Moduli spaces of complexes of coherent sheaves

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Dedication

This thesis is dedicated to all of those who have given me the happy life that I have, especially my mother Suzanne Wolfenson. This includes my loving sister Danielle Gross and family members Annette Wolfenson, Ariel Wolfenson-Bannon, and Gabrielle Wolfenson-Bannon and step-family members Alex Ward, Gerry Ward, and Nick Ward who helped make the childhood that I remember fondly. To my father Claude Gross and my grandparents Karen Bycer-Wolfenson, Milton Gross, and Sarah Suzanne Gross: I wish you were here to read this.

I appreciate the support of my lifelong friends: Benjamin Butz-Weidener, Kai Feinberg, Eli Laban, Jon Loeb, Alistair McDonald, Kobi Oxman, Justin Plasmeier, and Sorrel Thompson. And the friends I have met along the road: Jess Cerra, Jon Doctorick, Joe Dutton, Shahnaz Hoque, Alexander Oldenziel, Anya Perse, Alan Sweitzer, and Kevin Young.

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Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or qualification at the University of Oxford or at any other university or institution.

Section 4.4 is based on joint work with Yalong Cao and Dominic Joyce. Sections 5.3 and 5.4 are based on joint work with Markus Upmeier.

Abstract

In this thesis we consider problems related to Joyce's vertex algebra construction and the topology of stabilized moduli spaces. We first compute the homology of the moduli stack of objects in the derived category of a smooth complex projective variety X in class D, showing that the rational cohomology ring is freely generated by tautological classes. This is used to identify Joyce's construction with a generalized super-lattice vertex algebra on the rational K-theory of X^{an}. Then, we prove orientability of moduli spaces of (complexes of) coherent sheaves on projective Calabi–Yau 4-folds–this has applications to defining a $\mathbb{C}$-linear enumerative invariant theory for Calabi–Yau 4-folds. This result is based on joint work with Yalong Cao and Dominic Joyce. Lastly, we consider a connective even complex-oriented homology theory E with associated formal group law F and show that, given a Künneth isomorphism, replacing ordinary homology with E in Joyce's construction yields a vertex F-algebra–this result is based on joint work with Markus Upmeier.

Contents

Chapter 1

Introduction

A vertex algebra is a complicated algebraic structure that originates from conformal field theory. The mathematical definition was first given by Borcherds and used in his proof of the monstrous moonshine conjectures [36]. Vertex algebras can be thought of as either

a. infinitary Lie algebras, having Lie brackets $(-)_n(-)$ for each $n \in \mathbb{Z}$ which satisfy infinitely many identities that resemble anti-symmetry and the Jacobi identity, or

b. commutative rings with derivations (or shift operators) whose products are permitted to have singularities (see [37]).

In [112] Joyce constructs a graded vertex algebra structure on the homology of moduli stacks $\mathcal{M}$ of objects in certain dg-categories $\mathcal{A}$. Examples of dg-categories for which Joyce's construction produces a graded vertex algebra include $\mathrm{Coh}(X), \mathrm{Rep}(Q)$, $\mathrm{Perf}(X)$, and $D^b\mathrm{Rep}(Q)$ where X is a smooth projective variety and Q is a finite quiver. One motivation for this theory is to prove wall-crossing formulas for $\mathbb{C}$-linear[1] enumerative invariant theories such as Donaldson–Thomas type invariants of

[1] An enumerative invariant theory 'counts' subspaces of semistable objects of a fixed topological type $\mathcal{M}_\alpha^{\mathrm{ss}} \subset \mathcal{M}$ in a moduli space $\mathcal{M}$ by integrating a virtual class $[\mathcal{M}_\alpha^{\mathrm{st}}]^{\mathrm{vir}}$ against tautological

Calabi–Yau 4-folds (see Borisov–Joyce [34] and Cao–Leung [47]), Donaldson–Thomas invariants of Fano 3-folds (see Thomas [212]), and Donaldson invariants of algebraic surfaces (see Mochizuki [172]). Note that these invariants are non-motivic.

It is already known that the wall-crossing formula for generalized Donaldson–Thomas invariants of Calabi–Yau 3-folds can be expressed solely in terms of the Lie bracket of a Ringel–Hall algebra of stack functions [114, Thm. 3.14]. However, the formalism of Ringel–Hall algebras of stack functions is only useful for motivic invariants.

Vertex algebras are another source of Lie algebras. There is a series of conjectures to the effect that the wall-crossing formula for general motivic invariants in [110, p. 31]–written entirely in terms of the Lie bracket induced by the motivic Ringel–Hall algebra–is a universal wall-crossing formula in the sense that for general $\mathbb{C}$-linear enumerative invariant theories counting objects in a $\mathbb{C}$-linear category $\mathcal{A}$ it is possible to define strictly semistable counting invariants such that the formula [110, p. 31] holds for the Lie bracket induced by the vertex algebra structure on the homology of the moduli stack of objects in $\mathcal{A}$ (see Gross–Joyce–Tanaka [93] for more precise conjectures and a proof in the case of $\mathcal{A} = \mathrm{Rep}(Q)$ for Q a quiver without oriented cycles). This is one motivation for studying Joyce's vertex algebras.

The present document contains the following results:

- We consider Joyce's construction in the case of the rational Betti homology of the moduli stack of objects in the derived category of coherent sheaves on a smooth complex projective variety X. In certain cases, we are able to identify Joyce's construction with an explicit vertex algebra: a generalized super-lattice vertex algebra on the rational complex K-theory of X^{an}.

cohomology classes or 'insertions.' An enumerative theory is said to be $\mathbb{C}$-linear if $\mathcal{M}$ is the moduli space of objects in a $\mathbb{C}$-linear additive category $\mathcal{A}$. Non-examples of $\mathbb{C}$-linear enumerative invariant theories are Gromov–Witten invariants of symplectic manifolds (including gauged and equivariant Gromov–Witten invariants).

- Techniques used to compute the homology of "stabilized" moduli spaces (such as moduli spaces of complexes or virtual bundles) can also be used to prove orientability. Indeed, we are able to prove that the moduli space of coherent sheaves on a projective Calabi–Yau 4-fold is orientable. This has applications to defining Donaldson–Thomas type enumerative invariants of Calabi–Yau 4-folds. This result is based on joint with Yalong Cao and Dominic Joyce.

- Given a connective even complex-oriented homology theory E with associated formal group law F, we build a generalized graded vertex algebra called a *graded vertex F-algebra*, which depends on a choice of elliptic operator, on the E-homology of the moduli space of $U(n)$-connections or virtual $U(n)$-connections on a compact manifold.[2] This result is based on joint work with Markus Upmeier.

Chapter 2 begins with a discussion of graded Hopf algebras, which is the natural structure appearing on the rational homology of a connected or group-like H-space. This also includes the introduction of Sweedler's coalgebra notation and a bialgebraic cap product.

In Section 2.1, we introduce graded (nonlocal) vertex algebras. A graded nonlocal vertex algebra is a graded R-module V together with a graded *state-to-field correspondence* $Y(-,z) : V \to \mathrm{End}(V)((z))$, a *vacuum vector* $|0\rangle \in V_0$, and a graded *shift operator* $\mathcal{D}(w) : V \to V[[w]]$ such that $Y(a,z)|0\rangle_{|z=0} = a$, $\mathcal{D}(w) \circ \mathcal{D}(z) = \mathcal{D}(z+w)$, $\mathcal{D}(w) \circ \mathcal{D}(-w) = \mathrm{id}$, $\mathcal{D}(w)(|0\rangle) = |0\rangle$, $Y(\mathcal{D}(w)(a), z) = i_{z,w} Y(a, z+w)$, and for all $a, b, c \in V$ there exists $N \gg 0$ such that

$$(z+w)^N Y(Y(a,z)b,w)c = (z+w)^N i_{z,w} Y(a,z+w)Y(b,w)c. \tag{1.0.1}$$

[2] As this result is proved using an F-bicharacter construction, it is necessary that there is a Künneth isomorphism $E_*(X \times X) \cong E_*(X) \otimes E_*(X)$. This is true, for example, if $E_*(X)$ is a free or flat $E^*(\{\mathrm{pt}\})$-module [31, Thm. 4.2]. We doubt that this restrictive condition is actually necessary. But, we like the F-bicharacter construction because it provides an "explanation" for the otherwise mysterious existence of a vertex F-algebra structure.

Equation (1.0.1) may be interpreted as the statement that the fields $Y(Y(a,z)b,w)c$ and $i_{z,w}Y(a,z+w)Y(b,w)$ are both expansions of the same element of the R-module $V[[z,w]][z^{-1},w^{-1},(z+w)^{-1}]$. This is the next best thing to them being equal since they lie in different spaces. A graded nonlocal vertex algebra is said to be a *graded vertex algebra* if the symmetry condition $Y(a,z)b = (-1)^{ab}\mathcal{D}(z) \circ Y(b,-z)a$ holds.

If the fields $Y(a,z)$ are all holomophic then a graded (resp. nonlocal) vertex algebra structure on V is equivalent to the structure of a graded commutative (resp. non-commutative) bialgebra with a compatible shift operator.

We give two non-holomorpic examples of graded vertex algebras: a generalized[3] lattice vertex algebra associated to a finitely generated abelian group with a symmetric $\mathbb{Z}$-bilinear form and a generalized fermionic vertex algebra associated to a finitely generated abelian group with an anti-symmetric $\mathbb{Z}$-bilinear form.

Section 2.3 discusses a generalization of vertex algebras previously studied by both Borcherds [37] and Li [148]. Recall that a (1-dimensional commutative) formal group law is a bivariate formal power series $F(X,Y) \in R[[X,Y]]$ such that $F(X,0) = F(0,X) = X$, $F(X,Y) = F(Y,X)$, and $F(F(X,Y),Z) = F(X,F(Y,Z))$. There is a graded Hopf algebra $R[F]$ called the *formal group ring* of F associated to any formal group law. It can be shown that what we call F-shift operators[4] are equivalent to representations of $R[F]$. We define graded (nonlocal) vertex F-algebras in terms of F-shift operators.

Next, we review a method of Borcherds for constructing many examples of vertex algebras from simpler algebraic data: a bialgebra with a compatible derivation and a compatible bicharacter, although we prefer to perform the construction with a shift operator in place of a derivation. This construction also makes senses for F-shift

[3] These vertex algebras are generalized in the sense that we permit a finitely generated abelian group in place of a lattice. This is an extremely mild generalization because constructions of lattice vertex algebras typically begin by rationalizing the lattice.

[4] This term comes from Rota-style algebraic combinatorics (see Di Bucchianico–Loeb [56] or Roman–Rota [189] for further background reading on this subject).

operators. One further caveat is that our "F-bicharacters" $r : B \otimes_R B \to R((z))$ are not graded with respect to the natural grading on our bialgebras B. Instead, they are graded with respect to a grading $\hat{B}$ induced on B by an abelian monoid M together with a quadratic form $Q : M \to \mathbb{Z}$.

Theorem 1.0.1 (see Theorem 2.4.8)**.** *Let $(B, m, \Delta, \eta, 1_B)$ be a graded bialgebra, let M be an abelian monoid, and let $Q : M \to \mathbb{Z}$ be a quadratic form. Suppose that B decomposes as $B = \coprod_{\alpha \in M} B_\alpha$ and that the projection $B \to M$ is a monoid homomorphism. Let $\hat{B}$ denote the Q-shift of B (Definition 2.4.5). Let F be a formal group law over R with formal inverse $\iota(z) \in R[[z]]$ and $\mathcal{D}(z) : B \to B[[z]]$ an F-shift operator in the sense that $\mathcal{D}(z) \circ \mathcal{D}(w) = \mathcal{D}(F(z,w))$ and $\mathcal{D}(z) \circ \mathcal{D}(\iota(z)) = \mathrm{id}$. Let $r_z : \hat{B} \otimes_R \hat{B} \to R((z))$ be an (even) F-bicharacter in the sense that*

$$
\begin{aligned}
r_z(a \otimes 1) = r_z(1 \otimes a) &= \eta(a), \\
r_z(\mathcal{D}(w)(a) \otimes b) &= i_{z,w} r_{F(z,w)}(a \otimes b), \\
r_z(a \otimes \mathcal{D}(w)(b)) &= i_{z,w} r_{F(z,\iota(w))}(a \otimes b), \\
r_z(a \cdot b \otimes c) &= (-1)^{bc'} r_z(a \otimes c') \cdot r_z(b \otimes c''), \text{ and} \\
r_z(a \otimes b \cdot c) &= (-1)^{a''b} r_z(a' \otimes b) \cdot r_z(a'' \otimes c).
\end{aligned}
$$

Define $Y(-,z) : B \otimes B \to B((z))$ by the following composition

$$
\begin{tikzcd}
B \otimes B \arrow[d, "\Delta_{B\otimes B}"'] \arrow[ddr, "Y(-,z)"] & \\
B \otimes B \otimes B \otimes B \arrow[d, "(1\otimes 1\otimes r)\circ(1\otimes\tau\otimes 1)"'] & \\
B \otimes B((z)) \arrow[r, "m\circ(\mathcal{D}(z)\otimes 1)"'] & B((z)).
\end{tikzcd}
$$

Then $(B, \mathcal{D}(z), Y(-,z), \eta(1_R))$ is a nonlocal vertex F-algebra. If $r : \hat{B} \otimes \hat{B} \to R((z))$ is symmetric and graded of degree $Q(\alpha) + Q(\beta) - Q(\alpha+\beta)$, then $(\hat{B}, \mathcal{D}(z), Y(-,z), \eta(1_R))$ is a graded vertex F-algebra.

Chapter 3 contains the necessary background from algebraic topology and gauge theory. We begin by introducing spectra. There are many different definitions of spectra, all of which induce the same stable homotopy theory. The simplest model of spectra are *sequential spectra*. A sequential spectrum is a sequence of CW complexes $\{X_n\}$ together with *bonding maps* $\Sigma X_n \to X_{n+1}$. Spectra can be thought of as generalized topological spaces.

We are forced to introduce spectra for two reasons: First, all generalized cohomology groups can be represented by mapping spaces into spectra. For an infinite CW complex, the definition of complex topological K-theory as the Grothendieck group of complex vector bundles does not yield a generalized cohomology theory in the Eilenberg–Steenrod sense. Instead, one must define complex topological K-theory of an infinite CW complex directly via its representing spectrum. The moduli spaces that we will consider are infinite CW complexes. Second, (connective Ω-)spectra themselves are equivalent to a type of topological space called an infinite loop space; moduli spaces of complexes of coherent sheaves and of virtual connections have the structure of infinite loop spaces.

There is a notion of an orientation of a vector bundle V with respect to a spectrum E or an *E-Thom class of V*. Thom classes can be used to define "umkehr" or "wrong-way" maps, which are generalizations of fiber integration maps. There is in fact a general Riemann–Roch theorem in this context proved by Dold, which we recall in Section 3.2. If a spectrum E admits an orientation for the universal complex line bundle, then it is said to be *complex-oriented*. Complex-oriented spectra E have interesting properties such as

- a canonical[5] isomorphism

$$E^*(BU) \cong E^*(\mathrm{pt})[[c_1^E, c_2^E, \dots]],$$

[5]This isomorphism is canonical up to a choice of complex orientation.

the classes $c_k^E \in E^{2k}(BU)$ are called *universal Conner–Floyd Chern classes*, and

- for two complex line bundles $L, M \to X$

$$c_1^E(L \otimes M) = F(c_1^E(L), c_1^E(M))$$

  is a formal group law in $c_1^E(L)$ and $c_1^E(M)$–this is called the *formal group law associated to* E.

It can be shown that if E is complex-oriented then $E_*(BU(1))$ is isomorphic, as a graded Hopf algebra, to the formal group ring $R[F]$ of the formal group law F associated to E (Proposition 3.2.7). In particular, spaces with $BU(1)$ actions have F-shift operators acting on their E-homology.

Section 3.3 is concerned with rational homotopy theory. Rational homotopy types of nilpotent spaces are encoded by commutative differential graded $\mathbb{Q}$-algebras called *Sullivan algebras*. A homotopy type is said to be *formal* if the entire rational homotopy type is determined by its rational cohomology. Put differently, it admits a Sullivan model with vanishing differential. Examples of rationally formal spaces are compact Kähler manifolds, symmetric spaces, and H-spaces. Based on work of Brown–Szczarba, Haefliger, and Sullivan we are able to get rational homotopy-theoretic models for evaluation maps $X \times \mathrm{Map}_{C^0}(X, BU) \to BU$ for any rationally formal space X. This is used later since it implies that the rational cohomology ring of $\mathrm{Map}_{C^0}(X, BU)$ is freely generated by tautological classes.

Section 3.4 is a review of a framework from Joyce–Tanaka–Upmeier [115] for studying orientability problems in gauge theory. Given a compact manifold X we can regard the moduli space of $U(n)$-connections on X as a topological stack $\mathcal{B}$, which has a classifying space $\mathcal{B}^{\mathrm{cla}}$. Given rank r real vector bundles $E_0, E_1 \to X$ and an elliptic operator $D : C^\infty(E_0) \to C^\infty(E_1)$ there is a determinant line bundle $L^D \to \mathcal{B}^{\mathrm{cla}}$ whose fiber at a connection ∇ is the determinant line of the twisted elliptic operator D^∇.

The *orientation bundle* $O^D \to \mathcal{B}^{\mathrm{cla}}$ is the principal $\mathbb{Z}_2$-bundle of orientations along the fibers of L^D. An *orientation* of $\mathcal{B}^{\mathrm{cla}}$, with respect to D, is a trivialization of O^D.

Section 3.4 discusses two classes of manifolds which will be of interest to us: Calabi–Yau manifolds and Spin(7)-manifolds. These are manifolds with special holonomy which, in particular, are spin. If X is any 8-dimensional Riemannian spin manifold there is a positive Dirac operator $\not{D}_+ : C^\infty(\mathbb{S}_+) \to C^\infty(\mathbb{S}_-)$. We state the theorem that moduli spaces of unitary and special unitary connections on 8-dimensional Riemannian spin manifolds are orientable with respect to the positive Dirac operator (see Cao–Gross–Joyce [45, Thm. 1.11]). There is also some discussion of a well-known Hodge star operator that is unique to Calabi–Yau $4m$-folds.

Chapter 4 is dedicated to results about the topology of moduli spaces of complexes of coherent sheaves. Section 4.4 is based on joint work with Yalong Cao and Dominic Joyce.

In Section 4.1, we briefly review the theories of higher and derived stacks as well as shifted symplectic structures on derived stacks. This includes Borisov–Joyce's definition of orientations of $(2-4m)$-shifted symplectic derived stacks.[6]

Section 4.2 is about (co)homology theories of higher stacks. There are several notions of (co)homology for stacks such as sheaf cohomology, Borel–Moore homology, and algebraic K-theory. In Section 4.2, we explain the *Betti homology* of higher $\mathbb{C}$-stacks. This is just the ordinary cohomology of a topological space, called the *Betti realization* or *topological realization*, naturally associated to a higher $\mathbb{C}$-stack.

In Section 4.3 we consider a smooth complex projective variety X and the moduli stack $\mathcal{M}_{\mathrm{Perf}(X)}$ of objects in Perf(X). We combine work of Blanc [28] and of Antieu–Heller [8] to conclude that the Betti realization of $\mathcal{M}_{\mathrm{Perf}(X)}$ has the homotopy type of the semi-topological K-theory space $\Omega^\infty K^{\mathrm{sst}}(X)$ of X. From this, we are able to apply

[6]Derived moduli stacks of (complexes of) coherent sheaves on Calabi–Yau 4-folds are examples of -2-shifted symplectic derived stacks.

the Milnor–Moore theorem to compute the rational Betti homology of $\mathcal{M}_{\mathrm{Perf}(X)}$[7]–obtaining an isomorphism

$$H_*(\mathcal{M}_{\mathrm{Perf}(X)}, \mathbb{Q}) \cong \mathbb{Q}[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{SSym}_{\mathbb{Q}}[\bigoplus_{i>0} K^i_{\mathrm{sst}}(X)], \tag{1.0.2}$$

of graded Hopf algebras, where $\mathbb{Q}[-]$ denotes the group algebra over $\mathbb{Q}$ and $\mathrm{SSym}_{\mathbb{Q}}[-]$ denotes the free super-symmetric (i.e. commutative-graded) algebra over $\mathbb{Q}$.

Unfortunately, computing semi-topological K-groups is often difficult. There is a certain class of smooth complex projective varieties, however, for which $K^i_{\mathrm{sst}}(X)$ is isomorphic to $K^i_{\mathrm{top}}(X^{\mathrm{an}})$ for all $i > 0$. We call this class D (Definition 4.3.6). Projective varieties in class D include curves, surfaces, toric varieties, flag varieties, and rational 3- and 4-folds. Varieties with non-trivial Griffiths groups, such as general Calabi–Yau 3-folds, cannot be in class D. For varieties that are in class D, there is a connected-component-wise homotopy equivalence between the Betti realization of the moduli space of objects in their derived categories and the complex topological K-theory space of their underlying analytic spaces. We find that the rational cohomology ring of the moduli stack of perfect complexes of coherent sheaves on a variety in class D is freely generated by Künneth components of Chern classes of the universal complex (Theorem 4.3.12).

