f(p)). \qquad (3.14)$$

These maps have the additional property that they are affine in $\nabla^k f(p)$ where $k \geqslant 1$. Consider the scaling behaviour of both sides when the cone is rescaled by $\gamma > 0$. In other words, we replace f by f_γ, such that $f_\gamma(p) = f(\gamma \cdot p)$. Equation (3.14) still holds for f_γ, and we can relate the norms of both sides to the corresponding terms for f as follows:

$$\begin{aligned}
\gamma \cdot \nabla^k f^*(\chi^*\tau)(\gamma \cdot p) &= \nabla^k f^*(\gamma \cdot \chi^*\tau)(p) \\
&= \nabla^k f_\gamma^*(\chi^*\tau)(p) \\
&= E_k(f_\gamma(p), \nabla f_\gamma(p), \ldots, \nabla^{k+1} f_\gamma(p)) \\
&= E_k(\gamma \cdot f(p), \gamma \cdot \nabla f(p), \ldots, \gamma \cdot \nabla^{k+1} f(p))
\end{aligned}$$

As the maps E_k are affine in the (higher) covariant derivatives of f, we see that:

$$\gamma^k |\nabla^k f^*(\chi^*\tau)| \lesssim |f(p)| + \gamma|\nabla f(p)| + \cdots + \gamma^{k+1}|\nabla^{k+1} f(p)|. \tag{3.15}$$

Let us now estimate the norms of f and its derivatives. We have:

$$\begin{aligned}
f(p,s) = (\Theta^i_{\bar{A}} - \iota)(p,s) = (1 - \varphi_{\text{cut}})&(2t_i^{-\nu} - 1)(\Theta^i_{\text{AC}}(p,s) - \iota(p,s)) \\
&+ \varphi_{\text{cut}}(2t_i^{-\nu} - 1)(\Theta_{\text{CS}}(p,s) - \iota(p,s)).
\end{aligned} \tag{3.16}$$

Our bounds on f should be unchanged when varying $\bar{A}$. Changes with fixed scales can be dealt with by increasing the constant, as such variations form a compact space. Thus we are only concerned with rescalings.

To begin, we apply the AC-condition to the $t_i^{-1} A_i$, and rescale to obtain:

$$|\nabla^k(\Theta_{\text{AC}}(p,s) - \iota(p,s))| = O(t_i^{-\lambda+1} s^{\lambda-k}), \tag{3.17}$$

where the constant is independent of the scale. Analogously, we get from the CS condition that:

$$|\nabla^k(\Theta_{\text{CS}}(p,s) - \iota(p,s))| = O(s^{\mu-k}). \tag{3.18}$$

Taken together we obtain the bound:

$$|(\Theta_t - \iota)(p,s)| \lesssim t_i^{-\lambda+1} s^\lambda + s^\mu.$$

One can obtain bounds for the derivative of $\Theta_t - \iota$ in a similar manner. To be more explicit, the covariant derivatives applied k times to (3.16) will hit both $\varphi_{\text{cut}}(2t_i^{-\nu} - 1)$ and $\Theta_{\text{AC/CS}} - \iota$. If it hits φ_{cut} a total of l times in a term, we obtain a bound of the form $O(t_i^{-l\nu} \partial^l \varphi_{\text{cut}} |\nabla^{k-l}(\Theta_{\text{AC/CS}} - \iota)|)$. An explicit calculation leads us to the general formula:

$$|\nabla^k(\Theta_{\bar{A}} - \iota)(p,s)| = O\left(\sum_{j+l=k} (t_i^{-\lambda+1} s^{\lambda-j} + s^{\mu-j}) t_i^{-l\nu} \partial^l \varphi_{\text{cut}}\right).$$

Thus we can plug this into our estimate (3.15) to obtain the bound:

$$|\nabla^k(\Theta_{\bar{A}} - \iota)^*\chi^*\tau| = O\left(\sum_{j+l=k}(t_i^{-\lambda+1}\rho^{\lambda-j-1} + \rho^{\mu-j-1})t_i^{-l\nu}\partial^l\varphi_{\mathrm{cut}}\right).$$

From this, we obtain the claimed bounds by noting that either $r_0 t_i \leqslant \rho \leqslant \frac{1}{4}t_i^\nu$, where $\partial\varphi_{\mathrm{cut}} = 0$, or $\rho > \frac{1}{4}t_i^\nu$, so that $\rho = O(t_i^\nu)$. □

Proposition 3.6 (Initial Error estimate)**.** *For a sufficiently small global scale $t > 0$ and for $s \in \mathcal{S}$ sufficiently close to our initial* Spin(7)*-structure, $p > 4$, $\delta \in \mathbb{R}$, $\nu = \frac{\lambda-1}{\lambda-\mu}$, $k \in \mathbb{N}$, we have:*

$$\begin{aligned}\|F_{\bar{A}}(0,s)\|_{L^p_{k,\delta-1,\bar{A}}} &< C_F(t^{-\delta\nu}(t^{\nu\mu} + t^{(\nu-1)\lambda+1}) + d(s,s_0)) \\ &< C_F(t^{\nu(\mu-\delta)} + d(s,s_0)).\end{aligned} \tag{3.19}$$

Here $C_F > 0$ is a constant that only depends on the geometry of $N \subset (M,\Phi)$ and $\mathcal{S}$, but not on $\bar{A}$.

Proof. Let $0 \leqslant j \leqslant k$. Subdivide $N_m^{A_i} = N_{m,1}^{A_i} \cup N_{m,2}^{A_i}$, where $N_{m,1}^{A_i}$ is the region where

$\rho \leqslant \frac{1}{4}t_i^\nu$ and $N^{A_i}_{m,2}$ the rest. We then have that:

$$
\begin{aligned}
\int_{N^{\bar{A}}} |\rho^{-\delta+1+j}\nabla^j F_{\bar{A}}|^p \rho^{-4}\,\mathrm{dvol} &= \int_{N^{\bar{A}}} |\rho^{-\delta+1+j}\nabla^j \tau|_{N^{\bar{A}}}|^p \rho^{-4}\,\mathrm{dvol} \\
\lesssim \sum_{i=0}^{l} \int_{N_l^{A_i}} (\rho^{-\delta+1+j}|\chi_i^*\tau|_{N^{A_i}}|)^p \rho^{-4}\,\mathrm{dvol} \\
+ \sum_{i=0}^{l} \int_{N_m^{A_i}} (\rho^{-\delta+1+j}|\nabla^k \tau|_{N_m^{A_i}}|)^p \rho^{-4}\,\mathrm{dvol} + \mathrm{vol}(N_u^{\bar{A}}) d^p(s,s_0) \\
\lesssim \sum_{i=0}^{l} \int_{N_l^{A_i}} (\rho^{-\delta+1+j} t_i^{-j+1})^p \rho^{-4}\,\mathrm{dvol} \\
+ \sum_{i=0}^{l} \int_{N_{m,1}^{A_i}} (\rho^{-\delta+1+j}(\rho^{\mu-j-1} + t_i^{-\lambda+1}\rho^{\lambda-j-1}))^p \rho^{-4}\,\mathrm{dvol} \\
+ \sum_{i=0}^{l} \int_{N_{m,2}^{A_i}} (t_i^{-\delta\nu+j\nu+\nu} t_i^{-j\nu}(t_i^{\nu(\mu-1)} + t_i^{(\nu-1)(\lambda-1)}))^p \rho^{-4}\,\mathrm{dvol} + d^p(s,s_0) \\
\lesssim \sum_{i=0}^{l} t_i^{p(2-\delta)} \int_{N_l^{A_i}} \rho^{-4}\,\mathrm{dvol} + \sum_{i=0}^{l} \int_{N_{m,1}^{A_i}} \rho^{-p\delta}(\rho^\mu + \rho^{(1-\frac{1}{\nu})\lambda + \frac{1}{\nu}})^p \rho^{-4}\,\mathrm{dvol} \\
+ \sum_{i=0}^{l} t_i^{-p\nu\delta}(t_i^{\nu\mu} + t_i^{(\nu-1)\lambda+1})^p \int_{N_{m,2}^{A_i}} \rho^{-4}\,\mathrm{dvol} + d^p(s,s_0) \\
\lesssim \sum_{i=0}^{l} (t_i^{p(2-\delta)} + t_i^{-p\nu\delta}(t_i^{\nu\mu} + t_i^{(\nu-1)\lambda+1})^p) \\
+ \sum_{i=0}^{l} \int_{N_{m,1}^{A_i}} \rho^{p(\mu-\delta)} \rho^{-4}\,\mathrm{dvol} + d^p(s,s_0) \\
\lesssim \sum_{i=0}^{l} (t_i^{-p\nu\delta}(t_i^{\nu\mu} + t_i^{(\nu-1)\lambda+1})^p + t_i^{-p\nu\delta} t_i^{p\nu\mu}) + d^p(s,s_0) \\
\lesssim \sum_{i=0}^{l} t_i^{-p\nu\delta}(t_i^{\nu\mu} + t_i^{(\nu-1)\lambda+1})^p + d^p(s,s_0).
\end{aligned}
$$

Here we used all the various bounds from Proposition 3.5 as well as the fact that ρ can be uniformly bound from above by $2t_i^\nu$ on both $N_l^{A_i}$ and $N_m^{A_i}$. Furthermore we can also bound ρ from below by $\frac{1}{4}t_i^\nu$ on $N_{m,2}^{A_i}$, and by $r_0 t_i$ on $N_{m,1}^{A_i}$ and $N_l^{A_i}$. The integral:

$$
\int_{N_l^{A_i}} \rho^{-4}\,\mathrm{dvol} \lesssim \int_\epsilon^{r_0} s^{-4}s^3 \mathrm{d}s \leqslant C
$$

is bounded independently of t_i, as is:

$$\int_{N_{m,2}^{A_i}} \rho^{-4}\,\mathrm{dvol} \lesssim \int_{\frac{1}{4}t_i^\nu}^{2t_i^\nu} s^{-1}\mathrm{d}s = \log(2t_i^\nu) - \log\left(\frac{1}{4}t_i^\nu\right) \leqslant C.$$

Finally, we compute that

$$\int_{N_{m,1}^{A_i}} \rho^{p(\mu-\delta)}\rho^{-4}\,\mathrm{dvol} \lesssim \int_{r_0 t_i}^{\frac{1}{4}t_i^\nu} s^{p(\mu-\delta)-1}\mathrm{d}s \lesssim t_i^{p\nu(\mu-\delta)} - t_i^{p(\mu-\delta)} \lesssim t_i^{p\nu(\mu-\delta)}.$$

The bound now follows as the exponent of the t_i is positive, and thus the biggest one dominates, which is the global scale t. For the second line in (3.19) we use our choice of $\nu = \frac{\lambda-1}{\lambda-\mu}$, which is chosen exactly so that $\nu\mu = (\nu-1)\lambda + 1$. □

Estimates for $\not{D}_{\bar{A}}$

Recall that in our construction of $N^{\bar{A}}$, we have assumed identical cones (as subsets of $\mathbb{R}^8$) for the pieces A_i and N, given the choice of a Spin(7)-coordinate system. Here the interpolation happened between the radii $\frac{1}{2}t^\nu$ and t^ν, where $0 < \nu < 1$ is a constant. To derive estimates similar to the bounds in Propositions 2.44 and 2.36 for $\not{D}_{\bar{A}}$, we use a partition of unity to combine the results for the parts. For this we need further constants $0 < \nu'' < \nu' < \nu < 1$. Let $\varphi_{\mathrm{cut}} : \mathbb{R} \to [0,1]$ be a smooth cut-off function, such that:

$$\varphi_{\mathrm{cut}}|_{(-\infty,\nu'']} = 0, \quad \varphi_{\mathrm{cut}}|_{[\nu',+\infty)} = 1.$$

Using φ_{cut} we define a partition of unity on $N^{\bar{A}}$ as follows. Let $t > 0$ be the global scale of $N^{\bar{A}}$ and suppose that t_i is the local scale of A_i. We then define:

$$\alpha(p) = \begin{cases} \varphi_{\mathrm{cut}}\left(\frac{\log(\rho(p))}{\log(t_i))}\right), & \text{if } p \in \Psi_i^{\bar{A}}(L_i \times (r_0 t_i, R_0)) \\ 0, & \text{if } p \in N_u^{\bar{A}}, \\ 1, & \text{if } p \in N_l^{\bar{A}}, \end{cases} \tag{3.20}$$

Then $\alpha(p) = 0$ on $\Psi_i^{\bar{A}}(L_i \times (r_0 t_i, R_0))$ if $\rho(p) \geqslant t_i^{\nu''}$ and $\alpha(p) = 1$ if $\rho(p) \leqslant t_i^{\nu'}$. Thus α is supported in $N_{\mathrm{AC}}^{A_i} = \Psi_{\bar{A}}^i(L_i \times (r_0 t_i, t_i^{\nu''})) \cup N_l^{\bar{A}}$ and $1-\alpha$ is supported in $N_{\mathrm{CS}}^{\bar{A}} = \Psi_i^{\bar{A}}(L_i \times (t_i^{\nu'}, R_0)) \cup N_u^{\bar{A}}$. We also define $N_{\mathrm{AC}}^{\bar{A}} = \bigsqcup_{1\leqslant i\leqslant k} N_{\mathrm{AC}}^{A_i}$. In particular the gluing region $N_m^{\bar{A}}$ is entirely contained in $N_{\mathrm{AC}}^{\bar{A}}$. We would now like to relate the operator $\not{D}_{\bar{A}}|_{N_{\mathrm{AC}}^{A_i}}$ to $\not{D}_{\mathrm{AC}}^i$ on a perturbation of A_i and the operator $\not{D}_{\bar{A}}|_{N_{\mathrm{CS}}^{\bar{A}}}$ to $\not{D}_{\mathrm{CS}}$ on N. To do this we define a pseudo-kernel $\kappa_{\bar{A}} \subset C^\infty(\nu_\epsilon(N^{\bar{A}}))$ for the glued operator, the analogue of κ_{CS} and κ_{AC} from Propositions 2.43 and 2.35 respectively. We will be working with a rate $\lambda < \delta < \mu, \delta \neq 1$ which automatically means that $\delta \notin \mathscr{D}_{L_i}$ for all the links of N. The space $\kappa_{\bar{A}}$ will be defined as a direct sum of contributions from both pieces. First, the treatment of the conically singular piece is immediate. The elements of κ_{CS} all have support in a compact subset of N. Thus for t sufficiently small, we can consider them as section of $N_{\mathrm{CS}}^{\bar{A}}$ directly, since the $N_{\mathrm{CS}}^{\bar{A}}$ exhaust N as the global scale decreases to 0. In particular, we can then also consider them as sections over $N^{\bar{A}}$ after extending by 0 over $N^{\bar{A}} \setminus N_{\mathrm{CS}}^{\bar{A}}$. Even more,

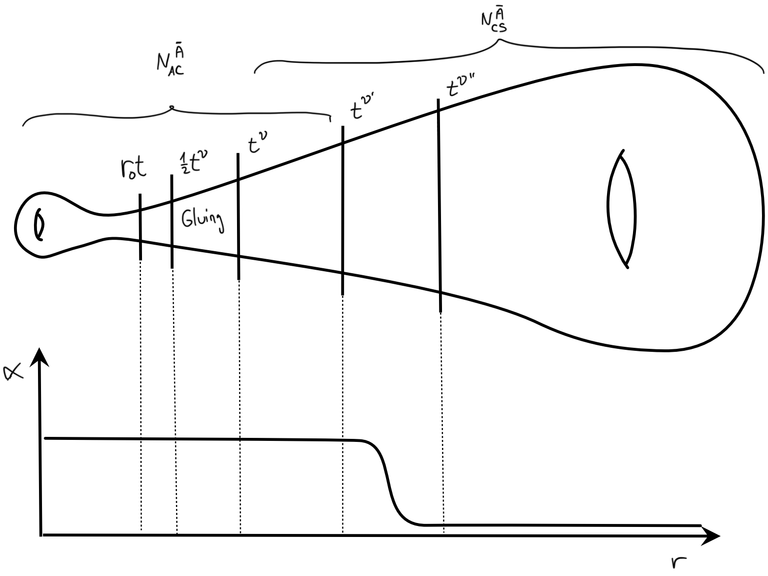

Figure 3.2: Decomposition of the glued cone

for small enough t, the elements of κ_{CS}, seen as sections on $N^{\bar{A}}$, vanish on $N^{\bar{A}}_{\mathrm{AC}}$. Similarly the operators $\not{D}_{\mathrm{CS}}$ and $\not{D}_{\bar{A}}|_{N^{\bar{A}}_{\mathrm{CS}}}$ can be identified and the bounds for $\not{D}_{\mathrm{CS}}$ carry over.

Next, the interpretation of κ_{AC} is more delicate, as the gluing region for a given conical singularity is a perturbation of A_i and not exactly Cayley. We first find an identification between $N^{A_i}_{\mathrm{AC}}$ and an open subset of A_i as follows. For technical purposes, we fix a further rate $0 < \tilde{\nu} < \nu''$. Then there is a diffeomorphism between an open subset $A^t_i \subset A_i$ and $\chi(K_{A_i}) \sqcup \Theta^i_{\bar{A}}(L_i \times (r_0 t_i, t_i^{\tilde{\nu}}))$, given by sending:

$$
\begin{aligned}
p \in K_{A_i} &\longmapsto \chi(p), \\
\Psi^i_{\mathrm{AC}}(p, s) &\longmapsto \Theta^i_{\bar{A}}(p, s).
\end{aligned}
$$

Let us call this map $\Psi^i_{\bar{A}} : A^t_i \to N^{A_i}_{\mathrm{AC}}$, which as usual factors as $\Psi^i_{\bar{A}} = \chi \circ \Theta^i_{\bar{A}}$. As the operator $\not{D}^i_{\mathrm{AC}}$ not only takes into account the metric structure of A_i, but also the ambient Spin(7)-structure we now thicken the map $\Theta^i_{\bar{A}}$. Let $U^t_{A_i}$ be a tubular neighbourhood of A^t_i in $\mathbb{R}^8$, so that every $q \in U^t_{A_i}$ can be written uniquely as $q = p + v$, where $p \in A^t_i$ and $v \in (\nu_{\epsilon,\Phi_0}(A^t_i))_p$. We then define:

$$
\begin{aligned}
\tilde{\Theta}^i_{\bar{A}} : U^t_{A_i} &\longrightarrow \mathbb{R}^8 \\
(p, v) &\longmapsto \Theta^i_{\bar{A}}(p) + v
\end{aligned}
$$

Then clearly $\tilde{\Theta}^i_{\bar{A}}|_{\tilde{A}_i} = \Theta^i_{\bar{A}}$. We now transport the Cayley form in a vicinity of $N^{A_i}_{\mathrm{AC}}$ over to A_i. Consider first $\chi^*\Phi$, which is a four-form on $B_{r_0}(0) \subset \mathbb{R}^8$. Via pullback, we obtain a form, which we define pointwise as:

$$(\tilde{\Phi}^i_{\bar{A}})_p = t_i^{-4}\tilde{\Theta}^i_{\bar{A}}\chi^*\Phi(t_i p) \in \Omega^4(U^t_{A_i}).$$

We introduced the factor t_i^{-4} to counteract the rescaling by t_i. With this normalisation we have $\tilde{\Phi}^i_{\bar{A}} \to \Phi_0$ uniformly on $U^t_{A_i}$ as $t \to 0$. This follows from Taylor's theorem and the fact that $U^t_{A_i} \subset B_{2r_0 t^{\tilde{\nu}-1}}(0)$, since it gives $\chi^*\Phi(t_i p) - \Phi_0 = O(t_i|p|)$ as we have $\chi^*\Phi(0) = \Phi_0$. We now extend $\tilde{\Phi}^i_{\bar{A}}$ to a form $\Phi^i_{\bar{A}}$ defined on all of $\mathbb{R}^8$. For this recall the smooth cut-off function $\varphi : \mathbb{R} \to [0,1]$ which we used in the construction of $N^{\bar{A}}$. It vanishes for negative values and is equal to 1 for values $\geqslant 1$, as in (3.1). The space of Cayley forms on $\mathbb{R}^8$ is a smooth submanifold $\mathcal{C} \subset \Lambda^4\mathbb{R}^8$ of dimension 43. Choose local coordinates $c : B_1(0) \subset \mathbb{R}^{43} \to \mathcal{C}$ such that $c(0) = \Phi_0$. As we have uniform convergence $\tilde{\Phi}^i_{\bar{A}} \to \Phi_0$ on $U^t_{A_i}$, we will eventually have $\tilde{\Phi}^i_{\bar{A}}(p) \in \operatorname{im} c$ for t sufficiently small and all $p \in \mathbb{R}^8$. The uniform convergence forces the linear Cayley forms at each point in $\mathbb{R}^8$ to be simultaneously close to the standard form Φ_0, thus in the image of the parametrisation c. We then interpolate between $\tilde{\Phi}_t$ and Φ_0 between the radii $\frac{1}{2}t^{\tilde{\nu}-1}$ and $t^{\tilde{\nu}-1}$ as follows:

$$\Phi^i_{\bar{A}}(\Theta_{\mathrm{AC},t_i^{-1}A_i}(r,p)+v) = c(c^{-1}(\tilde{\Phi}^i_{\bar{A}}(\Theta^i_{\mathrm{AC}}(r,p)+v))\varphi(2rt_i^{-1+\tilde{\nu}}-1)). \tag{3.21}$$

We now have a family of forms $\Phi^i_{\bar{A}}$ on U_A. If we choose the global scale sufficiently small, we can extend these forms to all of $\mathbb{R}^8$. For sufficiently small $t > 0$, we have that A_i is almost Cayley. These forms $\Phi^i_{\bar{A}}$ form a continuous family with respect to the parameter $\bar{A}$, and as $t \to 0$, we get uniform convergence $\Phi^i_{\bar{A}} \to \Phi_0$. In fact, we even have C^∞_η-convergence.

Lemma 3.7. *The family* $(\mathbb{R}^8, \Phi^i_{\bar{A}})$ *for varying* $\bar{A}$ *is a continuous family of* C^∞_η *perturbations of the standard* Spin(7) *form* Φ_0. *The rate* $\eta < 1$ *only depends on the constant* $0 < \tilde{\nu} < 1$ *chosen for the gluing, and* $\eta \to -\infty$ *as* $\tilde{\nu} \to 1$. *For* $\eta < \lambda$ *and* $1 \leqslant i \leqslant k$, *we have that* A_i *is an* AC_λ*-submanifold for the* Spin(7)*-structure* $\Phi^i_{\bar{A}}$.

Proof. Note that the family $\Phi^i_{\bar{A}}$ is flat at large radii, but the cutoff radius $Ct_i^{\tilde{\nu}-1}$ depends on $\bar{A}$. Thus the deformations at non-zero global scale are compactly supported near a fixed $\bar{A}_0$, and in particular also in C^∞_η for any $\eta \leqslant 1$. In particular, for any $\eta < \lambda$ the submanifold $A_i \subset \mathbb{R}^8$ will be AC_λ for $\Phi^i_{\bar{A}}$ because it already is for Φ_0. From Equation (3.21) and $|\Phi_p - \Phi_{z_i}| = O(\rho)$ on M we see that:

$$|(\Phi^i_{\bar{A}} - \Phi_0)r^{-\eta+1}| \leqslant Ct_i^{\tilde{\nu}}(t_i^{\tilde{\nu}-1})^{-\eta+1} = Ct_i^{\eta(1-\tilde{\nu})+2\tilde{\nu}-1}.$$

Thus we have C^0_η convergence as $t \to 0$ when $\eta > \frac{2\tilde{\nu}-1}{\tilde{\nu}-1}$. Similar reasoning for higher derivatives shows that:

$$|\nabla^k\Phi^i_{\bar{A}}r^{-\eta+k+1}| \leqslant Ct_i^{(-\eta+1)(\tilde{\nu}-1)+k}.$$

Thus C^∞_η convergence follows immediately whenever we have C^0_η convergence. □

Let us return to the question of defining the analogue of κ_{AC} for $N^{A_i}_{\mathrm{AC}}$. After composing $\tilde{\Theta}^i_{\bar{A}}$ with χ we have an identification $\tilde{\Psi}^i_A$ of open neighbourhoods of A_i and $N^{A_i}_{\mathrm{AC}}$. We can hence pull back the elements of κ_{AC} for A_i, or κ_{AC,A_i}, to sections of $TM|_{N^{\bar{A}}_{\mathrm{AC}}}$ for t sufficiently small. Note that we cannot in general require that they be normal sections. To remedy this we will first project κ_{AC,A_i} onto $\nu_{\Phi^i_{\bar{A}}}(A_i)$, i.e. the normal bundle of A_i with respect to the Cayley form $\Phi^i_{\bar{A}}$. Note that $\nu_{\Phi^i_{\bar{A}}}(A_i)$ is identified with $\nu(N^{A_i}_{\mathrm{AC}})$ under $\tilde{\Psi}^i_{A}$. Thus we define the space of sections $\kappa_{\mathrm{AC},\bar{A},i}$ as κ_{AC,A_i} projected onto $\nu_{\Phi^i_{\bar{A}}}(A_i)$ and then transported to $\nu(N^{A_i}_{\mathrm{AC}})$. For $t > 0$ sufficiently small the elements of $\kappa_{\mathrm{AC},\bar{A},i}$ can be extended to sections on all of $N^{\bar{A}}$, and after further reducing t the sections in κ_{CS} and $\kappa_{\mathrm{AC},\bar{A}} = \bigoplus_{1\leqslant i\leqslant k} \kappa_{\mathrm{AC},\bar{A},i}$ will have disjoint support. In this case, we define:

$$\kappa_{\bar{A}} = \kappa_{\mathrm{CS}} \oplus \kappa_{\mathrm{AC},\bar{A}}. \tag{3.22}$$

Assuming unobstructedness, this is a family of pseudo-kernels for the family of operators $\not{D}_{\bar{A}}$, as we will see in Proposition 3.14. Note that κ_{CS} contains all the contributions which have rate $> \delta$, where $\lambda < \delta < \mu$ was the rate of the operator $F_{\bar{A}}$ in Proposition 3.4, and $\kappa_{\mathrm{AC},\bar{A}}$ all the ones which have rate $< \delta$. As δ is by assumption not critical, this accounts for every possible deformation exactly once. Note also that while the non-linear deformation operator of an AC Cayley does not have geometrical meaning when the rate $\lambda > 1$, the linearised operator can be defined for any rate.