Chapter 4 concludes with a proof of the orientability of moduli stacks of coherent sheaves on projective Calabi–Yau 4-folds.

Theorem 1.0.2 (see Theorem 4.4.2)**.** *Let X be a projective Calabi–Yau 4-fold and let $\underline{\mathcal{M}}$ denote the derived stack of objects in Perf(X), which has a -2-shifted symplectic structure ω. Let $\mathcal{M}$ denote the classical truncation of $\underline{\mathcal{M}}$ and let $\Gamma : \Omega^\infty K^{\mathrm{sst}}(X) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ denote the natural K-theory comparison map (Definition 3.1.11). Let $O^\omega \to \mathcal{M}$ denote the orientation bundle induced by ω (Definition 4.1.9). The un-*

[7] This requires that the identity component $\mathcal{M}^0_{\mathrm{Perf}(X)}$ of $\mathcal{M}_{\mathrm{Perf}(X)}$ has finite Betti numbers. This is true in all cases that we will be interested in.

derlying analytic space X^{an} of X is a spin 8-manifold with a positive Dirac operator $\not{D}_+ : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$. Let $O^{\not{D}_+} \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ denote the principal $\mathbb{Z}_2$-bundle induced by $\not{D}_+$ (Definition 3.4.6). Then there is an[8] *isomorphism*

$$\gamma : O^{\omega,\mathrm{Betti}} \xrightarrow{\sim} \Gamma^*(O^{\not{D}_+})$$

of principal $\mathbb{Z}_2$-bundles.

The idea behind this proof is as follows: The data of a principal $\mathbb{Z}_2$-bundle is equivalent to the data of the homotopy class of a map into the classifying space $B\mathbb{Z}_2$. Note that $B\mathbb{Z}_2$ is a group-like H-space. Therefore, by the universal property of homotopy-theoretic group completions (see Proposition 3.1.2), the orientation bundle $O^{\not{D}_+} : \mathcal{B}^U \to B\mathbb{Z}_2$ on the moduli space $\mathcal{B}^U$ of all unitary connections on X^{an} extends to an orientation bundle on $\Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ which is the homotopy-theoretic group completion of $\mathcal{B}^U$. There is an algebraic principal $\mathbb{Z}_2$-bundle $O^\omega \to \mathcal{M}$ which parameterizes étale local choices of orientations in the sense of Borisov–Joyce. The Betti realization of O^ω extends to a topological principal $\mathbb{Z}_2$-bundle over $\mathcal{M}^{\mathrm{Betti}}$. By an analysis of certain $\mathbb{C}$-antilinear involutions $\heartsuit_0, \heartsuit_1$ compatible with the Calabi–Yau Hodge star of X^{an} one can prove that the following diagram (weakly) homotopy commutes

$$\begin{tikzcd} \mathcal{T}^{\mathrm{an}} \arrow[r] \arrow[d, "O^{\omega,\mathrm{Betti}}"'] & \mathcal{B}^U \arrow[dl, "O^{\not{D}_+}"] \\ B\mathbb{Z}_2, & \end{tikzcd}$$

where $\mathcal{T}$ is a $\mathbb{C}$-ind-scheme which can be thought of as a moduli space of globally generated algebraic vector bundles of X. This gives that the diagram of group-like H-spaces

$$\begin{tikzcd} \mathcal{M}^{\mathrm{Betti}} \simeq \Omega^\infty K^{\mathrm{sst}}(X) \arrow[r, "\Gamma"] \arrow[d, "O^{\omega,\mathrm{Betti}}"'] & \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \arrow[dl, "O^{\not{D}_+}"] \\ B\mathbb{Z}_2 & \end{tikzcd}$$

[8] As $O^{\not{D}_+}$ is trivializable (Theorem 3.5.8), Lemmas 3.4.12 and 4.4.1 imply that this isomorphism is actually canonical.

(weakly[9]) homotopy commutes. The orientation bundle over the moduli stack $\mathcal{M}_{\mathrm{Coh}(X)} \subset \mathcal{M}$ of coherent sheaves on X is given by the restriction of $O^{\omega,\mathrm{Betti}} : \mathcal{M}^{\mathrm{Betti}} \to B\mathbb{Z}_2$ along the Betti realization of the inclusion $\mathcal{M}_{\mathrm{Coh}(X)} \hookrightarrow \mathcal{M}$. Therefore the fact that $O^{\not{D}_+}$ is trivializable over $\mathcal{B}^U$ when X is a Calabi–Yau 4-fold implies that $\mathcal{M}_{\mathrm{Coh}(X)}$ is orientable as a -2-shifted symplectic stack.

It is also possible to apply the fact that Γ is a morphism of H-spaces to compare the behavior of orientations under direct sum on $\Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ to those on $\mathcal{M}^{\mathrm{Betti}}$.

Theorem 1.0.3 (see Theorem 4.4.3)**.** *Let $X, \mathcal{M}, O^\omega$, and $O^{\not{D}_+}$ be as in Theorem 1.0.2. Given $\alpha \in K^0(X^{\mathrm{an}})$, there is an open and closed substack $\mathcal{M}_\alpha \subset \mathcal{M}$ of perfect complexes of X of class α. Let $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ denote the $\mathbb{C}$-stack morphism induced by direct sum of complexes and let $\Psi : \overline{\mathcal{B}}^U \times \overline{\mathcal{B}}^U \to \overline{\mathcal{B}}^U$ denote the H-map induced by direct sum of connections, where $\overline{\mathcal{B}}^U := \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$. Recall that there are isomorphisms*

$$\phi : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \xrightarrow{\ \sim\ } \Phi^*(O^\omega), \qquad \psi : O^{\not{D}_+} \boxtimes_{\mathbb{Z}_2} O^{\not{D}_+} \xrightarrow{\ \sim\ } \Psi^*(O^{\not{D}_+})$$

of principal $\mathbb{Z}_2$-bundles [45, Thm. 1.15.c] [115, Def. 2.22].

By Theorems 3.4.10, 3.5.8, and 4.4.2, an orientation $o_\alpha^{\not{D}_+}$ of $\overline{\mathcal{B}}_\alpha^U$ for $\alpha \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$ induces an orientation o_α^ω of $\mathcal{M}_\alpha$. Then, for all $\alpha, \beta \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$ and $\epsilon_{\alpha,\beta} \in \{\pm 1\}$, if

$$\psi_*(o_\alpha^{\not{D}_+} \boxtimes o_\beta^{\not{D}_+}) = \epsilon_{\alpha,\beta} \cdot \Psi^*(o_{\alpha+\beta}^{\not{D}_+})$$

then

$$\phi_*(o_\alpha^\omega \boxtimes o_\beta^\omega) = \epsilon_{\alpha,\beta} \cdot \Phi^*(o_{\alpha+\beta}^\omega).$$

Chapter 5 concerns itself with the construction of vertex F-algebras on the E-homology of "singular ring"[10] moduli spaces. Sections 5.3 and 5.4 are based on joint work with Markus Upmeier.

[9]Because principal $\mathbb{Z}_2$-bundles are determined by their monodromy maps, here the difference between homotopy and weak homotopy is not important.

[10]This terminology comes from Borcherds' view that vertex algebra objects are commutative ring

In Section 5.1, we review Joyce's geometric construction of vertex algebras on the homologies of certain moduli stacks of algebro-geometric and representation-theoretic interest. In Section 5.2 we compute, when X is in class D, an explicit graded vertex algebra structure on the rational E-homology of the moduli stack of objects in $\mathrm{Perf}(X)$ and identify this with a known vertex algebra.

Theorem 1.0.4 (see Theorem 5.2.2). *Let X be a smooth complex projective variety in class D. Let E be a complex-oriented spectrum such that $R := E^*(\mathrm{pt})$ is a $\mathbb{Q}$-algebra. Let $\mathcal{M}$ denote the moduli stack of objects in Perf(X). Let $\mathbb{E}^\bullet$ denote the universal complex over $X \times \mathcal{M}$, let $\mathcal{E}xt^\bullet := \mathbb{R}\pi_*(\pi^*(\mathbb{E}^\bullet)^\vee \otimes^{\mathbb{L}} \pi^*(\mathbb{E}^\bullet))$, let $\chi : K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}) \to \mathbb{Z}$ denote the Euler form*

$$\chi(v, w) = \int_{X^{\mathrm{an}}} \mathrm{ch}(v)^\vee \cdot \mathrm{ch}(w) \cdot \mathrm{Td}(X^{\mathrm{an}}),$$

and let $Q(v) := \chi(v, v)$ be the associated quadratic form. Similarly, define

$$\chi^{\mathrm{sym}}(v, w) := \chi(v, w) + \chi(w, v)$$

and $Q^{\mathrm{sym}}(v) := \chi^{\mathrm{sym}}(v, v)$. Then

1. *The Q^{sym}-shift $\hat{E}_*(\mathcal{M})$ of $E_*(\mathcal{M})$ can be made into a graded vertex algebra by taking $K(\mathrm{Perf}(X)) = K^0_{\mathrm{sst}}(X)$, $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)$, and $\epsilon_{\alpha,\beta} = (-1)^{\chi(\alpha,\beta)}$ in (5.2.2). In this case, $\hat{E}_*(\mathcal{M})$ is isomorphic, as a graded vertex algebra, to*

$$R[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{Sym}(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-1}R[t^{-1}]) \otimes \bigwedge(K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-\frac{1}{2}}R[t^{-1}]), \quad (1.0.3)$$

where (1.0.3) is given the structure of a generalized super-lattice vertex algebra associated to $(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}), \chi^{\mathrm{sym}})$ and the inclusion $K^0_{\mathrm{sst}}(X) \hookrightarrow$

objects with respect to a singular tensor product. There may be some reason to believe that the moduli spaces themselves could be considered singular ring objects in the homotopy category of higher stacks–see Meinhardt [166].

$K^0_{\mathrm{top}}(X^{\mathrm{an}})$.

2. *Suppose X is $2n$-Calabi–Yau and for all $\alpha, \beta \in K^0_{\mathrm{sst}}(X)$ we are given signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ such that the collection $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$ is a solution of the equations (5.1.1)-(5.1.3). Then the Q-shift $\hat{E}_*(\mathcal{M})$ of $E_*(\mathcal{M})$ can be made into a graded vertex algebra by taking $K(\mathrm{Perf}(X)) = K^0_{\mathrm{sst}}(X)$, $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, and signs $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$. In this case, $\hat{E}_*(\mathcal{M})$ is isomorphic, as a graded vertex algebra, to*

$$R[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{Sym}(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-1} R[t^{-1}]) \otimes \bigwedge (K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-\frac{1}{2}} R[t^{-1}]), \quad (1.0.4)$$

where (1.0.4) is given the structure of a generalized super-lattice vertex algebra associated to $(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}), \chi)$ and the inclusion $K^0_{\mathrm{sst}}(X) \hookrightarrow K^0_{\mathrm{top}}(X^{\mathrm{an}})$. Up to isomorphism this graded vertex algebra is independent of the representative of the group cohomology class $[\epsilon] \in H^2(K^0_{\mathrm{sst}}(X), \mathbb{Z}_2)$ that $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$ defines.

Example 1.0.5. Let X be a K3 surface. Then $K^0_{\mathrm{top}}(X^{\mathrm{an}})$ is a lattice and $K^1_{\mathrm{top}}(X^{\mathrm{an}}) \cong 0$. In particular, the fermionic piece of (1.0.4) vanishes. We then get that $\hat{E}_*(\mathcal{M})$ is the graded lattice vertex algebra associated to the Mukai lattice (with restricted group algebra $R[K^0_{\mathrm{sst}}(X)] \subset R[K^0_{\mathrm{top}}(X^{\mathrm{an}})]$).

The generalization from rational homology to complex-oriented homology with rational coefficient rings is a mild one. Our reason for stating this generalization is that virtual cycles for refined Vafa–Witten invariants of algebraic surfaces take values in rational K-theory [88] [211]. Shen has also defined interesting invariants of 3-folds taking values in cobordism [200] and we expect that a generalization of these invariants, which is defined in the presence of strictly semistables, would take values in rational cobordism. Wall-crossing formulas for these invariants may turn out to be stated in terms of Lie algebras on the rational complex-oriented homology of moduli spaces.

In Section 5.2, we build an F-bicharacter compatible with the bialgebra structure and shift operator on $\hat{E}_*(\mathcal{M})$. This implies the existence of a graded vertex F-algebra structure on $\hat{E}_*(\mathcal{M})$. To do so, we first build a natural transformation $(-)\cap C_z^E(-):K^0_{\mathrm{top}}(-)\Longrightarrow \mathcal{F}(E^*(-))$ uniquely characterized by the following two properties

- *Whitney sum:* $(-)\cap C_z^E(V\oplus W)=((-)\cap C_z^E(V))\cap C_z^E(W)$, and
- *normalization:* $(-)\cap C_z^E(\mathcal{L})$ is given by cap product with $F(z,c_1^E(\mathcal{L}))$, where $\mathcal{L}\to BU(1)$ is the universal complex line bundle.

Theorem 1.0.6 (see Theorem 5.3.7)**.** *Let $\mathcal{X}$ be an H-space with H-map $\Phi:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$. Let $\Psi:BU(1)\times\mathcal{X}\to\mathcal{X}$ be a $BU(1)$-action on $\mathcal{X}$ which is also a morphism of H-spaces. Suppose that we are given the following data*

1. *a quotient $K(\mathcal{X})$ of $(\pi_0(\mathcal{X}))^+$, and*
2. *a complex topological K-theory class $[D^\nabla]\in K^0_{\mathrm{top}}(\mathcal{X}\times\mathcal{X})$*

such that for all $\alpha,\beta,\gamma\in K(\mathcal{X})$ there are equalities

$$(\Phi_{\alpha,\beta}\times\mathrm{id})^*([D^\nabla_{\alpha+\beta,\gamma}])=\pi_{1,3}^*([D^\nabla_{\alpha,\gamma}])+\pi_{2,3}^*([D^\nabla_{\beta,\gamma}]),\tag{1.0.5}$$

$$(\mathrm{id}\times\Phi_{\beta,\gamma})^*([D^\nabla_{\alpha,\beta+\gamma}])=\pi_{1,2}^*([D^\nabla_{\alpha,\beta}])+\pi_{1,3}^*([D^\nabla_{\alpha,\gamma}]),\tag{1.0.6}$$

$$(\Psi_\alpha\times\mathrm{id})^*([D^\nabla_{\alpha,\beta}])=\pi_1^*([\mathcal{L}])\cdot\pi_{2,3}^*([D^\nabla_{\alpha,\beta}]),\tag{1.0.7}$$

$$(\pi_2,\Psi_\beta\circ\pi_{1,3}^*)([D^\nabla_{\alpha,\beta}])=\pi_1^*([\mathcal{L}^\vee])\cdot\pi_{2,3}^*([D^\nabla_{\alpha,\beta}])\tag{1.0.8}$$

of K-theory classes over any finite sub-complex $C\subset\mathcal{X}$. Define a symmetric $\mathbb{Z}$-bilinear form $\chi^D:K(\mathcal{X})\otimes K(\mathcal{X})\to\mathbb{Z}$ by

$$\chi^D(\alpha,\beta):=\mathrm{rk}([D^\nabla_{\alpha,\beta}])+\mathrm{rk}([D^\nabla_{\beta,\alpha}]),$$

and let $Q^D(\alpha) := \chi^D(\alpha,\alpha)$ be the associated quadratic form. Let E be a connective even complex-oriented spectrum, let $\mathcal{D}(z)$ denote the F-shift operator induced by Ψ and E, let $\hat{E}_*(\mathcal{X})$ be the Q^D-shift of $E_*(\mathcal{X})$, let $R := E^*(\mathrm{pt})$, and let

$$(-)\cap G_z^E([D^\nabla_{\alpha,\beta}]) := ((-)\cap(-1)^{\mathrm{rk}([D^\nabla_{\beta,\alpha}])}C_z^E([D^\nabla_{\alpha,\beta}]))\cap C^E_{\iota(z)}(\sigma^*[D^\nabla_{\beta,\alpha}]).$$

Then the R-linear map $r : \hat{E}_*(\mathcal{X})\otimes\hat{E}_*(\mathcal{X})\to R((z))$ defined by

$$r_z(a\otimes b) := (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta\otimes\eta)\circ(\mathcal{D}(z)\otimes\mathrm{id})(a\otimes b\cap G_z^E([D^\nabla])) \tag{1.0.9}$$

is a graded even symmetric F-bicharacter. Therefore, if there is a Künneth isomorphism

$$E_*(\mathcal{X}\times\mathcal{X})\cong E_*(\mathcal{X})\otimes E_*(\mathcal{X})$$

(1.0.9) endows $\hat{E}_*(\mathcal{X})$ with the structure of a graded vertex F-algebra.

A similar version of the above theorem holds for moduli spaces of orthogonal connections.

Theorem 1.0.7 (see Theorem 5.4.4)**.** *Let $\mathcal{Y}$ be an H-space with H-map $\Phi:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$. Let $\Psi : BO(1)\times\mathcal{Y}\to\mathcal{Y}$ be a $BO(1)$-action on $\mathcal{Y}$ which is also a morphism of H-spaces. Suppose that we are given the following data*

1. *a quotient $K(\mathcal{Y})$ of $(\pi_0(\mathcal{Y}))^+$, and*
2. *a KO-theory class $[D^\nabla]\in KO^0(\mathcal{Y}\times\mathcal{Y})$*

such that for all $\alpha,\beta,\gamma\in K(\mathcal{Y})$ there are equalities in KO-theory

$$(\Phi_{\alpha,\beta}\times\mathrm{id})^*([D^\nabla_{\alpha+\beta,\gamma}]) = \pi_{1,2}^*([D^\nabla_{\alpha,\gamma}])+\pi_{2,3}^*([D^\nabla_{\beta,\gamma}]), \tag{1.0.10}$$

$$(\mathrm{id}\times\Phi_{\beta,\gamma})^*([D^\nabla_{\alpha,\beta+\gamma}]) = \pi_{1,2}^*([D^\nabla_{\alpha,\beta}])+\pi_{1,3}^*([D^\nabla_{\alpha,\gamma}]), \tag{1.0.11}$$

$$(\Psi_\alpha\times\mathrm{id})^*([D^\nabla_{\alpha,\beta}]) = \pi_1^*([\mathcal{L}])\cdot\pi_{2,3}^*([D^\nabla_{\alpha,\beta}]), \tag{1.0.12}$$

$$(\pi_2, \Psi_\beta \circ \pi_{1,3}^*)([D_{\alpha,\beta}^\nabla]) = \pi_1^*([\mathcal{L}^\vee]) \cdot \pi_{2,3}^*([D_{\alpha,\beta}^\nabla]) \tag{1.0.13}$$

over any finite sub-complex $C \subset \mathcal{Y}$, where $\mathcal{L} \to BO(1)$ denotes the universal real line bundle. Let $\chi^D : K(\mathcal{M}) \times K(\mathcal{M}) \to \mathbb{Z}$ be the symmetric $\mathbb{Z}$-bilienar form given by $\chi^D(\alpha, \beta) := \mathrm{rk}([D_{\alpha,\beta}^\nabla])$, let $Q^D(\alpha) := \chi^D(\alpha, \alpha)$ be the associated quadratic form, and let $\hat{H}_*(\mathcal{Y}, \mathbb{Z}_2)$ be the Q^D-shift of $H_*(\mathcal{Y}, \mathbb{Z}_2)$. Then the $\mathbb{Z}_2$-linear map defined by

$$r_u(a \otimes b) := (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})(a \otimes b \cap W_u([D^\nabla])), \tag{1.0.14}$$

where $(-) \cap W_u(-) : KO^0(-) \Rightarrow \mathcal{F}(H_*(-, \mathbb{Z}_2))$ is an operator defined in Section 5.4 (see Theorem 5.4.2), is a graded even symmetric bicharacter on $\hat{H}_*(\mathcal{Y}, \mathbb{Z}_2)$ with respect to the shift operator $\mathcal{D}(u) : \hat{H}_*(\mathcal{Y}, \mathbb{Z}_2) \to \hat{H}_*(\mathcal{Y}, \mathbb{Z}_2)[[u]]$ induced by the action $BO(1) \times \mathcal{Y} \to \mathcal{Y}$. In particular, this makes $\hat{H}_*(\mathcal{Y}, \mathbb{Z}_2)$ into a graded vertex algebra with state-to-field correspondence given by

$$Y(a, u)b = \Phi_*(\mathcal{D}(u) \otimes \mathrm{id})(a \otimes b \cap W_u([D^\nabla])).$$

As stated, Sections 4.4, 5.3, and 5.4 are based on joint work and this is indicated when it occurs. Where there is joint work, the author has emphasized his own contributions.

Notation.

- Unless otherwise stated, R denotes a fixed commutative unital ring.
- Unless otherwise stated, the adjective "graded" will mean $\mathbb{Z}$-graded.
- For an R-module V, $V((z)) := V[[z]][z^{-1}]$.
- For a graded R-module V, $(-1)^a := (-1)^{\mathrm{degree}(a)}$ for $a \in V$.

Chapter 2

Vertex algebras

We begin in Section 2.1 with a survey of basic graded bialgebra and Hopf algebra theory. This leads into our definition of graded vertex algebras, examples, and reconstruction theory in Section 2.2. In Section 2.3 we discuss (nonlocal) vertex algebras twisted by a formal group law and in Section 2.4 we give details on the corresponding bicharacter construction. Part of Section 2.2 is based on a background section from the author's paper [92].

2.1 Graded Hopf algebras

As a precursor to graded vertex algebras, we discuss graded Hopf algebras. Hopf algebras first appeared as a structure on the rational homology of a connected[1] H-space. The reason it is important to take something like rational coefficients is the lack of a Künneth theorem in general. Without a Künneth splitting, there will be no coalgebra structure. For further reading on Hopf algebras see Abe [1], Cartier [49], Milnor–Moore [170], and Sweedler [207].

Definition 2.1.1. An *associative graded R-algebra* is a graded R-module A together with R-linear maps $m : A \otimes_R A \to A$ and $1_A : R \to A$ called *product* and *unit* such

[1]This is also true for general group-like H-spaces.

that the diagrams

$$\begin{tikzcd} A\otimes_R A\otimes_R A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"] & A\otimes_R A \arrow[d, "m"] \\ A\otimes_R A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "1_A\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes 1_A"] \arrow[dr, "\mathrm{id}"] & A\otimes_R A \arrow[d, "m"] \\ A\otimes_R A \arrow[r, "m"] & A \end{tikzcd}$$

commute. A *commutative* graded algebra is an associative graded algebra such that the diagram

$$\begin{tikzcd} A\otimes_R A \arrow[r, "\sigma"] \arrow[d, "m"] & A\otimes_R A \arrow[dl, "m"] \\ A & \end{tikzcd}$$

commutes where $\sigma(a\otimes b) := (-1)^{ab} b\otimes a$. A *morphism of associative graded R-algebras* $(A, m_A, 1_A) \to (B, m_B, 1_B)$ is an graded R-module homomorphism $f : A \to B$ such that $f(1_A) = 1_B$ and $f(m_A(x, y)) = m_B(f(x), f(y))$ for all $x, y \in A$.

For algebras, we use *infix notation* $m(a, b) := a \cdot b$ for the product when no confusion arises from doing so. All our graded algebras are assumed to be commutative.

There is a natural action $A \otimes A^* \to A^*$ of an algebra A on its dual A^* as follows: let $a \in A, \phi \in A^*$ then $a \cdot \phi(x) = \phi(a \cdot x)$ for all $x \in A$.

Definition 2.1.2. A *coassociative graded R-coalgebra* is a graded R-module C together with R-linear maps $\Delta : C \to C \otimes_R C$ and $\eta : C \to R$ called *coproduct* and *counit* such that the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C\otimes_R C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C\otimes_R C \arrow[r, "\Delta\otimes\mathrm{id}"] & C\otimes_R C\otimes_R C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & C\otimes_R C \arrow[d, "\mathrm{id}\otimes\eta"] \\ C\otimes_R C \arrow[r, "\eta\otimes\mathrm{id}"] & C \end{tikzcd}$$

commute. A *cocommutative graded R-coalgebra* is a coassociative graded R-coalgebra such that the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C\otimes_R C \\ C\otimes_R C \arrow[ur, "\sigma"] & \end{tikzcd}$$

commutes. A *morphism of coassociative graded R-coalgebras* $(C, \Delta_C, \eta_C) \to (D, \Delta_D, \eta_D)$ is a graded R-module homomorphism $g : C \to D$ such that $\Delta_D \circ g = g \otimes g \circ \Delta_C$.

The *Sweedler notation* is a helpful notational system for coalgebra. Given a coalgebra (C, Δ, η), for $a \in C$ one writes

$$\Delta(a) = \sum_{(a)} a' \otimes a'' \tag{2.1.1}$$

for the decomposition of the coproduct of a into basic tensors. The summation is sometimes omitted as it can be inferred when a', a'' appear in a formula. For example, counitality $(1_C \otimes \eta) \circ \Delta = 1_C = (\eta \otimes 1_C) \circ \Delta$ is written as

$$a'\eta(a'') = a = \eta(a')a'', \tag{2.1.2}$$

coassociativity $\Delta_2 := (1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ as

$$\sum_{\Delta_2(a)} a^{(1)} \otimes a^{(2)} \otimes a^{(3)} = \sum_{(a),(a'')} a' \otimes (a'')' \otimes (a'')'' = \sum_{(a),(a')} (a')' \otimes (a')'' \otimes a'',$$

and cocommutativity as

$$a' \otimes a'' = (-1)^{a' \cdot a''} a'' \otimes a'. \tag{2.1.3}$$

If (C, Δ_C, η_C) is a coalgebra, then $C \otimes C$ is a coalgebra with coproduct $\Delta_{C \otimes C}$ defined by

$$\begin{tikzcd} C \otimes C \arrow[r, dashed, "\Delta_{C\otimes C}"] \arrow[d, "\Delta_C \otimes \Delta_C"'] & C \otimes C \otimes C \otimes C \\ C \otimes C \otimes C \otimes C \arrow[ur, "\mathrm{id} \otimes \sigma \otimes \mathrm{id}"'] & \end{tikzcd}$$

and counit given by $\eta_{C \otimes C} := \eta_C \otimes \eta_C$. The coproduct $\Delta_C : C \to C \otimes C$ is a coalgebra morphism if and only if C is cocommutative. All our coalgebras are assumed to be cocommutative.

Fact 2.1.3. Coassociativity of $(\Delta_{C\otimes C}, \Delta_{C\otimes C}, \eta_{C\otimes C})$ gives the identity

$$(-1)^{a^{(2)}b^{(1)}} a^{(1)} \otimes b^{(1)} \otimes a^{(2)} \otimes b^{(2)} \otimes a^{(3)} \otimes b^{(3)} = (-1)^{a^{(3)}b^{(2)}} a^{(1)} \otimes b^{(1)} \otimes a^{(2)} \otimes b^{(2)} \otimes a^{(3)} \otimes b^{(3)}. \tag{2.1.4}$$

Proof. Explicitly we have

$$\Delta_{C\otimes C}(a \otimes b) = (-1)^{a''b'} a' \otimes b' \otimes a'' \otimes b''.$$

For three copies of C we have the identity

$$\Delta_{C\otimes C\otimes C}(a \otimes b \otimes c) = (-1)^{a''b' + c'(a''+b'')} a' \otimes b' \otimes c' \otimes a'' \otimes b'' \otimes c''.$$

The isomorphism $C \otimes (C \otimes C) \cong (C \otimes C) \otimes C$ is reflected in the symmetry

$$a''b' + c'(a'' + b'') = b''c' + a''(b' + c'). \qquad \square$$

Given a coalgebra C, there is a natural (two-sided) action $\cap : C \otimes C^* \to C$ of the dual C^* on C as follows: given $c \in C$ and $\phi \in C^*$ define $c \cap \phi = c'\phi(c'')$. The suggestive cap product notation is used because the cap product $H_*(X, \mathbb{Q}) \otimes H^*(X, \mathbb{Q}) \to H_*(X, \mathbb{Q})$ on the rational homology of a topological space X is an example of this kind of natural action.[2] The usual cap product identities hold for this action in general.

Proposition 2.1.4. *Let (A, Δ_A, η_A) and (B, Δ_B, η_B) be coalgebras and let $f : A \to B$ be a coalgebra homomorphism. Then for $x \in A, \phi \in B^*$*

$$f(x) \cap \phi = f(x \cap f^*\phi) \tag{2.1.5}$$

[2] We took rational coefficients so that $H_*(X, \mathbb{Q})$ was a coalgebra. At the chain level, $C_*^{\text{sing}}(X)$ is an internal coalgebra for arbitrary coefficient ring. The cap product can be defined at the chain level and is an example of this kind of natural action.

and

$$x \cap \eta_A = x. \tag{2.1.6}$$

Proof. We compute

$$\begin{aligned} f(x) \cap \phi &= f(x')\phi(f(x'')) \\ &= f(x'\phi(f(x''))) \\ &= f(x' f^* \phi(x'')) \\ &= f(x \cap f^* \phi) \end{aligned}$$

to establish (2.1.5). Equation (2.1.6) is just a re-statement of co-unitality. □

A graded R-module M is said to be *finite type* if the degree k piece $M_k \subset M$ is finitely generated for all $k \in \mathbb{Z}$. Let A be a graded R-module which is projective of finite type. Then $(A, m, 1_A)$ is a graded algebra if and only if $(A^*, m^*, 1_A^*)$ is a graded coalgebra [170, Prop. 3.1].

Definition 2.1.5. A *graded bialgebra* is a graded R-module B that is both a graded R-algebra $(B, m, 1_B)$ and a graded R-coalgebra (B, Δ, η) with compatible such structures in the sense that the diagrams

$$\begin{tikzcd} B \otimes_R B \arrow[r, "m"] \arrow[d, "\Delta\otimes D"] & B \arrow[r, "\Delta"] & B \otimes_R B \\ B \otimes_R B \otimes_R B \otimes_R B \arrow[rr, "\mathrm{id}\otimes\sigma\otimes\mathrm{id}"] & & B \otimes_R B \otimes_R B \otimes_R B \arrow[u, "m\otimes m"'] \end{tikzcd}$$

$$\begin{tikzcd} B \otimes_R B \arrow[r, "m"] \arrow[d, "\eta\otimes\eta"] & B \arrow[dl] \\ R & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[d, "\eta\otimes\eta"] \arrow[dr, "\eta"] & \\ B \otimes_R B & B \arrow[l, "\Delta"] \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta"] \arrow[d, "\mathrm{id}"] & B \arrow[dl, "\eta"] \\ B & \end{tikzcd}$$

commute. A *graded Hopf algebra* is a graded bialgebra H together with an *antipode* $S : H \to H$ such that

$$
\begin{tikzcd}
H \otimes_R H \arrow[rr, "S\otimes\mathrm{id}"] \arrow[d, "\Delta"] & & H \otimes_R H \arrow[d, "m"] \\
H \arrow[r, "\eta"] & R \arrow[r, "\eta"] & H \\
H \otimes_R H \arrow[rr, "\mathrm{id}\otimes S"] \arrow[u, "\Delta"] & & H \otimes_R H \arrow[u, "m"']
\end{tikzcd}
$$

commutes. A bialgebra admits at most one antipode.

Example 2.1.6. The *binomial Hopf algebra* is the ring of polynomials in one variable $R[x]$ with product given by ordinary multiplication, coproduct given by

$$\Delta(x) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

counit given by

$$\eta(x^n) = \delta_{n,0}$$

and antipode given by

$$S(x^n) = (-1)^n x^n.$$

The *divided power Hopf algebra* $R\{x\}$ is the free graded R-module with basis symbols $x^{(n)}$ for $n \geq 0$ with product defined by

$$x^{(n)} \cdot x^{(m)} = \binom{n+m}{m} x^{(n+m)}$$

copoduct given by

$$\Delta(x^{(n)}) = \sum_{k=0}^{n} x^{(k)} \otimes x^{(n-k)}$$

counit given by

$$\eta(x^{(n)}) = \delta_{n,0}$$

and antipode given by

$$S(D^{(n)}) = (-1)^n D^{(n)}.$$

There is a Hopf algebra morphism $R[x] \to R\{x\}$ given by $x^n \mapsto n!x^{(n)}$ which is injective if R has no torsion.

If $(H, m, \delta, \eta, 1_H)$ is a graded Hopf algebra an element $x \in H$ is said to be *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$ and *group-like* if $\Delta(x) = x \otimes x$. Note that group-like elements are necessarily of degree zero.

Proposition 2.1.7 (see [1]). *Let $(B, m, \Delta, \eta, 1_B)$ be a graded bialgebra and let $G(B)$ denote the set of all group-like elements of B. Then B is a graded Hopf algebra if and only if all elements of $G(B)$ are invertible. In particular, the localization map $B \to B[G(B)]^{-1}$ is universal for all bialgebra morphisms from B into a graded Hopf algebra.*

Example 2.1.8. Let M be a commutative monoid. The *monoid algebra* $R[M]$ is the free R-module with basis symbols e^α for $\alpha \in M$. This is a bialgebra with product $e^\alpha \cdot e^\beta = e^{\alpha+\beta}$, unit given by e^0, counit $\eta(e^\alpha) = 1_R$, and coproduct $\Delta(e^\alpha) = e^\alpha \otimes e^\alpha$. Tautologically, all elements are group-like. By Proposition 2.1.7, the monoid algebra is a Hopf algebra if and only if M is a group. In this case we call $R[M]$ the *group algebra* of M and the antipode is given by $S(e^\alpha) = e^{-\alpha}$. Ordinary group completion $M \to M^+$ induces a completion map $R[M] \to R[M^+]$ is the sense of Proposition 2.1.7.

Example 2.1.9. An (associative) *H-space* is a topological space X together with a *product* $\mu : X \times X \to X$, defined up to homotopy, and a *unit* $1_X : 1 \to X$ such that the diagrams

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mathrm{id}_X \times \mu"] \arrow[d, "\mu \times \mathrm{id}_X"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}_X \times 1_X"] \arrow[d, "1_X \times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

homotopy commute. The H-space is said to be *commutative* if $\mu \circ \sigma \simeq \mu$, where $\sigma : X \times X \to X \times X$ denotes exchange of factors. All our H-spaces are assumed to be commutative. An H-space X is said to be *group-like* if $\pi_0(X)$ is a group. It can be shown through the shearing construction that an H-space is group-like if and only if X has admits a homotopy inverse map with respect to μ (see May–Ponto [162, Lem. 9.2.2]).

If X is a topological space and k is a field then the diagonal $\Delta : X \to X \times X$ endows $H_*(X,k)$ with a graded coproduct

$$H_*(X,k) \xrightarrow{\Delta_*} H_*(X \times X, k) \cong H_*(X,k) \otimes H_*(X,k)$$

and counit $H_*(\pi) : H_*(X,\mathbb{Q}) \to H_*(\mathrm{pt},\mathbb{Q}) \cong \mathbb{Q}$. If X is moreover an H-space, then $H_*(X,k)$ is further a graded bialgebra. There is a graded Hopf algebra structure on $H_*(X,k)$ if and only if X is a group-like H-space. Localization

$$H_*(X,k) \longrightarrow H_*(X,k)[\pi_0(X)]^{-1}$$

by the natural action of $\pi_0(X)$ on $H_*(X,k)$ is a completion map in the sense of Proposition 2.1.7.

2.2 Graded vertex algebras

Vertex algebras are a mathematical structure originating in conformal field theory. The first mathematically rigorous definition of vertex algebra was supplied by Richard E. Borcherds, and was used in his proof of the monstrous moonshine conjectures [35]. So far as the author is aware, the first vertex algebra that appeared in the mathematical literature is the vertex algebra associated to a lattice. Kac and Frenkel constructed interesting representations of affine Kac–Moody algebras acting by vertex operators on lattice vertex algebras [76]. [3]

[3] The Kac–Frenkel results actually pre-date Borcherds' general mathematical definition of a vertex algebra.

In this section we first define graded (nonlocal) vertex algebras and give two ways to think about them. We also review Borcherds' method for extracting a Lie algebra from a vertex algebra in addition to the reconstruction theorem. The reconstruction theorem allows one to take a "generators and relations" approach to building vertex algebras. This is followed up with some non-trivial examples: vertex algebras associated to generalized lattices and vertex algebras associated to generalized super-lattices. For further background on vertex algebras the reader is referred to Frenkel–Ben-Zvi [75], Frenkel–Lepowsky–Meurman [74], and Kac [120]. The reader may observe that these sources deal with non-graded or super vertex algebras whereas we deal with $\mathbb{Z}$-graded vertex algebras, although the theories are extremely similar.

One way to think of a graded nonlocal vertex algebra is as a generalized graded non-commutative ring. Similarly, one can think of a graded vertex algebra as a generalization of a graded commutative ring. We begin with a toy example.

Definition 2.2.1. Let A be a graded R-module. A *shift operator* is a graded R-linear map $\mathcal{D}(z) : A \to A[[z]]$ such that

$$\mathcal{D}(z) \circ \mathcal{D}(w) = \mathcal{D}(z + w) \text{ and } \mathcal{D}(z) \circ \mathcal{D}(-z) = \mathrm{id}.$$

If A is an algebra, we say that $\mathcal{D}(z)$ is *compatible with* A if

$$\mathcal{D}(z)(a \cdot b) = \mathcal{D}(z)(a) \cdot \mathcal{D}(z)(b) \text{ and } \mathcal{D}(z)(1_A) = 1_A,$$

where $a, b \in A$ and $1_A \in A$ is the algebra unit of A.

We can rewrite the data of a graded commutative algebra with compatible shift operator as follows.

Definition 2.2.2. A *graded nonlocal holomorphic vertex algebra* is a graded R-module V together with a degree-preserving R-linear map called a *state-to-field correspon-*

dence $Y(-,z) : V \to \mathrm{End}(V)[[z]]$, a distinguished element $|0\rangle \in V_0$ of degree zero called a *vacuum vector*, and a graded shift operator $\mathcal{D}(z) : V \to V[[z]]$ such that for all $a, b \in V$

1. $Y(a,z)|0\rangle|_{z=0} = a$, $Y(|0\rangle, z) = \mathrm{id}$,
2. $Y(\mathcal{D}(w)(a), z)b = Y(a, z+w)b$, $\mathcal{D}(z)(|0\rangle) = |0\rangle$, and
3. $Y(Y(a,z)b, w) = Y(a, z+w)Y(b,w)$.

A graded nonlocal holomorphic vertex algebra is said to be a *graded holomorphic vertex algebra* if

4. $Y(a,z)b = (-1)^{ab}\mathcal{D}(z) \circ Y(b,-z)a$ for all $a, b \in V$.

Note that 2.2.2.1. makes sense because the fields of a holomorphic vertex algebra have no poles and 2.2.2.3. makes sense because $V[[w]][[z+w]] \cong V[[z]][[w]]$. Given a holomorphic graded nonlocal vertex algebra one writes

$$Y(a,z) = \sum_{n \leq -1} a_n z^{-n-1}.$$

Proposition 2.2.3. *Let $(V, Y(-,z), \mathcal{D}(z), |0\rangle)$ be a graded (resp. nonlocal) holomorphic vertex algebra. Then $a \cdot b = a_{-1}b$ defines a graded associative (resp. commutative) product on V. Moreover, $\mathcal{D}(z)$ is compatible with this algebra structure on V. Conversely, given a graded associative (resp. commutative) algebra V with unit $|0\rangle$ and compatible shift operator $\mathcal{D}(z)$ the algebra V is a graded (resp. nonlocal) holomorphic vertex algebra with state-to-field correspondence $Y(a,z)b = \mathcal{D}(z)(a) \cdot b$ and vacuum vector $|0\rangle$.*

Proof. First, suppose that $(V, Y(-,z), \mathcal{D}(z), |0\rangle)$ is a graded nonlocal vertex algebra. Comparing constant terms of 2.2.2.3 shows that $a_{-1}b$ is an associative product. When $Y(a,z)b = (-1)^{ab}Y(b,-z)a$, comparing constants terms shows that $a_{-1}b$ is a graded commutative product. To prove compatibility of $\mathcal{D}(z)$ with this product we proceed as follows. Write $\mathcal{D}(z)$ as a generating series $\mathcal{D}(z) = \sum_{k\geq 0} D^{(k)} z^k$ of R-linear endomorphisms of V. We claim that

$$Y(a,w)|0\rangle = \mathcal{D}(w)(a) \tag{2.2.1}$$

for all $a \in V$. Equation (2.2.1) follows from 2.2.2.2 at $z=0$ with $b = |0\rangle$

$$\begin{aligned}\mathcal{D}(w)(a) &= Y(\mathcal{D}(w)a, z)|0\rangle|_{z=0}\\ &= Y(a, z+w)|)\rangle|_{z=0}\\ &= Y(a,w)|0\rangle.\end{aligned}$$

The identity 2.2.2.3 implies

$$\begin{aligned}Y(a,z+w)\mathcal{D}(w)(b) &= Y(a,z+w)Y(b,w)|0\rangle\\ &= Y(Y(a,z)b,w)|0\rangle\\ &= \mathcal{D}(w)\circ Y(a,z)b.\end{aligned}$$

Setting $z=0$ in the above gives

$$\begin{aligned}\mathcal{D}(w)(a_{-1}b) &= \sum_{n\geq 0} a_{-n-1}(\mathcal{D}(w)(b))w^n\\ &= \mathcal{D}(w)(a)_{-1}\mathcal{D}(w)(b).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{D}(w)|0\rangle &= Y(\mathcal{D}(w)|0\rangle, z)|0\rangle\\ &= Y(|0\rangle, z+w)|0\rangle\end{aligned}$$

$$= |0\rangle.$$

Conversely, let V be an algebra with unit $|0\rangle$ and compatible shift operator $\mathcal{D}(z)$. First

$$Y(a,z)|0\rangle|_{z=0} = \mathcal{D}(z)(a)\cdot|0\rangle|_{z=0} = \mathcal{D}(z)(a)|_{z=0} = a$$

and

$$Y(|0\rangle,z)a = \mathcal{D}(z)(|0\rangle)\cdot a = a.$$

Then, from the definition of a shift operator

$$Y(\mathcal{D}(w)(a),z)b = \mathcal{D}(z)(\mathcal{D}(w)(a))\cdot b = \mathcal{D}(z+w)(a)\cdot b = Y(a,z+w)b.$$

Similarly,

$$\begin{aligned}
Y(Y(a,z)b,w)c &= Y(\mathcal{D}(z)(a)\cdot b,w)c \\
&= \mathcal{D}(w)(\mathcal{D}(z)(a)\cdot b)\cdot c \\
&= \mathcal{D}(z+w)(a)\cdot\mathcal{D}(w)(b)\cdot c \\
&= Y(a,z+w)Y(b,w)c.
\end{aligned}$$

Finally, if the product on V is graded commutative then

$$\begin{aligned}
(-1)^{ab}\mathcal{D}(z)\circ Y(b,-z)a &= (-1)^{ab}\mathcal{D}(z)(\mathcal{D}(-z)(b)\cdot a) \\
&= (-1)^{ab}b\cdot\mathcal{D}(z)(a) \\
&= Y(a,z)b.
\end{aligned}$$

□

The idea of a (non-holomorphic) nonlocal vertex algebra is to allow finite order singularities in the state-to-field correspondence $Y(-,z) : V \longrightarrow \mathrm{End(V)}((z))$. A slight problem, however, then surfaces: the equation $Y(Y(a,z)b,w)c = Y(a,z+w)Y(b,w)c$ no longer makes sense because $V((w))((z+w)) \neq V((z))((w))$. For example, $\delta(z+w)_- := \sum_{n\in\mathbb{Z}} z^{-n-1}(-w)^n$ is an element of $V((z))((w))$ but not of $V((w))((z+w))$. One modifies 2.2.2.3 by demanding instead that $Y(Y(a,z)b,w)$ and $Y(a,z+w)Y(b,w)$ are both expansions of the same rational function in different variables.