We now show the analogue of Propositions 2.36 and 2.44 for the glued manifold $N^{\bar{A}}$, using both results as ingredients. We first introduce an inner product that interpolates between $L^2_{\delta-\epsilon}$ on the AC region and $L^2_{\delta+\epsilon}$ on the CS region, where $\epsilon > 0$ is a small parameter (necessary in Propositions 2.36 and 2.44 to apply the Sobolev embedding theorem 1.24). So we define for $u, v \in C^\infty(\nu(N^{\bar{A}}))$:

$$\langle u, v\rangle_{\delta\pm\epsilon} = \int_{N^{\bar{A}}} \langle u, v\rangle \, \rho^{w-4} \, \mathrm{dvol}\,. \tag{3.23}$$

Here $w(p) = \delta-\epsilon$ whenever $\rho(p) \leqslant \frac{1}{2}t_i^\nu$ and $w(p) = \delta+\epsilon$ whenever $\rho(p) \geqslant t_i^\nu$. By combining the Propositions 2.36 and 2.44 we conclude:

Proposition 3.8. *For t sufficiently small there is a constant C_{AC}, independent of $\bar{A}$ such that for $v \in L^p_{k+1,\delta,\bar{A}}(\nu(N^{\bar{A}}))$ with $\mathrm{supp}(v) \subset N^{\bar{A}}_{\mathrm{AC}}$ which is $L^2_{\delta\pm\epsilon}$-orthogonal to $\kappa_{\mathrm{AC},\bar{A}}$ we have:*

$$\|v\|_{L^p_{k+1,\delta,\bar{A}}} \leqslant C_{\mathrm{AC}} \|\not{D}_{\bar{A}} v\|_{L^p_{k,\delta-1,\bar{A}}}. \tag{3.24}$$

We now turn back to our task of combining the bounds on $\not{D}_A$ and $\not{D}_N$ to get bounds on the inverse of $\not{D}_{\bar{A}}$ modulo the pseudo-kernel. Recall the cut off function $\alpha : N^{\bar{A}} \to [0,1]$ we defined in (3.20). It has the following decay properties:

Lemma 3.9. *Let $l \geqslant 1$ be given. Then:*

$$\|\nabla^l \alpha\|_{C^0} \in O\left(\rho^{-l} \log(t_i)^{-1}\right). \tag{3.25}$$

Proof. As the cutoff function φ_{cut} is smooth and only varies on a compact set of fixed uniform size, all of its derivatives up to a given order l remain bounded on all of $\mathbb{R}$. Similarly, all derivatives up to order l of ρ are bounded on $N^{\bar{A}}_{\mathrm{CS}}$, independent of the scale, since ρ agrees with a radius function on the conically singular N on this part. Finally, through an argument similar to the one in Proposition 3.8 the same holds on $N_{\mathrm{AC}}^{A_i}$, except close to the radius $r_0 t_i$, where the smoothing happens. We will see however that this is not an issue. In geodesic normal coordinates $\{x^i\}_{i=1,\dots,4}$ around $p \in N^{\bar{A}}$ we have for $v \in T_p N^{\bar{A}}$:

$$
\begin{aligned}
(\mathcal{L}_v \alpha)(p) &= \mathcal{L}_v \varphi_{\mathrm{cut}} \left(\frac{\log \rho(p)}{\log t_i} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \varphi_{\mathrm{cut}} \left(\frac{\log \rho(\exp_p(sv))}{\log t_i} \right) \\
&= \frac{1}{\rho \log t_i} \cdot \varphi_{\mathrm{cut}}' \cdot (\mathcal{L}_v \rho)(p).
\end{aligned}
$$

From this we see that $\mathcal{L}_v \alpha$ is bounded by $\frac{C}{\rho \log t_i}$, where C is independent of p and t_i. This is because whenever the derivative of ρ might become unbounded, the derivative of φ_{cut} vanishes. Similarly we obtain for $v, w \in T_p N^{\bar{A}}$:

$$
\begin{aligned}
\nabla^2 \alpha &= \nabla \left(\mathrm{d}x^i \otimes \mathcal{L}_{\partial_i} \varphi_{\mathrm{cut}} \left(\frac{\log \rho}{\log t_i} \right) \right) \\
&= (\mathrm{d}x^i \otimes \mathrm{d}x^j) \mathcal{L}_{\partial_j} \left(\frac{1}{\rho \log t_i} \cdot \varphi_{\mathrm{cut}}' \cdot (\mathcal{L}_{\partial_i} \rho) \right) \\
&= (\mathrm{d}x^i \otimes \mathrm{d}x^j) \frac{\varphi_{\mathrm{cut}}'' \rho \log(t_i) - \varphi_{\mathrm{cut}}' \mathcal{L}_{\partial_j} \log t_i}{(\rho \log t_i)^2} \frac{\mathrm{d}^2}{\mathrm{d}s \mathrm{d}r} \rho(\exp_p(s\partial_i + r\partial_j)) \\
&= (\mathrm{d}x^i \otimes \mathrm{d}x^j) \frac{1}{\rho^2 \log(t_i)} C(\varphi_{\mathrm{cut}}'', \varphi_{\mathrm{cut}}', \partial_i \partial_j \rho, \partial_i \rho).
\end{aligned}
$$

This proves the statement for $l = 2$. The general statement follows in a similar way. □

To combine the bounds on $\not{D}_A$ and $\not{D}_N$ using a partition of unity argument we need two further technical lemmas about the norms of αu and $\nabla \alpha \diamond u$, where $\diamond$ is a bilinear map.

Lemma 3.10. *Let B be a bundle of tensors over $N^{\bar{A}}$. Then there is a constant C_0 which is independent of $\bar{A}$, such that for sufficiently small global scale t and a section $u \in L^p_{k,\delta,\bar{A}}(B)$ the following holds:*

$$
\|\alpha u\|_{L^p_{k,\delta,\bar{A}}} \leqslant C_0 \|u\|_{L^p_{k,\delta,\bar{A}}}. \tag{3.26}
$$

Proof. We have:

$$
\begin{aligned}
2^{-k}\|\alpha u\|^p_{L^p_{k,\delta,\bar{A}}} &= 2^{-k}\sum_{i=0}^{k}\int_{N^{\bar{A}}} |\rho^{i-\delta}\nabla^i(\alpha u)|^p\rho^{-4}\,\mathrm{dvol}\\
&\leqslant \sum_{0\leqslant j\leqslant i\leqslant k}\int_{N^{\bar{A}}} \rho^{p(i-\delta)}|\nabla^j\alpha|^p|\nabla^{i-j}u|^p\rho^{-4}\,\mathrm{dvol}\\
&= \alpha^p\|u\|^p_{L^p_{k,\delta,\bar{A}}} + \sum_{0\leqslant j\leqslant i\leqslant k-1}\int_{N^{\bar{A}}} \rho^{p(1+i-\delta)}|\nabla^{j+1}\alpha|^p|\nabla^{i-j}u|^p\rho^{-4}\,\mathrm{dvol}\\
&\leqslant \|u\|^p_{L^p_{k,\delta,\bar{A}}} + \sum_{0\leqslant j\leqslant i\leqslant k-1}\int_{N^{\bar{A}}} |\rho^{j+1}\nabla^{j+1}\alpha|^p|\rho^{i-j-\delta}\nabla^{i-j}u|^p\rho^{-4}\,\mathrm{dvol}\\
&\leqslant \|u\|^p_{L^p_{k,\delta,\bar{A}}} + C\left(\sum_{i=0}^{k-1}\int_{N^{\bar{A}}} |\rho^{i-\delta}\nabla^i u|^p\rho^{-4}\,\mathrm{dvol}\right)\left(\sum_{j=0}^{k-1}\|\nabla^{j+1}\alpha\cdot\rho^{j+1}\|^p_{C^0}\right)\\
&\leqslant \|u\|^p_{L^p_{k,\delta,\bar{A}}} + \frac{C}{|\log t|^p}\left(\sum_{i=0}^{k-1}\int_{N^{\bar{A}}} |\rho^{i-\delta}\nabla^i u|^p\rho^{-4}\,\mathrm{dvol}\right)\\
&\leqslant \left(1+\frac{C}{|\log t|^p}\right)\|u\|^p_{L^p_{k,\delta,\bar{A}}}.
\end{aligned}
$$

Here we used the asymptotic behaviour of $\nabla^l\alpha$ from Proposition (3.9) in the second to last line. □

Lemma 3.11. *Let B be a bundle of tensors over $N^{\bar{A}}$. Let $\diamond : T^*N^{\bar{A}}\otimes B\to B$ be a family of bilinear pairings which have bounded norms as $\bar{A}$ varies, seen as sections of $T^*N^{\bar{A}}\otimes B\otimes B^*$. Then there is a constant $C_1>0$, independent of $\bar{A}$, such that for small enough global scale t and for any section $u\in L^p_{k,\delta,\bar{A}}(B)$ we have:*

$$\|\nabla\alpha\diamond u\|_{L^p_{k,\delta-1,\bar{A}}}\leqslant\frac{C_1}{|\log t|}\|u\|_{L^p_{k,\delta,\bar{A}}}. \tag{3.27}$$

Proof. Using Proposition 3.9 the statement reduces to proving the following:

$$\|\nabla\alpha\diamond u\|_{L^p_{k,\delta-1,\bar{A}}}\leqslant C\left(\sum_{i=0}^{k-1}\|\nabla^{i+1}\alpha\rho^{i+1}\|^p_{C^0}\right)^{1/p}\|u\|_{L^p_{k,\delta,\bar{A}}}.$$

This in turn is proven similarly to the previous proposition.

$$\begin{aligned}\|\nabla\alpha\diamond u\|^p_{L^p_{k,\delta+1,\bar A}} &= \sum_{i=0}^{k}\int_{N^{\bar A}} |\rho^{i-\delta+1}\nabla^i\left(\nabla\alpha\diamond u\right)|^p\rho^{-4}\,\mathrm{dvol}\\ &\leqslant C\sum_{0\leqslant j\leqslant i\leqslant k}\int_{N^{\bar A}} |\rho^{j-\delta}\nabla^j u|^p|\rho^{i-j+1}\nabla^{i-j+1}\alpha|^p\rho^{-4}\,\mathrm{dvol}\\ &\leqslant C\left(\sum_{i=0}^{k-1}\|\nabla^{i+1}\alpha\rho^{i+1}\|^p_{C^0}\right)\|u\|^p_{L^p_{k,\delta,\bar A}}.\end{aligned}$$

In the second line, we used the bound on the norm of the $\diamond$-product. □

Next, we show that the operator $\not{D}_{\bar A}$ can be inverted modulo the pseudo-kernel $\kappa_{\bar A}$, with uniformly bounded norm independent of $\bar A$. This is the key fact that will allow us to perform the desingularisation via an iteration argument in the next section.

Proposition 3.12. *There is a constant C_D, independent of $\bar A$, such that for any $u \in L^p_{k+1,\delta,\bar A}(\nu(N^{\bar A}))$ which is $L^2_{\delta\pm\epsilon}$-orthogonal to $\kappa_{\bar A}$ we have:*

$$\|v\|_{L^p_{k+1,\delta,\bar A}} \leqslant C_D\|\not{D}_{\bar A}v\|_{L^p_{k,\delta-1,\bar A}} \tag{3.28}$$

Proof. Write $u \in L^p_{k+1,\delta,\bar A}(\nu(N^{\bar A}))$, using the cut off function α from (3.20) as:

$$u = \alpha u + (1-\alpha)u.$$

Then clearly $\|u\|_{L^p_{k+1,\delta,\bar A}} \leqslant \|\alpha u\|_{L^p_{k+1,\delta,\bar A}} + \|(1-\alpha)u\|_{L^p_{k+1,\delta,\bar A}}$. Let us consider the term $\|\alpha u\|_{L^p_{k+1,\delta,\bar A}}$ first. Note that αu is supported in $N^{\bar A}_{\mathrm{AC}}$, and that on the support of $\kappa_{\mathrm{AC},\bar A}$, the cut off function α is in fact equal to 1. Thus αu is orthogonal to $\kappa_{\bar A}$ by our orthogonality assumption on u. Using Proposition 3.8 we see that:

$$\|\alpha u\|_{L^p_{k+1,\delta,\bar A}} \leqslant \tilde C_A\|\not{D}_{\bar A}(\alpha u)\|_{L^p_{k,\delta-1,\bar A}}$$

Now as $\not{D}_{\bar A}$ is a first-order operator whose coefficients depend pointwise on the Spin(7)-structure as in Proposition 2.5, we see that $\not{D}_{\bar A}(\alpha u) = \alpha\not{D}_{\bar A}u + (\nabla\alpha)\diamond u$, where $\diamond$ is a family of bilinear products $\diamond : T^*N^{\bar A}\otimes E \to E$ which is uniformly bounded in t. Thus we may apply Lemma 3.11 to see that in fact:

$$\|\not{D}_{\bar A}(\alpha u)\|_{L^p_{k,\delta-1,\bar A}} \leqslant \tilde C_A\|\alpha\not{D}_{\bar A}u\|_{L^p_{k,\delta-1,\bar A}} + \frac{C_1\tilde C_A}{\log(t)}\|u\|_{L^p_{k+1,\delta,\bar A}}.$$

In other words, we have, if we also apply Lemma 3.10:

$$\begin{aligned}\left(1-\frac{C_1\tilde C_A}{\log(t)}\right)\|\alpha u\|_{L^p_{k+1,\delta,\bar A}} &\leqslant \tilde C_A\|\alpha\not{D}_{\bar A}u\|_{L^p_{k,\delta-1,\bar A}}\\ &\leqslant \tilde C_A C_0\|\not{D}_{\bar A}u\|_{L^p_{k,\delta-1,\bar A}}.\end{aligned}$$

In particular, for t sufficiently small, setting $C_D = 2\frac{\tilde{C}_A C_0}{\left(1-\frac{C_1\tilde{C}_A}{\log(t)}\right)}$, we get that:

$$\|\alpha u\|_{L^p_{k+1,\delta,\bar{A}}} \leqslant \frac{C_D}{2}\|\not{D}_{\bar{A}} u\|_{L^p_{k,\delta-1,\bar{A}}}. \tag{3.29}$$

We now note that the auxiliary Lemmas 3.10 and 3.11 can equally well be proven for $1-\alpha$. Furthermore, the analogue for $N^{\bar{A}}_{\mathrm{CS}}$ of Proposition 3.8 is true. To see this note that the $L^p_{r,\sigma,\bar{A}}$ norms on $N^{\bar{A}}$ agree with the $L^p_{r,\sigma}$ norm for sections supported in $N^{\bar{A}}_{\mathrm{CS}}$. Furthermore, since $N^{\bar{A}}_{\mathrm{CS}}$ is already Cayley, $\not{D}_{\bar{A}}|_{N^{\bar{A}}_{\mathrm{CS}}} = \not{D}_{\mathrm{CS}}|_{N^{\bar{A}}_{\mathrm{CS}}}$, and so the result follows from Proposition 2.44, noting that the $L^2_{\delta\pm\epsilon}$ norm is identical to the $L^2_{\delta+\epsilon}$ norm on $N^{\bar{A}}$. We can therefore prove:

$$\|(1-\alpha)u\|_{L^p_{k+1,\delta,\bar{A}}} \leqslant \frac{C_D}{2}\|\not{D}_{\bar{A}} u\|_{L^p_{k,\delta-1,\bar{A}}}. \tag{3.30}$$

Equations (3.29) and (3.30) taken together now give us:

$$\begin{aligned}\|u\|_{L^p_{k+1,\delta,\bar{A}}} &\leqslant \|\alpha u\|_{L^p_{k+1,\delta,\bar{A}}} + \|(1-\alpha)u\|_{L^p_{k+1,\delta,\bar{A}}}\\ &\leqslant C_D\|\not{D}_{\bar{A}} u\|_{L^p_{k,\delta-1,\bar{A}}}.\end{aligned}$$

□

Quadratic estimates

We conclude this section on estimates by proving the quadratic estimates, which are consequences of the estimates in the compact and conically singular setting.

Proposition 3.13. *Let $\delta > 0$, $p > 4$ and $k \geqslant 1$. There are constants $E_Q > 0$ and $C_Q > 0$, independent of $\bar{A}$ and the* Spin(7)*-structure, and an open neighbourhood of $s_0 \in U \subset \mathcal{S}$, such that for sufficiently small global scale $t > 0$, $s \in U$ and $v, w \in L^p_{k+1,\delta,\bar{A}}$ with $\|v\|_{L^p_{k+1,\delta,\bar{A}}}, \|w\|_{L^p_{k+1,\delta,\bar{A}}} < E_Q$ we have:*

$$\|Q_{\bar{A}}(v,s) - Q_{\bar{A}}(w,s)\|_{L^p_{k,\delta-1,\bar{A}}} \leqslant C_Q\|v-w\|_{L^p_{k+1,\delta,\bar{A}}}(\|v\|_{L^p_{k+1,\delta,\bar{A}}} + \|w\|_{L^p_{k+1,\delta,\bar{A}}}). \tag{3.31}$$

Proof. Let $u, v \in L^p_{k+1,\delta,\bar{A}}(\nu_\epsilon(N))$ be given. By the Sobolev embedding Theorem 1.24 for weighted spaces we see that there are embeddings $L^p_{k+1,\delta,\bar{A}} \hookrightarrow C^k_{\delta,\bar{A}}$. Here the Sobolev constants are bounded independent from $\bar{A}$ as it is invariant under rescaling of the AC pieces. Thus we have that u and v are C^1 and that their $C^1_{\delta,\bar{A}}$-norms are bounded by $C\cdot E_Q$. In particular we thus have that $|v|, |\nabla v| < C \cdot E_Q$ independently of $\bar{A}$. Hence we can invoke Lemma 2.10 to obtain a pointwise bound of the form:

$$\begin{aligned}|Q_{\bar{A}}(v,s) - Q_{\bar{A}}(w,s)|_{C^{k+1}} \leqslant & C(1+|TN^{\bar{A}}|_{C^{k+1}})\Big(|v-w|_{C^{k+1}}(|v|_{C^k}+|w|_{C^k})\\ &+|v-w|_{C^k}(|v|_{C^{k+1}}+|w|_{C^{k+1}})\Big).\end{aligned} \tag{3.32}$$

In a similar fashion to how we prove the initial error estimates on $F_{\bar{A}}$, we can also show that we have:

$$|TN^{\bar{A}}|_{C^{k+1}_{1,\bar{A}}} \leqslant t^{C_{TN}},$$

where $C_{TN} > 0$ is a positive constant, independent of $\bar{A}$ (but dependent on δ). Thus the same reasoning as in the conically singular case in Proposition 2.40 gives us the desired weighted bound, with the constant independent of $\bar{A}$. □

We now show that if all the pieces involved in the gluing are unobstructed at their respective rates, then the same is true for the glued manifolds. This relies on the fact that increasing the rate of $\not{D}$ on an AC manifold and decreasing the rate on a CS manifold respectively preserve unobstructedness of the operator. Since $\lambda < \delta < \mu$, both operators will hence still be unobstructed at rate δ.

Proposition 3.14. *Let $4 < p < \infty$ and $k \geqslant 1$. Assume that both the A_i and N are unobstructed as Cayley manifolds at rate $\lambda < 1$ and $1 < \bar{\mu} < 2$ respectively. Assume that $[\lambda, 1) \cap \mathscr{D}_i = \emptyset$, $(1, \mu_i] \cap \mathscr{D}_i = \emptyset$ and that all the cones which are glued in are unobstructed. Let $1 < \delta < \mu_i$ be fixed. We then have that for sufficiently small $t > 0$ the linearised deformation operator $\not{D}_{\bar{A}}$ is surjective. In particular, for any $w \in L^p_{k,\delta-1,\bar{A}}(E_{\mathrm{cay}})$ there is a unique $v \in \kappa^{\perp}_{\bar{A}}$ such that $\not{D}_{\bar{A}} v = w$.*

Proof. We have that the operators $\not{D}_{\mathrm{AC}}$ and $\not{D}_{\mathrm{CS}}$ are surjective as maps from $L^p_{k+1,\delta} \to L^p_{k,\delta-1}$, as increasing/decreasing the rate in the AC/CS case cannot introduce a cokernel by Theorem 1.32. In particular they admit bounded right-inverses P_{AC} and P_{CS} respectively, which map $L^p_{k,\delta-1}$ into $L^p_{k+1,\delta}$. We would first like to show that $\not{D}_{\bar{A}}$ is surjective for sufficiently small values of t.
Claim: If there is a bounded linear map $P_{\bar{A}} : L^p_{k,\delta-1} \to L^p_{k+1,\delta}$ such that the operator norm of $\mathrm{id} - \not{D}_{\bar{A}} P_{\bar{A}}$ satisfies $\| \mathrm{id} - \not{D}_{\bar{A}} P_{\bar{A}} \| < 1$, then $\not{D}_{\bar{A}}$ is surjective.
Proof: By the continuous functional calculus in Banach spaces, the operator $\not{D}_{\bar{A}} P_{\bar{A}} = \mathrm{id} - (\mathrm{id} - \not{D}_{\bar{A}} P_{\bar{A}})$ has the bounded inverse $\sum_{i=0}^{\infty} (\mathrm{id} - \not{D}_{\bar{A}} P_{\bar{A}})^i$, as this sum converges by the assumption on the operator norm of $\mathrm{id} - \not{D}_{\bar{A}} P_{\bar{A}}$. Thus in particular $\not{D}_{\bar{A}}$ is surjective. ■

We now construct such a $P_{\bar{A}}$ by joining together P_{AC} and P_{CS}, seen as operators on $N^{\bar{A}}_{\mathrm{AC}}$ and $N^{\bar{A}}_{\mathrm{CS}}$ respectively. Note that P_{CS} takes sections on N to sections on N. Thus in particular, if $s \in L^p_{k,\delta-1,\bar{A}}(E_{\mathrm{cay}})$ is a section on all of $N^{\bar{A}}$, then $(1-\alpha) P_{\mathrm{CS}}((1-\alpha)s)$ defines a well-defined section which is supported on $N^{\bar{A}}_{\mathrm{CS}}$. Similarly, we have an identification of sections on $N^{\bar{A}}_{\mathrm{AC}}$ with sections on A_i via the map $\Psi^i_{\bar{A}}$. This allows us to define the operator $\alpha P_{\mathrm{AC},\bar{A}}$ on $N^{\bar{A}}$, which takes section supported in $N^{A_i}_{\mathrm{AC}}$ to sections on A_i, applies P_{A_i}, and transports them back to section on $N^{\bar{A}}_{\mathrm{AC}}$. It has the noticeable property that $\not{D}_{\bar{A}} P_{\mathrm{AC},\bar{A}} = \mathrm{id}$. We thus define:

$$P_{\bar{A}}(s) = (1-\alpha) P_{\mathrm{CS}}((1-\alpha)s) + \alpha P_{\mathrm{AC},\bar{A}}(\alpha s). \tag{3.33}$$

When precomposed with $\not{D}_{\bar{A}}$, we obtain:

$$\not{D}_{\bar{A}} P_{\bar{A}}(s) - s = (2\alpha(1-\alpha))s + \nabla(1-\alpha) \diamond_1 P_{\mathrm{CS}}((1-\alpha)s) + \nabla\alpha \diamond_2 P_{\mathrm{AC},\bar{A}}(\alpha s), \tag{3.34}$$

where $\diamond_1, \diamond_2$ are two bilinear products. Notice that $2\alpha(1-\alpha) \leqslant \frac{1}{2}(\alpha+1-\alpha)^2 = \frac{1}{2}$, thus to prove the proposition we need to find $0 < K < \frac{1}{2}$ such that:

$$\|\nabla(1-\alpha) \diamond_1 P_{\mathrm{CS}}((1-\alpha)s)\|_{L^p_{k,\delta-1,\bar{A}}} \leqslant \frac{K}{2}\|s\|_{L^p_{k,\delta-1,\bar{A}}}, \text{ and}$$
$$\|\nabla\alpha \diamond_2 P_{\mathrm{AC},\bar{A}}(\alpha s)\|_{L^p_{k,\delta-1,\bar{A}}} \leqslant \frac{K}{2}\|s\|_{L^p_{k,\delta-1,\bar{A}}}.$$

Let us consider the second inequality for concreteness. Proposition 3.11 and the uniform boundedness of $P_{\mathrm{AC},\bar{A}}$ allow us to write:

$$\begin{aligned}\|\nabla\alpha \diamond_2 P_{\mathrm{AC},\bar{A}}(\alpha s)\|_{L^p_{k,\delta-1,\bar{A}}} &\leqslant \frac{C}{\log(t)}\|P_{\mathrm{AC},\bar{A}}(\alpha s)\|_{L^p_{k+1,\delta,\bar{A}}}\\ &\leqslant \frac{C}{\log(t)}\|P_{\mathrm{AC},\bar{A}}(\alpha s)\|_{L^p_{k+1,\delta,\bar{A}}}\\ &\leqslant \frac{C\tilde{C}_A}{\log(t)}\|\alpha s\|_{L^p_{k,\delta-1,\bar{A}}}\\ &\leqslant \frac{C_1\tilde{C}_A}{\log(t)}\|s\|_{L^p_{k,\delta-1,\bar{A}}}.\end{aligned}$$

In the last line, we applied Proposition 3.10. Now note that for t sufficiently small we can arrange that $\frac{C_1\tilde{C}_A}{\log(t)} < \frac{1}{4}$. The same reasoning applies to P_{CS}, hence we have shown that $\not{D}_{\bar{A}}$ surjects onto $L^p_{k,\delta-1,\bar{A}}(E_{\mathrm{cay}})$. Using Proposition 3.12 we see that in fact $\not{D}_{\bar{A}} : \kappa_{\bar{A}}^{\perp} \to L^p_{k,\delta-1,\bar{A}}(E_{\mathrm{cay}})$ is an isomorphism. □

As a consequence of the previous proposition, we can conclude that $\kappa_{\bar{A}} \simeq \ker \not{D}_{\bar{A}}$.

3.3 Finding a nearby Cayley

Theorem 3.15 (Gluing Theorem)**.** *Let (M, Φ) be a* Spin(7)*-manifold and N a $\mathrm{CS}_{\bar{\mu}}$-Cayley in (M, Φ) with singular points $\{z_i\}_{i=0,\dots,l}$ and rates $1 < \mu_i < 2$, modelled on the cones $C_i = \mathbb{R}_+ \times L_i \subset \mathbb{R}^8$. Assume that N is unobstructed in $\mathcal{M}^{\bar{\mu}}_{\mathrm{CS},cones}(N, \{\Phi\})$, i.e. in the moduli space with fixed points but allowing the cone to deform. For a fixed $k \leqslant l$, assume for each $i \leqslant k$ that the L_i are unobstructed as associatives (i.e. that the C_i are unobstructed cones), and that $\mathscr{D}_{L_i} \cap (1, \mu_i] = \emptyset$. For $1 \leqslant i \leqslant k$, suppose that A_i is an unobstructed AC_λ-Cayley with $\lambda < 1$, such that $\mathscr{D}_{L_i} \cap [\lambda, 1) = \emptyset$. Let $\{\Phi_s\}_{s\in\mathcal{S}}$ be a smooth family of deformations of $\Phi = \Phi_{s_0}$ as* Spin(7) *structures. Then there are open neighbourhoods U_i of $C_i \in \overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A_i)$, an open neighbourhood $s_0 \in \mathcal{U} \subset \mathcal{S}$ and a continuous map:*

$$\Gamma : \mathcal{U} \times \mathcal{M}^{\bar{\mu}}_{\mathrm{CS},cones}(N, \{\Phi\}) \times \prod_{i=1}^{k} U_i \longrightarrow \bigcup_{I\subset\{1,\dots,k\}} \mathcal{M}^{\bar{\mu}_I}_{\mathrm{CS}}(N_I, \mathcal{S}). \tag{3.35}$$

Here we denote by $\bar{\mu}_I$ the subsequence, where we removed the i-th element for $i \in I$ from $\bar{\mu}$. Moreover, N_I denotes the isotopy class of the manifold obtained after desingularising *the points z_i for $i \in I$ by a connected sum with A_i.*

This map is a local diffeomorphism of stratified manifolds. Thus in particular, simultaneously away from all cones in $\overline{\mathcal{M}}^{\lambda}_{\mathrm{AC}}(A_i)$ it is a local diffeomorphism onto the nonsingular Cayley submanifolds in $\mathcal{M}(N_{\{1,\dots,k\}}, \mathcal{S})$. It maps the point $(s, \tilde{N}, \tilde{A}_1, \dots, \tilde{A}_k)$ into $\mathcal{M}^{\bar{\mu}_I}_{\mathrm{CS}}(N_I, \mathcal{S})$, where I is the collection of indices for which $\tilde{A}_i = C_i$. This corresponds to partial desingularisation.

Remark 3.16. In the above Theorem, we consider all the deformations of rates 0, the translations, to be part of the moduli space of AC Cayleys. When gluing, we do not however simply glue translated versions of our AC Cayleys onto a static CS Cayley. This would result in too large an error coming from the partition of unity to apply the iteration scheme. Hence we always implicitly consider the CS Cayley in $\mathcal{M}^{\mu}_{\mathrm{CS}}(N)$ which underwent the same translation as the AC Cayley when gluing. This gives us an approximation up to order $O(r)$ included, which allows us to work in $L^p_{k,\gamma,\bar{A}}$ with $\gamma > 1$,

Proof. Let $k \geqslant 1$ and $p > 4$ be fixed. We first find a solution to the equation $F_{\bar{A}}(v) = 0$ for a fixed Spin(7)-structure via an iteration scheme. For this, fix an $\sigma > 0$ such that $\sigma < \frac{C_Q}{2}$. We will construct sections $v_i^{\bar{A}} \in L^p_{k+1,\delta,\bar{A}}$ with $i \in \mathbb{N}$ which satisfy:

$$\begin{aligned} \not{D}_{\bar{A}} v^{\bar{A}}_{i+1} &= -F_{\bar{A}}(0) - Q_{\bar{A}}(v_i^{\bar{A}}), \\ v_i^{\bar{A}} \perp^{L^2_{\delta\pm\epsilon}} \kappa_{\bar{A}} &\text{ and } \|v_i^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} < \sigma. \end{aligned} \tag{3.36}$$

For this, define first $v_0^{\bar{A}} = 0$ for any $\bar{A}$ with sufficiently small t and $\delta > 1$. Then Proposition 3.14 allows us to find a unique pre-image $v_i^{\bar{A}}$ of $-F_{\bar{A}}(0) - Q_{\bar{A}}(v_0^{\bar{A}}) = -F_{\bar{A}}(0)$. From our estimate on the inverse of $\not{D}_{\bar{A}}$ on sections which are orthogonal to the approximate kernel $\kappa_{\bar{A}}$ from Proposition 3.12 we see that:

$$\begin{aligned} \|v_1^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} &\leqslant C_D \|\not{D}_{\bar{A}} v_1^{\bar{A}}\|_{L^p_{k,\delta-1,\bar{A}}} \\ &\leqslant C_D \|F_{\bar{A}}(0)\|_{L^p_{k,\delta-1,\bar{A}}} \\ &\leqslant C_D C_F t^{\nu(\mu-\delta)}. \end{aligned}$$

Here we used the bound (3.19) on our initial error estimate in the last line. We see that for sufficiently small $1 < \delta < \mu$ the initial error will become arbitrarily small. Thus for $t_0 > 0$ sufficiently small we have:

$$\|v_1^{\bar{A}}\|_{L^p_{k+1,\delta+1,\bar{A}}} \leqslant C_D C_F t^{\nu(\mu-\delta)} < \frac{\sigma}{4},$$

for all $t \in (0, t_0]$. Suppose now that we have constructed $v_i^{\bar{A}}$ for some $i \in \mathbb{N}$, such that $\|v_1^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} < \sigma$. We then find the pre-image $v_{i+1}^{\bar{A}}$ of $-F_{\bar{A}}(0) - Q_{\bar{A}}(v_i^{\bar{A}})$ and use our estimate on $Q_{\bar{A}}$ from Proposition 3.31 to show the following:

$$\begin{aligned}\|v_{i+1}^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} &\leqslant C_D \|\not{D}_{\bar{A}} v_i^{\bar{A}}\|_{L^p_{k,\delta-1,\bar{A}}} \\ &\leqslant C_D(\|F_{\bar{A}}(0)\|_{L^p_{k,\delta-1,\bar{A}}} + \|Q_{\bar{A}}(v_i^{\bar{A}})\|_{L^p_{k,\delta-1,\bar{A}}}) \\ &\leqslant C_D C_F t^{\nu(\mu-\delta)} + C_Q \|v_i^{\bar{A}}\|^2_{L^p_{k+1,\delta+1,\bar{A}}} \\ &\leqslant \frac{\sigma}{4} + C_Q\sigma^2 < \sigma.\end{aligned}$$

We can now iterate this procedure to obtain a sequence $\{v_i^{\bar{A}}\}_{i\in\mathbb{N}}$ for every $t \in (0, t_0]$ which satisfies our requirements (3.36). Note that we are free to choose $0 < \sigma < \frac{C_Q}{2}$. This family of sequences converges uniformly in $\bar{A}$, as we have the bounds:

$$\begin{aligned}\|v_{i+1}^{\bar{A}} - v_i^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} &\leqslant C_D \|Q_{\bar{A}}(v_i^{\bar{A}}) - Q_{\bar{A}}(v_{i-i}^{\bar{A}})\|_{L^p_{k,\delta-1,\bar{A}}} \\ &\leqslant C_D C_Q (\|v_i^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} + \|v_{i-1}^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}}) \|v_i^{\bar{A}} - v_{i-1}^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} \\ &\leqslant 2C_D C_Q \sigma \|v_i^{\bar{A}} - v_{i-1}^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}}.\end{aligned}$$

If we choose σ small enough, we can ensure that:

$$\|v_{i+1}^{\bar{A}} - v_i^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} < \tfrac{1}{2}\|v_i^{\bar{A}} - v_{i-1}^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}}.$$

Thus $\{v_i^{\bar{A}}\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^p_{k+1,\delta,\bar{A}}(\nu(N^{\bar{A}}))$ for each $\bar{A}$ simultaneously. We can thus find limits $v_\infty^{\bar{A}} \in L^p_{k+1,\delta,\bar{A}}(\nu(N^{\bar{A}}))$. Since both $\not{D}_{\bar{A}}$ and $Q_{\bar{A}}$ are continuous maps of Banach manifolds, we have:

$$\begin{aligned}\not{D}_{\bar{A}} v_\infty^{\bar{A}} &= \lim_{i\to\infty} \not{D}_{\bar{A}} v_{i+1}^{\bar{A}} \\ &= \lim_{i\to\infty} -F_{\bar{A}}(0) - Q_{\bar{A}}(v_i^{\bar{A}}) \\ &= -F_{\bar{A}}(0) - Q_{\bar{A}}(v_\infty^{\bar{A}}).\end{aligned}$$

Thus $F_{\bar{A}}(v_\infty^{\bar{A}}) = 0$. We then immediately get smoothness for $v_\infty^{\bar{A}}$ by Proposition 2.15. By Theorem 2.16 we can conclude that $\tilde{N}^{\bar{A}} = \exp_{v_\infty^{\bar{A}}}(N^{\bar{A}})$ is a family of smooth Cayley submanifolds, as the family clearly only varies in a compact subset of M. The manifold $\tilde{N}^{\bar{A}}$ has the same topological type as $N^{\bar{A}}$ and together the $\tilde{N}^{\bar{A}}$ form the desired desingularisation.

Thus we can define a map Γ as above on the slice $\{\Phi\} \times \mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \{\Phi\}) \times \prod_{i=1}^k U_i$. We would now like to extend this map when the ambient Spin(7)-structure is allowed to vary. For this, we first choose a trivialisation $T : U \times \mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \{\Phi\}) \simeq \mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, U)$, where $s_0 \in U \subset \mathcal{S}$ is an open neighbourhood, which can be done by unobstructedness of N, using Theorem 2.42. Now we can repeat the above iteration scheme for $\Phi' \in U$, where we now glue $\bar{A}$ onto $N' = T(\Phi', N)$. From this, we see that smoothly varying the Spin(7)-structure leads to a smooth change in the resulting submanifold.

Note that:

$$\|v_\infty^{\bar{A}}\|_{L^p_{k+1,\delta,\bar{A}}} \leqslant 2\|v_0^{\bar{A}}\|_{L^p_{k+1,\delta+1,\bar{A}}} \leqslant Ct^{\nu(\mu-\delta)}, \tag{3.37}$$

and thus as the scale t tends to 0, the resulting Cayley will converge in $L^p_{k+1,\delta,\bar{A}}$ (thus in C^k_{loc}) to $N^{\bar{A}}$, which in turn converges in the sense of currents to the conically singular N. As we also have C^k_{loc} convergence for any $k \geqslant 1$, we get C^∞_{loc} convergence as well. Moreover, there is nothing special about reducing the global scale as opposed to reducing only a subset of the scales to 0. In this case, the same argument localised to the singular points in question gives the C^∞_{loc} convergence to the partially desingularised N.

Finally, this construction is smooth in the gluing pieces away from cones. Indeed, varying the pieces gives rise to a smooth change of the p.d.e. $F_{\bar{A}}(v) = 0$, and all the constants involved in the iteration scheme remain valid. Thus the result will also vary smoothly. □

Remark 3.17. We would like to point out that Theorem 3.15 is not the only possible gluing result in this setting. What is needed in the construction are the following three ingredients. Whenever these are true, we can prove a corresponding gluing result.

- The initial error $\|F_{\bar{A}}(0)\|_{L^p_{k+1,\delta,\bar{A}}}$ needs to go to zero as the global neck size $t \to 0$.
- The quadratic estimate (3.31) needs to hold for some constant C_Q.
- The linearised operator needs to be invertible orthogonal to its kernel, and has to have uniformly bounded norm.

The first two items above are true as long as our initial approximation gets better in a C^1 sense as $t \to 0$, and we know how to handle the local model of the noncompact piece (in this case a cone). In particular, we do not need the unobstructedness of the AC and CS pieces for these two items. We do however need it for the last item, where it is crucial that the glued operator is surjective and has a well-understood behaviour in the $L^p_{k,\delta,\bar{A}}$ norms as $t \to 0$. In Theorem 3.15 we chose the rates of both pieces to be near 1 and then included the slightly tricky rate 1 into the moduli space of CS Cayleys. However, provided that $A_i \in \mathcal{M}^{-\epsilon}_{\mathrm{AC}}$ and $\mathcal{M}^{1+\epsilon}_{\mathrm{CS}}$ are unobstructed (where we now allow the points to move in the CS moduli space), we can define a gluing map:

$$\tilde{\Gamma} : \mathcal{U} \times \mathcal{M}^{\bar{\mu}}_{\mathrm{CS}}(N, \{\Phi\}) \times \prod_{i=1}^{k} U_i \longrightarrow \bigcup_{I \subset \{1,\dots,k\}} \mathcal{M}^{\bar{\mu}_I}_{\mathrm{CS}}(N_I, \mathcal{S}). \tag{3.38}$$

Here $U_i \subset \overline{\mathcal{M}}^{-\epsilon}_{\mathrm{AC}}$ are now excluding the translations. They are included in the CS moduli space. Essentially we can define a gluing map whenever we have rates $\lambda < 1 < \mu$ for which the pieces are unobstructed, and we can include the translations and rotations manually on the conically singular side.

Note however that if we are missing some critical rates, in the sense that there is a critical rate $\delta \in \mathscr{D}_L$ which is not accounted for on either the AC or the CS piece, then the gluing map will not be surjective. So for instance, if we are given a cone with no critical rates in the range $(0, 1)$, we still have surjectivity of the map $\tilde{\Gamma}$.

3.4 Desingularising immersed Cayley submanifolds

Two positively intersecting Cayley planes cannot be desingularised by a minimal surface, as they are already area-minimizing by the sufficiency part of the angle criterion, proven by Nance in [40]. More concretely, two complex planes Π_1, Π_2 intersecting transversally are an example of positively intersecting Cayley planes. Now it is a consequence of Hartog's phenomenon that no nonsingular complex surface S can exist that is AC to two such planes. Indeed, over $\Pi_1 \setminus \{0\}$, such a surface can be seen as the graph of a holomorphic function $f : \Pi_1 \setminus \{0\} \to \Pi_2$. According to Hartog's phenomenon such a function must extend holomorphically to all of Π_1, which is in contradiction to the fact that f must diverge to infinity as one approaches $0 \in \Pi_1$.

As an immediate consequence of Lemma 2.29 and Theorem 3.15 we obtain the following desingularisation result, which is optimal by this discussion. We note that if a Cayley N is unobstructed as an immersed Cayley, then it is also unobstructed as a CS Cayley with moving points and cones. This can be seen by comparing the two deformation operators:

$$F : C^\infty_0(\nu(N)) \to C^\infty(E), \quad F : C^\infty_\mu(\nu(N)) \oplus \mathcal{F} \to C^\infty(E).$$

Here $\mathcal{F}$ contains the zeroth and first order deformations, which is the only difference between C^∞_0 and C^∞_μ (for the cone given by two transversal planes) with $1 < \mu < 2$.

Theorem 3.18 (Desingularisation of immersions)**.** *Let N be an unobstructed immersed Cayley submanifold which admits a negative self-intersection at $p \in N$. Then there is a family of Cayley submanifolds with one fewer singular point $\{N_t\}_{t\in(0,\epsilon)}$ such that $N_t \to N$ in the sense of currents and also in C^∞_{loc} away from the singularity as $t \to 0$.*

Example 3.19. Consider the Spin(7)-manifold (T^8, Φ_0), which is obtained as a quotient of $(\mathbb{R}^8, \Phi_0)$ by the lattice of integer points. Consider any affine plane in $\mathbb{R}^8$ which descends to a closed manifold in the quotient. Take for instance the special Lagrangian plane $\mathbb{R}^4 \subset \mathbb{C}^4$. It admits a 16-dimensional space of Cayley deformations, however, a 12-dimensional subset of these is generated by rotations and thus not preserved in the quotient (as the image will be of a different topological type). What remains are the 4-dimensional family of translation, which descend to the obvious translations of a $T^4 \times \{0\} \subset T^8$. Its Cayley moduli space however has expected dimension $\frac{1}{2}(\sigma(T^4) - \chi(T^4)) = 0$ by Example 2.19. Thus this four-torus is obstructed as a Cayley in the moduli space $\mathcal{M}(T^8, \Phi)$. We can modify the Spin(7)-structure near T^4 so that the submanifold becomes unobstructed in the new moduli space $\mathcal{M}(T^8, \tilde{\Phi})$. In particular, if we take the union of a finite number of such tori that each intersect each other negatively, we can construct a Spin(7)-structure in which we can desingularise the union of tori using our gluing theorem 3.15 to obtain a connected sum of tori in a (T^8, Φ), where Φ is a small perturbation of the usual flat structure Φ_0.

Example 3.20. Consider the CY fourfold $M = \{z_0^6 + z_1^6 + z_2^6 + z_3^6 + z_4^6 + z_5^6 = 0\} \subset \mathbb{C}P^5$. In this manifold, we can construct special Lagrangian and complex submanifolds which intersect at a point. For the complex surface we take $N = \{z_1 = iz_2, z_3 = iz_4\}$. For the special Lagrangian, we choose the fixed-point locus of the following anti-holomorphic involution:

$$\sigma([z_0, z_1, z_2, z_3, z_4, z_5]) = [\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3, e^{i\frac{\pi}{3}}\bar{z}_5].$$

We have that $L = \mathrm{Fix}(\sigma)$ is a special Lagrangian submanifold by [19, Prop. 12.5.2]. They intersect negatively, however it turns out that the special Lagrangian is obstructed. Thus as in the previous example, we can only say that there is a Cayley in a nearby Spin(7)-structure. More generally, special Lagrangians tend to be obstructed, as we see from Example 2.19. There we show that the obstruction space of a special Lagrangian L in a CY fourfold M is given by:

$$\mathcal{O}(L) \simeq H^0(L) \oplus H^{2,-}(L).$$

In particular, if L is connected we then have $\dim \mathcal{O}(L) = 1 + b^{2,-}$. We remark that these obstructions seem to stem from freedom in the choice of parameters in the Cayley form, at least in the torsion-free setting where calibrated submanifolds are minimal. Recall the formula for the Cayley form:

$$\Phi_{\varphi,\omega} = \mathrm{Re}(e^{i\varphi}\Omega) + \tfrac{1}{2}\omega \wedge \omega.$$

Here any choice of $\varphi \in \mathbb{R}$ and any choice of ω in the Kähler cone K of (M, J, ω, g) gives rise to a valid Cayley form. However note that if L is special Lagrangian in M, i.e. $\mathrm{Re}(\Omega)|_L = \mathrm{dvol}_L$, then the moduli space $\mathcal{M}(L, \Phi_\varphi)$ with $\varphi \neq 2\pi k$ is necessarily empty, for by Stokes' theorem whenever $\tilde{L}$ is homologous to L:

$$\int_{\tilde{L}} \Phi_\varphi = \int_L \Phi_\varphi = \int_L \mathrm{Re}(e^{i\varphi}\Omega) < \int_L \mathrm{Re}(\Omega) = \mathrm{vol}(L),$$

And thus no calibrated submanifolds in the homology class of L can exist for Φ_φ, in the torsion-free setting. We can remove the obstructions associated to φ manually by quotienting M by an antiholomorphic involution. The only Spin(7)-structures that descend to the quotient must satisfy $\varphi = 2\pi k$. One could feasibly remove the obstructions coming from ω by working in the family of Spin(7) structures $\{\Phi_{\varphi,\tilde{\omega}}\}_{\tilde{\omega}\in K}$, which may be enough to ensure surjectivity of the family operator $\not{D}_{\mathcal{S}}$. While the gluing Theorem 3.15 was only be proved for Cayleys that are unobstructed for a fixed Spin(7)-structure, an analogous iteration scheme involving Φ is conceivable. This would lead to a true Cayley in one of the structures $\Phi_{\varphi,\tilde{\omega}}$.

Chapter 4

Cayley fibrations

In this chapter, we focus our attention on the question of stability for Cayley fibrations of compact Spin(7)-manifolds. That is, if (M, Φ) is a Spin(7)-manifold fibred by a collection of Cayleys $\overline{\mathcal{M}}(N, \Phi)$ (which will contain both singular and nonsingular Cayleys), under which conditions does the Spin(7)-manifold $(M, \tilde{\Phi})$ for $\tilde{\Phi}$ a small perturbation of Φ also admit a fibration by Cayley submanifolds, now for the Spin(7)-structure $\tilde{\Phi}$?

Our answer will be twofold. First, we discuss the notion of weak fibration, which is homological in nature, and prove that weak Cayley fibrations are stable if their singularities are at worst conical and the locus of singular fibres in the base is of codimension at least 2. This is a direct extension of the work in Chapters 2 and 3. When the Spin(7)-structure is perturbed smoothly, unobstructed Cayleys (both nonsingular and conically singular) deform smoothly because of the structure results for their family moduli spaces 2.16 and 2.42. We will work under the assumption that the fibres have at worst conical degenerations. In other words, adjoining the conically singular Cayleys provides a compactification of the moduli space of nonsingular compact Cayleys. We then use the gluing theorem 3.15 to show that the fibration remains continuous even at the interface between compact and CS Cayleys.

Then, building on top of the weak stability result, we prove that a strong fibration (satisfying some reasonable assumptions) remains C^1 with a uniform bound on the derivative of the fibration in the base direction, even when approaching the singular fibres. This prevents the fibres from starting to intersect as the Spin(7)-structure is deformed. The proof relies on a gluing argument, where we glue solutions to the linearised Cayley equation (these give exactly the deformations to nearby fibres in the fibration) on the desingularised manifolds from Chapter 3.

4.1 Strong and weak fibrations

Let (M, Φ) be a fixed Spin(7)-manifold, and assume that N is a compact, unobstructed Cayley submanifold such that every element of the moduli space $\mathcal{M}(N, \Phi)$ is unobstructed. Then $\mathcal{M}(N, \Phi)$ is a smooth manifold, which in general will be noncompact. Various kinds of behaviours could in principle arise, but one can find examples where at worst conically singular degenerations occur. Under these assumptions and using the gluing map Γ from

Section 3.3, we expect $\mathcal{M}(N,\Phi)$ to decompose as:

$$\mathcal{M}(N,\Phi) = K \sqcup \bigsqcup_{i=1}^{n} \Gamma(\mathcal{M}^{\bar{\mu}_i}_{\mathrm{CS}}(N_i,\Phi), \{\bar{A} \in \mathcal{M}^{\lambda}_{\mathrm{AC}}(\bar{A}) : 0 < t(\bar{A}) < \epsilon\}, \Phi).$$

Here K is a compact set of nonsingular Cayley submanifolds, and the rest is given as desingularisations of a collection of conically singular Cayleys N_i ($1 \leqslant i \leqslant n$) with rates $1 < \bar{\mu}_i < 2$ by appropriate AC_λ Cayleys ($\lambda < 1$). The constant $\epsilon > 0$ is chosen sufficiently small, and we write $\mathcal{M}^{\lambda}_{\mathrm{AC}}(\bar{A})$ (where $\bar{A} = (A_1, \ldots, A_l)$) for the product space $\mathcal{M}^{\lambda}_{\mathrm{AC}}(A_1,\Phi_0) \times \cdots \times \mathcal{M}^{\lambda}_{\mathrm{AC}}(A_l,\Phi_0)$. We expect that for generic Spin(7)-structures both the conically singular and the asymptotically conical manifolds are unobstructed and we may thus apply our gluing theorem 3.15. We can include the conically singular Cayleys to form the **completed moduli space**:

$$\overline{\mathcal{M}}(N,\Phi) = \mathcal{M}(N,\Phi) \sqcup \bigsqcup_{i=1}^{n} \mathcal{M}^{\bar{\mu}_i}_{\mathrm{CS}}(N_i,\Phi).$$

The topology is induced from the completed moduli space of asymptotically conical manifolds. In other words if $N_k = \Gamma(A_k, \hat{N}_k)$ is a sequence with A_k limiting to a cone C as $k \to \infty$ and $\hat{N}_k \to \hat{N}$, then also $N_k \to \hat{N}$ in the completed moduli space. This gives $\overline{\mathcal{M}}(N,\Phi)$ a well-defined topology by the continuity of the gluing map Γ.

In fact, this space is a stratified manifold where the full-dimensional open stratum is exactly $\mathcal{M}(N,\Phi)$. The lower-dimensional strata are the $\mathcal{M}^{\bar{\mu}_i}_{\mathrm{CS}}(N_i,\Phi)$ which by unobstructedness are of codimension $\dim \mathcal{M}^{\lambda}_{\mathrm{AC}}(\bar{A}_i)$. From this discussion, it is natural to define the following concept of a Cayley fibration.

Definition 4.1. A **strong Cayley fibration** or simply **Cayley fibration** of a compact Spin(7)-manifold (M,Φ) is a homeomorphism $\mathrm{ev} : \mathrm{Univ}(\overline{\mathcal{M}}(N,\Phi)) \simeq M$, for some smooth Cayley submanifold N. Here $\mathrm{Univ}(\mathcal{M})$ is the **universal family** of a moduli space of submanifolds $\mathcal{M}$. As a topological space, it is the union of all $\tilde{N} \in \mathcal{M}$ with the topology induced from the embeddings of the $\tilde{N}$ into the ambient manifold. Furthermore, ev is the evaluation map that sends a point in a Cayley to itself, seen as a point of M.

Ideally, we would like Cayley fibrations to not contain any singular fibres at all. However, this assumption seems to be unrealistic in practice, as Cayley fibrations coming from complex fibrations of Calabi–Yau fourfolds need to admit a topologically determined number of singular fibres when counted with multiplicity. In Remark 5.15 for example, where we investigate a particular example of a fibration, the number of singular fibres equals the number of solutions to a system of polynomial equations on complex projective space, which by Bézout's theorem is just the product of the degrees of the polynomials.

Fibres with singularities complicate proving the stability of Cayley fibrations under small smooth perturbations of the ambient Spin(7)-structure, essentially because it is harder to compare nearby Cayleys with different singularities than it is to compare Cayleys of the same topological type. For strong stability one needs to make sure that no two nearby fibres deform too rapidly relative to one another as the Spin(7)-structure varies, which requires a C^1 estimate on the fibration. This is explained in more detail in Section 4.3.

To remedy this we introduce a weakened version of the fibration property. Here, stability under change of the Spin(7)-structure relies only on showing continuity of the fibration under perturbation, which is a direct extension of the desingularisation theory we developed in the previous chapter.