There are maps

$$i_{z,w} : V[[z,w]][z^{-1},w^{-1},(z+w)^{-1}] \longrightarrow V((z))((w))$$

and

$$i_{w,z} : V[[z,w]][z^{-1},w^{-1},(z+w)^{-1}] \longrightarrow V((w))((z))$$

defined by

$$i_{z,w}(\frac{1}{(z+w)^k}) = \sum_{n\geq 0} \binom{n}{k} z^{-n-1}(-w)^{n-k}$$

and

$$i_{z,w}(\frac{1}{(z+w)^k}) = -\sum_{n\leq -1} \binom{n}{k} z^{-n-1}(-w)^{n-k}.$$

We define the *formal delta* to be $\delta(z+w) := \sum_{n\in\mathbb{Z}} z^{-n-1}(-w)^n$. One has

$$\begin{aligned}\partial_w^k \delta(z+w) &= \sum_{n\in Z} \binom{n}{k} z^{-n-1}(-w)^{n-k} \\ &= (\iota_{z,w} - \iota_{w,z})(\frac{1}{(z+w)^k})\end{aligned}$$

and

$$(z+w)^k \partial_w^k \delta(z+w) = 0.$$

In particular, the condition that two series $A(z,w) \in V((z))((w))$ and $B(z,w) \in$

$V((w))((z))$ are expansions of the same element of $V[[z,w]][z^{-1},w^{-1},(z+w)^{-1}]$ is equivalent to the statement that there exists $N \gg 0$ such that

$$(z+w)^N A(z,w) = (z+w)^N B(z,w).$$

We prove this fact with $z+w$ replaced by a general formal group law in Section 2.4.

If V is an R-module, a *field* on V is an R-linear map $f : V \to V((z))$. The set of fields on V is written $\mathcal{F}(V)$. If V is graded, then $\mathcal{F}(V)$ inherits a natural grading by declaring a field f to be of degree n if the map $f : V \to V((z))$ is graded of degree n with the degree of z taken to be -2.

Definition 2.2.4. A *graded nonlocal vertex algebra* is a graded R-module, a degree-preserving R-linear map $Y(-,z) : V \to \mathcal{F}(\mathrm{V})$ called a *state-to-field correspondence*, a distinguished element $|0\rangle \in V_0$ of degree 0 called a *vacuum vector*, and a graded shift operator $\mathcal{D}(z) : V \to V[[z]]$ such that

1. for all $a \in V$, $Y(a,z)|0\rangle$ is holomorphic with $Y(a,z)|0\rangle|_{z=0} = a$ and $Y(|0\rangle, z) = \mathrm{id}$,
2. for all $a, b \in V$, $Y(\mathcal{D}(w)(a), z)b = i_{z,w}Y(a, z+w)b$, $\mathcal{D}(w)(|0\rangle) = |0\rangle$ and
3. for all $a, b, c \in V$, there exists $N \gg 0$ such that

$$(z+w)^N i_{z,w} Y(a, z+w)Y(b,w)c = (z+w)^N Y(Y(a,z)b, w)c.$$

Condition 2.2.4.1 is usually called *vacuum and creation*, condition 2.2.4.2 is related to an axiom that is called *translation covariance*, and 2.2.4.3 is called *weak associativity.*

Definition 2.2.5. Given a graded nonlocal vertex algebra $(V, Y(-,z), \mathcal{D}(z), |0\rangle)$. Then there is an *opposite* graded nonlocal vertex algebra structure on V with state-to-field correspondence given by

$$Y^{\mathrm{op}}(a,z)b := (-1)^{ab}\mathcal{D}(z) \circ Y(b,-z)a.$$

A graded nonlocal vertex algebra is said to be a *graded vertex algebra* if $Y^{\mathrm{op}}(-,z) = Y(-,z)$.

Remark 2.2.6. Nonlocal vertex algebras can be regarded as non-commutative singular rings in a certain relaxed multilinear category (see Borcherds [37]). Vertex algebras can be regarded as commutative singular ring objects in that same category. Nonlocal vertex algebras have been studied by Bakalov–Kac [23], Li [145] [146], and Kim [124].

It is often more useful to describe a vertex algebra in terms of a generating set of fields, rather than by describing all fields. This is made possible by the Reconstruction Theorem.

Definition 2.2.7. Let $(V, |0\rangle)$ be a pair consisting of a graded R-module V and distinguished element $|0\rangle \in V_0$. Let $\{a^i(z) = \sum_{n\in\mathbb{Z}} a^i_n z^{-n-1}\}_{i\in I}$ be a set of $\mathrm{End}_R(V)$-valued fields. Let $a^i = a^i(z)|0\rangle|_{z=0}$. Then $\{a^i(z)\}_{i\in I}$ is said to *generate* $(V, |0\rangle)$ if the collection of vectors of the form $a^{i_1}_{n_1} \dots a^{i_s}_{n_s}|0\rangle$ span V.

Proposition 2.2.8 (Reconstruction Theorem [120, Thm. 4.5])**.** *Let R be a field of characteristic zero and let V be a graded R-module with a distinguished vector $|0\rangle \in V$. Let $\{a^i(z)\}_{i\in I}$ be a mutually local collection of $\mathrm{End}_R(V)$-valued fields that generate $(V, |0\rangle)$. Let $T : V \to V$ be a derivation of degree 2 such that $T|0\rangle = 0$ and such that $[T, a^i(z)] = \partial_z a^i(z)$. Then there is a unique graded vertex R-algebra structure on V such that $|0\rangle$ is the vacuum vector and such that the state-to-field correspondence maps $a^i(z)|0\rangle|_{z=0} \mapsto a^i(z)$.*

Another way to think of a vertex algebra is as a Lie algebra with with infinitely many Lie brackets, one for each integer, which satisfy infinitely many identities that resemble Jacobi and anti-symmetry. This is the spirit of the following definition, equivalent to Definition 2.2.5, which was actually Borcherds' original definition.

Definition 2.2.9. A *graded vertex algebra* is a graded R-module V together with R-linear maps $D^{(n)} : V \to V$ of degree $2n$ for $n = 0, 1, 2, \dots$ with $D^{(0)} = \mathrm{id}$, R-linear maps $a_n : V \to V$ of degree $\deg(a) - 2n - 2$ for $a \in V, n \in \mathbb{Z}$ that are R-linear in a, and a distinguished element $|0\rangle \in V_0$ of degree zero such that

1. for $a, b \in V$, $a_n(b) = 0$ for $n \gg 0$,

2. for $a \in V$ $|0\rangle_{-1}(a) = a$ and $|0\rangle_n(a) = 0$ for $n \neq 1$,

3. for $a \in V$
$$a_n(|0\rangle) = D^{(-n-1)}(a),$$
when $n < 0$ and $a_n(|0\rangle) = 0$ when $n \geq 0$,

4. for $a, b \in V$
$$a_n(b) = \sum_{k \geq 0} (-1)^{ab+n+k+1} D^{(k)}(a_{n+k}(b)),$$
and

5. for $a, b, c \in V$ and $m, \ell \in \mathbb{Z}$
$$(a_\ell(b))_m(c) = \sum_{n \geq 0} (-1)^n \binom{\ell}{n} (a_{\ell-n}(b_{n+m})(c)) - (-1)^{\ell+ab} b_{\ell+m+n}(a_n(c)).$$

Given a vertex algebra we write $D(V) := \bigoplus_{n>0} D^{(n)}(V)$. The the quotient module $V/D(V)$ inherits the structure of a graded Lie algebra by

$$((v \bmod DV), (w \bmod DV)) \mapsto (v_0(w) \bmod DV).$$

This is how Ringel–Hall type Lie algebras are produced to prove a wall-crossing formula for $\mathbb{C}$-linear enumerative invariants of quivers in Gross–Joyce–Tanaka [93].

We now provide some examples of non-trivial (i.e. non-holomorphic) graded vertex algebras.

Definition 2.2.10. Let $(V, Y_V, |0\rangle_V)$ and $(W, Y_W, |0\rangle_W)$ be graded vertex R-algebras. Their *tensor product* is a graded vertex algebra $(V \otimes W, Y_{V\otimes W}, |0\rangle_{V\otimes W})$ whose space of states is the graded tensor product $V \otimes W$, whose vacuum vector is given by $|0\rangle_{V\otimes W} = |0\rangle_V \otimes |0\rangle_W$, and whose state-to-field correspondence is given by

$$Y_{V\otimes W}(a \otimes b, z)(c \otimes d) = (-1)^{(a+b)c} Y_V(a, z)c \otimes Y_W(b, z)d.$$

A *generalized integral lattice* is a finitely generated abelian group equipped with a symmetric $\mathbb{Z}$-bilinear form. Let (A^+, χ^+) be a generalized integral lattice and let $\iota : B^+ \to A^+$ be a morphism of finitely generated abelian groups. Let $\mathfrak{h} := A^+ \otimes_{\mathbb{Z}} R$. Then χ^+ extends to an R-valued symmetric bilinear form on $\mathfrak{h}$ which, by abuse of notation, we will also call χ^+. Write $\hat{\mathfrak{h}} = \mathfrak{h} \otimes R[t, t^{-1}] \oplus R\underline{K}$ for the Lie algebra with bracket

$$[h \otimes t^n, h' \otimes t^m] = m\delta_{m,-n}\chi^+(h, h')\underline{K}, \quad [h \otimes t^m, \underline{K}] = 0$$

for all $h, h' \in \mathfrak{h}, n, m \in \mathbb{Z}$. There is a representation $\rho_1 : \hat{\mathfrak{h}} \to \mathrm{Sym}_R(A^+ \otimes t^{-1}R[t^{-1}])$ such that $\mathfrak{h} \otimes t^{-1}R[t^{-1}]$ acts by multiplication, $\mathfrak{h}[t] \cdot 1 = 0$, and $\underline{K}$ acts as the identity. There is another Lie algebra representation $\rho_2 : \hat{\mathfrak{h}} \to R[B^+]$ where $\underline{K}$ acts by zero and $h \otimes t^n$ acts on e^α as multiplication by $\delta_{n,0}\chi^+(\iota(\alpha), h)$. Write $V_{A^+,B^+} := R[B^+] \otimes \mathrm{Sym}_R(A^+ \otimes t^{-1}R[t^{-1}])$. There is a representation $\rho : \hat{\mathfrak{h}} \to V_{A^+,B^+}$ defined by $\rho = \rho_1 \otimes 1 + 1 \otimes \rho_2$. Given $v \in \mathfrak{h}$ write

$$v(z) := \sum_{n\in\mathbb{Z}} \rho(v \otimes t^n) z^{-n-1}$$

and

$$\Gamma_\alpha(z) := e^\alpha z^{\alpha_0} \exp(-\sum_{j<0} \frac{z^{-j}}{j}\alpha_j) \exp(-\sum_{j>0} \frac{z^{-j}}{j}\alpha_j) c_\alpha,$$

where c_α are operators $c_\alpha : V_{A^+,B^+} \to V_{A^+,B^+}$ such that

$$c_0 = 1, \quad c_\alpha|0\rangle = |0\rangle, \quad [v_n, c_\alpha] = 0 \quad (v \in \mathfrak{h}, n \in \mathbb{Z}), \tag{2.2.2}$$

and given $\alpha \in B^+$

$$e^\alpha c_\alpha e^\beta c_\beta = (-1)^{\chi^+(\iota(\alpha),\iota(\beta))+\chi^+(\iota(\alpha),\iota(\alpha))\chi^+(\iota(\beta),\iota(\beta))} e^\beta c_\beta e^\alpha c_\alpha. \tag{2.2.3}$$

Then the collection of fields $v(z)$ and $\Gamma_\alpha(z)$ generate a graded vertex R-algebra structure on V_{A^+,B^+}. For any given solution of (2.2.2) and (2.2.3) there is a unique such graded vertex R-algebra structure (cf. [120, Thm. 5.4]). The only solutions of (2.2.2) and (2.2.3) that concern us are scaling operators

$$c_\alpha(e^\beta \otimes x) = \epsilon_{\alpha,\beta} e^\beta \otimes x, \quad (\alpha, \beta \in B^+)$$

with $\epsilon_{\alpha,\beta} \in \{\pm 1\}$. For such c_α, equations (2.2.2) and (2.2.3) are satisfied if and only if for all $\alpha, \beta, \gamma \in B^+$

$$\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 0 \tag{2.2.4}$$

$$\epsilon_{\alpha,\beta} = (-1)^{\chi^+(\iota(\alpha),\iota(\beta))+\chi^+(\iota(\alpha),\iota(\alpha)))\chi^+(\iota(\beta),\iota(\beta))} \epsilon_{\beta,\alpha} \tag{2.2.5}$$

$$\epsilon_{\beta,\gamma}\epsilon_{\beta+\gamma,\alpha} = \epsilon_{\gamma,\alpha+\beta}\epsilon_{\beta,\alpha}. \tag{2.2.6}$$

There always exist solutions to (2.2.4)-(2.2.6) [112, Lem. 4.5] (see also [120, Cor. 5.5] [4]). Note that (2.2.4) and (2.2.6) imply $\epsilon : B^+ \times B^+ \to \mathbb{Z}_2$ is a group 2-cocycle.

One makes V_{A^+,B^+} into a *graded* vertex R-algebra by declaring $e^\alpha \otimes (v \otimes t^{-n})$ to be of degree $2n - \chi^+(\iota(\alpha), \iota(\alpha))$. This is called the *generalized lattice vertex R-algebra*

[4] Note that Kac proves any solution of (2.2.4)-(2.2.6) will be unique up to equivalence. However, Kac is working with lattices. In general, the ambiguity in choosing a solution of (2.2.4)-(2.2.6) is controlled by the 2-torsion in B^+ [115, Thm. 2.27].

associated to (A^+, χ^+) and ι. When $B^+ = A^+$ with $\iota = \mathrm{id}_{A^+}$ and A^+ is torsion-free this is called the (graded) lattice vertex R-algebra associated to the integral lattice (A^+, χ^+) (see [120, § 5.4], [143, § 6.4-5]).

Given a finitely generated abelian group A^- equipped with an anti-symmetric integral bilinear form χ^- one can construct a similar graded vertex R-algebra. Consider the Lie algebra $\mathfrak{h}^- := A^- \otimes t^{\frac{1}{2}} R[t, t^{-1}] \oplus R\underline{K}$ with commutation relations

$$[v \otimes t^{m+\frac{1}{2}}, w \otimes t^{n+\frac{1}{2}}]_+ = m\chi^-(v, w)\delta_{m,-n}\underline{K}, \quad [\underline{K}, \mathfrak{h}^-] = 0.$$

Let $\mathcal{A} = U(\mathfrak{h}^-)/(\underline{K} - 1)$ and let $\mathcal{A}_{\geq 0}$ denote the ideal of $\mathcal{A}$ generated by elements of the form $(v \otimes t^{m+\frac{1}{2}}) \cdot 1$ with $v \in A^-$ and $m \geq 0$. Then there is a natural Lie algebra representation ρ^- of $\mathfrak{h}^-$ on $\mathcal{A}/\mathcal{A}_{\geq 0} \cong \bigwedge(A^- \otimes t^{-\frac{1}{2}} R[t^{-1}])$. Make $\bigwedge(A^- \otimes t^{-\frac{1}{2}} R[t^{-1}])$ into a graded R-algebra by declaring $v \otimes t^{-i-\frac{1}{2}}$ to be of degree $2i + 1$. For $v \in A^-$ write $v(z) = \sum_{n \in \mathbb{Z}} \rho^-(v \otimes t^{n+\frac{1}{2}}) z^{-n-1}$. The collection of fields $\{v(z)\}_{v \in A^+}$ generate a graded vertex R-algebra structure on $\bigwedge(A^- \otimes t^{-\frac{1}{2}} R[t^{-1}])$. When A^- is torsion-free and χ^- is non-degenerate, this construction gives Abe's symplectic fermionic vertex operator algebra [1].[5]

A *generalized integral super-lattice* is a finitely generated abelian group A, which is written as a direct sum $A = A^+ \oplus A^-$ of finitely generated abelian groups, equipped with a $\mathbb{Z}$-valued bilinear form χ such that $\chi^+ := \chi|_{A^+}$ is symmetric and $\chi^- := \chi|_{A^-}$ is anti-symmetric.

Definition 2.2.11. Let $(A = A^+ \oplus A^-, \chi)$ be a generalized integral super-lattice. Let $\iota : B^+ \to A^+$ be a map of finitely generated abelian groups. Let V_{A^+,B^+} be the generalized lattice vertex R-algebra associated to (A^+, χ^+) and ι. Let V_{A^-} denote the graded vertex R-algebra associated to (A^-, χ^-). Write $V_A = V_{A^+,B^+} \otimes V_{A-}$ for the tensor product graded vertex R-algebra. This is called the *generalized super-lattice*

[5] Abe's symplectic fermionic vertex algebras have state space of the form $\bigwedge(A^- \otimes t^{-1} R[t^{-1}])$ rather than $\bigwedge(A^- \otimes t^{-\frac{1}{2}} R[t^{-1}])$. We use the factor $t^{-\frac{1}{2}}$ so that all generators are in odd degrees.

vertex R-algebra associated to $(A = A^+ \oplus A^-, \chi)$ *and* ι. It is defined up to a choice of solution to (2.2.4)-(2.2.6) .

Let $(A = A^+ \oplus A^-, \chi)$ be an generalized integral super-lattice, $\iota : B^+ \to A^+$ be a map of finitely generated abelian groups, Q^+, Q^- be additive bases for $A^+ \otimes_{\mathbb{Z}} R, A^- \otimes_{\mathbb{Z}} R$ respectively, and $Q = Q^+ \cup Q^-$. Then the map

$$e^\alpha \otimes (v^+ \otimes t^{-i})^{n_{v^+,i}} \otimes (v^- \otimes t^{-j-\frac{1}{2}})^{m_{v^-,j}} \mapsto e^\alpha \otimes u_{v^+,i}^{n_{v^+,i}} \cdot u_{v^-,j}^{m_{v^-,j}} \tag{2.2.7}$$

defines an isomorphism of graded R-algebras

$$V_A \cong R[B^+] \otimes \mathrm{SSym}_R[u_{v,i} : v \in Q, i \geq 1] \tag{2.2.8}$$

where the right hand side of (2.2.8) is graded by declaring $e^\alpha \otimes \prod_{v \in Q, i \geq 1} u_{v,i}^{n_{v,i}}$ to be of degree $\sum_{v^+ \in Q^+, i \geq 1} 2in_{v^+,i} + \sum_{v^- \in Q^-, i \geq 1} (2i+1)n_{v,i} - \chi^+(\iota(\alpha), \iota(\alpha))$.
For $v^+ \in A^+ \otimes R$ one writes

$$v^+(z) := \sum_{n \in \mathbb{Z}} \rho^+(v^+ \otimes t^n) \otimes \mathrm{id}\ z^{-n-1}$$

and for $v^- \in A^- \otimes R$ one writes

$$v^-(z) := \sum_{n \in \mathbb{Z}} \mathrm{id} \otimes \rho^-(v^- \otimes t^{n+\frac{1}{2}}) z^{-n-1}.$$

The following is a consequence of the reconstruction theorem.

Proposition 2.2.12. *Let* $A, A^+, A^-, \chi^+, B^+, \iota, Q, Q^+,$ *and* Q^- *be as above. Then any graded vertex R-algebra with space of states* $R[B^+] \otimes \mathrm{SSym}_R[u_{v,i} : v \in Q, i \geq 1]$ *such that*

$$(u_{0,v,1})_n = v(z)_n \tag{2.2.9}$$

for all $v \in Q, n \in \mathbb{Z}$ *is isomorphic to the generalized super-lattice vertex R-algebra* *associated to* (A, χ) *and* ι.

Proof. The proof is nearly identical to that of [120, Thm. 5.4]. The collection of fields $\{v(z), \Gamma_\alpha(z)\}_{v \in Q, \alpha \in B^+}$ is a set of mutually local generating fields. And (2.2.9) implies the state-to-field correspondence maps $e^\alpha \otimes 1 \otimes 1 \mapsto \Gamma_\alpha(z)$ (see [120, p. 102-3]). By Proposition 2.2.8, (2.2.9) determines a graded vertex R-algebra structure on $R[B^+] \otimes \mathrm{SSym}_R[u_{v,i} : v \in Q, i \geq 1]$ unique up to isomorphism.

To show that the super-lattice vertex algebra $V_A \cong R[B^+] \otimes \mathrm{SSym}_R[u_{v,i} : v \in Q, i \geq 1]$ satisfies (2.2.9), suppose that $v \in Q^+$. Then, as $e^0 \otimes (v \otimes t^{-1}) \otimes 1$ is sent to $u_{0,v,1}$ under (2.2.9), we have

$$\begin{aligned}
(u_{0,v,1})_n &= (e^0 \otimes (v \otimes t^{-1}) \otimes 1)_n \\
&= \mathrm{Coeff}_{-n-1} Y(e^0 \otimes (v \otimes t^{-1}) \otimes 1, z) \\
&= \mathrm{Coeff}_{-n-1} Y_+(e^0 \otimes (v \otimes t^{-1}), z) \otimes Y_-(1, z) \\
&= \mathrm{Coeff}_{-n-1} (\sum_{n \in \mathbb{Z}} \rho^+(v \otimes t^n) \otimes \mathrm{id} z^{-n-1}) \\
&= v(z)_n.
\end{aligned}$$

The $v \in Q^-$ case is similar. $\square$

2.3 Formal group rings and vertex F-algebras

We begin this section with a discussion of formal group laws and formal group rings. For further background on formal group laws the reader is referred to Hazewinkel [99].

As in Borcherds [35] and Li [148] one can use formal group laws to define generalized vertex algebras, called vertex F-algebras. Moreover, the Borcherds' bicharacter construction–where one constructs a vertex algebra from the data of a bialgebra with a shift operator and a bicharacter–also works for vertex F-algebras. In Chapter 5 we show that Joyce's construction, over $\mathbb{Q}$,[6] comes from a bicharacter constrution and that it can be modified to obtain a vertex F-algebra on the complex-oriented homology of appropriate moduli spaces.

Definition 2.3.1. A *formal group law*[7] over R is a formal power series $F(X,Y) \in R[[X,Y]]$ such that

1. $F(X,Y) = F(Y,X)$,
2. $F(0,Y) = Y$ and $F(X,0) = X$, and
3. $F(F(X,Y),Z) = F(X,F(Y,Z))$.

For any formal group law F there exists a unique power series $\iota_F = \iota \in R[[X]]$ such that $F(X,\iota(X)) = 0$. This series is called the *inverse of F*. The inverse of the additive formal group law is $\iota(X) = -X$. The inverse of a general formal group law often behaves formally like an additive inverse.

Example 2.3.2. The *multiplicative formal group law* is defined by $F(X,Y) = X + Y + XY$ and has $\iota(X) = (1+X)^{-1} - 1 = -X + X^2 - X^3 + \ldots$. In fact, one can show that all formal group laws are of the form

$$F(X,Y) = X + Y + O(XY)$$

and all inverses are of the form

$$\iota(X) = -X + O(X^2).$$

[6]The reason we have to take rational coefficients is that without a Künneth decomposition, the homology of a topological space may not be a coalgebra. This problem could be skirted by lifting the entire construction to chains or spectra. We believe that at the spectral level, Joyce's vertex algebra always comes from a spectral Borcherds bicharacter construction.

[7]Technically, this is the definition of a commutative 1-dimensional formal group law.

Proposition 2.3.3. *Let F be a formal group law and let $\iota(X)$ be the inverse of F. Then*

$$\iota(\iota(X)) = X, \tag{2.3.1}$$

and

$$\iota(F(\iota(X), Y)) = F(X, \iota(Y)). \tag{2.3.2}$$

Proof. For (2.3.1) we have

$$F(\iota(X), X) = F(X, \iota(X)) = 0$$

so that $\iota(\iota(X)) = X$ by uniqueness of formal group law inverses. Moreover

$$\begin{aligned} F(F(X,\iota(Y)), F(\iota(X),Y)) &= F(F(\iota(X),Y), F(X,\iota(Y))) \\ &= F(\iota(X), F(Y, F(X,\iota(Y)))) \\ &= F(\iota(X), F(F(X,\iota(Y)),Y)) \\ &= F(\iota(X), F(X, F(\iota(Y),Y))) \\ &= F(\iota(X), F(X,0)) \\ &= 0 \end{aligned}$$

which gives (2.3.2) by uniqueness of formal group law inverses. □

Definition 2.3.4. Let $F(X,Y) \in R[[X,Y]]$ be a formal group law. The *formal group ring* $R[F]$ of F is the following Hopf R-algebra. As an R-module $R[F]$ is the free R-module with basis the set of symbols $D^{(k)}$ for $k > 0$. The coproduct on $R[F]$ is defined by $D^{(i)} \mapsto \sum D^{(i-j)} \otimes D^{(j)}$. The product is defined by

$$D^{(i)} \cdot D^{(j)} = \sum_{k \geq 0}^{i+j} F_{kij} D^{(k)},$$

where F_{kij} are defined as the coefficients in the expansion

$$F(X,Y)^k = \sum_{i,j\geq 0} F_{kij} x^i y^j.$$

The antipode is defined by $D^{(i)} \mapsto (-1)^i D^{(i)}$. If F is moreover a *graded formal group law* (see Rudyak [190, Ch. VII, Def. 5.17]) then $R[F]$ can be promoted to a graded Hopf R-algebra by declaring $D^{(i)}$ to be of degree $2i$.

Remark 2.3.5. The product $\cdot$ on $R[F]$ can be understood as follows: The dual of $R[F]$ is a formal power series ring $R[[z]]$. A formal group law induces a coproduct $\Delta : R[[z]] \mapsto R[[z_1]]\hat{\otimes} R[[z_2]] \cong R[[z_1, z_2]]$ by $z \mapsto F(z_1, z_2)$. Then the following diagram commutes

$$\begin{tikzcd}
R[F] \otimes R[F] \arrow[r, "\cdot"] \arrow[d, hook] & R[F] \arrow[d, hook] \\
(R[F] \otimes R[F])^{**} \arrow[r, "\Delta^*"] & R[F]^{**},
\end{tikzcd}$$

which gives a formula for the natural action of $R[F]$ on its dual $R[[z]]$

$$D^{(m)} \cdot \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} (\sum_{k=0}^{n+m} F_{knm} a_k) z^n.$$

Formal group law gives rise to a generalized notion of shift operator.

Definition 2.3.6. Let V be a (graded) R-module and let F be a formal group law over R. An *F-shift operator* is a (graded[8]) R-linear map $\mathcal{D}(z) : V \to V[[z]]$ such that

$$\mathcal{D}(z) \circ \mathcal{D}(w) = \mathcal{D}(F(z,w)). \tag{2.3.3}$$

and

$$\mathcal{D}(z) \circ \mathcal{D}(\iota(z)) = \mathrm{id}. \tag{2.3.4}$$

[8] We take z to be graded of degree -2.

Example 2.3.7. Let $V = \mathbb{Q}[x]$ be the ring of $\mathbb{Q}$-coefficient polynomials in one variable. Define

$$\mathcal{D}(z)(f(x)) = e^{z\frac{d}{dx}} f(x).$$

By the exponential law $\mathcal{D}(z)$, is a $\mathbb{G}_a$-shift operator. Moreover, for a fixed $a \in \mathbb{Q}$,

$$\mathcal{D}(a)(f(x)) = f(a + x). \tag{2.3.5}$$

Equation (2.3.5) motives the terminology 'F-shift operator.'

F-shift operators give examples of representations of formal group rings.

Proposition 2.3.8. *Let V be a graded R-module, let F be a formal group law over R, and let $\mathcal{D}(z) : V \to V[[z]]$ be an F-shift operator. Then V is canonically a representation of the formal group ring $R[F]$.*

Proof. First we expand $\mathcal{D}(z)$ in a power series

$$\mathcal{D}(z) = \sum_{k \geq 0} D_F^{(k)} z^k.$$

Define an action of $R[F]$ on V by $D^{(i)} \cdot a = D_F^{(i)}(a)$. Expanding the identity $\mathcal{D}(z) \circ \mathcal{D}(w)(a) = \mathcal{D}(F(z, w))$ gives

$$(\sum_{n \geq 0} D_F^{(n)} z^n)(\sum_{m \geq 0} D_F^{(m)} w^m)(a) = (\sum_{k \geq 0} D_F^{(k)} (F(z, w))^k)(a) = \sum_{k \geq, i, j \geq 0} F_{kij} D_F^{(k)}(a). \tag{2.3.6}$$

Fixing $n, m \geq 0$ and comparing coefficients of $z^n w^m$ on the left and right hand sides of (2.3.6) we get

$$(D_F^{(n)} \circ D_F^{(k)})(a) = \sum_{k \geq 0}^{n+m} F_{knm} D_F^{(k)}(a). \qquad \square$$

Conversely, a representation of $R[F]$ on an R-module V determines an F-shift operator on V.

Proposition 2.3.9. *Let V be a graded R-module and let $\rho : R[F] \otimes V \to V$ be a representation of $R[F]$. Then the R-linear map $\mathcal{D}(z) : V \to V[[z]]$ by*

$$\mathcal{D}(z)(a) = \sum_{k \geq 0} (D^{(k)} \cdot a) z^n$$

is an F-shift operator.

Proof. Compute

$$\begin{aligned}
\mathcal{D}(z) \circ \mathcal{D}(w)(a) &= \sum_{n \geq 0} D^{(n)} \cdot (\sum_{m \geq 0} D^{(m)} \cdot a) w^m z^n \\
&= \sum_{n,m \geq 0} \sum_{k=0}^{n+m} (F_{knm} D^{(k)} \cdot a) w^m z^n \\
&= \sum_{k \geq 0} \sum_{n,m \geq 0} (F_{knm} D^{(k)} \cdot a) w^m z^n \\
&= \sum_{k \geq 0} (D^{(k)} \cdot a) F(z, w)^k.
\end{aligned}$$

□

Definition 2.3.10. Let $(B, m, \Delta, \eta, 1_B)$ be a graded bialgebra over R, let $F(X, Y) \in R[[X, Y]]$ be a formal group law, and let $\mathcal{D}(z) : V \to V[[z]]$ be an F-shift operator. We say that $\mathcal{D}(z)$ is *compatible with B* if $\mathcal{D}(z)$ is compatible with the underlying algebra of B and if for all $a, b \in B$

$$\begin{aligned}
&\Delta(\mathcal{D}(z)(a)) = \mathcal{D}(z)(a') \otimes \mathcal{D}(z)(a''), \text{ and} \\
&\mathcal{D}(z)(\eta(a)) = \eta(a).
\end{aligned}$$

Example 2.3.11. Consider the dual $R[[z]]$ of the formal group ring $R[F]$. The coproduct

$$R[[z]] \longrightarrow R[[z, w]] \cong R[[z]] \hat{\otimes} R[[w]]$$

$z \mapsto F(z, w)$ defines an F-shift operator on $R[[z]]$. The F-shift property is equivalent to associativity of the formal group law. This F-shift operator is tautologically compatible with the bialgebra structure on $R[[z]]$.

A *morphism* $\theta : F \to G$ of formal group laws over R is a power series θ such that

$$\theta(F(X, Y)) = G(\theta(X), \theta(Y)).$$

When R is a $\mathbb{Q}$-algebra each formal group law $F(X, Y) \in R[[X, Y]]$ admits a *logarithm* $\log_F(X) \in R[[X]]$ with the property that

$$\log_F(F(X, Y)) = \log_F(X) + \log_F(Y).$$

The logarithm is an isomorphism. This means all formal group laws over a $\mathbb{Q}$-algebra are isomorphic to the additive formal group law and all formal group rings over a $\mathbb{Q}$-algebra are isomorphic to polynomial rings.

2.4 Borcherds' bicharacter construction

In [35] Borcherds gave a method for constructing many examples of vertex algebras from simpler algebraic data: a bialgebra with a derivation and a bicharacter. Borcherds' construction of "G-vertex algebras" works in the setting of a general symmetric (or even braided) monoidal category and uses a complicated theory of singular tensor products.

We give a simpler presentation of Borcherds' bicharacter construction in the special case of vertex F-algebras. The reader wishing to learn more about Borcherds' bicharacter construction should also consult Angelova [7], Patnaik [180], and Snydal [202].

Definition 2.4.1. Because all formal group laws are of the form $F(z,w) = z + w + O(zw)$ we have that the constant term of $F(z,w)$ in $R((z))((w))$ is z, which is a unit. So, $F(z,w)$ has an inverse in $R((z))((w))$ which we denote $i_{z,w}F(z,w)^{-1}$. By symmetry, $F(z,w)$ also has an inverse in $R((w))((z))$ which we denote $i_{w,z}F(z,w)^{-1}$. This yields, for any R-module V, R-linear maps

$$i_{z,w} : V[[z,w]][z^{-1}, w^{-1}, F(z,w)^{-1}] \longrightarrow V((z))((w))$$

and

$$i_{w,z} : V[[z,w]][z^{-1}, w^{-1}, F(z,w)^{-1}] \longrightarrow V((w))((z)).$$

Proposition 2.4.2. *Let V be an R-module and let F be a formal group law over R. Then two series $A(z,w) \in V((z))((w))$ and $B(z,w) \in V((w))((z))$ satisfy the property that there exists some series $C(z,w) \in V[[z,w]][z^{-1}, w^{-1}, F(z,w)^{-1}]$ such that*

$$A(z,w) = i_{z,w}C(z,w)$$

and

$$B(z,w) = i_{w,z}C(z,w)$$

if and only if there exists $N \gg 0$ such that

$$F(z,w)^N A(z,w) = F(z,w)^N B(z,w).$$

Proof. Suppose there exists $N \gg 0$ such that

$$F(z,w)^N A(z,w) = F(z,w)^N B(z,w).$$

Let $D(z,w) := F(z,w)^N A(z,w)$. Note that

$$V((z))((w)) \cap V((w))((z)) = V[[z,w]][z^{-1},w^{-1}].$$

This implies $D(z,w) \in V[[z,w]][z^{-1},w^{-1}]$. Let $C(z,w) := F(z,w)^{-N}D(z,w)$ which lies in $V[[z,w]][z^{-1},w^{-1},F(z,w)^{-1}]$. Then $i_{z,w}C(z,w) = A(z,w)$ and $i_{w,z}C(z,w) = B(z,w)$.

Conversely, suppose that there exists some $C(z,w) \in V[[z,w]][z^{-1},w^{-1},F(z,w)^{-1}]$ such that $i_{z,w}C(z,w) = A(z,w)$ and $i_{w,z}C(z,w) = B(z,w)$. Then there exists some $N \gg 0$ such that $F(z,w)^N C(z,w) \in V[[z,w]][z^{-1},w^{-1}]$. Since both $i_{z,w}$ and $i_{w,z}$ are the identity on $V[[z,w]][z^{-1},w^{-1}] \subset V[[z,w]][z^{-1},w^{-1},F(z,w)^{-1}]$, we have

$$\begin{aligned} F(z,w)^N A(z,w) &= i_{z,w}F(z,w)^N C(z,w) \\ &= i_{w,z}F(z,w)^N C(z,w) \\ &= F(z,w)^N B(z,w). \end{aligned}$$

□

Definition 2.4.3. Let $F(X,Y) \in R[[X,Y]]$ be a graded formal group law. A *graded nonlocal vertex F-algebra* is a graded R-module V together with a *vacuum* $|0\rangle \in V_0$, an F-shift operator $\mathcal{D}(z) : V \to V[[z]]$, and a graded R-linear map $Y(-,z) : V \to \mathcal{F}(V)$ called a *state-to-field correspondence* such that

- *vacuum and creation:* for all $a \in V$, $Y(a,z)|0\rangle \in V[[z]]$ with $Y(a,z)|0\rangle|_{z=0} = a$ and $Y(|0\rangle, z) = \mathrm{id}$,
- *F-translation covariance:* for all $a \in V$

$$Y(\mathcal{D}(w)(a), z) = i_{z,w}Y(a, F(z,w)) \text{ and } \mathcal{D}(z)(|0\rangle) = |0\rangle,$$

and

- *weak F-associativity:* for all $a, b, c \in V$ there exists some $N \gg 0$ such that

$$F(z,w)^N Y(Y(a,z)b,w)c = F(z,w)^N i_{z,w} Y(a, F(z,w))Y(b,w)c.$$

A graded nonlocal vertex F-algebra is said to be a *graded vertex F-algebra* if

$$Y(a,z)b = (-1)^{ab}\mathcal{D}(z) \circ Y(b, \iota(z))a$$

for all $a, b \in V$.

We can prove the following general properties of nonlocal vertex F-algebras.

Lemma 2.4.4. *Let V be a graded nonlocal vertex F-algebra. Then for all $a \in V$*

$$Y(a,z)|0\rangle = \mathcal{D}(w)(a) \tag{2.4.1}$$

and

$$\mathcal{D}(w) \circ Y(a,z) = i_{z,w} Y(a, F(z,w)) \circ \mathcal{D}(w). \tag{2.4.2}$$

Proof. Vacuum and creation and F-translation covariance give

$$\begin{aligned}\mathcal{D}(w)(a) &= Y(\mathcal{D}(w)(a), z)|0\rangle|_{z=0} \\ &= Y(a, F(z,w))|0\rangle|_{z=0} \\ &= Y(a,z)|0\rangle.\end{aligned}$$

From this

$$\begin{aligned}i_{z,w} Y(a, F(z,w))\mathcal{D}(w)(b) &= i_{z,w} Y(a, F(z,w))Y(b,w)|0\rangle \\ &= Y(Y(a,z)b,w)|0\rangle \\ &= \mathcal{D}(w) \circ Y(a,z)b.\end{aligned}$$

□

In [7] Angelova constructs $\mathbb{Z}_2$-graded vertex algebras using $\mathbb{Z}_2$-graded bicharacters. This construction carries over wholesale to the construction of graded vertex algebras. However, we will need to consider something more general as the bicharacters coming from moduli theoretic examples in Chapter 5 are not graded. However, they are graded with respect to a shift induced by a quadratic form.

Definition 2.4.5. Let M be an abelian monoid and let $Q : M \to \mathbb{Z}$ be a quadratic form. Let B be a graded R-bialgebra which decomposes as

$$B = \bigoplus_{\alpha \in M} B_\alpha$$

such that the projection $B \to M$ is a monoid homomorphism. Given a graded R-module B the Q-*shift* of B is the graded R-module $\hat{B}$ defined by

$$(\hat{B}_\alpha)_n := (B_\alpha)_{n-Q(\alpha)}.$$

Note that the product $\hat{B} \otimes \hat{B} \to \hat{B}$ may no longer be graded.

Definition 2.4.6. Let $(B, \mu, \Delta, \eta, 1_B)$ be a graded bialgebra and let $\mathcal{D}(z) : B \to B[[z]]$ be a compatible F-shift operator. Let M be an abelian monoid such that $B = \bigoplus_{\alpha \in M} B_\alpha$ with $B \to M$ a monoid homomorphism, let $Q : M \to \mathbb{Z}$ be a quadratic form, and let $\hat{B}$ be the Q-shift of B. An F-*bicharacter* (compatible with $B, \hat{B}$, and $\mathcal{D}(z)$) is an R-linear map $r_z : B \otimes B \to R((z))$ such that

1. $r_z(a \otimes 1) = r_z(1 \otimes a) = \eta(a)$,
2. $r_z(a \cdot b \otimes c) = (-1)^{bc'} r_z(a, c') r_z(b, c'')$,
3. $r_z(a \otimes b \cdot c) = (-1)^{a''b} r_z(a', b) r_z(a'' \otimes c)$,
4. $r_z(\mathcal{D}(w)(a) \otimes b) = i_{z,w} r_{F(z,w)}(a \otimes b)$, and

5. $r_z(a \otimes \mathcal{D}(w)(b)) = i_{z,w} r_{F(z,\iota(w))}(a \otimes b)$,

where $i_{z,w}, i_{w,z}$ are the expansion maps of Definition 2.4.1. An F-bicharacter is said to be *symmetric* if

6. $r_z(a \otimes b) = (-1)^{\hat{a}\hat{b}} r_{\iota(z)}(b \otimes a)$,

where $(-1)^{\hat{x}} := (-1)^{\deg(x) - Q(\gamma)}$ for $x \in B_\gamma$.