We use the notion of **pseudo-cycles** from [33, Section 7.1]. They allow us to define the degree of the evaluation map $\mathrm{ev} : \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)) \to M$, even when the domain is not a compact manifold. To work with pseudo-cycles, we need the singular stratum of $\overline{\mathcal{M}}(N, \Phi)$ to be of codimension at least 2 so that the push-forward of the fundamental class can still be defined.

More precisely, a pseudo-cycle from a smooth (possibly noncompact) manifold X of dimension n to a compact smooth manifold M is a smooth map $f : X \to M$ such that the boundary of $f(X)$ is of dimension at most $n-2$. Here we define the boundary as the set of all limit points (in M) of sequences $f(x_k)$, such that x_k does not have a convergent subsequence in X. In our situation, we will take $X = \mathrm{Univ}(\mathcal{M}(N, \Phi))$ and $f = \mathrm{ev}$, so that the boundary of $f(X)$ consists of all the points in M which lie in a conically singular Cayley. We say that two pseudo-cycles $f : X \to M$ and $g : Y \to M$ of dimension n are **bordant** if there is a further pseudo-cycle with boundary $h : W \to M$ of dimension $n+1$ such that the boundary of W is exactly $X \sqcup Y$, and h restricts to f and g on X and Y respectively. Pseudo-cycles of a given dimension n, taken up to bordism, form a group, which we denote $B^n(M)$. It is related to the homology of M by a group morphism $[\cdot] : B^n(M) \to H^n(M)$. In other words, each pseudo-cycle defines a homology class. More specifically when $n = \dim M$, we can define the **degree** of a pseudo-cycle $f : X \to M$ as $\deg f = k$ where $[f] = k \cdot [M]$, $[M]$ being the fundamental class of the compact smooth manifold M. This corresponds to the usual definition of the degree when X is a smooth compact manifold. We are now able to define weak Cayley fibrations.

Definition 4.2. A **weak Cayley fibration** of a compact Spin(7)-manifold (M, Φ) is a well defined pseudo-cycle $\mathrm{ev} : \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)) \to M$, for some smooth Cayley submanifold N, where ev is required to have degree 1. Here ev is the evaluation map that sends a point in a Cayley submanifold to the corresponding point in the ambient manifold M and $\overline{\mathcal{M}}(N, \Phi)$ is the moduli space of compact Cayleys together with conically singular degenerations.

Note that requiring the evaluation map ev to be a pseudo-cycle puts some restrictions on the possible local models near the singular fibres. Indeed, the singular Cayleys need to be of codimension at least 2 in $\overline{\mathcal{M}}(N, \Phi)$. Thus for the unobstructed case, this means that $\dim \mathcal{M}^{\lambda}_{\mathrm{AC}}(A, \Phi_0) \geqslant 2$. This is for instance satisfied for the asymptotically conical model $A_\epsilon = \{x^2 + y^2 + z^2 = \epsilon, w = 0\}$ in $\mathbb{C}^4$ from Remark 2.25, which has $\mathcal{M}^{\lambda}_{\mathrm{AC}}(A_\epsilon) \simeq \mathbb{C} \setminus \{0\}$.

4.2 Stability of weak fibrations

In order to discuss the stability of Cayley fibrations, we need to revisit the iteration scheme for almost Cayley submanifolds from Section 3.3. In Equation (3.36) the scheme is described for weighted Sobolev spaces, but it is easiest to understand in the unweighted setting. Assume that $N \subset (M, \Phi)$ be a compact almost Cayley with a well-defined elliptic deformation operator $F(v) = F(0) + \not{D}v + Q(v)$ and a pseudo-kernel $\kappa \subset C^\infty(\nu(N))$, further assuming that $\not{D}|_{\kappa^\perp}$ is an isomorphism (we call such N **unobstructed**). Now consider a sequence of normal vector fields $v_i \in C^\infty(\nu(N))$ such that $v_0 = 0$ and for all $i \geqslant 0$:

$$\not{D}v_{i+1} = -F(0) - Q(v_i), \quad v_i \perp^{L^2} \kappa. \tag{4.1}$$

By going through the proof of the gluing theorem 3.15 as well as the preliminary lemmas we see that also, in this case, there are constants $C_D, C_Q > 1$ such that the following holds for normal vector fields $u, v \in C^\infty(\nu(N))$:

$$\begin{aligned} \|u\|_{L^p_{k+1}} &\leqslant C_D \|\not{D}u\|_{L^p_k} \text{ whenever } u \perp^{L^2} \kappa, \\ \|Q(u) - Q(v)\|_{L^p_k} &\leqslant C_Q \|u - v\|_{L^p_{k+1}} (\|u\|_{L^p_{k+1}} + \|v\|_{L^p_{k+1}}), \\ &\text{whenever } \|u\|, \|v\| \text{ are sufficiently small.} \end{aligned} \tag{4.2}$$

The iteration scheme then converges if:

$$C_D C_Q \|F(0)\|_{L^p_k} < \epsilon \tag{4.3}$$

for a fixed $\epsilon > 0$. This will still be true for an L^p_k-neighbourhood of submanifolds around N, where the pseudo-kernel is κ parallelly transported and suitably projected. In particular, given sufficiently small initial data, we will have a bound for the norm of the limiting vector field $v_\infty = \lim_{i\to\infty} v_i$:

$$\|v_\infty\|_{L^p_{k+1}} \leqslant C_I \|F(0)\|_{L^p_k}, \tag{4.4}$$

where C_I is a fixed constant for nearby almost Cayleys.

Consider now a smooth family of nearby almost Cayleys $\{N_t\}_{t\in\mathcal{T}}$ with pseudo-kernels that satisfy the convergence conditions. We would like to investigate the dependence of the resulting Cayleys on the initial almost Cayley. For this, note that we can recast the deformation problem on the nearby submanifold N_t as a deformation problem on N, but where the smooth differential operator F is perturbed smoothly to F_t. Similarly, we have that perturbing a Spin(7)-structure Φ in a family $\{\Phi_s\}_{s\in\mathcal{S}}$ gives rise to a further perturbation of the differential operator to $F_{s,t}$, where we set $F = F_{s_0,t_0}$.

Lemma 4.3. *Assume that $p > 4$ and $k \geqslant 1$. Let N be a compact, unobstructed, nonsingular almost Cayley submanifold of (M, Φ), with deformation operator $F : U \subset L^p_{k+1}(\nu(N)) \to L^p_k(E_{\mathrm{cay}})$, where U is an open neighbourhood of $0 \in L^p_{k+1}(\nu(N))$. Assume that κ is a pseudo-kernel such that (N, κ) satisfies the convergence criteria for the iteration scheme (4.1). Let $F_{s,t} : U \to L^p_k(E_{\mathrm{cay}})$ be a family of smooth perturbations for $s \in \mathcal{S}, t \in \mathcal{T}$ as described above. Then there is an open neighbourhood $V \subset \mathcal{S} \times \mathcal{T}$ of (s_0, t_0) such that for any $(s,t) \in U$ there is a unique element $v_{s,t} \in U$ such that $v_{s,t} \perp \kappa_{s,t}$ and $F_{s,t}(v_{s,t}) = 0$, which depends smoothly on s, t.*

Proof. First of all, note that $F_{s,t} : U \to L^p_k(E_{\mathrm{cay}})$ is a smooth family of Banach maps. Hence the convergence criteria will also be satisfied for $(F_{s,t}, \kappa_{s,t})$ with slightly larger constants C_D, C_Q, provided that (s,t) vary in a sufficiently small neighbourhood V of (s_0, t_0). Thus the iteration converges to a unique solution $v_{s,t} \in \kappa^\perp_{s,t} \cap U$ to $F_{s,t}(v_{s,t}) = 0$.

As the constants are only slightly increased in this neighbourhood, we also see by the bound (4.4) that $\|v_{s,t}\|_{L^p_{k+1}} \leqslant 2C_I \|F(0)\|_{L^p_k}$, independent of s, t.

We now use an implicit function argument to show that $v_{s,t}$ varies smoothly in s and t when it exists. We first note that we can assume $\kappa_{s,t} = \kappa$ are all equal, by precomposing $F_{s,t}$ with a suitably chosen automorphism of $L^p_{k+1}(\nu(N))$ that varies smoothly in s, t. We still call the resulting maps $F_{s,t}$ and the constants C_D and C_Q remain unchanged. We then look at the smooth map:

$$\begin{aligned} A : (\kappa^\perp \cap U) \times \mathcal{S} \times \mathcal{T} &\longrightarrow \kappa^\perp \cap U \\ (v, s, t) &\longmapsto (\not{D}_{s,t}|_{\kappa^\perp})^{-1}(-F_{s,t}(0) - Q_{s,t}(v)) - v. \end{aligned}$$

We clearly have $A(w, s, t) = 0$ exactly when $w = v_{s,t}$. To prove the smoothness of $v_{s,t}$ it is thus sufficient to show that $\partial_v A(v_{s,t}, s, t) : \kappa^\perp \to \kappa^\perp$ is an isomorphism. One can show explicitly that:

$$\partial_v A(v_{s,t}, s, t) = (\not{D}_{s,t}|_{\kappa^\perp})^{-1} \partial_v Q_{s,t}(v_{s,t}) - \mathrm{id}\,.$$

From the quadratic bound on $Q_{s,t}$ we see that

$$\|\partial_v Q_{s,t}(v_{s,t})\|_{\mathrm{op}} \leqslant 2C_Q \|v_{s,t}\|_{L^p_{k+1}} \leqslant 4C_Q \|F(0)\|_{L^p_k}.$$

From the bound on $\not{D}$, we see that

$$\|(\not{D}_{s,t}|_{\kappa^\perp})^{-1} \partial_v Q_s(v(s))\|_{\mathrm{op}} \leqslant 4C_D C_Q \|F(0)\|_{L^p_k}.$$

In particular, if we further reduce V so that $4C_D C_Q \|F(0)\|_{L^p_k} < \epsilon$ (recall that $F_{s,t}(0) = 0$ for $s = s_0$), we can assure that $\partial_v A(v(s), s)$ is an isomorphism. □

The previous result shows that a collection of compact and nonsingular unobstructed Cayley submanifolds varies smoothly under change of the ambient Spin(7)-structure, even in a quantitative way. We now need to analyse how nearly singular Cayleys are perturbed. For this, consider an unobstructed CS_μ $(1 < \mu < 2)$ Cayley $N \subset (M, \Phi)$ with one singular point. Assume that we have a matching AC_λ $(\lambda < 1)$ Cayley in $\mathbb{R}^8$ of sufficiently small scale so that we may glue as in Theorem 3.15. Then nonsingular Cayleys in $\mathcal{M}(N \sharp A)$ near $N \subset M$ are given as $\Gamma(\tilde{N}, \tilde{A}, \Phi)$ for $\tilde{N} \in \mathcal{M}^\mu_{\mathrm{CS}}(N, \Phi)$ and $\tilde{A} \in \mathcal{M}^\lambda_{\mathrm{AC}}(A, \Phi_0)$. If $\{\Phi_s\}_{s \in \mathcal{S}}$ is a small perturbation of the Spin(7)-structure, we may also consider $\Gamma(N_s, A, \Phi_s)$, where N_s is the family of deformations of N. We consider this to be the deformation of $\Gamma(N, A, \Phi)$ in the new Spin(7)-structure Φ_s.

Let $v_s \in C^\infty(\nu(\Gamma(N, A, \Phi)))$ be the normal vector field that describes the perturbation of $\Gamma(N, A, \Phi)$ to $\Gamma(N_s, A, \Phi_s)$. We claim that it can be decomposed into two contributions as follows:

$$v_s = v_{\mathrm{CS},s} + \tilde{v}_s. \tag{4.5}$$

Here $v_{\mathrm{CS},s}$ is the deformation between the two pre-glued manifolds $N^{s_0,A}$ and $N^{s,A}$. It can be thought of as a gluing of the perturbation vector field that takes N to N_s with the perturbation vector field that takes A to translated and rotated A, which is determined by how the conical point of N deforms as we pass to N_s.

The remaining error term $\tilde{v}_s$ is the sum of the perturbations from $\Gamma(N, A, \Phi)$ to $N^{s_0,A}$ and from $N^{s,A}$ to $\Gamma(N, A, \Phi_s)$. Now by our gluing theorem 3.15 we know that $\|\tilde{v}_s\|_{L^p_{k+1,\delta,A}} < Ct^\alpha$ for some constants $1 < \delta < \mu, \alpha > 0, C > 0$. In particular, since $t^\alpha \to 0$ as $t \to 0$, we know that the dominant term must be $v_{\mathrm{CS},s}$, which is of order $O(1)$. We are now ready to prove the stability result for weak Cayley fibrations.

Theorem 4.4 (Stability of weak fibrations)**.** *Let (M, Φ) be a* Spin(7)*-manifold that is weakly fibred by* $\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi))$*, and suppose that $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of* Spin(7)*-structures with $\Phi = \Phi_{s_0}$. Assume that all the Cayleys in $\overline{\mathcal{M}}(N, \Phi)$ are unobstructed and that the cones in the conically singular degenerations of N are semi-stable and unobstructed. Then there is an open set $s_0 \in U \subset \mathcal{S}$ such that M is weakly fibred by* $\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi_s))$ *for any $s \in U$.*

Proof. Note first that all the Cayleys in $\overline{\mathcal{M}}(N, \Phi)$ persist under a sufficiently small perturbation of the Spin(7)-structure. To see this, we apply the iteration scheme from the proof of Theorem 3.15 simultaneously to all the Cayleys in $\overline{\mathcal{M}}(N, \Phi)$ in the following way. First, we fix pseudo-kernels $\kappa_{\mathrm{CS}}(\hat{N})$ for all the conically singular Cayleys $\hat{N} \in \overline{\mathcal{M}}(N, \Phi) \setminus \mathcal{M}(N, \Phi)$. As we assumed all the CS manifolds to be unobstructed, their moduli spaces are of strictly lower dimension than $\mathcal{M}(N, \Phi)$ (as $\dim \mathcal{M}^{\mathrm{AC}}(A) \geqslant 1$ since rescaling is always a possible deformation). These moduli spaces can be noncompact as well, but only in that further conical singularities can appear. Thus only finitely many conical singularities can appear, and both $\overline{\mathcal{M}}(N, \Phi)$ and $\overline{\mathcal{M}}(N, \Phi) \setminus \mathcal{M}(N, \Phi)$ must be compact. In particular, we can bound the values of the constants C_D, C_Q uniformly for all conically singular Cayleys that appear. The same is true for the nonsingular Cayleys that are a fixed distance away from the singular points, as they form a compact set as well. Finally by the estimates (3.12) and (3.13) we see that the remaining nonsingular Cayleys, which are desingularisations of the conically singular ones have bounded C_D and C_Q as well.

This is exactly because we adapt our Banach spaces $L^p_{k,\delta,A}$ to the scale of the glued manifold. In conclusion, the values of the constants C_D and C_Q are uniformly bounded for all Cayleys in the weak fibration. In particular, for sufficiently small perturbations of the Spin(7)-structure that fix the singular points, we can ensure that $2C_D$ and $2C_Q$ are still valid constants and that the initial error $\|F(0)\|_{L^p_{k+1,\delta,A}}$ is arbitrarily small as well.

Hence for small perturbations of the ambient manifold, all Cayleys persist simultaneously, and we get a family of vector fields $v_s \in \mathrm{Map}(\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)), TM)$ for $s \in \mathcal{S}$. These vector fields need not be continuous a priori, as they are defined separately on each Cayley as the limit vector field v_∞ obtained in the iteration scheme. However by Lemma 4.3 above we immediately see that they fit together to form a smooth vector field on the open subset of $\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi))$ given by the union of all nonsingular Cayleys. Similarly, we see that on a singular stratum of $\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi))$ with fixed kinds of conical singularities the vector fields also fit together to form a single smooth vector field along that stratum.

What is not a priori known is the regularity of the global vector field along the normal direction of a singular stratum, i.e. what happens as a Cayley degenerates towards a more singular Cayley. We can now use the bounds on the desingularisations in Equation (3.37) to show the continuity of the deformation vector fields.

Consider for this a conically singular Cayley $N \subset M$ together with its family of deformations N_s for small deformations $s \in \mathcal{S}$ of s_0. The Cayleys close to $N_s \subset (M, \Phi_s)$ in the moduli spaces $\mathcal{M}(N, \mathcal{S})$ are given by its desingularisations. We now choose an identification $I : \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)) \simeq \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi_s))$ as topological spaces so that $\Gamma(N, \bar{A}, \Phi)$ is sent to $\Gamma(N_s, \bar{A}, \Phi_s)$. Next, we analyse the behaviour of the vector fields $v_s \in \mathrm{Map}(\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)), TM$ near the singular fibres. As we have seen from Equation (4.5), the vector field $v_{\Gamma(N,\bar{A},\Phi)}$ that describes the perturbation of $\Gamma(N, \bar{A}, \Phi)$ decomposes as follows:

$$v_{\Gamma(N,\bar{A},\Phi)} = v_{\mathrm{CS},s} + \tilde{v}_s. \tag{4.6}$$

Here $v_{\mathrm{CS},s}$ is a glued vector field, obtained by combining the vector fields $v_{N,s}$ that take N to N_s and A to a rotated and translated A. In particular, this component approaches $v_{N,s}$ as $t \to 0$. The other component, $\tilde{v}_s$, satisfies $\|\tilde{v}_s\|_{L^p_{k+1,\delta,\bar{A}}} < Ct^\alpha$ from our gluing theory, and hence also $|\tilde{v}_s|_{C^0} < t^\alpha \rho^\delta$. Thus ev_{Φ_s} is a continuous map, even as one approaches the singular Cayleys, and the vector fields $v_s \in \mathrm{Map}(\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi)), TM)$ are in fact continuous, and vary continuously with s.

We showed that I is a smooth map on the nonsingular stratum, and maps the singular strata homeomorphically to singular strata with the same singularities. Since we also showed that ev_{Φ_s} are continuous maps and the boundaries of ev_{Φ_s} remain of codimension at least 2, we see that $\mathrm{ev}_{\Phi_s}|_{\mathrm{Univ}(\mathcal{M}(N,\Phi_s))}$ remain pseudo-cycles.

Fix now a smooth path $\gamma : [0, 1] \to \mathcal{S}$ joining $\gamma(0) = s_0$ and $\gamma(1) = s$. Define the manifold $W = \mathrm{Univ}(\mathcal{M}(N, \{\Phi_{\gamma(t)}\}_{t\in[0,1]}))$ and consider the evaluation map $\mathrm{ev}_W : W \to M$. We see that ev_W forms a bordism pseudo-cycle between ev_Φ and ev_{Φ_s}. So in particular, if the degree of ev_Φ was 1, it is also 1 for ev_{Φ_s}. □

4.3 Stability of strong fibrations

We showed in the previous section that weak fibrations are stable under perturbation of the Spin(7)-structure. This relied on the fact that the perturbation vector fields (which describe how a given Cayley perturbs under change of the Spin(7)-structure to a nearby Cayley for the new structure) remain continuous under the collapse of nearly singular Cayleys to their conically singular limits. In other words, the nearly singular Cayleys deform with the singular Cayleys. This means that by perturbing the Spin(7)-structure, the entire completed moduli space (including the conically singular Cayleys) varies continuously, even at the singular fibre. Proving the stability of strong fibrations means improving this result by showing that these vector fields, which are continuously differentiable, have bounded C^1 norm as the neck size shrinks to zero and one approaches a singular limit (as we will see later, the region away from the singularities, as well as the conically singular Cayleys themselves, are easy to handle, essentially because their moduli space are compact). As a toy example, we consider a fibration of $\mathbb{R}^2$ by lines, which we see as the projection map:

$$\begin{aligned} f :\ \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (r, t) &\longmapsto t. \end{aligned}$$

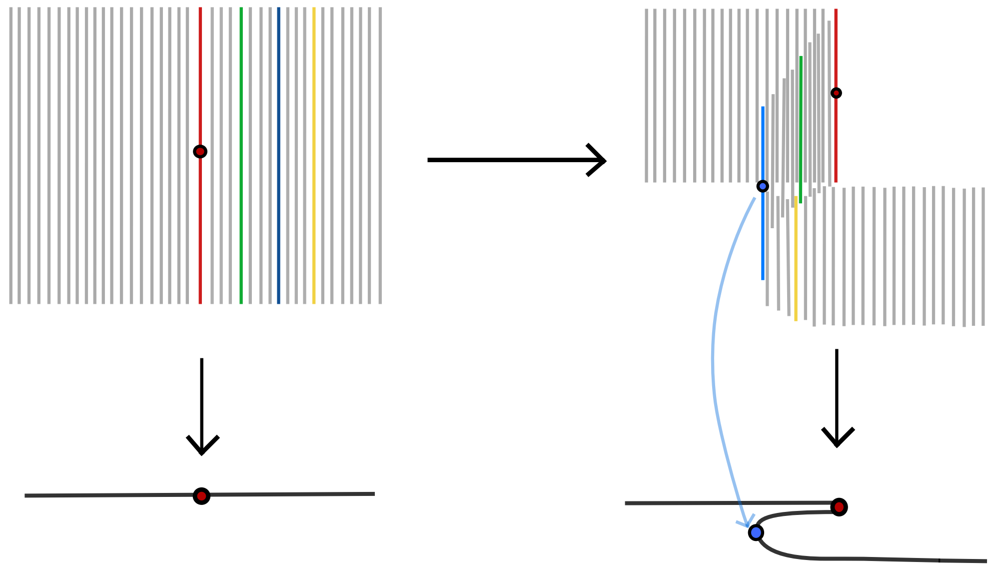

Figure 4.1: Folding over for arbitrarily small time. The fold has width $O(s^{1/(1-\alpha)})$.

Here we think of t as the distance from the "singular fibre" $f^{-1}(0)$, and of r as the radial distance from the "singular point" $(0,0) \in \mathbb{R}^2$. The corresponding weak fibration would be the evaluation map:

$$\begin{aligned} \mathrm{ev}: \quad \mathrm{Univ}(\mathcal{L}) &\to \mathbb{R}^2 \\ (L,(r,t)) &\longmapsto (r,t). \end{aligned}$$

Here $L = \{(s,t), s \in \mathbb{R}\} = f^{-1}(t)$ is a straight line in $\mathbb{R}^2$, and $(r,t) \in L$ is a point on this line. We denote by $\mathcal{L}$ the "moduli space of lines" in this example. In analogy to the weak stability result, Theorem 4.4, we consider a perturbation of this fibration (as a weak fibration) given by a homotopy $h: (-\epsilon,\epsilon) \times \mathbb{R}^2 \to \mathbb{R}$. Note that here the new fibres in the weak fibration are not the pre-images $h_s^{-1}(t)$ for $t \in \mathbb{R}$, but the images $h(s,t,\cdot)$ for some fixed $s,t \in \mathbb{R}$. Let's say that the singular fibre and point remain fixed so that $h(s,r,0) = 0$ for all $s \in (-\epsilon,\epsilon)$. In our analogy, we proved above that for $t \in \mathbb{R}$ sufficiently close to 0, the value of $h(s,r,t)$ remains close to $h(0,r,t)$ in that $|h(s,r,t) - h(0,r,t)| \leqslant s|t|^\alpha |r|^\gamma$ $(0 < \alpha < 1 < \gamma)$. We realise quickly then that this does not imply that $h(s,\cdot,\cdot)$ is C^1 on all of $\mathbb{R}^2$ for $s \neq 0$, even if we assume that it is smooth at time $s = 0$ and smooth away from the singular fibre for all time. Indeed, we consider:

$$h(s,r,t) = t - s|t|^\alpha |r|^\gamma.$$

Then clearly $|\partial_t h| \to \infty$ as $t \to 0$ for some fixed $r \neq 0$. Thus the fibres in this fibration start to move very quickly relative to one another, even though they do not perturb very much after any finite time. Indeed the fibration property is not preserved, as for $s \neq 0$ the weak fibration h_s admits fibres that intersect. Just consider the initial fibres $t = \eta$ and $t = \epsilon$ for $0 < \epsilon < \eta < s^{1/(1-\alpha)}$. One can see that there is $r_{\eta,\epsilon}$ such that $h_s(\eta, r_{\eta,\epsilon}) = h_s(\epsilon, r_{\eta,\epsilon})$.

Thus we need to investigate the equivalent of $\partial_t h$ for the Cayley fibration problem, which are the **infinitesimal deformation vector fields**. For a given Cayley fibration $f : M^8 \to B^4$, they are the normal vector fields to a Cayley $N = f^{-1}(b)$ which are lifts of tangent vector $v \in T_b B$ in the base. Seen differently, they are the first-order variation of a family of Cayleys parametrised by curves in B. Finally, they can also be seen as solutions to the linearised Cayley problem $\not{D}w = 0$ for $w \in C^\infty(\nu(N))$, and this is the perspective we will use. To show the boundedness of the infinitesimal deformation vector fields, we solve the linearised Cayley equation $\not{D}w = 0$ on the desingularisations $\Gamma(N, \bar{A}, \Phi)$ via another gluing argument. In the following, we assume:

- $N \subset (M, \Phi)$ is an unobstructed CS_μ Cayley ($1 < \mu < 2$) with a unique singular point, with semistable (as in Definition 1.36) cone $C \subset \mathbb{R}^8$.
- $A \subset \mathbb{R}^8$ is an unobstructed AC_λ Cayley ($\lambda < 0$) with asymptotic cone C.
- There is a critical rate $\zeta = \max\{\mathscr{D}_C \cap (-\infty, 0)\}$ such that $\not{D}_{\mathrm{AC}}$ is an isomorphism at rates just below ζ.

Under these assumptions, the deformation vector fields of $\Gamma(N, \bar{A}, \Phi)$ split into two classes:

- The deformations of rate ζ, which come from varying A, i.e. $w_{\mathrm{AC}} = \partial_t \Gamma(N, \bar{A}_t, \Phi)$ for a family $\{\bar{A}_t\}_{t\in(-\epsilon,\epsilon)}$. These correspond to moving *orthogonal* to the singular locus in B.
- The deformations of rate 0, which come from varying N, i.e. $w_{\mathrm{CS}} = \partial_t \Gamma(N_t, \bar{A}, \Phi)$ for a family $\{N_t\}_{t\in(-\epsilon,\epsilon)}$. These correspond to moving *parallel* to the singular locus in B.

We will now show in turn that these infinitesimal deformation fields remain bounded in suitable weighted Sobolev spaces.

Deformations in the normal directions

First, we look at the deformations of nearly singular Cayley submanifolds that are coming from variations in A, i.e. deformations of rate r^ζ with $\zeta < 0$. Note that as the neck size $t \to 0$, we can find vector fields $w_{\mathrm{AC},t}$ as above with $\min |w_{\mathrm{AC},t}| \to 1$ but $\max |w_{\mathrm{AC},t}| = O(t^\zeta)$. In this sense, they are fundamentally different from vector fields describing parallel movement, which are of constant magnitude as we approach the singular limit.

Suppose that $\{\Phi_s\}_{s\in\mathcal{S}}$ is a smooth family of Spin(7)-structure on $\mathbb{R}^8$ which are all AC_η ($\eta < 1$) to the flat Φ_0. For a fixed $s_0 \in \mathcal{S}$, let $A \subset (\mathbb{R}^8, \Phi_{s_0})$ be an unobstructed AC_λ α-Cayley submanifold ($\eta < \lambda < 1$) asymptotic to the cone $C = \mathbb{R}^+ \times L$. Suppose that α is sufficiently close to 1 so that A admits a linearised deformation operator $\not{D}_{\mathrm{AC}}$. For a given weight $\zeta \in \mathbb{R}$, denote by $\mathcal{I}^\zeta_{\mathrm{AC}}(A)$ the solutions $w \in C^\infty_\zeta(\nu(A))$ to $\not{D}_{\mathrm{AC}} w = 0$, i.e.

which have decay rate at most r^ζ. More precisely, fix an identification of the end of A with $(R_0, \infty) \times L$. We can then define for $w \in C^\infty_\zeta(\nu(A))$:

$$\partial_\zeta w = \lim_{r \to \infty} r^{-\zeta+1} M_r^*(w|_{\{r\}\times L}). \tag{4.7}$$

Here M_r denotes the map from the unit sphere in $\mathbb{R}^8$ to the sphere of radius r, also in $\mathbb{R}^8$. This rescales the normal vector field by a factor of r and explains the shift by 1 above. Hence $\partial_\zeta w \in C^\infty(\nu(L \subset S^7))$ is extracting the component $r^\zeta \sigma$ of exactly rate r^ζ. The vector field w is called ζ-**non-zero** if $\partial_\zeta w \neq 0$. Similarly, we call it ζ-**nowhere-vanishing** if $\partial_\zeta w$ is nowhere vanishing. Note that on the end we can write $\not{D}_{\mathrm{AC}} = \frac{\mathrm{d}}{\mathrm{d}r} - r^{-1}B(r)$, with $B(r) \to B_\infty$, the limiting operator on the link. If $w \in \mathcal{I}^\zeta_{\mathrm{AC}}(A)$, then $\partial_\zeta w$ is well-defined and an eigensection of the limiting operator B_∞ with eigenvalue ζ. By Proposition 2.37, the asymptotic behaviour of $B(r)$ is precisely:

$$\|B(r) - B_\infty\|_{\mathrm{op}} = O(r^{\lambda-1}). \tag{4.8}$$

Here the operator norm is taken with regard to an arbitrary Sobolev norm on the cross section. From this, we deduce the asymptotic expansion of infinitesimal deformation vector fields.

Proposition 4.5. *Let $w \in \mathcal{I}^\zeta_{\mathrm{AC}}(A)$ with $\not{D}_{\mathrm{AC}}$ unobstructed at sufficiently large rates $\zeta - \epsilon < \zeta$ for $\epsilon > 0$. Then there is $\epsilon, R > 0$ such that for $r > R$ and $p \in L$ we have:*

$$w(r,p) = (\partial_\zeta w)(p) r^\zeta \alpha(r) + \delta w,$$

where $\delta w \in C^\infty_{\zeta-\epsilon}(\nu(A))$ and $\alpha : \mathbb{R}_{>0} \to [0,1]$ is a cut-off function such that $\alpha = 1$ for large radii and $\alpha = 0$ for small radii.

Proof. Recall that the AC_λ condition gives us an identification of the end $A \setminus K$ with $L \times (r_0, \infty)$, where $K \subset A$ is compact. Then define $\delta w = w - \alpha(r)\sigma r^\zeta$ for $\sigma = \partial_\zeta w$ and a cut-off function $\alpha : A \to [0,1]$ such that $\alpha = 1$ for large radii and $\alpha = 0$ for small radii. We then compute, using the fact that $\not{D}_{\mathrm{AC}} = \frac{\mathrm{d}}{\mathrm{d}r} - r^{-1}(B_\infty + \delta B(r))$ with $\|\delta B(r)\|_{\mathrm{op}} = O(r^{\lambda-1})$ by Equation (4.8):

$$\begin{aligned} 0 = \not{D}_{\mathrm{AC}} w &= \not{D}_{\mathrm{AC}}(\alpha(r)\sigma r^\zeta + \delta w) \\ &= r^{\zeta-1}(r\partial_r \alpha\sigma - \delta B(r)[\alpha\sigma]) + \not{D}_{\mathrm{AC}}\delta w. \end{aligned}$$

In particular, this implies that for r large we have:

$$\not{D}_{\mathrm{AC}}\delta w = \delta B(r)\sigma r^{\zeta-1} \in C^\infty_{\zeta-1+(\lambda-1)},$$

so that from unobstructedness at the rate $\zeta + \lambda - 1 < \tilde{\zeta} < \zeta$ we see that there is $\tilde{w} \in L^p_{k,\tilde{\zeta}}(\nu(A))$ with $\not{D}_{\mathrm{AC}}\tilde{w} = \not{D}_{\mathrm{AC}}\delta w$ (not necessarily unique). In particular this means that $\tilde{w} = \delta w + u$, where $\not{D}_{\mathrm{AC}} u = 0$. However, there are no non-zero infinitesimal deformation vector fields with rate in $(\tilde{\zeta}, \zeta]$ which satisfy $\partial_\zeta u = 0$, hence δw itself must have decay in $O(r^{\zeta-\epsilon})$ for sufficiently small $\epsilon > 0$. From elliptic regularity for the operator $\not{D}_{\mathrm{AC}}$ at rate $\zeta - \epsilon$ we can now deduce that $\delta w \in C^\infty_{\zeta-\epsilon}(\nu(A))$. □

We can now prove that both the existence and ζ-nowhere-vanishing of infinitesimal deformation vector fields are stable under AC_λ perturbations with $\lambda < 0$.

Proposition 4.6. *Let $\{A_t\}_{t\in\mathcal{T}}$ be a smooth family of AC_λ perturbations of $A = A_{t_0}$ where $t_0 \in \mathcal{T}$. Assume that all the elements of $\mathcal{I}^\zeta_{\mathrm{AC}}(A, \Phi)$ are ζ-nowhere-vanishing and that the operator $\not{D}_{\mathrm{AC}}$ is an isomorphism at rate $\zeta - \epsilon$ for some small $\epsilon > 0$. Then there is an open neighbourhood $U \subset \mathcal{S} \times \mathcal{T}$ of (s_0, t_0) such that all the elements of $\mathcal{I}^\zeta_{\mathrm{AC}}(A_t, \Phi_s)$ are still ζ-nowhere-vanishing for $(s, t) \in U$.*

Proof. Take a solution $w \in \mathcal{I}^\zeta_{\mathrm{AC}}(A, \Phi)$, which we can write as follows, using Proposition 4.5:

$$w = (\partial_\zeta w) r^\zeta \alpha + \delta w,$$

for some $\delta w \in L^p_{k,\zeta-\epsilon}(\nu(A))$ for some small $\epsilon > 0$ and a cut-off function $\alpha : A \to \mathbb{R}$ that is zero for $r \leqslant R_1$ and one for $r \geqslant R_2$. We make the ansatz $w_{s,t} = ((\partial_\zeta w)+\sigma)r^\zeta\alpha+\delta w+\delta w_{s,t}$, where $\sigma \in L^p_{k+1}(\nu(L))$ and $\delta w \in L^p_{k,\zeta-\epsilon}(\nu(A))$. The linearised Cayley equations for (A_t, Φ_s) then becomes:

$$\begin{aligned} 0 = \not{D}_{\mathrm{AC},s,t} w_{s,t} &= (\not{D}_{\mathrm{AC}} + r^{-1}\delta B_{s,t}(r))[w + \sigma r^\zeta \alpha + \delta w_{s,t}] \\ &= \not{D}_{\mathrm{AC}}[\sigma r^\zeta \alpha + \delta w_{s,t}] + r^{-1}\delta B_{s,t}(r)[w_{s,t}]. \end{aligned} \tag{4.9}$$

Here $\delta B_{s,t}$ is the error term introduced by (A_t, Φ_s) compared to (A, Φ). In particular $\delta B_{s_0,t_0} = 0$. From Proposition 2.37 we see that $\|\delta B_{s,t}(r)\|_{\mathrm{op}} \in O(r^{\lambda-1})$, where $\lambda < 1$ is the asymptotic rate of A. Thus from Equation (4.9) we see that solving the Cayley equation on (A_t, Φ_s) amounts to solving the Cayley equation on A with the error term $-r^{-1}\delta B_{s,t}(r)[w_{s,t}] \in L^p_{k,\zeta-1+(\lambda-1)}(\nu(A))$. In particular, this implies $\sigma = 0$ because any solution needs to have decay at least $\zeta + (\lambda - 1) < \zeta$. Hence any solution will be ζ-nowhere-vanishing, since $\partial_\zeta w$ is nowhere vanishing by assumption. Finally, the existence of solutions follows the same argument as the proof of the previous Proposition 4.5. □

Let $N \subset (M, \Phi)$ be an unobstructed CS_μ Cayley $(1 < \mu < 2)$ and $A \subset (\mathbb{R}^8, \Phi_0)$ an unobstructed AC_λ Cayley $(\lambda < 0)$ which satisfy the assumptions of the gluing theorem 3.15 in the form of Remark 3.17. Hence we include all the positive rates in the CS moduli space and all the strictly negative rates in the AC moduli space.

In particular both Cayleys admit the same asymptotic cone $C \subset \mathbb{R}^8$ and the same critical rates $\mathscr{D}_C \subset \mathbb{R}$. We thus get a family N^{tA} of compact almost Cayley submanifolds of M for $t > 0$ small, obtained by gluing a copy of A rescaled by a factor t onto the conical singularity on N.

By Proposition 3.6 this approximate Cayley satisfies $\|\tau|_{N^{tA}}\|_{L^p_{k,\gamma,tA}} \leqslant t^{\nu(\mu-\gamma)}$, where $0 < \nu < 1$ is an additional gluing parameter and $1 < \gamma < \mu$ is a chosen weight. Furthermore, there are perturbations $\Gamma(N, tA, \Phi)$ of N^{tA} which are truly Cayley and that satisfy $\Gamma(N, tA, \Phi) = \exp(v_t)$ for normal vector fields $v_t \in C^\infty(\nu(N^{tA}))$ with $\|v_t\|_{L^p_{k,\gamma,tA}} \leqslant 2t^{\nu(\mu-\gamma)}$. Since we know that N^{tA} is $L^p_{k,\gamma,tA}$-close to the cone in the intermediate region, we can deduce from Proposition 2.6 that the linearised deformation operator satisfies:

$$\not{D} = \frac{\mathrm{d}}{\mathrm{d}r} - r^{-1}(B_\infty + \delta B_t(r)). \tag{4.10}$$

Here $\|\delta B_t(r)\|_{\mathrm{op}} = O(r^{\nu(\mu-\gamma)})$ by the a priori gluing estimates (3.19).

We will now perform an additional gluing construction for the infinitesimal deformation vector fields defined on the glued manifolds $\Gamma(N, tA, \Phi)$. Let $w_{\mathrm{CS}} \in \mathcal{I}^{\zeta}_{\mathrm{CS}}(N)$ be an infinitesimal deformation of N of rate $\zeta \in \mathscr{D}_C$, where $\zeta = \max\{(-\infty, \lambda) \cap \mathscr{D}\}$. This is a solution to $\not{D}_{\mathrm{CS}}[w_{\mathrm{CS}}] = 0$ so that $\partial_\zeta w_{\mathrm{CS}} = \sigma$ is non-zero, except if $w_{\mathrm{CS}} = 0$. Similarly let $w_{\mathrm{AC}} \in \mathcal{I}^{\zeta}_{\mathrm{AC}}(A)$ be an infinitesimal deformation of A of rate ζ that shares the same limiting eigensection σ. We then claim that a suitably glued vector field w_t on N^{tA} is a good approximation of a true solution to the equation $\not{D}_{N^{tA}} w = 0$. For brevity, we denote the operator $\not{D}_{N^{tA}}$ also by $\not{D}$ when the neck size is evident. For this, we recall that N^{tA} can be divided into three pieces as follows:

$$N^{tA} = N^{tA}_u \sqcup N^{tA}_m \sqcup N^{tA}_l.$$

Here N^{tA}_u, the upper region, is just $\{p \in N : \rho(p) > tr_0\}$, a truncated version of N. Then we have the lower region, N^{tA}_l, which is $\{p \in tA : |p| < tR_0\}$ embedded into M via a Spin(7)-parametrisation $\chi : \mathbb{R}^8 \to M$. Finally the middle region N^{tA}_m is interpolating the two pieces between the radii tr_0 and R_0. The gluing parameter $0 < \nu < 1$ determines where the interpolation happens, namely in between the radii $\frac{1}{2}t^\nu$ and t^ν. We can however also think about this decomposition in a different way, namely:

$$N^{tA} = N^{tA}_{\mathrm{CS}} \cup N^{tA}_{\mathrm{AC}}.$$

Here $N^{tA}_{\mathrm{CS}} = \{p \in N : \rho(p) > t^\nu\}$ extends the upper region from before down to radius t^ν and still agrees with N, but N^{tA}_{AC} now includes everything up until radius $t^{\nu''}$, where $0 < \nu'' < \nu' < \nu < 1$ are two further parameters. The gluing of the infinitesimal deformation vector fields will be performed between $t^{\nu'}$ and $t^{\nu''}$. If we now consider $t^{-1}\chi^{-1}(N^{tA}_{\mathrm{AC}}) \subset (\mathbb{R}^8, \Phi_0)$, we see that it is a noncompact almost Cayley, that agrees with A for radii below $\frac{1}{2}t^{\nu-1}$, but extends to radius $t^{\nu'-1}$. In fact, the CS_μ condition implies that this noncompact Cayley can be extended to an AC_λ Cayley extending all the way to infinity, such that the resulting family $A_t \subset \mathbb{R}^8$ converges to A in C^∞_λ. We can do the same for the family of Spin(7)-structures $t^{-1} \cdot \chi^{-1}(\Phi|_{B_r(p)})$, and they will form an AC_η family for $\eta < \lambda$. We refer to Lemma 3.7 and its proof for a more precise description of how this is achieved.

First, since A is unobstructed, the Proposition 4.6 shows that we can find a smooth family of perturbations $w_{\mathrm{AC},t} \in C^\infty_\zeta(\nu(A_t))$ of w_{AC} with $\partial_\zeta w_{\mathrm{AC},t} = \partial_\zeta w_{\mathrm{AC}}$, so that $w_{\mathrm{AC},t} \to w_{\mathrm{AC}}$ in C^k_ζ as $t \to 0$. Hence we also get infinitesimal deformation vector fields over N^{tA}_{AC} which we will also denote by $w_{\mathrm{AC},t}$. Now choose a smooth cut-off function $\varphi_{\mathrm{cut}} : \mathbb{R} \to [0, 1]$, such that:

$$\varphi_{\mathrm{cut}}|_{(-\infty,\nu'']} = 0, \quad \varphi_{\mathrm{cut}}|_{[\nu',+\infty)} = 1.$$

Using φ_{cut} we define a partition of unity on N^{tA} as follows:

$$\varphi(p) = \begin{cases} \varphi_{\mathrm{cut}}\left(\frac{\log(\rho(p))}{\log(t)}\right), & \text{if } p \in \Psi^{tA}(L \times (r_0 t_i, R_0)) \\ 0, & \text{if } p \in N^{tA}_u, \\ 1, & \text{if } p \in N^{tA}_l, \end{cases} \tag{4.11}$$

Here Ψ^{tA} is a parametrisation that identifies $L \times (r_0 t, R_0)$ with the gluing region of N^{tA}. We can now define w_t as the interpolation:

$$w_t = \varphi w_{\mathrm{CS}} + (1-\varphi) t^{\zeta} w_{\mathrm{AC},t}.$$

This normal vector field is a good approximation to an infinitesimal deformation on N^{tA} in that it is almost a solution to the linearised Cayley equation:

Proposition 4.7. *We have for A, N, w_t and ζ as above that there is some $\epsilon, \alpha > 0$ such that:*

$$\|\not{D} w_t\|_{L^p_{k,\zeta+\epsilon-1,tA}} \lesssim t^{\alpha} \|w_t\|_{L^p_{k+1,\zeta,tA}}.$$

Here we note that $L^p_{k+1,\zeta,tA}$ is the highest weight space such that the family w_t has sub-polynomial volume growth. Indeed, we have:

$$\|w_t\|_{L^p_{k+1,\tilde{\zeta},tA}} = \begin{cases} O(1), & \tilde{\zeta} < \zeta, \\ \Theta(\log t), & \tilde{\zeta} = \zeta, \\ O(t^{\zeta-\tilde{\zeta}}), & \tilde{\zeta} > \zeta \end{cases}.$$

To see this, we note that the intermediate conical regional, which has mass proportional to $\log t$, is the dominant term. Proposition 4.7 shows that the decay of $\not{D} w_t$ is faster than the expected $O(r^{\zeta-1})$ decay. In fact, it shows that the decay rate is $\zeta - 1 + \epsilon$, for some small $\epsilon > 0$. For the proof, we use the following auxiliary result comparing the Cayley operator on N^{tA} to the conical operator on the gluing region of w_t. We denote this region by $G_t = (t^{\nu'}, t^{\nu''}) \times L \subset N^{tA}_m$.

Lemma 4.8. *We have for A, N, w_t and ζ, γ as above that there is some $\epsilon, \alpha > 0$ such that if $s \in C^{\infty}(\nu(N^{tA}))$ is a normal vector field then:*

$$|(\not{D} - \not{D}_{\mathrm{con}}) s|_{G_t}|_{C^k_{\zeta+\epsilon-1,tA}} \leqslant t^{\alpha} |s|_{G_t}|_{C^{k+1}\zeta,tA}$$

Proof. We note that for $(r,p) \in (t^{\nu'}, t^{\nu"}) \times L$ we have that the perturbation vector field v taking the cone to N^{tA} satisfies:

$$|\nabla^k v| \leqslant r^{\mu-\eta-k},$$

for any $\eta > 0$. Since we have $\mu > 1$ we see from Corollary 2.37 that for $\epsilon = \frac{1}{2}(\mu-1)$ and $\alpha = \frac{1}{2}(\mu-1)\nu''$ we get:

$$\begin{aligned} r^{-\zeta-\epsilon+1} |(\not{D} - \not{D}_{\mathrm{con}}) s|_{G_t}| &= r^{-\zeta-\frac{1}{2}(\mu-1)+1} |(\not{D} - \not{D}_{\mathrm{con}}) s|_{G_t}| \\ &\leqslant r^{-\zeta-\frac{1}{2}(\mu-1)+1} r^{\mu-1} |s|_{G_t}|_{C^1_1} \\ &\leqslant r^{-\zeta+\frac{1}{2}(\mu-1)+1} |s|_{G_t}|_{C^1_1} \leqslant t^{\alpha} |s|_{G_t}|_{C^1_{\zeta}}. \end{aligned}$$

This is the case $k = 0$, and the higher-order cases are entirely analogous. □

Next, we need to take a second look at the asymptotic expansion of Lemma 4.5 for w_{AC} and adapt it to a suitable estimate on N^{tA}.

Lemma 4.9. *For $w_{\mathrm{AC},t}$ and w_{CS} as above, we can write them on the gluing region G_t as follows:*

$$w_{\mathrm{AC},t} = r^{\zeta}\partial_{\zeta} w_{\mathrm{AC}} + \delta w_{\mathrm{AC},t}, \quad w_{\mathrm{CS}} = r^{\zeta}\partial_{\zeta} w_{\mathrm{CS}} + \delta w_{\mathrm{CS}}.$$

Then there is $\epsilon > 0$ such that $\delta w_{\mathrm{AC},t} \in C^{\infty}_{\zeta-\epsilon}(\nu(G_t))$ (with $C^k_{\zeta-\epsilon}$-norms bounded uniformly in t) and $\delta w_{\mathrm{CS}} \in C^{\infty}_{\zeta+\epsilon}(\nu(G_t))$. On N^{tA} we have furthermore for some $\epsilon' < \epsilon$:

$$\begin{aligned}
\|\delta w_{\mathrm{AC},t}|_{G_t}\|_{L^p_{k,\zeta+\epsilon',tA}} &\leqslant t^{\epsilon(1-\nu')-\nu'\epsilon'}\|w_t\|_{L^p_{k,\zeta,tA}},\\
\|t^{\zeta}\delta w_{\mathrm{CS}}|_{G_t}\|_{L^p_{k,\zeta+\epsilon',tA}} &\leqslant t^{\nu'(\epsilon-\epsilon')}\|w_t\|_{L^p_{k,\zeta,tA}}.
\end{aligned}$$

Proof. From Lemma 4.5 it is clear that we can find $\epsilon > 0$ and $\delta w_{\mathrm{AC},t}$ such that:

$$w_{\mathrm{AC},t} = r^{\zeta}\partial_{\zeta} w_{\mathrm{AC}} + \delta w_{\mathrm{AC},t}.$$

The uniform boundedness (in t) of the $C^k_{\zeta-\epsilon}$ norms of $w_{\mathrm{AC},t}$ follow from the fact that the construction in the proof of Lemma 2.37 is continuous with respect to the operator $\not{D}$. Since we are working with a family $\{A_t\}_{t\in[0,\epsilon)}$ where $A_0 = A$, at least for sufficiently small t we will have boundedness. The result in the conically singular case is completely analogous to the AC case, so that $\delta w_{\mathrm{CS}} \in C^{\infty}_{\zeta+\epsilon}$ can be constructed. Note that in the CS case, stronger decay means a higher rate, whereas in the AC case, stronger decay means a lower rate. Now we move onto the bounds on N^{tA}. In the following we set $\mathrm{vol}(G_t) = \int_{G_t} r^{-4}\,\mathrm{dvol}$.

$$\begin{aligned}
\|t^{\zeta}\delta w_{\mathrm{AC},t}\|_{L^p_{k,\zeta+\epsilon',tA}} &\lesssim \mathrm{vol}(G_t)t^{\zeta}\max_{G_t}|\delta w_{\mathrm{AC},t}|_{C^k_{\zeta+\epsilon',tA}}\\
&\lesssim \mathrm{vol}(G_t)t^{\zeta}\max_{G_t}|\delta w_{\mathrm{AC},t}(t^{-1}\cdot)|_{C^k_{\zeta+\epsilon',tA}}\\
&\leqslant \mathrm{vol}(G_t)t^{\zeta}\left(\frac{r}{t}\right)^{\zeta-\epsilon} r^{-\zeta-\epsilon'}\\
&\leqslant \mathrm{vol}(G_t)t^{\epsilon(1-\nu')-\nu'\epsilon'} \lesssim t^{\epsilon(1-\nu')-\nu'\epsilon'}\|w_t\|_{L^p_{k,\zeta,tA}}.
\end{aligned}$$

Here we used the fact that $\mathrm{vol}(G_t) = O(\|w_t\|_{L^p_{k,\zeta,tA}})$. Note that the exponent of t is positive for sufficiently small ϵ'. Finally, the calculation for the conically singular case is more direct:

$$\begin{aligned}
\|\delta w_{\mathrm{CS}}\|_{L^p_{k,\zeta+\epsilon',tA}} &\lesssim \mathrm{vol}(G_t)\max_{G_t}|\delta w_{\mathrm{CS}}|_{C^k_{\zeta+\epsilon',tA}}\\
&\lesssim \mathrm{vol}(G_t)r^{\zeta+\epsilon-\epsilon'} \lesssim t^{\nu'(\epsilon-\epsilon')}\|w_t\|_{L^p_{k,\zeta,tA}}.
\end{aligned}$$

□

Proof of Prop. 4.7. Note first that $\not{D}w_t$ is only non-zero in the gluing annulus G_t since w_t

is interpolating between two exact solutions in this region. From the expression:

$$w_t = \varphi w_{\mathrm{CS}} + (1-\varphi) t^\zeta w_{\mathrm{AC},t}$$

and using the fact that $\not{D} = \not{D}_{\mathrm{con}} + \delta\not{D}$ with $\delta\not{D}$ a small perturbation (see Lemma 2.37) we can compute the following:

$$\begin{aligned}\not{D} w_t &= (\not{D}_{\mathrm{con}} + \delta\not{D}) w_t \\ &= \not{D}_{\mathrm{con}}(\varphi w_{\mathrm{CS}} + (1-\varphi) t^\zeta w_{\mathrm{AC},t}) + \delta\not{D} w_t\end{aligned}$$

Already, we see from Proposition 4.8 that for $\epsilon > 0$ sufficiently small there is a $\alpha > 0$ such that:

$$\|\delta\not{D} w_t\|_{L^p_{k,\zeta+\epsilon,tA}} \leqslant t^\alpha \|w_t\|_{L^p_{k+1,\zeta+1,tA}}.$$

Furthermore, if we use the asymptotic expansions from Lemma 4.9 and the fact that $\partial_\zeta w_{\mathrm{CS}} = \partial_\zeta w_{\mathrm{AC},t}$ we see that:

$$\begin{aligned}\not{D}_{\mathrm{con}}(\varphi w_{\mathrm{CS}} + (1-\varphi) t^\zeta w_{\mathrm{AC},t}) &= \not{D}_{\mathrm{con}}(r^\zeta(\partial_\zeta w_{\mathrm{CS}})\varphi + \delta w_{\mathrm{CS}} + (1-\varphi) t^\zeta \delta w_{\mathrm{AC},t}) \\ &= \not{D}_{\mathrm{con}}(\varphi \delta w_{\mathrm{CS}} + (1-\varphi) t^\zeta \delta w_{\mathrm{AC},t}),\end{aligned}$$

since $\partial_\zeta w_{\mathrm{CS}}$ is by definition a ζ-eigensection, and thus $r^\zeta(\partial_\zeta w_{\mathrm{CS}})$ an infinitesimal Cayley deformation of the cone. Now, since both δw_{CS} and $t^\zeta \delta w_{\mathrm{AC},t}$ have $L^p_{k,\zeta+\epsilon',tA}$ norms bounded by $t^{\alpha'}\|w_t\|_{L^p_{k,\zeta,tA}}$ for some $\alpha', \epsilon' > 0$, we see that we get the desired expression:

$$\|\not{D} w_t\|_{L^p_{k,\zeta+\min\{\epsilon,\epsilon'\}-1,tA}} \lesssim t^{\min\{\alpha,\alpha'\}} \|w_t\|_{L^p_{k+1,\zeta,tA}}.$$

□

We found a solution up to order r^ζ to the linearised Cayley equation. We next solve the equation in $L^p_{k,\zeta+\epsilon,tA}$, for which we recall that the inverses of the Cayley operators $\not{D}_{N^{tA}}$ (up to the kernel) have operator norms uniformly bounded in t as in Lemma 3.12. More precisely there are subspaces $\kappa_t \subset C^\infty_c(\nu(N^{tA}))$ such that for any $u \in L^p_{k,\zeta+\epsilon,tA}(\nu(N^{tA}))$ with $u \perp \kappa_t$ (for a suitably chosen inner product) we have:

$$\|u\|_{L^p_{k,\zeta+\epsilon,tA}} \lesssim \|\not{D} u\|_{L^p_{k-1,\zeta+\epsilon-1,tA}} \tag{4.12}$$

This relies on the fact that $\zeta + \epsilon$ is not a critical rate and that both $\not{D}_{\mathrm{AC}}$ and $\not{D}_{\mathrm{CS}}$ are unobstructed at rate $\zeta + \epsilon$. We proved this in Proposition 3.28. We also already showed that when both operators on the pieces are surjective, then so is $\not{D}$ on N^{tA}. This is the contents of Proposition 3.14. In particular, this means that there is a unique $u_t \perp \kappa_t$ such that:

$$\not{D} u_t = \not{D} w_t,$$

and furthermore:

$$\begin{aligned}\|u_t\|_{L^p_{k,\zeta+\epsilon,tA}} &\lesssim \|\not{D} u_t\|_{L^p_{k-1,\zeta+\epsilon-1,tA}}\\ &= \|\not{D} w_t\|_{L^p_{k-1,\zeta+\epsilon-1,tA}}\\ &\lesssim t^\alpha \|w_t\|_{L^p_{k,\zeta,tA}}.\end{aligned}$$

Thus in particular we get a normal vector field $w_t - u_t$, which is an infinitesimal deformation vector field on N^{tA} and which we understand up to second order (orders ζ and $\zeta+\epsilon$). We now need to make the leap to a deformation vector field on the true Cayley $\Gamma(N, tA, \Phi)$.