Remark 2.4.7. An F-bicharacter is said to be *even* if $r(a \otimes b) = 0$ whenever the parity of a is not equal to the parity of b. In many cases, there are no non-trivial non-even bicharacters (see Angelova [7, Rem. 3.4]). We will typically assume our bicharacters are even.

Theorem 2.4.8. *Let $B, Q, \hat{B}, \mathcal{D}(z)$, and r_z be as in Definition 2.4.6 with r_z even. Define $Y(-,z) : B \otimes B \to B((z))$ by the following composition*

$$\begin{tikzcd}
B \otimes B \arrow[d, "\Delta\otimes\Delta"'] \arrow[ddr, "Y(-{,}z)"] & \\
B \otimes B \otimes B \otimes B \arrow[d, "(\mathrm{id}\otimes\mathrm{id}\otimes r_z)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})"'] & \\
B \otimes B((z)) \arrow[r, "m\circ(\mathcal{D}(z)\otimes\mathrm{id})"'] & B((z)).
\end{tikzcd}$$

Then $(B, \mathcal{D}(z), Y(-,z), \eta(1_R))$ is a nonlocal vertex F-algebra.

If $r : \hat{B} \otimes \hat{B} \to R((z))$ is symmetric and graded of degree $Q(\alpha) + Q(\beta) - Q(\alpha + \beta)$, then $(\hat{B}, \mathcal{D}(z), Y(-,z), \eta(1_R))$ is a graded vertex F-algebra.

Proof. First

$$\begin{aligned}
Y(a,z)\eta(1_R) &= \sum_{(a)} \mathcal{D}(z)(a') r_z(a'', \eta(1_R)) \\
&= \sum_{(a)} \mathcal{D}(z)(a')\eta(a'') \\
&= \mathcal{D}(z)(\sum_{(a)} a'\eta(a''))
\end{aligned}$$

$$= \mathcal{D}(z)(a),$$

which implies vacuum and creation. For F-translation

$$\begin{aligned}
Y(\mathcal{D}(w)(a), z)b &= (-1)^{a''b'}\mathcal{D}(z)(\mathcal{D}(w)(a')) \cdot b' \cdot r_z(\mathcal{D}(w)(a''), b'') \\
&= (-1)^{a''b'}\mathcal{D}(F(z,w))(a') \cdot b' \cdot i_{z,w} r_{F(z,w)}(a'', b'') \\
&= i_{z,w} Y(a, F(z,w))b.
\end{aligned}$$

For weak F-associativity we first compute

$$\begin{aligned}
i_{z,w}Y(a, F(z,w))Y(b,w)c &= i_{z,w}((-1)^{b''c'}\mathcal{D}(w)(b') \cdot c' \cdot r_w(b'', c'')) \\
&= (-1)^{b''c' + a''(b^{(1)} + c^{(1)}) + b^{(2)}c^{(1)}}\mathcal{D}(F(z,w))(a') \cdot \\
&\quad \mathcal{D}(w)(b^{(1)}) \cdot c^{(1)} \cdot i_{z,w} r_{F(z,w)}(a'', \mathcal{D}(w)(b^{(2)}) \cdot c^{(2)}) \cdot r_w(b^{(3)}, c^{(3)}) \\
&= (-1)^{b''c' + a''(b^{(1)} + c^{(1)}) + b^{(2)}c^{(1)} + a^{(3)}b^{(2)}}\mathcal{D}(F(z,w))(a^{(1)}) \cdot \\
&\quad \mathcal{D}(w)(b^{(1)}) \cdot c^{(1)} \cdot i_{z,w} r_{F(z,w)}(a^{(2)}, \mathcal{D}(w)(b^{(2)})) \\
&\quad \cdot r_{F(z,w)}(a^{(3)}, c^{(2)}) \cdot c^{(2)}) \cdot r_w(b^{(3)}, c^{(3)}) \\
&= (-1)^{b''c' + a''(b^{(1)} + c^{(1)}) + b^{(2)}c^{(1)} + a^{(3)}b^{(2)}}\mathcal{D}(F(z,w))(a^{(1)}) \cdot \\
&\quad \mathcal{D}(w)(b^{(1)}) \cdot c^{(1)} \cdot r_z(a^{(2)}, b^{(2)}) \cdot i_{z,w} r_{F(z,w)}(a^{(3)}, c^{(2)}) \\
&\quad \cdot r_w(b^{(3)}, c^{(3)}) \\
&= (-1)^{b''c' + a''(b^{(1)} + c^{(1)}) + b^{(2)}c^{(1)} + a^{(3)}b^{(2)} + a^{(2)}a^{(3)} + b^{(2)}b^{(3)}} \\
&\quad \mathcal{D}(F(z,w))(a^{(1)}) \cdot \mathcal{D}(w)(b^{(1)}) \cdot c^{(1)} \cdot r_z(a^{(3)}, b^{(3)}) \\
&\quad \cdot i_{z,w} r_{F(z,w)}(a^{(2)}, c^{(2)}) \cdot r_w(b^{(2)}, c^{(3)}).
\end{aligned}$$

On the other hand

$$
\begin{aligned}
Y(Y(a,z)b,w)c &= (-1)^{a''b'}Y(\mathcal{D}(z)(a')\cdot b',w)c\cdot r_z(a'',b'') \\
&= (-1)^{a''b'+a^{(2)}b^{(1)}+(a^{(1)}+b^{(1)})c'}\mathcal{D}(w)(\mathcal{D}(z)(a^{(1)})\cdot b^{(1)})\cdot c' r_w \\
&\quad (\mathcal{D}(z)(a^{(2)})\cdot b^{(2)},c'')\cdot r_z(a^{(3)},b^{(3)}) \\
&= (-1)^{a''b'+a^{(2)}b^{(1)}+(a^{(1)}+b^{(1)})c'+b^{(2)}c^{(2)}}\mathcal{D}(F(z,w))(a^{(1)})\cdot\mathcal{D}(w)(b^{(1)}) \\
&\quad \cdot c' r_w(\mathcal{D}(z)(a^{(2)}),c^{(2)})\cdot r_w(b^{(2)},c^{(3)})\cdot r_z(a^{(3)},b^{(3)}) \\
&= (-1)^{a''b'+a^{(2)}b^{(1)}+(a^{(1)}+b^{(1)})c'+b^{(2)}c^{(2)}}\mathcal{D}(F(z,w))(a^{(1)})\cdot\mathcal{D}(w)(b^{(1)}) \\
&\quad \cdot c^{(1)}\cdot i_{w,z} r_{F(z,w)}(a^{(2)},c^{(2)})\cdot r_w(b^{(2)},c^{(3)})\cdot r_z(a^{(3)},b^{(3)}).
\end{aligned}
$$

which completes the proof by evenness of r and Proposition 2.4.2. Finally, if r is symmetric then

$$
\begin{aligned}
(-1)^{\hat{a}\hat{b}}\mathcal{D}(z)(Y(b,\iota(z))a) &= (-1)^{\hat{a}\hat{b}+b''a'}b'\cdot\mathcal{D}(z)(a')\cdot r_{\iota(z)}(b'',a'') \\
&= (-1)^{\hat{a}\hat{b}+b''a'+\hat{a''}\hat{b''}+a''b'}\mathcal{D}(z)(a')\cdot b'\cdot r_z(a'',b'') \\
&= Y(a,z)b,
\end{aligned}
$$

where the last equality is a consequence of evenness. $\square$

Chapter 3

Algebraic topology and gauge theory

In this chapter we give some background on the algebraic topology that will be used in later sections. We begin in Section 3.1 with a discussion of homotopy-theoretic group completions of H-spaces. We discuss Segal's Γ-spaces, which are machines for producing completions of suitable H-spaces. Completed Γ- (or E_∞-)spaces determine connective spectra.

On reason that spectra are useful is that they represent generalized cohomology theories. From this point of view, one can define many algebro-topological products on generalized (co)homology groups. Furthermore, generalized Thom classes allow one to define generalized fiber integration or *umkehr* maps.

Cohomology theories represented by spectra with *complex orientations* have associated generalized Chern classes commonly called *Conner–Floyd Chern classes.* We review some properties of Conner–Floyd Chern classes in Section 3.2. Section 3.3 covers a few facts from rational homotopy theory that are used in Chapter 4 to compute the rational cohomology of an evaluation map.

We then switch gears in Section 3.4 to reviewing material from Joyce–Tanaka–

Upmeier [115] on moduli spaces $\mathcal{B}$ of connections on a compact manifold X. Each elliptic operator D on X gives a principal $\mathbb{Z}_2$-bundle $O^D \to \mathcal{B}$ called an *orientation bundle*. An *orientation* of $\mathcal{B}$ (with respect to D) is defined to be a trivialization of O^D. Often one will consider spaces $\mathcal{M}^{\text{inst}}$ of irreducible instanton connections whose curvature satisfies some condition. In such cases, an orientation of $\mathcal{B}$ with respect to an elliptic operator D related to the instanton curvature condition gives an orientation of $\mathcal{M}^{\text{inst}}$ as an ordinary manifold (or as a derived manifold). Finally, in Section 3.5, we discuss Calabi–Yau manifolds and Spin(7)-manifolds. This includes a statement of orientability of moduli spaces of Spin(7)-instantons on Spin(7)-manifolds. Portions of sections 3.1, 3.2, and 3.3 are based on a background section from the author's currently unpublished paper [92].

3.1 Spectra and homotopy-theoretic group completion

Recall that one can complete a bialgebra to a Hopf algebra by inverting group-like elements. The analogue of that procedure at the level of spaces is called homotopy-theoretic group completion. In the presence of an additional structure called a Γ-*space* structure or an E_∞-*space* structure the homotopy-theoretic group completion is guaranteed to exist. Homotopy-theoretic group completions of E_∞-spaces have the structure of *infinite loop spaces*.

Infinite loop spaces are equivalent to connective spectra. Spectra are generalized topological spaces that can be used to represent exotic cohomology theories. In fact, the formalism of Γ- and E_∞-spaces can be used to construct new cohomological invariants. One example of a theory built in this way is semi-topological K-theory–an example that will be of central importance in Chapter 4. Representing cohomology theories by spectra also allows us to easily define numerous algebraic products. Also, for infinite CW complexes the usual description of complex topological K-theory as the group completion of the monoid of isomomorphism classes of complex vector bundles fails to give a generalized cohomology theory. Instead, one must define complex topological K-theory directly by its representing spectrum.

Definition 3.1.1. Let X be an H-space. Then a *homotopy-theoretic group completion of* X is an H-map $\Xi : X \to X^+$ to a group-like H-space X^+ such that

- $\pi_0(\Xi) : \pi_0(X) \to \pi_0(X^+)$ is group completion of the monoid $\pi_0(X)$, and
- $H_*(\Xi, A) : H_*(X, A) \to H_*(X^+, A)$ is module localization by the natural $\pi_0(X)$ action, for every abelian group A.

Note that homotopy-theoretic group completions are not defined by a universal property at the space level. Caruso–Cohen–May–Taylor have demonstrated that homotopy-theoretic group completions of H-spaces have a *weak* universal property [50]. Two continuous maps $f, g : X \to Y$ are said to be *weakly homotopy equivalent*, written $f \simeq^w g$ if for every continuous map $h : Z \to X$ from a finite CW complex Z there is a homotopy $f \circ h \simeq g \circ h$.

Proposition 3.1.2 (see Caruso–Cohen–May–Taylor [50, Prop. 1.2]). *Let $\Xi : X \to X^+$ be a homotopy-theoretic group completion such that $\pi_0(X)$ contains a countable cofinal*[1] *sequence. Then for any group-like H-space Z and any weak H-map $f : X \to Z$ there exists a weak H-map $g : X^+ \to Z$, unique up to weak homotopy, such that $g \circ \Xi \simeq^w f$.*

There are many models of spectra: sequential spectra, symmetric spectra [103], orthogonal spectra [194], etc., all of which present the same stable homotopy theory [154].

[1] The countable cofinality condition means that there exists a countable sequence $\{a_i\}$ of elements of $\pi_0(X)$ such that there exists $b_i \in \pi_0(X)$ such that $a_i = b_i \cdot a_{i+1}$ and for all $c \in \pi_0(X)$ there exists $d \in \pi_0(X)$ such that $d \cdot c = a_i$. This condition is satisfied in all of our examples.

Definition 3.1.3. A *(sequential) spectrum* is a sequence of based topological spaces $\{X_n\}$ together with *bonding maps* $\Sigma X_n \to X_{n+1}$. A spectrum is said to be an Ω-*spectrum* if all bonding maps are equivalences. All our spectra are assumed to be Ω-spectra. The 0^{th} space of an Ω-spectrum satisfies $X_0 \simeq \Omega^n X_n$. There is a functor Ω^∞ from Ω-spectra to infinite loop spaces defined by $\{X_n\} \mapsto X_0$.

Definition 3.1.4. Let X be a topological space and let X_+ denote the based space consisting of the disjoint union of X and a basepoint. The *suspension spectrum* $\Sigma^\infty(X_+)$ is the sequential spectrum with $\Sigma^\infty(X_+)_n := \Sigma^n(X_+)$, where $\Sigma^n(-)$ denotes the n-fold reduced suspension. The *sphere spectrum* $\mathbb{S}$ is defined to be $\Sigma^\infty(S^0)$.

The homotopy category of spectra is symmetric monoidal with respect to a functor $\wedge : \mathrm{Ho}(Sp) \times \mathrm{Ho}(Sp) \to \mathrm{Ho}(Sp)$ called the *smash product* (see, for example, Shipley [201]). The sphere spectrum $\mathbb{S}$ is the monoidal unit for $\wedge$.

The notion of the homotopy group of a spectrum generalizes the notion of the stable homotopy group of a topological space.

Definition 3.1.5. The k^{th} homotopy group of a spectrum $\{X_n\}$ is defined to be

$$\pi_k(X) := \varinjlim_n \pi_{n+k}(X_n).$$

A spectrum is said to be *connective* if its k^{th} homotopy group vanishes whenever k is negative.

There are equivalences between the homotopy categories of connective spectra and infinite loop spaces [38, 1.4], [195, 5.1]. One common method of producing spectra are the delooping machines of May [159] and Segal [195].

Definition 3.1.6. Let Γ denote the skeletal category of finite pointed sets. A Γ-*space* is a functor $S : \Gamma^{\text{op}} \to \mathrm{sSet}$ such that

- $S(0)$ is contractible, and

- the *Segal maps* $p_n : S([n]) \to S([1]) \times \cdots \times S([1])$ induced by the sets functions $\iota_k : [1] \to [n]$ are homotopy equivalences for all $n > 0$.

Given a Γ-space $S : \Gamma^{\mathrm{op}} \to \mathrm{sSet}$ we regard $S([1])$ as the 'underlying space' of S. Γ-spaces simultaneously generalize both H-spaces and topological abelian groups.

Let q_2 be a homotopy inverse of p_2. Then the operation

$$S([1]) \times S([1]) \xrightarrow{q_2} S([2]) \xrightarrow{\iota_2} S([1])$$

makes $S([1])$ into an H-space. If we require the Segal maps to all be homeomorphisms, then $S([1])$ is a topological abelian group. The analogue of a group-like H-space is a *very special* Γ*-space.*[2] In particular, if S is a very special Γ-space then $S([1])$ is a group-like H-space. Given a Γ-space S, Segal constructs a map $S \to \Omega BS$ which is a homotopy-theoretic group completion.[3] There is then an associated connective Ω-spectrum $\{\Omega BS, BS, B^2S, \dots\}$ whose 0^{th} space is a homotopy-theoretic group completion of S. Note that this procedure endows ΩBS with the structure of an *infinite loop space.*

May's version of very special Γ-spaces are *group-like* E_∞*-spaces* [158]. May defines E_∞*-operads* and *actions* of E_∞-operads. If an E_∞-operad acts on a topological space X then X is an H-space with a homotopy-theoretic group completion, which is modelled on a two-sided bar construction (see May [160]). The connective spectra produced by May's construction agree with those produced by Segal's construction [163].

Example 3.1.7. The H-space $\coprod_{n\geq 0} BU(n)$ is moreover a Γ-space. The associated spectrum is called the *connective complex topological K-theory spectrum* and written bu. The 0^{th} space of bu is $BU \times \mathbb{Z}$.

Example 3.1.8. Let X be a finite CW complex. Then $\mathrm{Map}_{C^0}(X, \coprod_{n\geq 0} BU(n))$ is a Γ-space. The induced connective spectrum is written $K^{\mathrm{top}}(X)$ and it is the

[2] What we call a "Γ-space" is what Bousfield–Friedlander [38] call a "special Γ-space." Segal also calls these Γ-spaces. The term "very special Γ-space" comes from Bousfield–Friedlander.

[3] The functor $B(-)$ denotes a simplicial realization and $\Omega(-)$ denotes the loop space functor.

connective cover of the mapping spectrum $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty(X_+), bu)$ [82, Prop. 6.7]. In particular, the 0^{th} space of $K^{\text{top}}(X)$ is $\mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})$ and the natural map $\mathrm{Map}_{C^0}(X, \coprod_{n\geq 0} BU(n)) \to \mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})$ is a homotopy-theoretic group completion.

The ideas of spectra and homotopy-theoretic group completions can be used to define semi-topological K-theory. The semi-topological K-theory of a complex variety X is an invariant interpolating between algebraic K-theory of X and the complex topological K-theory of X^{an}. Semi-topological K-theory can be defined for arbitrary complex quasi-projective varieties [83]. However, the theory is simpler for weakly normal varieties. Smooth varieties are examples of weakly normal varieties [142, Lem. 2.5]. For a survey of semi-topological K-theory, the reader is referred to Friedlander–Walker [83].

Like algebraic K-theory and complex topological K-theory, the bundle theory $K^{\text{sst}}(-)$ has accompanying cycle and cocycle theories called *Lawson homology* and *morphic cohomology*, respectively (see [138] [139] [140]). There is always a spectral sequence relating the bundle theory to the cocycle theory.

K-theory	Spectral Sequence
topological	Atiyah–Hirzebruch spectral sequence [18]
semi-topological	Friedlander–Walker spectral sequence [79]
algebraic	motivic spectral sequence [80]

We follow the construction of semi-toplogical K-theory in [82]. Let $\mathrm{IndSch}_{\mathbb{C}}$ denote the category of $\mathbb{C}$-ind-schemes. If $T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \dots$ is a sequence of closed embeddings of $\mathbb{C}$-schemes then the direct limit $T := \varinjlim_k T_k$ is an example of a $\mathbb{C}$-ind-scheme. Infinite affine space $\mathbb{A}^\infty := \varinjlim_k \mathbb{A}_k$ is an ind-scheme of this kind. All $\mathbb{C}$-ind-schemes appearing in this thesis will be of this type. For further background on ind-schemes the reader may wish to consult Gaitsgory–Rozenblyum [85]. The underlying analytic space T^{an} of such a $\mathbb{C}$-ind-scheme is given as $\varinjlim_k T_k^{\mathrm{an}}$.[4]

Definition 3.1.9. For $n \geq 0$, write $\mathrm{Gr}(\mathbb{C}^n) = \coprod_{k\geq 0} \mathrm{Gr}_k(\mathbb{C}^n)$. There is a sequence of closed embeddings $\mathrm{Gr}(\mathbb{C}^n) \hookrightarrow \mathrm{Gr}(\mathbb{C}^{n+1})$ and the direct limit of this sequence is a $\mathbb{C}$-ind-scheme denoted $\mathrm{Gr}(\mathbb{C}^\infty)$. Direct sum of subspaces induces a map $\chi_n : \mathrm{Gr}(\mathbb{C}^n) \times \mathrm{Gr}(\mathbb{C}^n) \to \mathrm{Gr}(\mathbb{C}^{2n})$. Because the χ_n maps commute with closed embeddings into higher Grassmannians, they admit a direct limit $\chi_\infty : \mathrm{Gr}(\mathbb{C}^\infty) \times \mathrm{Gr}(\mathbb{C}^\infty) \to \mathrm{Gr}(\mathbb{C}^\infty)$. This makes $\mathrm{Gr}(\mathbb{C}^\infty)^{\mathrm{an}}$ an H-space [82, Prop. 2.8].