Proposition 4.10. *Let $N \subset (M, \Phi)$ be an unobstructed CS_μ Cayley ($1 < \mu < 2$) with a unique singular point, and assume that its cone $C \subset \mathbb{R}^8$ is semi-stable. Let $A \subset \mathbb{R}^8$ be an unobstructed AC_λ Cayley ($\lambda < 0$) with a matching cone and sufficiently small scale. Assume that the operator $\not{D}_{\mathrm{AC}}$ is an isomorphism just below the critical rate $\zeta = \max\{\mathscr{D}_C \cap (-\infty, \lambda)\}$ and that all the AC Cayley deformations of A below rate 0 are of rate exactly ζ. We then have that for any two matching infinitesimal deformation vector field $w_{\mathrm{CS}} \in \mathcal{I}^\zeta_{\mathrm{CS}}(N)$ and $w_{\mathrm{AC}} \in \mathcal{I}^\zeta_{\mathrm{AC}}(A)$ there are glued vector fields $w_{\mathrm{CS}} \sharp_t w_{\mathrm{AC}} \in \mathcal{I}(\Gamma(N, tA, \Phi))$ such that (after identifying $\nu(N^{tA}) \simeq \nu(\Gamma(N, tA, \Phi))$) we have:*

$$w_{\mathrm{CS}} \sharp_t w_{\mathrm{AC}} = w_t + \delta w_t. \tag{4.13}$$

Here $\|w_t\|_{L^p_{k,\zeta,tA}} = O(|\log t|)$ and $\|\delta w_t\|_{L^p_{k,\zeta+\epsilon,tA}} \leqslant t^\alpha \|w_t\|_{L^p_{k,\zeta,tA}}$, with $\alpha > 0$. In particular this implies that $|\delta w_t| \ll |w_t|$ as $t \to 0$.

Proof. We first note that by the a priori gluing estimates from Proposition 3.6 and Lemma 2.37 we have that:

$$\|\not{D}_{N^{tA}} - \not{D}_{\Gamma(N,tA,\Phi)}\|_{\mathrm{op}} \lesssim t^{\gamma-1},$$

for some $\gamma > 1$. In particular for sufficiently small t we also get:

$$\|(\not{D}_{N^{tA}}|_{\kappa^\perp})^{-1} - (\not{D}_{\Gamma(N,tA,\Phi)}|_{\kappa^\perp})^{-1}\|_{\mathrm{op}} \lesssim t^{\gamma-1}.$$

Thus the same procedure as above will allow us to prove that we can perturb our infinitesimal deformation vector field $w_t - u_t$ on N^{tA} to an infinitesimal deformation vector field $w_{\mathrm{CS}} \sharp_t w_{\mathrm{AC}}$ on $\Gamma(N, tA, \Phi)$, with a just a further $C^\infty_{\zeta+\epsilon}$-perturbation, whose norm we can bound in exactly the same way. This concludes the proof. □

Deformations in the parallel directions

We will now discuss the deformations of nearly singular Cayley submanifolds that can be interpreted as running parallel to the singular locus in the base of the fibration. Whereas in the previous section we looked at deformations of rate $\zeta < 0$ coming from the AC piece, we now look at the deformations of the next higher rate 0, which can be understood as coming from translations of the conically singular points in the CS piece. The key difference however compared to the previous section is that the Cayley operator $\not{D}_{\mathrm{CS}}$ on

N is never unobstructed at rates above 0. This is because we assume unobstructedness of $\not{D}_{\mathrm{CS}}$ at rates slightly below 0 where $\operatorname{ind} \not{D}_{\mathrm{CS}} \leqslant 4$ by the fibration property. However the multiplicity of the critical rate $0 \in \mathscr{D}_C$ is at least 8 whenever the Cayley cone is not a plane (in which case it is 4). Thus for all the conical models we are interested in we see by Theorem 1.32 that $\operatorname{ind}_\delta \not{D}_{\mathrm{CS}} < 0$ for any $\delta > 0$.

Now the following problem appears back on N^{tA}: if $\xi \in L^p_{k,\delta-1,tA}(E)$, then there is always a unique $w \in L^p_{k+1,\delta,tA}(\nu(N^{tA}))$ such that $\not{D} w = \xi$ with $w \perp^{L^2} \ker \not{D}$, independent of δ. Now for rates $\zeta < \delta < 0$ we have the important estimate from Proposition 3.12:

$$\|w\|_{L^p_{k+1,\delta,tA}} \leqslant C \|\xi\|_{L^p_{k,\delta-1,tA}} \tag{4.14}$$

with $C > 0$ independent of the neck size t. This crucially relies on the fact that both pieces of the gluing be unobstructed. However this is not true any more if δ falls outside this range. Indeed if $\delta > 0$ then $\not{D}_{\mathrm{CS}}$ admits non-trivial obstructions, i.e. there are elements $\xi_{\mathrm{ob}} \in L^p_{k,\delta-1}(E_N)$ which are not in the image of $\not{D}_{\mathrm{CS}}$. Since these obstructions disappear once δ crosses 0, this means that $\not{D}_{\mathrm{CS}} w_{\mathrm{ob}} = \xi_{\mathrm{ob}}$ for some $w_{\mathrm{ob}} \in C^{k+1}_0(\nu(N))$. Thus from the perspective of w_{ob}, the rate of decay of $\not{D}_{\mathrm{CS}} w_{\mathrm{ob}}$ is higher than expected ($-1+\delta$ instead of the weaker rate of -1). This is the reason why the estimate (4.14) cannot hold as is. Indeed, we see for δ just slightly positive that the kernel of $\not{D}_{\mathrm{AC}}$ is $(d(\zeta) + d(0))$-dimensional, while $\not{D}_{\mathrm{CS}}$ has trivial kernel. In particular from Theorem 3.12 we see that the bound (4.14) does hold, but only if we have $w \perp^{L^2_{\delta \pm \epsilon}} \kappa_t$ where κ_t is $(d(\zeta) + d(0))$-dimensional. Since $d(\zeta) + d(0) \geqslant 5$ that means that asymptotically there are some elements in $L^p_{k,\delta-1,tA}(E)$ on the glued manifold which simply do not admit a small pre-image under $\not{D}_{N^{tA}}$. Hence if we want to proceed as in the previous section we need to avoid $\not{D} w_t$ having too large a component in this "bad sector".

Example 4.11. Consider the model fibration:

$$\begin{aligned} f_0 : \mathbb{C}^4 &\longrightarrow \mathbb{C}^2 \\ (x,y,z,w) &\mapsto (x^2+y^2+z^2, w). \end{aligned}$$

It is modelled on the quadratic cone $C_q = \{x^2+y^2+z^2 = 0, w = 0\}$, for which we know from Example 1.35 that $d(-1) = 2, d(0) = 8, d(1) = 22$ and that there are no other critical rates in the range $[-1,1]$. If this local model were part of a fibration by unobstructed Cayleys of a compact Spin(7)-manifold, then we would have for $\epsilon > 0$ small:

$$\operatorname{ind}_{-\epsilon} \not{D}_{\mathrm{CS}} = \operatorname{ind}_{-\epsilon} \not{D}_{\mathrm{AC}} = 2.$$

Thus in particular $\operatorname{ind}_{+\epsilon} \not{D}_{\mathrm{CS}} = 2 + d(0) = 2 + 8 = 10$. This means that $\not{D} w_t$ needs to lie in a codimension $10 - \dim \ker \not{D}_{N^{tA}} = 6$ subspace of $L^p_{k,\epsilon-1}(E)$ in order to perturb w_t to a true solution with a small perturbation (i.e. using the bound (4.14)).

We now go back to the deformation theory of N as an unobstructed CS_μ Cayley with moving points and cones. Using the notation of Remark 2.45, by solving the deformation problem we get a smooth submanifold $P \subset \mathcal{U}$ of possible vertex locations and cone deformations of neighbouring CS_μ Cayleys. We remark that the higher-rate deformation of the cone in a given CS Cayley is already determined by the translation applied to the point (as there are no deformations that fix the singular point). Hence if N has singular point p and cone $C \subset \mathbb{R}^8$, then $T_{(p,C)}P$ can be identified with the possible translation directions of the conically singular point. This will be a subspace $\theta_N \subset T_pM$ of dimension $\dim \mathcal{M}^{\mu}_{\mathrm{CS}}(N)$. Now we can decompose the kernel of $\not{D}_{\mathrm{AC}}$ at rate $\epsilon > 0$ as follows:

$$\ker \not{D}_{\mathrm{AC}} = \mathcal{I}^{\zeta}_{\mathrm{AC}}(A) \oplus (\Theta_N \oplus \Theta_N^{\perp}). \tag{4.15}$$

Here $\mathcal{I}^{\zeta}_{\mathrm{AC}}(A)$ are the deformations of rate $\zeta < 0$ that we discussed in the previous section, Θ_N are the deformation vector fields corresponding to the unobstructed directions θ_N of the conically singular problem, and $\Theta_N^{\perp}$ is spanned by vector fields corresponding to obstructed translations of the cone. We can choose compactly supported approximations u_0 of elements $u \in \Theta_N^{\perp}$ such that $u \perp \Theta_N$ pointwise. This then gives rise to a splitting of the pseudokernel κ_t as follows:

$$\kappa_t = \kappa_{\zeta,t} \oplus \kappa_{0,t} \oplus \kappa_{\mathrm{ob},t}.$$

Note that all the sections from the family of pseudo-kernels are entirely supported on N^{tA}_{AC}. We can now consider, as in the previous section, two infinitesimal deformation vector fields $w_{\mathrm{AC}} \in \mathcal{I}^0_{\mathrm{AC}}(A), w_{\mathrm{CS}} \in \mathcal{I}^0_{\mathrm{CS}}(N)$ with matching boundary conditions $\sigma = \partial_0 w_{\mathrm{AC}} = \partial_0 w_{\mathrm{CS}}$. We then pre-glue them together as before to obtain $w_t \in C^{\infty}(\nu(N^{tA}))$ with:

$$\|\not{D} w_t\|_{L^p_{k,\epsilon-1,tA}} \lesssim t^{\alpha} \|w_t\|_{L^p_{k+1,\epsilon,tA}},$$

where $\epsilon > 0$ is a small constant. We now perturb slightly, so that $w_t \perp \kappa_{\mathrm{ob},t}$. This is possible, since by assumption w_{AC} must be aligned with an unobstructed direction for the CS problem, as the two vector fields have matching boundary conditions. Now, since $\kappa_{\mathrm{ob},t}$ consists of compactly supported normal vector fields which are pointwise orthogonal to the unobstructed perturbation directions such as w_{CS}, we must have $\|\pi_{\kappa_{\mathrm{ob},t}}[w_t]\|_{L^p_{k+1,\epsilon,tA}} \to 0$ as $t \to 0$, where $\pi_{\kappa_{\mathrm{ob},t}}$ is the $L^2_{\pm}$-orthogonal projection onto $\kappa_{\mathrm{ob},t}$. Thus we can perturb to $\tilde{w_t} = w_t - \pi_{\kappa_{\mathrm{ob},t}}[w_t]$ such that we still have $\|\not{D}\tilde{w_t}\|_{L^p_{k,\epsilon-1,tA}} \lesssim t^{\alpha}\|\tilde{w_t}\|_{L^p_{k+1,\epsilon,tA}}$ and additionally $\tilde{w_t} \perp \kappa_{\mathrm{ob},t}$. We are now in a position to run the argument from the previous section again using the bound (4.14) to obtain:

Proposition 4.12. *Let $N \subset (M, \Phi)$ be an unobstructed CS_{μ} Cayley ($1 < \mu < 2$) with a unique singular point, and assume that its cone $C \subset \mathbb{R}^8$ is semi-stable. Let $A \subset \mathbb{R}^8$ be an unobstructed AC_{λ} Cayley ($\lambda < 0$) with matching cone and sufficiently small scale. We then have that for any two matching infinitesimal deformation vector field $w_{\mathrm{CS}} \in \mathcal{I}^0_{\mathrm{CS}}(N)$ and $w_{\mathrm{AC}} \in \mathcal{I}^0_{\mathrm{AC}}(A)$ there are glued vector fields $w_{\mathrm{CS}} \sharp_t w_{\mathrm{AC}} \in \mathcal{I}(\Gamma(N, tA, \Phi))$ such that (after identifying $\nu(N^{tA}) \simeq \nu(\Gamma(N, tA, \Phi))$) we have:*

$$w_{\mathrm{CS}} \sharp_t w_{\mathrm{AC}} = w_t + \delta w_t. \tag{4.16}$$

Here $\|w_t\|_{L^p_{k,\zeta,tA}} = O(|\log t|)$ and $\|\delta w_t\|_{L^p_{k,\epsilon,tA}} \leqslant t^{\alpha}\|w_t\|_{L^p_{k,0,tA}}$. In particular this implies that $|\delta w_t| \ll |w_t|$ as $t \to 0$.

Stability of strong fibrations

Now we have shown that if the close to singular fibres of a Cayley fibration admit deformations of exactly two different asymptotic rates, namely the normal deformations at rate $\zeta < 0$ and the parallel deformations at rate 0, then under the change of Spin(7)-structure the infinitesimal deformation vector fields are perturbed by adding additional terms which are in $L^p_{k+1,\zeta+\epsilon,tA}(\nu(N^{tA}))$ and $L^p_{k+1,\epsilon,tA}(\nu(N^{tA}))$ respectively, and are always bounded uniformly in t. We can now use this to show that the strong fibration property is stable under perturbation, given some additional assumptions.

Let us assume that we have a strong Cayley fibration $f : (M, \Phi) \to B$ as in Definition 4.1 with discriminant $\Delta \subset B$ of dimension $l = 1$ or 2. Already this means that all compact and conically singular Cayley fibres of f are unobstructed in their respective moduli spaces in the cases coming from complex geometry we consider later on, see Propositions 5.3 and 5.4. The singular cones of the conically singular Cayleys share their set of weights $\mathscr{D} \subset \mathbb{R}$, and we let $\zeta = \max\{\mathscr{D} \cap (-\infty, 0)\}$. We then require additionally that each conically singular Cayley be **simple** in the following sense:

Definition 4.13. A conically singular Cayley $N \subset M$ is called **simple** if the Cayley operator $\not{D}_{\mathrm{CS}}$ has index 4 just below the critical weight ζ and is unobstructed.

Consider now an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of the base B, where $\varphi_\alpha : U_\alpha \to B_1(0) \subset \mathbb{R}^4$ is a diffeomorphism. If $U_\alpha \cap \Delta \neq 0$, then we further assume that this chart is compatible with the gluing map Γ in the sense that we identify $B_1(0) \simeq U_{\mathrm{CS},\alpha} \times U_{\mathrm{AC},\alpha}$, where $U_{\mathrm{CS},\alpha} \subset \mathcal{M}^\mu_{\mathrm{CS}}(N)$ and $U_{\mathrm{AC},\alpha} \subset \overline{\mathcal{M}}^\lambda_{\mathrm{AC}}(A)$ with the condition that $f^{-1}(\varphi_\alpha^{-1}(\tilde{N}, \tilde{A})) = \Gamma(\tilde{N}, \tilde{A})$. On top of this we consider the framings $\{e_{i,\alpha}\}_{i=1,2,3,4}$ of $TB|_{U_\alpha}$ such $e_{i,\alpha} = \partial_i$. For each $b \in B \setminus \Delta$ we thus get four infinitesimal deformation vector fields $w_{1,\alpha}, \dots, w_{4,\alpha}$, which are just the lifts of $e_{i,\alpha}(b)$ via f. For $b \in \Delta$, we can again find a local frame, such that $T_b(U_{\mathrm{CS},\alpha}) = \operatorname{span}\{e_{1,\alpha}(b), \dots, e_{l,\alpha}(b)\}$ and $T_b(U_{\mathrm{AC},\alpha})_b = \operatorname{span}\{e_{l+1,\alpha}(b), \dots, e_{4,\alpha}(b)\}$. Note that in this case, $e_{1,\alpha}, \dots, e_{l,\alpha} \in C^\infty_0(\nu(N^{tA}))$ and $e_{l+1,\alpha}, \dots, e_{4,\alpha} \in C^\infty_\zeta(\nu(N^{tA}))$. Note also that at or near a singular point $w_{1,\alpha}, \dots, w_{l,\alpha}$ are what we above called the parallel infinitesimal deformation vector fields and $w_{l+1,\alpha}, \dots, w_{4,\alpha}$ are the orthogonal deformation vector fields.

For a given $b \in U_\alpha$ consider the following function in $C^\infty(f^{-1}(b))$:

$$\det_{\alpha,b} = \det(w_1, \dots, w_l, \rho^{-\zeta} w_{l+1}, \dots, \rho^{-\zeta} w_4). \tag{4.17}$$

Our final assumption on the initial fibration will be a condition on $\det_{\alpha,b}$.

Definition 4.14. A Cayley fibration is called **nondegenerate** if there is a constant $C > 0$ such that for all $\alpha \in I, b \in B$ and $x \in f^{-1}(b)$:

$$C < \det_{\alpha,b}(x) < C^{-1}.$$

This means that the infinitesimal vector fields never vanish for any Cayley in the fibration, and that the deformations of the cone of rate ζ and 0 have no zeros as well.

Consider now a variation of the Spin(7)-structure $\{\Phi_s\}_{s \in \mathcal{S}}$ with $\Phi_{s_0} = \Phi$. For each $\alpha \in I, b \in U_\alpha$ we then get a family of determinant maps $\det_{\alpha,b,s}$, depending continuously on $s \in \mathcal{S}$. The key insight is the following:

Lemma 4.15. *If* $\det_{\alpha,b,s} > 0$ *for all* α, b *and* $s \in V$ *an open neighbourhood of* $s_0 \in \mathcal{S}$*, then the universal families* $\mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi_s))$ *form strong fibrations of* M*.*

Proof. As we have $\dim \Delta \leqslant 2$, we may apply the weak fibration theorem 4.4 to conclude that small perturbations of (M, Φ) are still weakly fibering. Thus if $N \subset M$ is any Cayley fibre then

$$\mathrm{ev}_s : \mathrm{Univ}(\overline{\mathcal{M}}(N, \Phi_s)) \longrightarrow M$$

are homotopic maps for $s \in V$ of degree one in the sense of pseudo-cycles. Each of these maps is stratified smooth, thus in particular a local diffeomorphism on the open stratum as $\det_{\alpha,b,s} > 0$ for every $s \in V$. Indeed, if the derivative $\mathrm{D}\,\mathrm{ev}_s$ were to fail to be an isomorphism at a point $p \in N$, this means that $\dim \mathrm{D}\,\mathrm{ev}_s(p)[\nu_p(N)] \leqslant 3$ (as N is nonsingular) and $\det_{\alpha,b,s}(p) = 0$. For the same reason, the maps ev_s remain orientation preserving.

But an orientation preserving local diffeomorphism of degree one must necessarily be a global diffeomorphism, as the algebraic count (which in this case is simply the naive count) of pre-images of a point equals the degree. □

We then have that far away from the singular fibres, the w_i perturb smoothly in $L^p_{k+1}(\nu(N))$. In particular, by compactness of M, we can ensure that $\det_{\alpha,b,s} > \frac{1}{2}C > 0$ for some open neighbourhood V of s_0, for all fibres that are a given distance away from the singularities simultaneously. Near the singular fibres, we see from the gluing results Proposition 4.10 and Proposition 4.12 that $w_1, \ldots, w_l$ perturb by continuously varying additional terms $\delta w_{1,s}, \ldots, \delta w_{l,s} \in L^p_{k,\epsilon,tA}(\nu(N^{tA}))$ and $w_{l+1}, \ldots, w_4$ additional terms $\delta w_{l+1,s}, \ldots, \delta w_{4,s} \in L^p_{k,\zeta+\epsilon,tA}(\nu(N^{tA}))$. Now since $L^p_{k,\delta+\epsilon,tA} \hookrightarrow C^0_{\delta,tA}$ are continuous embeddings with bounded embedding constants, we see that $\det(w_1, \ldots, w_l, \rho^{-\zeta} w_{l+1}, \ldots, \rho^{-\zeta} w_4)$ varies continuously in C^0, uniformly in t. In other words,

$$\|\det_{\alpha,b,s} - \det_{\alpha,b,s_0}\|_{C^0} \leqslant C_{\det} d(s, s_0),$$

where $C_{\det} > 0$ is independent of the neck size t. In particular, for a given chart we can find an open neighbourhood V_α of $s_0 \in \mathcal{S}$ such that $\det_{\alpha,b,s} > \frac{1}{2}C > 0$ for any $s \in V_\alpha$. Hence, since we can cover B with finitely many charts, this means that for $s \in V_{\mathrm{stab}} = V \cap \bigcap_\alpha V_\alpha$ we maintain the fibration property of the nonsingular fibres. From this, we can also deduce that the singular fibres do not intersect the nonsingular fibres.

Indeed, assume that for some $s \in V_{\mathrm{stab}}$ there is a singular fibre $\hat{N}$ intersecting a nonsingular fibre N. Then by what we just proved, the fibres near N are locally still fibering, thus in particular for $t > 0$ sufficiently small, $\Gamma(\hat{N}, tA, \Phi_s)$ will intersect another nonsingular fibre, which is, of course impossible, as the nonsingular fibres are still fibering for Φ_s. Finally, as the conically singular fibres are unobstructed, we have from Theorem 2.42 that their infinitesimal deformation vector fields deform smoothly under the variation of $\mathrm{Spin}(7)$-structure Φ_s. Since the moduli space of all singular fibres with all possible degenerations is a compact topological space, a similar argument with determinant maps can be applied to show that they too will remain intersection-free for s in an open neighbourhood of s_0. We thus proved:

Theorem 4.16 (Stability of strong fibrations)**.** *Let* (M, Φ) *be a* $\mathrm{Spin}(7)$*-manifold which is strongly fibred by conically singular Cayleys which are simple, such that all the Cayleys in the fibration are unobstructed. Assume that the fibration is nondegenerate as in Definition 4.14. Let* $\{\Phi_s\}_{s\in\mathcal{S}}$ *be a smooth family of deformations of the* $\mathrm{Spin}(7)$*-structure* $\Phi = \Phi_{s_0}$*. Then there is an open neighbourhood* $s_0 \in U \subset \mathcal{S}$ *such that for all* $s \in U$ *the manifold* (M, Φ_s) *can still be strongly fibred.*

Example 4.17. As we will see in Chapter 5, complex fibrations of Calabi–Yau fourfolds with Morse type singularities satisfy the conditions of Theorem 4.16. Hence such complex fibrations are stable under small deformations of the Calabi–Yau structure, as a Cayley submanifold in the homology class of a complex surface is necessarily a complex surface again by Stokes' theorem.

Chapter 5

Gluing construction of a Kovalev-Lefschetz fibration

In this last chapter we give examples of strong fibrations by Cayley submanifolds of a family of torsion-free Spin(7)-manifolds, which are products of the circle S^1 with twisted connected sum G_2-manifolds. The latter were first introduced by Kovalev [23] and are compact G_2-manifolds obtained by gluing two asymptotically cylindrical G_2-manifolds together along a sufficiently long neck. The construction was later extended to include more general gluing maps by Corti, Haskins, Pacini and Nordström [9]. The ACyl G_2-manifolds are constructed from Fano threefolds [8] in such a way that their link contains two copies of S^1 which may be interchanged or twisted when gluing. By their construction, the pre-glued approximations of compact G_2-manifolds come with coassociative fibrations that admit complex singularities.

By taking the product with S^1, we obtain Cayley fibrations on Spin(7)-manifolds with small torsion. Over either end, the fibration looks like a complex fibration by surfaces. However the entire Spin(7)-manifold does not admit a global complex structure. By choosing the Fano ingredients carefully we can ensure that the fibration, which locally is a complex fibration, has singularities which are at worst of Morse type.