For a complex projective variety X, $\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))^{\mathrm{an}}$ inherits an H-space structure from χ_∞. In fact, Friedlander–Walker construct the action of an E_∞-operad on $\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))^{\mathrm{an}}$ making it into an E_∞-space [82, Prop. 2.2]. Therefore, $\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))^{\mathrm{an}}$ admits a homotopy-theoretic group-completion which we denote $\Omega^\infty K^{\mathrm{sst}}(X)$ and call the *semi-topological K-theory space of X*.

Definition 3.1.10. Let X be a smooth complex projective variety. Then the *kth semi-topological K-theory group of X* is defined to be

$$K_n^{\mathrm{sst}}(X) := \pi_n(\Omega^\infty K^{\mathrm{sst}}(X)).$$

Definition 3.1.11. Note there is a natural map

$$\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))^{\mathrm{an}} \longrightarrow \mathrm{Map}_{C^0}(X^{\mathrm{an}}, \mathrm{Gr}(\mathbb{C}^\infty)^{\mathrm{an}}).$$

Functoriality of group completions gives a *K-theory comparison map*

$$\Gamma : \Omega^\infty K^{\mathrm{sst}}(X) \longrightarrow \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}).$$

We now discuss spectra in some further detail. One popular use of spectra is to

[4]$\mathbb{C}$-ind-schemes can also be regarded as non-Artin $\mathbb{C}$-stacks. The Betti realizations of $\mathbb{C}$-ind-schemes in this sense are homotopy equivalent to their underlying analytic spaces.

represent generalized cohomology theories. For further background on generalized cohomology theories see Adams [4] or Whitehead [225].

A *(generalized) reduced cohomology theory* is a collection of functors $\tilde{E}^i : \mathrm{CW} \to \mathrm{Ab}$ for $n \in \mathbb{Z}$ together with *suspension isomorphisms* $\tilde{E}^i(-) \cong \tilde{E}^{i+1}(\Sigma(-))$ satisfying homotopy invariance, exactness, and additivity. Given a reduced theory $\tilde{E}$ one can define the *unreduced theory* by $E^*(A, X) := \tilde{E}^*(X/A)$ and $E^*(X) := E^*(X, \emptyset) := \tilde{E}^*(X_+)$ where X_+ denotes X together with a disjoint basepoint.

It is worth pointing out that for every generalized cohomology theory there is a corresponding *Atiyah–Hirzebruch spectral sequence*

$$H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X)$$

relating E-cohomology groups to ordinary ones [18] [4, III.7].

Given two based spaces (or spectra) X, Y, we write $[X, Y]$ for the homotopy classes of based maps $X \to Y$. Given a spectrum $E = \{E_n\}$, the functors $\tilde{E}^n(-) = [-, E_n]$ define a reduced cohomology theory; in this case one says that E *represents* the reduced theory $\tilde{E}^*(-)$. The well-known Brown representability theorem states that for any reduced cohomology theory $\tilde{E}^*(-)$ there exists a spectrum[5] E that represents $\tilde{E}^*(-)$.

Example 3.1.12. Bott periodicity states that there are equivalences $\Omega(BU \times \mathbb{Z}) \cong U$ and $\Omega^2(BU \times \mathbb{Z}) \cong BU \times \mathbb{Z}$. This gives a non-connective spectrum $BU = \{\dots, BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, U, \dots\}$ that represents periodic complex topological K-theory $KU^*(-)$.

It is also true that $BU \times \mathbb{Z}$ is a group-like E_∞-space. The corresponding connective spectrum is the connective topological K-theory spectrum bu which represents connective complex topological K-theory $ku^*(-)$. As the 0^{th} space of both BU and bu is $BU \times \mathbb{Z}$, there is an equivalence $KU^0(-) \cong ku^0(-)$ in degree zero. Further, for

[5]By the suspension isomorphism axiom, one can assume without loss of generality that the representing spectrum is an Ω-spectrum (see Adams [3, p. 18-19])

a CW complex X, the suspension isomorphism gives

$$KU^i(X) \cong \tilde{KU}^0(\Sigma^i(X_+)) \cong \tilde{ku}^0(\Sigma^i(X_+)) \cong ku^i(X)$$

for all $i \geq 0$. However, $KU^{-i}(X)$ and $ku^{-i}(X)$ for $i > 0$ may not agree in general.

It is also worth noting a useful presentation of the homotopy type of $BU \times \mathbb{Z}$ in terms of Fredholm operators. Let $\mathcal{H}$ be any infinite-dimensional separable complex Hilbert space and let $\mathrm{Fred}(\mathcal{H})$ denote the H-space of Fredholm operators on $\mathcal{H}$. Then, by the Atiyah–Jänich theorem, there is a weak homotopy equivalence $\mathrm{Fred}(\mathcal{H}) \simeq BU \times \mathbb{Z}$ [13, App. 2].

Example 3.1.13. For any $n > 0$, given an abelian group A, there is an *Eilenberg–Maclane space* $K(A, n)$ with the property that $\pi_n(K(A, n)) \cong A$ and $\pi_i(K(n, A)) \cong 0$ for all $i \neq n$. There are equivalences $\Sigma K(A, n) \xrightarrow{\sim} K(A, n+1)$ giving an *Eilenberg–Maclane spectrum* HA. The reduced theory represented by HA is ordinary reduced cohomology with coefficients in A.

More generally, given a graded abelian group $A = \bigoplus_i A_i$ there is a *graded Eilenberg–Maclane spectrum* $HA := \prod_i \Sigma^i HA_i$ that represents reduced cohomology with coefficients in the graded abelian group A (see Boardman [30]).

The homotopy category of spectra $\mathrm{Ho}(Sp)$ is triangulated. Indeed, for any spectrum E and any $n \in \mathbb{Z}$ there is a spectrum $\Sigma^n E$ with the property that $\Sigma^n E_k = E_{n+k}$ for all $k \in \mathbb{Z}$. Then $\Sigma := \Sigma^1 : Ho(Sp) \to Ho(Sp)$ is a shift functor. In this language, the data of a degree n E-homology class is the same as the data of the homotopy class of a map $\mathbb{S} \to \Sigma^n(\Sigma^\infty X_+ \wedge E)$ and a degree n E-cohomology class is the same as the data of the homotopy class of a map $\Sigma^{-n}(\Sigma^\infty X_+) \to E$. E-homology and E-cohomology groups of spectra are defined in the same way. If E is a connective spectrum, then the homology groups represented by E vanish in negative degrees.

We now consider ring spectra. This is a class of spectra that represent cohomology theories with cup products.

Definition 3.1.14. A *ring spectrum* is an ordered triple (E, μ, η) where E is a spectrum, $\mu : E \wedge E \to E$ is a spectrum map called the *multiplication*, and $\eta : \mathbb{S} \to E$ is a spectrum map called the *unit* such that $\mu \circ (\mathrm{id} \wedge \mu) \simeq \mu \circ (\mu \wedge \mathrm{id})$ and $\mu \circ (\mathrm{id} \wedge \eta) \simeq \mathrm{id} \simeq \mu(\mu \wedge \mathrm{id})$.

If (E, μ, η) is a ring spectrum, then the cohomology theory represented by E has, in addition to cup products, external products, slant products, and cap products (see Whitehead [225, § 6]).

3.2 Thom classes and complex orientations

In this section we discuss Thom classes for general ring spectra before moving onto complex-oriented ring spectra. An excellent reference for all of the material in this section is Rudyak [190].

Let X be a connected finite CW complex and let $V \to X$ be a rank n real vector bundle. Choose a metric g on the fibers of V. Using g one can form the unit disc bundle $D(V)$ and the unit sphere bundle $S(V)$. The *Thom space* of V is the quotient bundle $\mathrm{Th}(V) = D(V)/S(V)$, which is independent of the choice of g. Pulling back $\mathrm{Th}(V) \to X$ along the inclusion of some point $x_0 \hookrightarrow X$ and applying Σ^∞ gives a map $\mathrm{fib} : \Sigma^n \mathbb{S} \to \Sigma^\infty \mathrm{Th}(V)$ whose homotopy class is independent of x_0. If (E, μ, η) is a ring spectrum, then an *E-Thom class* (or *E-orientation*) for V is a map $u_V : \Sigma^\infty \mathrm{Th}(V) \to \Sigma^n E$ such that $\Sigma^n(\eta) \simeq u_V \circ \mathrm{fib}$.

Now suppose that X is a compact connected k-manifold. Let $e : X \hookrightarrow \mathbb{R}^N$ be an embedding and let ν_e denote the normal bundle of e. Then the homotopy type of the spectrum $\mathrm{Th}(X) := \Sigma^{-N} \Sigma^\infty \mathrm{Th}(\nu_e)$ is independent of e and this spectrum is called the *Thom spectrum of X*. An *E-orientation of X* is an E-Thom class $u_X : \mathrm{Th}(X) \to \Sigma^{-k} E$ and X is said to be *E-orientable* if there exists an E-Thom class for X. The data of an E-Thom class for X is equivalent to the data of an E-orientation on its tangent bundle TX (see Dyer [68, §D.1 Prop. 5]).

There is a diagonal map $d : \mathrm{Th}(X) \to \mathrm{Th}(X) \wedge \Sigma^\infty X_+$ (see Rudyak [190, IV.5.36]). Given a map $v : \Sigma^\infty X_+ \to E$ and an E-orientation $u : \mathrm{Th}(X) \to \Sigma^{-k}E$ one gets a map $\mathrm{Th}(X) \to \Sigma^{-k}E$ by the composition

$$\mathrm{Th}(X) \xrightarrow{d} \mathrm{Th}(X) \wedge \Sigma^\infty X_+ \xrightarrow{u \wedge v} \Sigma^{-k}E \wedge E \xrightarrow{\mu} \Sigma^{-k}E.$$

This determines an equivalence $[\Sigma^\infty X_+, E] \simeq [\mathrm{Th}(X), \Sigma^{-k}E]$ which, by construction, induces the usual Thom isomorphism on homotopy groups.

The notion of umkehr map is a generalization of the notion of integration along the fibers of an oriented vector bundle. For further reading on umkehr maps, the reader is referred to Cohen–Klein [52] and Dold [57].

Let (E, μ, η) be a ring spectrum, let $f : X \to Y$ be a smooth map of compact connected E-oriented manifolds of dimensions k and ℓ. Then the *umkehr map* $f^!_E$: $[\Sigma^\infty X_+, E] \to [\Sigma^\infty Y_+, \Sigma^{\ell-k}E]$ is defined to be the composition

$$[\Sigma^\infty X_+, E] \simeq \left[\mathrm{Th}(X), \Sigma^{-k}E\right] \longrightarrow \left[\mathrm{Th}(Y), \Sigma^{-k}E\right] \cong \left[\Sigma^\infty Y_+, \Sigma^{\ell-k}E\right],$$

where the map $\mathrm{Th}(Y) \to \mathrm{Th}(X)$ is the one induced by $\Sigma^\infty X_+ \to \Sigma^\infty Y_+$ via Atiyah duality (see Atiyah [13] and Dold–Puppe [58]). If $f : X \to Y$ is merely a fibration whose homotopy fiber is a compact connected E-oriented k-manifold then there is a fiberwise Thom spectrum construction leading to an umkehr map $f^!_E : [\Sigma^\infty X_+, E] \to [\Sigma^\infty Y_+, \Sigma^{-k}E]$ (see Ando–Blumberg–Gepner [6]). We do not require umkehr maps in this generality. We need only consider projections $\pi : X \times Y \to Y$ where X is a compact connected E-oriented k-manifold–allowing us to avoid introducing parameterized spectra. In this case one has a Thom isomorphism $[\Sigma^\infty X_+ \wedge \Sigma^\infty Y_+, E] \to [\mathrm{Th}(X) \wedge \Sigma^\infty Y_+, \Sigma^{-k}E]$ by sending a map $v : \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \to E$ to the composition

$$\mathrm{Th}(X) \wedge \Sigma^\infty Y_+ \xrightarrow{d \wedge 1} \mathrm{Th}(X) \wedge \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \xrightarrow{u \wedge v} \Sigma^{-k}E \wedge E \xrightarrow{\mu} \Sigma^{-k}E.$$

One then gets an umkehr map on cohomology $E^*(X \times Y) \to E^*(Y)$ using these Thom isomorphisms.

Now let (E, μ_E, η_E) and (F, μ_F, η_F) be two ring spectra and let $\Phi : E \to F$ be a morphism of ring spectra. Let X be an E-oriented and F-oriented compact connected k-manifold. Then let $\mathrm{Td}^{E,F}(X)$ be the cohomology class given by the composition

$$\mathbb{S} \xrightarrow{1_E} [\Sigma^\infty X_+, E] \simeq \left[\mathrm{Th}(X), \Sigma^{-k} E\right] \xrightarrow{\Phi^*} \left[\mathrm{Th}(X), \Sigma^{-k} F\right] \simeq [\Sigma^\infty X_+, F] .$$

For example, when E is complex topological K-theory and F is rational ordinary cohomology then $\mathrm{Td}^{E,F}(X)$ is the usual Todd class of X.

We define the cohomology ring of an infinite CW complex to be the direct product of all cohomology groups. This is necessary, for example, to make sense of the universal Todd class of a complex line bundle $\mathrm{Td}(\mathcal{L}) \in H^*(BU(1), \mathbb{Q}) \cong \mathbb{Q}[[t]]$ which is a formal power series.[6]

There is a Riemann–Roch theorem for generalized cohomology theories. The following is a straightforward adaptation of Dold's Atiyah–Hirzebruch–Riemann–Roch theorem for maps between compact manifolds [57].

Lemma 3.2.1. *Let $\pi : X \times Y \to Y$ be a projection where X is a compact connected k-manifold. Let $\Phi : E \to F$ be a morphism of ring spectra and suppose that X is both E-oriented and F-oriented. Then for all $v : \Sigma^\infty X_+ \to E$ there is an equality*

$$F^*(\Phi)(\pi_E^!(v)) = \pi_F^!(E^*(\Phi)(v) \cdot \mathrm{Td}^{E,F}(X))$$

in $F^*(Y)$.

We now consider spectra that have orientations for the universal complex line bundle over $BU(1)$.

[6]In the section on rational homotopy theory we will actually be following the convention of defining the cohomology ring of an infinite CW complex to be the direct sum, not product, of its cohomology groups. This is in accordance with the conventions in rational homotopy theory literature. However, we use rational homotopy theory only to compute cohomology rings and one can easily recover the power series rings from the polynomial rings so there is no harm in doing this.

Definition 3.2.2. A *complex orientation* of a ring spectrum (E, μ, η) is a choice of E-Thom class for the universal complex line bundle. A ring spectrum together with a choice of complex orientation is said to be a *complex-oriented spectrum.*

An equivalent definition of a complex-oriented spectrum E is a spectrum together with an *orientation class* $c_1^E \in \tilde{E}(\mathbb{C}P^\infty)$ such that the image of c_1^E under the composition

$$\tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2)$$

is the canonical generator.

Cohomology theories represented by spectra with complex orientations include ordinary cohomology, complex topological K-theory, complex cobordism, Morava K-theories, and Peterson–Brown cohomologies. Note that an almost complex manifold is oriented with respect to any complex-oriented cohomology theory. For further reading on complex-oriented spectra, the reader may wish to consult Lurie [153], Ravenel [186], Rudyak [191], or Switzer [208].

It is well known that $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[c_1]/(c_1^{n+1})$ for all $n > 0$. Using the Atiyah–Hirzebruch spectral sequence one can compute that for any spectrum E with orientation class c_1^E there is a canonical isomorphism

$$E^*(\mathbb{C}P^n) \cong R[c_1^E]/((c_1^E)^{n+1}),$$

where $R = E^*(\mathrm{pt})$. Using the Milnor exact sequence [169] we then get

$$E^*(BU(1)) \cong R[[c_1^E]]$$

and similarly,

$$E^*(BU(n)) \cong R[[c_1^E, \ldots, c_n^E]],$$

where c_i^E is a formal variable of degree $2i$.

Definition 3.2.3. The tensor product map $\otimes : BU(1) \times BU(1) \to BU(1)$ makes $BU(1)$ into an H-space. Let $\pi_i : BU(1) \times BU(1) \to BU(1)$, $i = 1, 2$ denote projection onto the first and second factors respectively. Let $X := \pi_1^*(c_1^E)$ and $Y := \pi_2^*(c_1^E)$. Then $E^*(\otimes)(c_1^E) \in R[[X, Y]]$ is a formal group law in X and Y. We call this the *formal group law associated to* E.

Example 3.2.4. Let $E = H\mathbb{Z}$ be ordinary integral homology. The canonical generator $c_1 \in \tilde{H}^2(\mathbb{C}P^\infty) \cong \tilde{H}^2(\mathbb{C}P^1)$ defines an orientation class. The formal group law associated to $H\mathbb{Z}$ is the additive formal group law.

Example 3.2.5. Let $E = KU$ be (periodic) complex topological K-theory. Let $\mathcal{L} \to BU(1)$ denote the universal complex line bundle. Then $[\mathcal{L}] - 1 \in \tilde{K}^2(\mathbb{C}P^\infty)$ is an orientation class. The formal group law associated to KU is the multiplicative formal group law.

We can now naturally place formal group rings in the setting of algebraic topology.

Proposition 3.2.6. *Let E be a complex-oriented spectrum with associated formal group law F. Let $R = E^*(\mathrm{pt})$. Then there is a canonical isomorphism of graded Hopf R-algebras*

$$E_*(BU(1)) \cong R[F],$$

where $R[F]$ denotes the formal group ring of F (see Definition 2.3.4).

Proof. The diagonal map $\Delta : BU(1) \times BU(1) \to BU(1)$ along with the tensor product $BU(1) \times BU(1) \to BU(1)$ and duality $BU(1) \to BU(1)$ maps make $BU(1)$ a Hopf algebra object in the homotopy category of topological spaces. This makes $E_*(BU(1))$ and $E^*(BU(1))$ into graded Hopf R-algebras. Because $E_*(BU(1))$ is torsion-free as an R-module, by the generalized universal coefficients theorem [4, III.13.9], there is a canonical isomorphism $E_*(BU(1))^* \cong E^*(BU(1))$ of graded Hopf R-algebras.

The R-linear map

$$E^*(BU(1)) \cong R[[c_1^E]] \longrightarrow R[[z]] \cong (R[F])^*$$

determined by $c_1^E \to z$ is an isomorphism of graded R-algebras. The coproduct on $R[[z]]$ is given by $z \mapsto F(z,w)$ which, by definition of the formal group law associated to a complex-oriented spectrum, agrees with the coproduct on $E^*(BU(1))$ induced by tensor product of lines bundles. Because any two antipodes of a bialgebra must be equal, $E^*(BU(1))$ and $R[[z]]$ are isomorphic as graded Hopf R-algebras. Taking (restricted) duals gives a graded Hopf R-algebra isomorphism $E_*(BU(1)) \cong R[F]$. $\square$

Corollary 3.2.7. *Let $\mathcal{M}$ be a topological space equipped with a $BU(1)$-action $\rho : BU(1) \times \mathcal{M} \to \mathcal{M}$ defined up to homotopy. Then $E_*(\mathcal{M})$ is canonically a representation of the formal group ring $R[F]$. This representation gives an operator $\mathcal{D}(c_1^E) : E_*(\mathcal{M}) \to E_*(\mathcal{M})[[c_1^E]]$. If, moreover, $\mathcal{M}$ is an H-space and ρ is a morphism of H-spaces, then $\mathcal{D}(c_1^E)$ is a shift operator compatible with the H-space induced bialgebra[7] structure on $E_*(\mathcal{M})$.*

In [53] Conner–Floyd define generalized Chern classes in cobordism. Generalized Chern classes actually exist for any cohomology theory represented by a complex-oriented spectrum, which are called *Conner–Floyd Chern classes.*

Definition 3.2.8. Let E be a complex-oriented spectrum. Then the *E-Conner–Floyd Chern classes* are the components of the unique natural transformation

$$c^E : \mathrm{Vect}_{\mathbb{C}}(-) \Longrightarrow E^*(-)$$

such that

[7]This assumes that we have a Künneth isomorphism $E_*(\mathcal{M} \times \mathcal{M}) \cong E_*(\mathcal{M}) \hat{\otimes} E_*(\mathcal{M})$

- *Whitney sum formula:* if V and W are two complex vector bundles then

$$c^E(V \oplus W) = c^E(V) \cup c^E(W),$$

  and

- *normalization:* $c^E(\mathcal{L}) = 1 + c_1^E$, where $\mathcal{L} \to BU(1)$ is the universal complex line bundle.

The cohomology class $c^E(V) = c_1^E(V) + \cdots + c_{\mathrm{rk}(V)}^E(V)$ is called the *total E-Conner–Floyd Chern class of V*.