5.1 The complex quadric

For a moment let us focus on the local model f_0 near a singular point, given by the following holomorphic fibration:

$$\begin{aligned} f_0 : \mathbb{C}^4 &\longrightarrow \mathbb{C}^2 \\ (x, y, z, w) &\longmapsto (x^2 + y^2 + z^2, w). \end{aligned}$$

Hence $f_0^{-1}(0, \eta) \simeq C_q$ is a quadric cone and the nearby nonsingular fibres are the asymptotically conical Cayleys $A_\epsilon = f_0^{-1}(\epsilon, 0)$. We note that the holomorphic normal bundle $\nu^{1,0}(A_\epsilon)$ is trivial. To see this explicitly, consider the following two nowhere vanishing sections of $\nu^{1,0}(A_\epsilon)$:

$$\begin{aligned} s_{1,\mathrm{AC}}(x,y,z,w) &= \partial_w, \\ s_{2,\mathrm{AC}}(x,y,z,w) &= \frac{\bar{\partial}_x + \bar{\partial}_y + \bar{\partial}_z}{|(x,y,z,0)|^2}. \end{aligned} \tag{5.1}$$

We remark that $s_{1,\mathrm{AC}}$ is an infinitesimal deformation corresponding to a translation $w \mapsto w + a$ and thus of rate 0, whereas $s_{2,\mathrm{AC}}$ is of rate -1 and corresponds to a variation in the parameter ϵ. In other words, the normal bundle of A_ϵ is trivial exactly because of the existence of the fibration f_0. Next, we prove a Liouville theorem for A_ϵ in order to compute $H^0(\nu^{1,0}(A_\epsilon))$:

Proposition 5.1 (Liouville theorem). *Any bounded holomorphic function on A_ϵ is constant.*

Proof. We can embed:

$$A_\epsilon \subset \bar{A}_\epsilon \subset \mathbb{C}P^4,$$

where $\bar{A}_\epsilon$ is the completion $\{x^2 + y^2 + z^2 = \epsilon u^2, w = 0\} \subset \mathbb{C}P^4$ (with homogeneous coordinates $[x : y : z : w : u]$ on $\mathbb{C}P^4$). Since $\bar{A}_\epsilon$ is compact and nonsingular, any bounded holomorphic function on $\bar{A}_\epsilon$ is automatically constant. Now, using a removable singularities theorem in higher dimensions (such as [46, Thm. 1.23]) and noting that $\bar{A}_\epsilon \setminus A_\epsilon \subset \mathbb{C}P^4$ is a non-singular subvariety, we can extend any bounded holomorphic function on A_ϵ to a holomorphic function on $\bar{A}_\epsilon$, which concludes the proof. □

We can now use this to prove the unobstructedness of A_ϵ as a Cayley.

Proposition 5.2. *The AC_λ Cayley $A_\epsilon \subset (\mathbb{R}^8, \Phi_0)$ for $\epsilon \in \mathbb{C} \setminus 0$ is unobstructed and has no infinitesimal deformations at rate $-2 < \lambda < -1$.*

Proof. Following [37, Prop. 3.5] we can write the Cayley operator on a complex AC_λ surface in $(\mathbb{R}^8, \Phi_0)$ as $\not{D}_{\mathrm{AC}} = \bar{\partial} + \bar{\partial}^*$ mapping between the spaces:

$$C^\infty_\lambda(\nu^{1,0}(A_\epsilon) \oplus (\Lambda^{0,2}A \otimes \nu^{1,0}(A_\epsilon))) \longrightarrow C^\infty_{\lambda-1}(\Lambda^{0,1}A_\epsilon \otimes \nu^{1,0}(A_\epsilon)).$$

Thus if $(u,v) \in C^\infty_\lambda(\nu^{1,0}(A_\epsilon) \oplus (\Lambda^{0,2}A_\epsilon \otimes \nu^{1,0}(A_\epsilon)))$ satisfies $\bar{\partial}u + \bar{\partial}^* v = 0$ then the pair (u,v) corresponds to an infinitesimal Cayley deformation vector field. If in addition, we have $\bar{\partial}u = 0$ and $\bar{\partial}^* v = 0$, then (u,v) is in fact an infinitesimal complex deformation [37, Cor. 4.7]. To start, we will prove that for $\lambda < -1$ any infinitesimal Cayley deformation is necessarily an infinitesimal complex deformation. For this note that if $\bar{\partial}u + \bar{\partial}^* v = 0$, then we automatically have $\bar{\partial}^* \bar{\partial} u = -\bar{\partial}^* \bar{\partial}^* v = 0$. Now, since $\bar{\partial}u$ has rate $\lambda - 1 < -2$ we also have $\bar{\partial}u \in L^2(\Lambda^{0,1}A_\epsilon \otimes \nu^{1,0}(A_\epsilon))$. This leads us to:

$$0 = \int_{A_\epsilon} \langle \bar{\partial}^* \bar{\partial} u, u \rangle \, \mathrm{dvol} = \int_{A_\epsilon} \langle \bar{\partial} u, \bar{\partial} u \rangle \, \mathrm{dvol} = \|\bar{\partial} u\|_{L^2}.$$

In particular $\bar{\partial}u = 0$, which entails $\bar{\partial}^* v = 0$, and thus any infinitesimal Cayley deformation is in fact also infinitesimal complex. Now, since there are no bounded and non-constant holomorphic functions on A_ϵ by Proposition 5.1, there are no other infinitesimal complex deformations of rate less than or equal to -1 besides constant multiples of $s_{2,\mathrm{AC}}$. Thus the kernel of the Cayley operator $\ker_\lambda \not{D}_{\mathrm{AC}}$ must be trivial at this rate.

Finally, we prove the surjectivity of the Cayley operator at rate $\lambda > -2$. This is equivalent to the injectivity of its formal adjoint $\not{D}^*_{\mathrm{AC}} = (\bar\partial^*, \bar\partial)$ which maps between the spaces:

$$C^\infty_{-4-\lambda}(\Lambda^{0,1}A_\epsilon \otimes \nu^{1,0}(A_\epsilon)) \longrightarrow C^\infty_{-5-\lambda}(\nu^{1,0}(A_\epsilon) \oplus (\Lambda^{0,2}A_\epsilon \otimes \nu^{1,0}(A_\epsilon))).$$

But we have $C^\infty_{-4-\lambda}(\Lambda^{0,1}A_\epsilon \otimes \nu^{1,0}(A_\epsilon)) = C^\infty_{-4-\lambda}(\Lambda^{0,1}A_\epsilon \otimes \mathbb{C}^2)$ since the normal bundle is trivial. So if $\bar\partial v = 0$ and $\bar\partial^* v = 0$, then in fact v is a harmonic 1-form with values in $\mathbb{C}^2$, as A_ϵ is Kähler. Now v is square-integrable (by our assumption on the rate), and thus we can invoke [28, Thm. 0.14], which says that in this situation square-integrable harmonic one-forms are in one-to-one correspondence with elements of $H^1(A_\epsilon) = 0$. Thus we get $v = 0$, and the Cayley operator is surjective. □

5.2 Complex fibrations of Calabi–Yau fourfolds

Proposition 5.3. *Let $f : M^8 \to B^4$ be a complex fibration, where M is a smooth Calabi–Yau fourfold and B is a smooth, complex two-dimensional base. If a fibre F is diffeomorphic to a nonsingular K3 surface then it is unobstructed as a Cayley submanifold and has a four-dimensional Cayley moduli space.*

Proof. First, we have from Proposition 2.17 that the index of a fibre F as above is given by:

$$\operatorname{ind} \not{D}_F = \tfrac{1}{2}(\sigma(F) + \chi(F)) - [F]\cdot[F] = \tfrac{1}{2}(-16 + 24) - 0 = 4.$$

Here the self-intersection number $[F]\cdot[F]$ vanishes by the fibration property. The fibre F admits at least 4 Cayley deformations by perturbing to nearby fibres, which is equal to the index of the elliptic problem. Hence, showing unobstructedness is equivalent to showing that there are exactly 4 infinitesimal Cayley deformations. Now by [36, Lemma 4.7], we have that infinitesimal Cayley deformations are necessarily infinitesimal complex deformations. However, because F is part of a complex fibration locally, the holomorphic normal bundle $\nu(F) = \mathcal{O}(F) \oplus \mathcal{O}(F)$ is trivial and we have $H^0(\nu(F)) \simeq \mathbb{C}^2$ by compactness of F. This concludes the proof, as holomorphic normal vector fields are exactly the infinitesimal Cayley deformations. □

Proposition 5.4. *Let $f : M^8 \to B^4$ be a complex fibration, where M is a smooth Calabi–Yau fourfold and B is a smooth, complex two-dimensional base. Suppose that the fibration is modelled near a singular point on the complex quadric fibration*

$$\begin{aligned} f_0 : \mathbb{C}^4 &\longrightarrow \mathbb{C}^2 \\ (x, y, z, w) &\longmapsto (x^2 + y^2 + z^2, w). \end{aligned}$$

Assume furthermore that each singular fibre contains at most two singular points and that the nonsingular fibres are diffeomorphic to nonsingular K3-surfaces. Finally the singular *locus $\Delta \subset B$ should take the form of a union of transversely intersecting smooth submanifolds. In that case $\dim \Delta = 2$ and each Cayley in the fibration is unobstructed in its moduli space, where we allow moving points and cones.*

Proof. We denote a nonsingular fibre of the fibration by F, a singular fibre with a unique singular point by F_s and a singular fibre with two singularities by F_{ss}. The expectation is that nonsingular fibres are generic, fibres with one singularity appear in codimension 2 and fibres with two singularities appear in codimension 4. We will now show more precisely that the indices of the deformation problems are given by:

$$\mathrm{ind} \not{D}_F = 4, \quad \mathrm{ind}_{1+\epsilon} \not{D}_{\mathrm{CS},F_s} = 2, \text{ and } \quad \mathrm{ind}_{1+\epsilon} \not{D}_{\mathrm{CS},F_{ss}} = 0.$$

Here $\epsilon > 0$ is small and the operators $\not{D}_{\mathrm{CS},F_s}$ and $\not{D}_{\mathrm{CS},F_s}$ take into account the deformations of the points and cones. We first note that the equality $\mathrm{ind} \not{D}_F = 4$ is the contents of Proposition 5.3. Next, the critical rates of the quadratic cone C_q in the range $(-2, 2)$ are known from Example 1.35, and have multiplicities:

$$d(-1) = 2, \quad d(0) = 8, \quad d(1) = 22, \quad d(-1+\sqrt{5}) = 6.$$

Thus using Theorem 1.32 we see that the index of the problem with varying cones and points at rate $1+\epsilon$ is equal to the index of the operator with fixed points and cones, but at rate $-1+\epsilon$.

Now, by gluing one or two matching AC-manifold A_ϵ onto the conically singular points with we obtain nonsingular $F \simeq \Gamma(F_s, A_\epsilon, \Phi) \simeq \Gamma(F_{ss}, (A_\epsilon, A_\epsilon), \Phi)$. Thus we have (using $\mathrm{ind}_{-1+\epsilon} \not{D}_{\mathrm{AC}} = 2$ from Proposition 5.2):

$$\mathrm{ind}_{1+\epsilon} \not{D}_{\mathrm{CS},F_s} = \mathrm{ind}_{-1+\epsilon} \not{D}^{\mathrm{fix}}_{\mathrm{CS},F_s} = \mathrm{ind}\, \not{D}_F - \mathrm{ind}_{-1+\epsilon} \not{D}_{\mathrm{AC}} = 4 - 2 = 2$$

and

$$\begin{aligned} \mathrm{ind}_{1+\epsilon} \not{D}_{\mathrm{CS},F_{ss}} &= \mathrm{ind}_{-1+\epsilon} \not{D}^{\mathrm{fix}}_{\mathrm{CS},F_{ss}} \\ &= \mathrm{ind}\, \not{D}_F - \mathrm{ind}_{-1+\epsilon} \not{D}_{\mathrm{AC}} - \mathrm{ind}_{-1+\epsilon} \not{D}_{\mathrm{AC}} \\ &= 4 - 2 - 2 = 0. \end{aligned}$$

From this it is also clear we we should not expect unobstructed fibres with three or more singularities, as they would have strictly negative virtual dimension. We have now proven the index claims.

In order to prove unobstructedness in the singular case (the compact case has been taken care of in Proposition 5.3) it is thus sufficient to prove that the spaces of infinitesimal Cayley deformations are exactly real two-dimensional and zero-dimensional respectively.

First, consider the a fibre with a single conical singularity $F_s = f^{-1}(b) \setminus \{p\}$, with the conically singular point p removed. Let $\partial_1, \partial_2 \in T_b B$ be two tangent vectors, where we assume $\partial_1 \in T_b \Delta$ and $\partial_2 \notin T_b \Delta$. As the differential $\mathrm{D}f$ only vanishes at the conically singular points, we see that the holomorphic sections of $\nu(F_s)$ given by:

$$s_1 = \mathrm{D}f^*[\partial_1],$$
$$s_2 = \mathrm{D}f^*[\partial_2],$$

are nowhere vanishing, and thus span $\nu(F_s) = \mathcal{O}(F_s) \oplus \mathcal{O}(F_s)$. We note that s_1 has rate $O(1)$ when approaching the vertex p, as it comes from deforming the conically singular manifold to a nearby conically singular one (i.e. moving within $\Delta \subset B$). However as $\operatorname{im} \mathrm{D}f(p) = T_b\Delta$, we see that $|s_2|$ must diverge as we approach the cone. Indeed from the local model f_0 we see that s_2 must be asymptotic to $s_{2,\mathrm{AC}}$ from Equation (5.1), and thus of rate $O(r^{-1})$.

Now we are in a position to repeat the proof of Proposition 5.2. We first note that F_s also has a Liouville theorem. Suppose that $h : F_s \to \mathbb{C}$ is a bounded holomorphic function. Then blowing up F_s at the conically singular point, we obtain a nonsingular $\pi : \tilde{F}_s \to F_s$ which is a biholomorphism away from a single exceptional and nonsingular curve $Q = \pi^{-1}(p)$. We can then apply a removable singularities theorem in higher dimension [46, Thm 1.23] to conclude that h extends to a holomorphic function on $\tilde{F}_s$. Thus h must be constant in the first place. Hence the only complex deformations of rate 0 or above are the deformations coming from moving F_s within the fibration. Now can use the same integration by parts argument that we used for the AC case to show that there are no further deformations which are Cayley but not complex.

For the singular fibres with two singularities we again see that $\nu(F_{ss}) = \mathcal{O}(F_{ss}) \oplus \mathcal{O}(F_{ss})$. However now $F_{ss} = f^{-1}(b)$, where $b \in \Delta$ is a transverse intersection point. Thus deforming $b \in \Delta$ within Δ results in one singularity persisting, with the other one being resolved. Thus our discussion from above shows that all normal sections of F_{ss} necessarily blow up with rate $O(r^{-1})$ near one of the singular points. In particular the conically singular fibres with two singularities are rigid and therefore unobstructed. □

5.3 Fibrations on twisted connected sums

In this section we introduce the twisted connected sum construction, first described by Kovalev [23], and later extended by Corti-Haskins-Nordström-Pacini [9]. It gives rise to torsion-free G_2-manifolds via perturbation of an explicit small torsion glued G_2-manifold. From their construction, these pre-glued manifolds M admit natural fibrations by coassociatives. Our stability theorem 4.16 allows us to perturb the induced Cayley fibration on $M \times S^1$, which ultimately allows us to prove the existence of coassociative fibrations of G_2-manifolds as well.

Cylindrical Calabi–Yau 3-folds

Let $(S, I, \omega_\infty, g_\infty, \Omega_\infty)$ be a K3 surface with a fixed hyperkähler structure. Assume that $(X^6, J, \omega, g, \Omega)$ is a noncompact Calabi–Yau threefold. We say that X is **asymptotically cylindrical** of rate $\lambda < 0$ or (ACyl_λ), limiting to the hyperkähler surface S if there is a compact subset $K \subset X$ and a diffeomorphism $f : X \setminus K \to \mathbb{R}_{>0} \times S^1 \times S$ with the following properties for all $k \geqslant 0$:

$$
\begin{aligned}
|\nabla^k(g - (g_\infty + \mathrm{d}t^2 + \mathrm{d}s^2))| &= O(e^{\lambda t}),\\
\nabla^k(\omega - (\omega_\infty + \mathrm{d}t \wedge \mathrm{d}s)) &= \mathrm{d}\sigma, \text{ where } |\nabla^k \sigma| = O(e^{\lambda t}),\\
\nabla^k(\Omega - (\mathrm{d}s - i\mathrm{d}t) \wedge \Omega_\infty) &= \mathrm{d}\Sigma, \text{ where } |\nabla^k \Sigma| = O(e^{\lambda t}).
\end{aligned}
$$

Here $\mathbb{R}_{>0} \times S^1$ has coordinates (t, s) and $|\cdot|, \nabla$ are defined with respect to the product metric on $\mathbb{R}_{>0} \times S^1 \times S$. Asymptotically cylindrical Calabi–Yau threefolds can be constructed from compact Fano three folds using the following theorem.

Theorem 5.5 (Thm. 2.6 in [8]). *Let Z be a compact Kähler threefold with a morphism $f : Z \to \mathbb{C}P^1$, with a smooth connected reduced fibre S that is an anticanonical divisor, and let $V = Z \setminus S$. If $(S, J, \omega_S, g_S, \Omega_S)$ is a hyperkähler structure on the complex surface (S, J) such that $[\omega_S] \in H^{1,1}(S)$ is the restriction of the Kähler class on Z, then there is a CY3 structure $(V, J, \omega, g, \Omega)$ on V which is asymptotically cylindrical to the CY3 cylinder $\mathbb{R} \times S^1 \times S$ induced by the hyperkähler structure on $(S, J, \omega_S, g_S, \Omega_S)$.*

Now we will discuss briefly how to obtain such f, Z and S as in the theorem above. In fact Corti, Haskins, Nordström and Pacini impose extra conditions on maps $f : Z \to \mathbb{C}P^1$ which make them more suitable for the twisted connected sum construction.

Definition 5.6 (Building block). A nonsingular complex algebraic threefold Z together with a projective morphism $f : Z \to \mathbb{C}P^1$ is called a **building block** if the following conditions are satisfied:

1. The anti-canonical class $-K_Z \in H^2(Z, \mathbb{Z})$ is primitive, i.e. not an integer multiple of another class in $H^2(Z, \mathbb{Z})$.
2. The pre-image $S = f^{-1}(\infty)$ is a nonsingular K3 surface and $S \sim -K_Z$ as divisors.
3. If $k : H^2(Z, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is the map induced by the embedding $S \hookrightarrow Z$, then $\operatorname{im} k \hookrightarrow H^2(S, \mathbb{Z})$ is primitive, i.e. $H^2(S, \mathbb{Z})/\operatorname{im} k$ is torsion-free as an abelian group.
4. The groups $H^3(Z, \mathbb{Z})$ and $H^4(Z, \mathbb{Z})$ are torsion-free.

There are multiple ways to construct building blocks. The first was introduced by Kovalev in [23] and starts with a Fano threefold as in Definition 1.4. This was later extended in [9] by Corti, Haskins, Pacini, Nordström to what they call semi-Fano threefolds, which can be thought of as desingularisations of certain mildly singular Fano varieties. They outnumber Fano threefolds by several orders of magnitude. Finally there is a different type of building block coming from K3 surfaces with non-symplectic involutions [25] which yields different examples still (however these will not be of interest to us from the point of view of fibrations).

In all these examples we obtain a building block $f : Z \to \mathbb{C}P^1$ where the generic fibre of f is a smooth K3 surface. Singular fibres appear in complex codimension 1, but in general we cannot say much about the kinds of singularities that appear. Hence we will go through the first construction of building blocks (starting from (semi-)Fano threefolds) and give an example where we can determine the singularities explicitly.

So suppose X is a Fano threefold, such as for instance a smooth quartic in $\mathbb{C}P^4$. Then a generic anticanonical divisor in X (which is effective by the Fano property) is a smooth K3 surface by a classical result of Šokurov [42]. We then make the assumption that the linear system $|-K_X|$ contains two nonsingular members S_0, S_∞ such that $C = S_0 \cap S_\infty$ is a transverse intersection, and thus a nonsingular curve. In this case the pencil described by S_0 and S_∞ exhausts X and has base locus exactly C. If we now blow up X at C to obtain a new manifold Z, the pencil generated by the proper transforms $\tilde{S}_0$ and $\tilde{S}_\infty$ of S_0 and S_∞ respectively will be base point free. Thus we obtain a holomorphic map $f : Z \to \mathbb{C}P^1$ with generically smooth K3 fibres such that $f^{-1}(0) = \tilde{S}_0$ and $f^{-1}(\infty) = \tilde{S}_\infty$.

Proposition 5.7 (Proposition 3.17 in [9])**.** *The map $f : Z \to \mathbb{C}P^1$ determines a building block.*

Example 5.8. Consider the following quartic polynomial on $\mathbb{C}P^4$ with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3 : x_4]$:

$$P = x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_3^3(x_0 + 10x_1 + 100x_2).$$

Consider the smooth complex submanifold $X = \{P = 0\} \subset \mathbb{C}P^4$. Then, using the adjunction formula, we can see that the canonical bundle ω_Q of Q is given by:

$$\omega_X = (\omega_{\mathbb{C}P^4} \otimes \mathcal{O}_{\mathbb{C}P^4}(Q))|_X = (\mathcal{O}_{\mathbb{C}P^4}(-5) \otimes \mathcal{O}_{\mathbb{C}P^4}(4))|_X = \mathcal{O}_{\mathbb{C}P^4}(-1)|_X.$$

In particular the anticanonical bundle $\omega_X^\star = \mathcal{O}(1)|_X$ is ample, and the anticanonical divisors are exactly the hyperplane sections of X. So we can take for instance:

$$\begin{aligned} S_0 &= \{x_3 = 0\} \cap X \simeq \{x_0^4 + x_1^4 + x_2^4 + x_4^4 = 0\} \subset \mathbb{C}P^3 \\ S_\infty &= \{x_4 = 0\} \cap X \\ &\simeq \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_3^3(x_0 + 10x_1 + 100x_2)\} \subset \mathbb{C}P^3. \end{aligned}$$

Both are smooth K3 surfaces. They intersect transversely in a curve

$$C \simeq \{x_0^4 + x_1^4 + x_2^4 = 0\} \subset \mathbb{C}P^2.$$

A general element of the pencil generated by S_0 and S_∞ is the intersection of X with the plane $\{ax_3 + bx_4\} = 0 \subset \mathbb{C}P^4$. The base point free pencil induced in Z can be described outside the exceptional divisor as the map:

$$\begin{aligned} f : Z \setminus E &\longrightarrow \mathbb{C}P^1 \\ [x_0 : x_1 : x_2 : x_3 : x_4] &\longmapsto [x_3 : x_4]. \end{aligned}$$

The fibres in Z are isomorphic to their images in X and thus we can restrict our search for singularities to the complement of the exceptional divisor, i.e. we can work in the original quartic X. A point $x = [x_0 : x_1 : x_2 : x_3 : x_4]$ on $X \setminus C$ will be singular for a hyperplane section exactly when $\mathrm{D}P(x) \in \mathrm{span}\{\mathrm{d}x_3, \mathrm{d}x_4\}$. Thus the singular points can be described as the subvariety $S \subset \mathbb{C}P^4$ defined by the set of equations:

$$\begin{cases} P = 0, \\ \partial_0 P = 0, \\ \partial_1 P = 0, \\ \partial_2 P = 0 \end{cases} \Leftrightarrow \begin{cases} x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_3^3(x_0 + 10x_1 + 100x_2) = 0, \\ 4x_0^3 + x_3^3 = 0, \\ 4x_1^3 + 10x_3^3 = 0, \\ 4x_2^3 + 100x_3^3 = 0. \end{cases}$$

By Bézout's theorem, the algebraic count of solutions to this system of equations (meaning that we count points with their scheme theoretic multiplicity) is the product of the degrees of the equations, hence $3^3 \cdot 4$. In this specific case, the full number of solutions is attained, hence all of them have multiplicity one. To see this, note first that any non-zero solution must have $x_3 \neq 0$. So we are free to set $x_3 = 1$, and solve the three equations $x_i^3 = c_i x_3^3$ ($c_i \in \mathbb{C} \setminus 0$) first. We thus get 3^3 distinct possibilities for the tuple (x_0, x_1, x_2). Now for each such choice we can solve the first equation for x_4 in exactly four different ways, as it reduces to an equation of the form $x_4^4 = c(x_0, x_1, x_2)$ where $c \neq 0$.

Notice also that no two solutions lie in the same hyperplane section, as they all have different values of $[x_3 : x_4]$. Indeed once x_3 is chosen, this determines x_0, x_1, x_2 up to a choice of a third root of unity. Now $x_0 + 10x_1 + 100x_2$ can never take identical values for x_0, x_1, x_2 differing only by a multiple of a root of unity. This explains the slightly odd choice of $x_0 + 10x_1 + 100x_2$ instead of something more symmetric like $x_0 + x_1 + x_2$ for instance. In the latter case permuting x_0, x_1, x_2 while keeping x_3, x_4 the same maps the singular set onto itself, and thus multiple singularities appear on one fibre.

Now as mentioned above, all points of S have multiplicity 1. That means that if $[x_0 : x_1 : x_2 : x_3 : x_4] \in S$ we have that $\dim \mathcal{O}_{S,p} = 1$, where $\mathcal{O}_{S,p}$ denotes the local ring of S at p. Now fix a singular point p of $X \cap \Pi$, where Π is a hyperplane in $\mathbb{C}P^4$. By choosing affine coordinates around the singular point $p \in \Pi$ we may assume that our singular fibre is given by $f^{-1}(0)$ for a polynomial map $f : \mathbb{C}^3 \to \mathbb{C}$ which additionally satisfies $f(0) = 0$ ($0_{\mathbb{C}^3}$ corresponds to $p \in S \cap \Pi$) and $\mathrm{D}f(0) = 0$ (thus every term in f is at least of second order). In this picture we see that:

$$\mathcal{O}_{S,p} = \frac{\mathbb{C}[x_0, x_1, x_2]}{(\partial_0 f, \partial_1 f, \partial_2 f)}.$$

We now claim that if the dimension of this local ring is 1, then we can choose coordinates such that $f = x_0^2 + x_1^2 + x_2^2 + O(x^3)$. In particular it suffices to show that if the quadratic terms of f do not form a nondegenerate quadratic form, then $\dim \mathcal{O}_{S,p} > 1$. Suppose that this is the case, so that after a linear change of coordinates we can assume that x_0^2 does not appear as a term in f. Then we clearly have $\partial_0 f = cx_0^2 + O(x_0^3, x_1, x_2)$, and similarly $\partial_1 f$ and $\partial_2 f$ do not contain a linear term proportional to x_0. Thus 1 and x_0 are non-zero and linearly independent elements of $\mathcal{O}_{S,p}$, and thus $\dim \mathcal{O}_{S,p} > 1$.

So in particular we have proven that all the singularities that appear in this example of a building block are isomorphic to the quadratic cone singularity $x^2 + y^2 + z^2 = 0$ in $\mathbb{C}^3$, as all the singularities have multiplicity one. This can alternatively also be checked explicitly by looking at the defining equations of S in more detail.

The property of only having Morse type singularities, all in separate fibres, is Zariski open, i.e. it is true for X in an open subset of its deformation type and for S_0, S_∞ in an open subset of the corresponding linear anticanonical system. Since this moduli space is irreducible as a complex variety, it is generically true for Fano threefolds arising from quartics in $\mathbb{C}P^4$. Thus we showed the same is true for a dense Zariski open subset of $\mathcal{F}^{N,A}$, where $A = -K_X \in N = H^2(S_0)$.

Twisted connected sum construction of G_2-manifolds

Now we have established the basic properties of building blocks, which by Theorem 5.5 can be used to construct ACyl Calabi–Yau threefolds. Starting from a building block $f_Z : Z \to \mathbb{C}P^1$ with chosen K3 fibre $K = f_Z^{-1}(\infty)$ (seen as a complex manifold), we can choose a hyperkähler structure $(\omega_\infty, g_\infty, \Omega_\infty)$ compatible with (K, I), under the condition that $[\omega_\infty] \in H^{1,1}(K)$ is the restriction of a Kähler class on the ambient Z. This is an open condition, but may be non-trivial. Using Theorem 5.5 we thus get an ACyl CY structure on $X = Z \setminus K$.

By taking the product with S^1 we get an asymptotically cylindrical G_2-manifold $M = X \times S^1$ with associative form φ, defined by $\varphi = \mathrm{dt} \wedge \omega + \mathrm{Re}\,\Omega$ as in Example 1.6. As $Z \setminus K$ and X are biholomorphic, we see that X is also fibred by generically smooth K3 surfaces via the same map $f_X = f_Z|_{Z\setminus S}\colon Z \setminus K \simeq X \to \mathbb{C}P^1 \setminus \{\infty\}$. On M, this induces a corresponding fibration $f : M \to S^1 \times (\mathbb{C}P^1 \setminus \infty)$ by coassociative submanifolds.

On the cylindrical end of M, the fibration is diffeomorphic to the projection map

$$\pi : \mathbb{R}_{>0} \times S^1 \times S^1 \times K \to \mathbb{R}_{>0} \times S^1 \times S^1.$$

By the ACyl_λ-condition (with $\lambda < 0$) the metric on the link converges exponentially to $g_{S^1} \times g_{S^1} \times g_\infty$.

The key idea of the twisted connected sum construction is to take two cylindrical G_2-manifolds M_+, M_- with isometric asymptotic hyperkähler K3s K_+, K_- and glue them together by a diffeomorphism for $T > 0$:

$$\begin{aligned} G : (T, T+1) \times S^1 \times S^1 \times K_+ &\longrightarrow (T, T+1) \times S^1 \times S^1 \times K_- \\ (t, \theta_a, \theta_b, p) &\longmapsto (2T + 1 - t, \theta_b, \theta_a, \mathfrak{r}(p)), \end{aligned} \tag{5.2}$$

where $\mathfrak{r} : K_+ \to K_-$ is a suitably chosen isometry. We exchange the two circles with the gluing diffeomorphism so that the fundamental group of the glued manifold becomes finite, and thus the holonomy will be exactly G_2 by a result of Joyce [15, Prop. 10.2.2]. In terms of the hyperkähler structure on $(K_\pm, \omega^1_\pm, \omega^2_\pm, \omega^3_\pm, I_\pm, g_\pm)$, where $\Omega_\pm = \omega^2_\pm + i\omega^3_\pm$ is the complex volume form, the asymptotic associative form can be written as:

$$\begin{aligned} \varphi_{\infty,\pm} &= \mathrm{d}\theta_a \wedge \omega_{\mathbb{C}\times K_\pm} + \mathrm{Re}\,\Omega_{\mathbb{C}\times K_\pm} \\ &= \mathrm{d}\theta_a \wedge (\mathrm{dt} \wedge \mathrm{d}\theta_b + \omega^1_\pm) + \mathrm{Re}(\mathrm{d}\theta_b - i\mathrm{dt}) \wedge (\omega^2_\pm + i\omega^3_\pm) \\ &= \mathrm{d}\theta_a \wedge \mathrm{dt} \wedge \mathrm{d}\theta_b + \mathrm{d}\theta_a \wedge \omega^1_\pm + \mathrm{d}\theta_b \wedge \omega^2_\pm + \mathrm{dt} \wedge \omega^3_\pm. \end{aligned}$$

In particular to ensure that $\varphi_{\infty,\pm}$ match up on the overlap, we need:

$$\mathfrak{r}^*\omega^1_- = \omega^2_+, \quad \mathfrak{r}^*\omega^2_- = \omega^2_+, \quad \mathfrak{r}^*\omega^3_- = -\omega^3_+,$$

which is equivalent to asking that $\mathfrak{r}$ is a hyperkähler rotation between $K_\pm$ as in Equation (1.7). Let the parametrisations of the ends of $M_\pm$ as cylinders be denoted by $\Psi_\pm : \mathbb{R}_{>0} \times S^1 \times S^1 \times K_\pm \hookrightarrow M$. For $T > 1$ we consider the following truncated manifolds:

$$M_{T,\pm} = M_\pm \setminus \Psi_\pm([T+1,\infty) \times S^1 \times S^1 \times K_\pm).$$

Over the cylindrical end $\mathbb{R}_{>0} \times L \simeq \mathbb{R}_{>0} \times S^1 \times S^1 \times K_\pm$ we have both the G_2-structure $\varphi_\pm$ induced from $M_\pm$ as well as the product G_2-structure $\varphi_{\infty,\pm}$. Define a smooth cut-off function $f_{\text{cut}} : \mathbb{R} \times [0,1]$ such that $f_{\text{cut}}|_{(-\infty,0]} = 0$ and $f_{\text{cut}}|_{[1,\infty)} = 1$. We can now define a (non torsion free) G_2-structure interpolating between the two on $M_{T,\pm}$ by declaring it equal to $\varphi_\pm$ away from the cylindrical end, and on the end by the formula:

$$\varphi_{T,\pm}(t,p) = f_{\text{cut}}(t-T)\varphi_\pm + (1 - f_{\text{cut}}(t-T))\varphi_{\infty,\pm}.$$

For $T \gg 1$ we will have $|\varphi_\pm - \varphi_{\infty,\pm}|_{(T,T+1)\times L} = O(e^{\lambda T})$ small. Hence $\varphi_{T,\pm}(t,p)$ will be a small perturbation of $\varphi_{\infty,\pm}$, and thus again a associative form. Notice that $\varphi_{T,\pm}$ is exactly equal to $\varphi_{\infty,\pm}$ on $(T+1,T+2)\times L$ and torsion-free everywhere except over the interpolation region $(T,T+1)\times L$, where $|\nabla \varphi_{T,\pm}| \in O(e^{\lambda T})$. Thus, after choosing a hyperkähler rotation matching up $M_\pm$ we can glue $M_{T,\pm}$ over the regions $(T+1,T+2)\times L \subset M_{T,\pm}$ using the gluing map G from (5.2) to obtain a G_2-manifold $(M_T, \varphi_{T,\mathfrak{r}})$. This can be perturbed to a torsion-free G_2-manifold.

Theorem 5.9 (Theorem 3.12 in [9]). *Let $(X_\pm, J_\pm, \omega_\pm, g_\pm, \Omega_\pm)$ be two asymptotically cylindrical Calabi–Yau 3-folds whose asymptotic ends are of the form $\mathbb{R}_{>0} \times S^1 \times K_\pm$ for a pair of hyperkähler K3 surfaces $K_\pm$, and suppose there exists a hyperkähler rotation $\mathfrak{r} : K_+ \to K_-$. Define closed G_2–structures $\varphi_{T,\mathfrak{r}}$ on the twisted connected sum $M_\mathfrak{r}$ as above. For sufficiently large T there is a torsion-free perturbation of $\varphi_{T,\mathfrak{r}}$ within its cohomology class.*

It can be shown that this perturbation will become arbitrarily small as T increases. The most difficult aspect of the gluing construction is certainly finding pieces with compatible Calabi–Yau cylindrical ends. This we call the **matching problem**. The asymptotically cylindrical Calabi–Yau threefolds we consider come from building blocks, which in turn come from (semi-)Fano threefolds with a choice of anticanonical K3 divisors.

We now give an outline of the matching procedure from [9]. Consider the deformation types of two Fano manifolds $Y_\pm$ which are polarised by the lattices $N_+ \subset \Lambda$ and $N_- \subset \Lambda$ respectively. Assume that $N_\pm$ has signature $(1, r_\pm)$. Recall the forgetful morphisms $s^{N_\pm} : \mathcal{F}^{N_\pm} \to \mathcal{K}^{N_\pm}$ which takes pairs $(\tilde{Y}_\pm, S)$ of Fano threefolds in the deformation type of $Y_\pm$ and anticanonical K3 divisors $S \subset \tilde{Y}_\pm$ to the polarised K3 moduli space $S \in \mathcal{K}^{N_\pm}$. We know from Proposition 1.5 that this morphism is dominant on each irreducible component of $\mathcal{F}^{N_\pm}$. This gives us a first restriction on K3 surfaces which can appear as asymptotic links for the gluing problem, as they must lie in open dense subsets $U_\pm \subset D_{N_\pm}$, which are determined by $s^{N_\pm}$ and a reference marking.

The next step is to consider the hyperkähler structure. To make the discussion simpler, we assume that the lattices $N_\pm$ have trivial intersection and are orthogonal to one another. This way we can avoid introducing the construction of an *orthogonal pushout* of two lattices and also have more concise notation. Define $T = (N_+ \oplus N_-)^\perp$. Consider the following subset of the hyperkähler K3 domain:

$$D = D^{\text{hk}}_{K3} \cap ((N_+ \otimes \mathbb{R})^+ \times (N_- \otimes \mathbb{R})^+ \times (T \otimes \mathbb{R})).$$

Here L^+ denotes the positive cone of a lattice L. The submanifold D is real 20-dimensional. Now as the period domain of $N_\pm$-polarised K3 surfaces can be identified with positive two-planes in $N_\pm^\perp \otimes \mathbb{R}$ as in Equation (1.3), there are two natural projection maps $\pi_\pm : D \to D_{N_\pm}$ given by:

$$\pi_\pm(\omega_+, \omega_-, \omega_0) = \text{span}\langle \omega_\mp, \pm\omega_0 \rangle \in D_{N_\pm}.$$

Recall from Equation (1.7) that the hyperkähler rotation of the triple $(\omega_+, \omega_-, \omega_0) \in D^{\text{hk}}_{K3}$ is exactly $(\omega_-, \omega_+, -\omega_0)$. Hence π_- is just the mapping to the complex structure of the hyperkähler rotated K3. So in particular candidates for asymptotic hyperkähler K3 surfaces must be contained in the subset $\pi_+^{-1}(U_\pm) \cap \pi_-^{-1}(U_-)$. It turns out that the image of $\pi_\pm$ in $D_{N_\pm}$ is a real $(20 - r_\pm)$-dimensional submanifold in a real $2(20 - r_\pm)$-dimensional space. So it is not clear a priori that $\operatorname{im} \pi_\pm$ and $U_\pm$ even intersect.

However one can show that $\operatorname{im} \pi_\pm$ is an embedded totally real submanifold. Since it is also of maximal dimension it must intersect any open Zariski dense subset, such as $U_\pm$.

Finally, we also need to take into account not only the complex geometry of the two K3 surfaces, but their Kähler geometry as well. Indeed if the hyperkähler structure is given by the triple $(\omega_+, \omega_-, \omega_0)$, then the complex geometry is determined by (ω_-, ω_0) (via π_+ as above), while the Kähler class will be ω_+, and similarly for the hyperkähler rotation. So if $\text{Amp}_\pm$ are the ample cones of the polarised K3 surfaces we need that the set:

$$\mathcal{A} = \{(\omega_+, \omega_-, \omega_0) : \omega_\pm \in \text{Amp}_\mp\}$$

is non-empty. In good cases, this can be shown to be a (Euclidean) open subset of D, see [8, Prop.6.9]. Thus, since $\pi_+^{-1}(U_\pm) \cap \pi_-^{-1}(U_-)$ is open dense, there must be a hyperkähler structure satisfying all the conditions, and thus the matching is possible. We note at this point that imposing a finite number of open dense conditions on either complex K3 surface does not impact the matching procedure.

Example 5.10. Consider the twisted connected sums of two building blocks in the deformation type of Example 5.8. In this case the matching is possible by [9, Prop. 6.18] (the matching is what they call *perpendicular*; in these cases the condition on the Kähler classes are automatically satisfied) and we can see from Table 5 in [9] that the resulting G_2-manifold will have $b^2 = 0$ and $b^3 = 155$. The example is also discussed in more detail as Example 1 in Section 7 of [9].

Coassociative fibrations on connected sums

We now turn our attention to the fibrations which naturally arise from the twisted connected sum construction of G_2-manifolds.

Proposition 5.11 (Prop. 2.18 in [24])**.** *The fibrations $f_\pm : M_\pm \to S^1 \times \mathbb{C}P^1 \setminus \infty$ join together to form a fibration $f_T : M_\mathfrak{r} \to S^3$.*

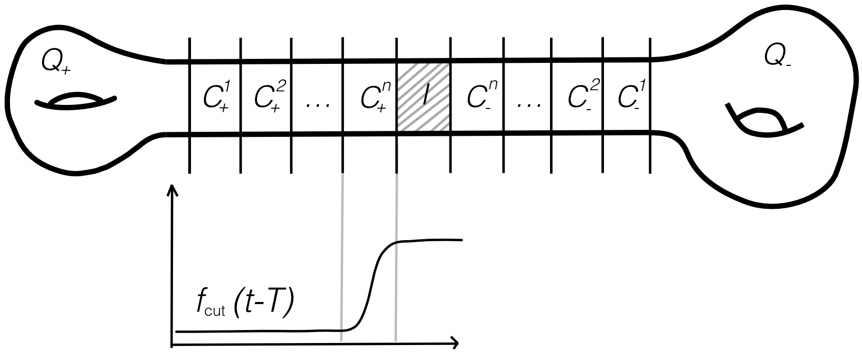

Figure 5.1: Decomposition of the twisted connected sum into $2n+3$ pieces.

Proof. Gluing $M_{T,\pm}$ identifies the K3 fibres of the two fibrations by construction. On the level of the base space, this reduces to gluing two solid tori $S^1 \times D^2$ (where D^2 is the two dimensional disk with boundary) along their boundaries via the map

$$S^1 \times S^1 \longrightarrow S^1 \times S^1, \quad (a,b) \longmapsto (b,a).$$

This gluing is diffeomorphic to S^3, and the decomposition into tori is in fact a Heegard splitting of S^3. Consider now a fixed K3 surface on the overlap $f_\pm^{-1}(t, \theta_a, \theta_b)$. It is coassociative with respect to $\varphi_\pm$ by construction and we easily see that it also is coassociative with respect to $\varphi_{\infty,\pm}$. Thus it remains coassociative for any linear combination $c\varphi_\pm + (1-c)\varphi_{\infty,\pm}$ with $0 \leqslant c \leqslant 1$. □

Let us now apply the twisted connected sum construction to building blocks with the additional property that the fibres of the map $f : Z \to \mathbb{C}P^1$ are either nonsingular K3 surfaces or have conical singularities modelled on the cone $C_q = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3$. A possible building block arises from a quartic in $\mathbb{C}P^4$ as explained in Example 5.8.

Ensuring the matching up of two building blocks $(Z_\pm, f_\pm)$ is an involved procedure, as we already mentioned above. But since the additional condition we impose is Zariski open dense on the moduli space $\mathcal{K}^{N,A}$, the matching goes through as without change. Thus we can find two matching building blocks so that the glued fibrations f_T also have the same kind of complex conical singularities. On the tubular intermediate region all the fibres will be smooth K3s.

The upshot is that we are given a smooth twisted connected sum G_2-manifold M^7 and for any $T \gg 1$ sufficiently large a G_2-structure φ_T such that $\|\nabla\varphi_T\|_{L^p_k} \leqslant e^{\lambda T}$ with $\lambda < 0$. These come with coassociative fibrations by (possibly singular) K3 surfaces, which over either end are products of complex fibrations of Calabi–Yau threefolds with a circle S^1. In the gluing region there is no complex structure for which this is true, however this region has finite volume, and converges to a piece of a G_2 cylinder.

If we then take the product with S^1 once more we obtain a Spin(7)-manifold $(X = M \times S^1, \Phi_T = \mathrm{dt} \wedge \varphi + \star\varphi)$, which admits a torsion-free deformation $\tilde{\Phi}_T$, also of product type, from Theorem 5.9. The deformations required to achieve torsion freeness become smaller as T increases, in the sense that $\|\Phi_T - \tilde{\Phi}_T\|_{L^p_k} = O(e^{\lambda T})$ where $\lambda < 0$.

Now geometrically either end of X can by construction be considered a Calabi–Yau fourfold with a fibration by complex surfaces, and the gluing region is converging towards a finite volume piece of a Spin(7)-cylinder. In particular we can use Propositions 5.3 and 5.4 to see that away from the gluing region the fibres, both compact and singular with either one or singularities, are unobstructed as Cayleys. Regarding the gluing region we note that unobstructedness is an open condition in the choice of Spin(7)-structure. Thus while we cannot apply Proposition 5.3 directly, we see that it applies to the limiting cylindrical Spin(7)-structure. Hence the fibres in the gluing region will be unobstructed for all $T \gg 1$.

Now we are in a position to prove the stability of the fibration as we pass from the Spin(7)-structure Φ_T to the torsion-free Spin(7)-structure $\tilde{\Phi}_T$. For this, we imagine cutting up (M_n, Φ_n) (for $n \in \mathbb{N}$), a manifold of diameter approximately $2n$ into $2n+3$ pieces. These pieces are first of all the two compact pieces $Q_\pm \subset X_\pm$. Next we have for either side the n pieces $C_{k,\pm} = \varphi_\pm^{-1}((k - \frac{1}{2}, k + 1\frac{1}{2}) \times L) \times S^1$ for $0 \leqslant k \leqslant n$. Finally we have the glued piece $I = \varphi_\pm^{-1}((n + \frac{1}{2}, n + 2\frac{1}{2}) \times L) \times S^1$. Notice that both $Q_\pm$ as well as the $C_{k,\pm}$ for $0 \leqslant k \leqslant n-1$ when seen as G_2-manifolds remain constant as n increases. The two pieces $C_{n,\pm}$ are where the interpolation between the ACyl_λ structure $\varphi_\pm$ and the exactly cylindrical G_2-structure $\varphi_{\infty,\pm}$ happens. Finally I is exactly cylindrical, independent of n.

Now, since we checked unobstructedness of all the fibres in the fibration we may apply Theorem 4.16 to each piece separately, as long as we can ensure nondegeneracy of the fibration. This is clearly satisfied for any piece without a singular fibre. The finitely many pieces with singular fibres can be considered as Calabi–Yau fourfolds with fibrations by complex surfaces and Morse-type singularities. From this we can conclude non-degeneracy, since we know the local model near the singular point.

Thus for each piece there is a maximal $s_{\max} \in (0,1]$ such that for each $0 \leqslant s < s_{\max}$ the fibration property of the fibres in just that piece is preserved for $\Phi_{n,s} = \Phi_n + s(\tilde{\Phi}_n - \Phi_n)$. Now, since the preglued Spin(7) structure on each piece is eventually constant, we see that as n increases, $s_{\max}$ for that piece increases and reaches 1 eventually. This is because $\|\tilde{\Phi}_n - \Phi_n\|_{L^p_k} \leqslant e^{\lambda n}$ with $\lambda < 0$. Eventually $\tilde{\Phi}_n$ will lie in the open neighbourhood about Φ_n for which stability of the fibration is given. Like this we see that for any choice of finitely many pieces, we can ensure the stability of the fibration on the union of these pieces for any sufficiently large T.

On the other hand we have $(C_{k,\pm}, \varphi|_{C_{k,\pm}}) \to (I, \varphi|_I)$ in C^∞. In fact if we consider the path of Spin(7)-structures $\gamma(T) = (-T + \frac{1}{2})^* \Phi_\pm|_{[T-\frac{1}{2}, T+\frac{3}{2}] \times L}$ (where $(-T + \frac{1}{2})^*$ is the pullback by translation) on $[0,2] \times L$, then the ACyl condition on $M_\pm$ gives us that $\|\gamma(T) - \Phi_\infty\|_{L^p_k} \leqslant e^{\lambda T}$, for $\lambda < 0$. Stability of the fibration is true in a quantitative sense, meaning that there is a ball $B(\Phi_\infty, \epsilon)$ around Φ_∞ where the moduli space of Cayleys for the given Spin(7)-structure is still fibering. Thus there will be a ball of radius $\epsilon - e^{\frac{\lambda}{2}T}$ around $\gamma(T)$ so that the same is true. But now, since the distance $\|\tilde{\Phi}_T - \Phi_T\|_{L^p_k} \leqslant e^{\lambda T} < 1 - e^{\lambda T}$ for T sufficiently large, the torsion-free Spin(7)-structure will stay within a ball of fibering Spin(7)-structures around $(C_{k,\pm}, \varphi_{k,\pm})$. In this way we can thus prove stability of the fibration for all pieces with index above a minimal $n_{\min}$. The previous argument then takes care of the finitely many remaining pieces. Thus we have proven the following:

Theorem 5.12 (Existence of strong Kovalev-Lefschetz fibrations on Spin(7)-manifolds). *There are compact, torsion-free* Spin(7)*-manifolds of holonomy* G_2 *which admit fibrations by generically smooth Cayley submanifolds. The singular Cayley submanifolds may have at most two conical singularities.*

From this we can deduce that the stability result also holds for the initial G_2-manifold, using the following auxiliary result.

Lemma 5.13. *Let* (M, g, τ) *be a manifold together with a calibration* τ*. Assume that* $\tau = \psi + \rho$*, where* ψ *is another calibration and* ρ *is a closed form. Let* $N \subset M$ *be* τ*-calibrated submanifold such that* $\int_N \rho = 0$*. Then* N *is* ψ*-calibrated.*

Proof. We have that by assumption $\mathrm{dvol}_N = \tau|_N$. Now note that:

$$\int_N \mathrm{dvol}_N = \int_N \tau|_N = \int_N \psi|_N + \rho|_N = \int_N \psi|_N \leqslant \int_N \mathrm{dvol}_N \,.$$

If there is a point with $\psi|_N(p) < \mathrm{dvol}_N(p)$, then we must have $\int_N \psi|_N < \int_N \mathrm{dvol}_N$, a contradiction. Thus $\psi|_N = \mathrm{dvol}_N$ and N is ψ-calibrated. □

Now we set $\tau = \Phi$, $\psi = \star\varphi$ and $\rho = \mathrm{d}s \wedge \varphi$ in the previous proposition, where $s \in S^1$ is a coordinate on the circle in $X = M \times S^1$. As a K3 fibre N of the initial fibration is contained in $M \times \{s\}$ for a single point $s \in S^1$, we clearly have $\int_N \mathrm{d}s \wedge \varphi = 0$. Next, as the perturbed Cayleys are continuous deformations of the initial (possibly conically singular) Cayleys and the new G_2-structure $\tilde{\varphi}$ is cohomologous to φ, we still have $\int_{\tilde{N}} \mathrm{d}s \wedge \tilde{\varphi} = 0$ by Stokes' Theorem. Hence the Cayleys for $\tilde{\Phi}$ are also calibrated by $\star\varphi$, meaning that their tangent planes are contained in $M \times \{s\}$ and they are in fact coassociative. Thus we have shown:

Corollary 5.14 (Existence of coassociative fibrations). *There are compact, torsion-free* G_2*-manifolds of full holonomy which admit fibrations by coassociative submanifolds.*

Example 5.15. Consider the G_2-manifold obtained by gluing two copies of the quartic building block from Example 5.8, as in Example 5.10. This G_2-manifold has Betti numbers $b_2 = 0$ and $b^3 = 155$. Furthermore, as the the conical singularities are stable and no fibre has more than one singular point, the resulting coassociative fibration will have $2 \cdot 3^3 \cdot 4 = 216$ connected components of singular coassociatives.

5.4 Full holonomy and further work

Even though we proved Theorem 4.3 in the Spin(7) setting, we concluded with the example of a fibration of a G_2-manifold by coassociative submanifolds in Example 5.15. This then induces an example of a non trivial fibration of a compact Spin(7)-manifold, which is however of product type. Thus it is in particular not an example of full holonomy and as such not using Theorem 4.3 to its fullest potential. This is due to a lack of known examples of fibrations on pre-glued Spin(7)-manifolds. We suggest that future work should both search for more examples of fibrations and try to widen the scope of Theorem 4.3, by for instance allowing more general kinds of singularities.

More concretely, we expect that similarly well-behaved fibrations in the G_2 setting can be constructed from many more semi-Fano threefolds, thanks to a discussion with Mark Haskins. This is because the anticanonical system is generally quite large (e.g. it has complex dimension 4 in the case of the quartic in $\mathbb{C}P^4$), and thus it should be possible to avoid problematic singularities by choosing a suitably generic pencil and invoking a Bertini-type theorem.

Slightly more speculatively one might find examples of suitable fibrations on the second construction of Spin(7)-manifolds, due to Joyce [14], whose starting point are Calabi–Yau orbifolds. In his thesis, Clancy [7, Section 7.4.4] gives an example of how this can be done, based on the twistor fibration $f : \mathbb{C}P^3 \to S^4$. Unfortunately, using his method directly, bad singularities like $(\star)$ will appear in codimension 4. However, it may be possible that a modified version of the construction could avoid non conical singularities. Alternatively, the stability Theorem 4.3 could potentially be extended to include non conical singularities. This requires us to develop the deformation theory of Cayley submanifolds with more complicated singularities, such as $(\star)$, and more advanced analytical tools than are currently available.

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