Note that, given existence, the total E-Conner–Floyd Chern class extends to a unique natural transformation $c^E : K^0_{\mathrm{top}}(-) \Longrightarrow E^*(-)$ of functors $\mathrm{CW}^{\mathrm{finite}} \to \mathrm{Ab}$ by

$$\begin{aligned}
c^E(V \ominus W) &= c^E(V) \cup c^E(W)^{-1} \\
&= c^E(V) \cup (1 + c_1^E(W) + \cdots + c_{\mathrm{rk}(W)}^E(W))^{-1} \\
&= c^E(V) \cup (\sum_{i \geq 0} (-1)^i [c^E(W) - 1]^{-i}).
\end{aligned}$$

To establish existence, note that any complex vector bundle $V \to X$ of rank n is classified by the homotopy class of a map $f_V : X \to BU(n)$. Taking E-cohomology gives a map

$$E^*(f_V) : E^*(BU(n)) \cong R[[c_1^E, \dots c_n^E]] \longrightarrow E^*(X).$$

One can then define

$$c_i^E(V) := E^*(f_V)(c_n^E).$$

The *splitting principle* is a useful technique for proving facts about Chern classes. In effect, the splitting principle allows one to assume that a complex vector bundle is the direct sum of complex line bundles. Our presentation of the splitting principle is inspired by that of May [161].

For $n \geq 0$ let $i_n : U(1) \times \cdots \times U(1) \hookrightarrow U(n)$ denote the inclusion of the maximal torus of $U(n)$. Let $V \to X$ be a rank n complex vector bundle on X classified by the homotopy class of a map $f_V : X \to BU(n)$. Consider the pullback square

$$\begin{array}{ccc} Y & \xrightarrow{q} & BU(1) \times \cdots \times BU(1) \\ \big\downarrow & & \big\downarrow{\scriptstyle Bi_n} \\ X & \xrightarrow{f_V} & BU(n). \end{array}$$

Then both rows of the E-cohomology square

$$\begin{array}{ccc} E^*(BU(n)) & \xhookrightarrow{E^*(Bi_n)} & E^*(BU(1))\hat{\otimes}\ldots\hat{\otimes}E^*(BU(1)) \\ \big\downarrow{\scriptstyle E^*(f_V)} & & \big\downarrow{\scriptstyle E^*(q)} \\ E^*(X) & \hookrightarrow & E^*(Y) \end{array}$$

are injective. Moreover, the injection

$$E^*(BU(n)) \hookrightarrow E^*(BU(1) \times \cdots \times BU(1))$$

is given by

$$c_k^E \mapsto \sigma_n((c_1^E)^{(1)}, \ldots, (c_1^E)^{(k)}),$$

where $\sigma_k(x_1, \ldots, x_n)$ denotes the k^{th} elementary symmetric polynomial in n variables.

We have that first E-Conner–Floyd Chern classes obey the tensor product law

$$c_1^E(L \otimes M) = F(c_1^E(L), c_1^E(M)),$$

for two complex line bundles L, M where F is the formal group law associated to E. However, computing E-Conner–Floyd Chern classes of tensor products of higher rank vector bundles is very complicated. E-Chern characters help solve this problem, if one is willing to work rationally.

Definition 3.2.9. Let E be a complex-oriented spectrum and let $R = E^*(\text{pt})$ be a $\mathbb{Q}$-algebra. For any CW complex X, there is a natural equivalence of ring spectra

$\mathrm{DoCh}^E(X) : [X, E] \to [X, HR]$ [191, Thm-Def. II.7.13] (see also Dold [57, Thm. 2] and Greenlees–May [90, Thm. A.1]). Taking homotopy groups gives a multiplicative natural isomorphism

$$\pi_*(\mathrm{DoCh}^E(X)) : E^*(X) \xrightarrow{\sim} H^*(X, R)$$

called the *E-Dold–Chern character*. We define the *E-Chern character* $\mathrm{ch}^E : K^*_{\mathrm{top}}(-) \Longrightarrow E^*(-)$ by

$$\mathrm{ch}^E(-) := (\pi_*(\mathrm{DoCh}^E)^{-1} \circ \mathrm{ch})(-).$$

One can write down an explicit formula for E-Chern characters.

Proposition 3.2.10. *Let E be a complex-oriented spectrum with $E^*(\mathrm{pt})$ a $\mathbb{Q}$-algebra. Let $L \to X$ be a complex line bundle over a CW complex X. Then*

$$\mathrm{ch}^E(L) = e^{\log_F(c_1^E(L))}. \tag{3.2.1}$$

Proof. By naturality of the E-Dold–Chern character, it suffices to prove the proposition when X is $BU(1)$ and L is the universal complex line bundle $\mathcal{L} \to BU(1)$. The map

$$\pi_*(\mathrm{DoCh}^E(BU(1))) : E^*(BU(1)) \cong R[[c_1^E]] \to R[[c_1]] \cong H^*(BU(1), R)$$

is given by $c_1^E \mapsto \exp_F(c_1)$, where $\exp_F$ denotes inverse of the formal logarithm of F [191, Thm. Vii.6.18]. Therefore $\pi_*(\mathrm{DoCh}^E(BU(1)))^{-1}$ sends $c_1 \mapsto \log_F(c_1^E)$. In particular, $\pi_*(\mathrm{DoCh}^E(BU(1)))^{-1} : \mathrm{ch}(\mathcal{L}) = e^{c_1} \mapsto e^{\log_F(c_1^E)}$. □

The splitting principle implies that (3.2.1) determines a formula for the E-Chern character of any complex vector bundle.

Corollary 3.2.11. *Then E-Chern characters obey the sum and tensor product rules of ordinary Chern characters*

$$\mathrm{ch}^E(V \otimes W) = \mathrm{ch}^E(V) \cup \mathrm{ch}^E(W), \tag{3.2.2}$$

and

$$\mathrm{ch}^E(V \oplus W) = \mathrm{ch}^E(V) + \mathrm{ch}^E(W) \tag{3.2.3}$$

for all complex vector bundles $V, W \to X$ on a CW complex X.

Proof. By the splitting principle, it suffices to prove (3.2.2) and (3.2.3) when V, W are complex line bundles. In this case (3.2.3) follows from linearity and (3.2.2) follows from the calculation

$$\begin{aligned}
\mathrm{ch}^E(V \otimes W) &= e^{\log_F(c_1^E(V \otimes W))} \\
&= e^{\log_F(F(c_1^E(V), c_1^E(W))} \\
&= e^{\log_F(c_1^E(V)) + \log_F(c_1^E(W))} \\
&= \mathrm{ch}^E(V) \cup \mathrm{ch}^E(W).
\end{aligned}$$ □

3.3 Rational homotopy theory

The subject of rational homotopy theory was initiated by the foundational works of Quillen [185] and Sullivan [206] on the algebraicization of the homotopy types of rational spaces. So far as the author is aware, the standard encyclopedic reference for this subject is Félix–Halpern–Thomas [71]. Some other references for rational homotopy theory are and Félix–Oprea–Tanré [72] and Hess [100]. Our only use of the theory will be to compute the rational cohomology of an evaluation map.

Definition 3.3.1. Let k be a field. A *differential graded k-algebra* (or *k-dga*) is an ordered pair (A, d) where $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded k-vector space with an associative and unital product $A_n \otimes A_m \to A_{n+m}$ and differential $d : A_n \to A_{n+1}$ such that

- $d(a \cdot b) = d(a) \cdot b + (-1)^{ab} a \cdot d(b)$, and
- $d \circ d = 0$.

One often just denotes (A, d) by A.

A k-dga is said to be a *commutative differential graded k-algebra* (or *k-cdga*) if the product $A \otimes A \to A$ is graded commutative. A k-dga is said to be *connective* if $A_n = 0$ for all $n < 0$ and is said to be *connected* if it is connective and $A_0 = k$. A *morphism* $f : (A, d_A) \to (B, d_B)$ of k-dgas is a family of k-linear maps $\{f_n : A_n \to B_n\}_{n \in \mathbb{Z}}$ such that $d_B \circ f = f \circ d_A$, $f(a \cdot b) = f(a) \cdot f(b)$, and $f(1_A) = 1_B$. Note that the condition $d \circ d = 0$ means (A, d) determines a chain complex. The *homology* of (A, d) is defined to be the homology of this chain complex. An *equivalence* of k-dgas is a quasi-isomorphism of chain complexes.

If X is a real manifold, there is an $\mathbb{R}$-cdga $(\Omega^*_{\mathrm{dR}}(X), d_{\mathrm{dR}})$ such that $H^*(\Omega^*_{\mathrm{dR}}) \cong H^*(X, \mathbb{R})$. If X is any topological space there is a $\mathbb{Q}$-cdga $(A_{\mathrm{PL}}(X), d)$ called the *algebra of rational polynomial forms* such that there is an isomorphism $H^*((A_{\mathrm{PL}}(X), d)) \cong H^*(X, \mathbb{Q})$ (see [71, § 10]). When X has the homotopy type of a real manifold there is an isomorphism $A_{\mathrm{PL}}(X) \otimes \mathbb{R} \cong \Omega^*_{\mathrm{dR}}(X)$.

Definition 3.3.2. A *Sullivan algebra* is a $\mathbb{Q}$-cdga of the form $(\mathrm{SSym}_{\mathbb{Q}}[V], d)$ that satisfies a nilpotency condition (see [71, p. 138]). A *Sullivan model* of a $\mathbb{Q}$-cdga (A, d_A) is a quasi-isomorphism $(\mathrm{SSym}_{\mathbb{Q}}[V], d_V) \to (A, d_A)$ from a Sullivan algebra. A *Sullivan model* of a $\mathbb{Q}$-cdga morphism $(A, d_A) \to (B, d_B)$ is a $\mathbb{Q}$-cdga morphism $(\mathrm{SSym}_{\mathbb{Q}}[V], d_V) \to (\mathrm{SSym}_{\mathbb{Q}}[W], d_W)$ making the diagram

$$\begin{array}{ccc} (\mathrm{SSym}_{\mathbb{Q}}[V], d_V) & \longrightarrow & (\mathrm{SSym}_{\mathbb{Q}}[W], d_W) \\ \downarrow & & \downarrow \\ (A, d_A) & \longrightarrow & (B, d_B) \end{array}$$

commute, where $(\mathrm{SSym}_{\mathbb{Q}}[V], d_V) \to (A, d_A)$ and $(\mathrm{SSym}_{\mathbb{Q}}[W], d_W) \to (B, d_B)$ are both Sullivan models. Every connected Sullivan algebra is isomorphic to a particularly nice Sullivan algebra called its *minimal model* [71, §12, Def. 1.3]. The minimal Sullivan model of a Sullivan algebra is unique up to isomorphism.

Definition 3.3.3. A *Sullivan model* of a topological space X is a Sullivan model of $(A_{\mathrm{PL}}(X), d)$ and a *minimal model* of X is a minimal model of $(A_{\mathrm{PL}}(X), d)$. A *model* of a topological space X is a $\mathbb{Q}$-cdga which admits a Sullivan model quasi-isomorphic to a Sullivan model of X. Models of continuous maps are defined analogously.

Nilpotency is a frequently occurring hypothesis in rational homotopy theory which should be thought of as a generalization of simply-connectedness.

Definition 3.3.4. A topological space X is said to be *nilpotent* if $\pi_1(X)$ is nilpotent and acts nilpotently on all higher homotopy groups of X.

A topological space X is said to be *simple* if $\pi_1(X)$ is abelian and acts trivially on $\pi_n(X)$ for all $n > 1$. All simple spaces are nilpotent. All H-spaces are simple [98, 4.3A]. Another useful notion in rational homotopy theory is that of being formal.

Definition 3.3.5. If X is a CW complex such that there exists a model (not necessarily minimal) of X with vanishing differential, then X is said to be *formal*.

H-spaces and compact Kähler manifolds are examples of formal spaces [55, p. 270], [72, Ex. 2.46]. In fact, H-spaces have a minimal model with vanishing differential [72, Ex. 2.46].

Definition 3.3.6. A graded R-module $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is said to be *finite type* if V_n is finitely generated for all $n \in \mathbb{Z}$. A CW complex X is said to be *finite type* if its rational cohomology $H^*(X, \mathbb{Q})$ is finite type.

Proposition 3.3.7 (see [71, Thm. 15.11]). *Let X be a connected nilpotent CW complex of finite type with minimal model* $(\mathrm{SSym}_{\mathbb{Q}}[V], d)$*. Then for all $n \geq 2$ there is a natural isomorphism*

$$V_n \cong \mathrm{Hom}(\pi_n(X) \otimes \mathbb{Q}, \mathbb{Q}) =: \pi^n(X) \otimes \mathbb{Q},$$

where V_n denotes the degree n graded piece of V.

Haefliger [95], following up on work of Sullivan [206, p. 314] constructed models of spaces of cross-sections of nilpotent bundles homotopic to a given section and their corresponding universal evaluation maps. These results were further refined by Brown–Szczarba [40]. We specialize to the case of trivial fibrations with nilpotent fiber.

The rational homotopy theory of function spaces and evaluation maps is now quite well-understood, having been studied by subsequent authors such as Buijs–Murillo [43], Félix [70], Félix–Tanré [73], Kotani [131], Kuribayshi [133], Smith [203], and Vigué-Poirrier [220].

One can compute the rational cohomology of the space of maps from a rationally formal space into $BU \times \mathbb{Z}$ using Haefliger's model of the evaluation map of connected components of a mapping space.

Proposition 3.3.8 (see Buijs–Murillo [43, Prop. 4.4, Thm. 4.5], Brown–Szczarba [40, Thms. 1.5, 6.1], Haefliger [95, Thm. 3.2]). *Let X be a finite CW complex and let Y be a connected and simply-connected finite type CW complex. Let* $(\mathrm{SSym}_{\mathbb{Q}}[\pi^*(Y)], d_Y)$ *be the minimal model for Y. Let (A, d_A) be a finite-dimensional model for X. Let A_* be the graded differential $\mathbb{Q}$-coalgebra whose i^{th} component is* $\mathrm{Hom}(A_{-i}, \mathbb{Q})$*. There is a canonical $\mathbb{Q}$-cdga morphism*

$$\epsilon' : \mathrm{SSym}_{\mathbb{Q}}[\pi^*(Y)] \longrightarrow A \otimes \mathrm{SSym}_{\mathbb{Q}}[A_* \otimes \pi^*(Y)]$$

given in terms of an additive basis $v_1, \dots, v_n$ *of* A *and dual basis* $v_1^\vee, \dots, v_n^\vee$ *of* A_* *by*

$$\gamma \mapsto \sum_i v_i \otimes (v_i^\vee \otimes \gamma).$$

There is a unique differential d *on* $\mathrm{SSym}_{\mathbb{Q}}[A_* \otimes \pi^*(Y)]$ *such that* ϵ' *is a* $\mathbb{Q}$*-cdga morphism. Note that* $A_* \otimes \pi^*(Y)$ *is not necessarily connective. Let* I *denote the ideal of* $A_* \otimes \pi^*(Y)$ *generated by elements of negative degree and their differentials. Let* W *be the quotient of* $A_* \otimes \pi^*(Y)$ *by* I*. Then*

- *the* $\mathbb{Q}$*-cdga* $(\mathrm{SSym}_{\mathbb{Q}}[W], d)$ *is a model for the connected component of the space of continuous maps* $X \to Y$ *containing the constant map, and*
- ϵ' *induces a* $\mathbb{Q}$*-cdga morphism* $\epsilon : \mathrm{SSym}_{\mathbb{Q}}[\pi^*(Y)] \to A \otimes \mathrm{SSym}_{\mathbb{Q}}[W]$ *which is a model for the evaluation map.*

Corollary 3.3.9. *Let* X *be a finite CW complex which is formal in the sense of rational homotopy theory. Then the rational cohomology* $H^*(\mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})^0, \mathbb{Q})$ *of the identity component of* $\mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})$ *is freely generated by Künneth components of the universal K-theory class over* $X \times \mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})^0$*.*

Proof. The universal K-theory class over $X \times \mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})^0$ is classified by the evaluation map $\mathrm{ev} : X \times \mathrm{Map}_{C^0}(X, BU)_0 \to BU$. As X is rationally formal we may choose $(H^*(X, \mathbb{Q}), 0)$ as a finite-dimensional model for X. Because BU is a simply-connected finite type H-space its minimal model is $(\pi^*(BU), 0)$. The differential induced on $\mathrm{SSym}_{\mathbb{Q}}[H_*(X, \mathbb{Q}) \otimes \pi^*(BU)]$ is therefore zero. This implies that the map

$$\begin{tikzcd}
\mathrm{SSym}_{\mathbb{Q}}[\pi^*(BU)] \arrow[r, "\epsilon"] \arrow[dr, dashed] & H^*(X, \mathbb{Q}) \otimes \mathrm{SSym}_{\mathbb{Q}}[H_*(X, \mathbb{Q}) \otimes \pi^*(BU)] \arrow[d, two heads] \\
& H^*(X, \mathbb{Q}) \otimes \mathrm{SSym}_{\mathbb{Q}}[W]
\end{tikzcd}$$

coincides with the rational cohomology of the evaluation map. By definition, slant products of Chern classes of the universal K-theory class are slant products with the image of $c_k \in H^*(BU, \mathbb{Q})$ under $H^*(\mathrm{ev}, \mathbb{Q})$. By Proposition 3.3.8, the slant product of $H^*(\mathrm{ev}, \mathbb{Q})(c_k)$ with a homology class $v^\vee \in H_*(X, \mathbb{Q})$ is of the form $c_k \otimes v^\vee \in W \subset \mathrm{SSym}_{\mathbb{Q}}[W]$ when non-zero. By definition, all elements of W are of this form. $\square$

3.4 Orientability of gauge-theoretic moduli spaces

In this section we mainly review material from Joyce–Tanaka–Upmeier [115]. We construct moduli spaces of connections and define orientations of them with respect to elliptic oeprators. We conclude with an example from Donaldson theory on 4-manifolds and a discussion of strong and weak H-principal $\mathbb{Z}_2$-bundles. For the initial background on connections, the reader may wish to consult Donaldson–Kronheimer [62] or Kobayashi–Nomizu [128].

Definition 3.4.1. Let G be a Lie group, let X be a compact connected manifold, and let $\pi : P \to X$ be a principal G-bundle. Let $\mathcal{G}_P$ denote the infinite-dimensional group of G-equivariant diffeomorphisms $\gamma : P \xrightarrow{\sim} P$ such that $\pi \circ \gamma = \pi$. We call $\mathcal{G}_P$ the *gauge group of* P.

The space of all connections on a principal G-bundle $\pi : P \to X$ is denoted $\mathcal{A}_P$. The gauge group $\mathcal{G}_P$ acts on $\mathcal{A}_P$. There is an inclusion $Z(G) \hookrightarrow \mathcal{G}_P$ of the center $Z(G)$ of G into the gauge group by taking some $z \in Z(G)$ and applying the principal action on the fibers of P. Because X is connected, $Z(G)$ is isomorphic to $Z(\mathcal{G}_P)$.

A connection ∇_P on P is said to be *irreducible* if $\mathrm{Stab}_{\mathcal{G}_P}(\nabla_P) = Z(G)$ i.e. if it has the smallest possible stabilizer. Let $\mathcal{A}_P^{\mathrm{irr}}$ denote the space of irreducible connections on P. The reduced gauge group $\mathcal{G}_P/Z(G)$ acts freely on $\mathcal{A}_P^{\mathrm{irr}}$ yielding a topological space $\hat{\mathcal{B}}_P^{\mathrm{irr}} = \mathcal{A}_P^{\mathrm{irr}}/(\mathcal{G}_P/Z(G))$.

In the literature, the space $\hat{\mathcal{B}}_P^{\mathrm{irr}}$ is usually considered. However, we will be considering direct sums of connections and a direct sum of irreducible connections is not necessarily irreducible. We are therefore forced to work with the *topological stack*[8] $\mathcal{B}_P = [\mathcal{A}_P/\mathcal{G}_P]$ of all connections modulo gauge. A (hoparacompact) topological stack S admits a classifying space S^{cla}, which is a paracompact topological space equipped with a map $\pi^{\text{cla}} : S^{\text{cla}} \to S$ which is a homotopy equivalence of topological stacks [177]. Classifying spaces of topological quotient stacks are particularly easy to describe: the classifying space of a topological stack $[X/G]$ is given by the Borel construction $(X \times EG)/G$. We write $\mathcal{B}_P^{\text{cla}}$ for the classifying space of $\mathcal{B}_P$ and

$$\mathcal{B}^U := \coprod_{\text{isomorphism classes of } U(n)\text{-bundles } P \to X, n \geq 0} \mathcal{B}_P^{\text{cla}}.$$

Proposition 3.4.2. *Let $\pi : P \to X$ be a principal $U(n)$-bundle. Then there is also a homotopy equivalence*

$$\Sigma_P : \mathcal{B}_P^{\text{cla}} \simeq \text{Map}_{C^0}(X, BU(n))_{[P]},$$

where $\text{Map}_{C^0}(X, BU(n))_{[P]}$ denotes the connected component of $\text{Map}_{C^0}(X, BU(n))$ corresponding to the isomorphism class of P.

Similarly, let $\pi : Q \to X$ be a principal $O(n)$-bundle. Then there is a homotopy equivalence

$$\mathcal{B}_Q^{\text{cla}} \simeq \text{Map}_{C^0}(X, BO(n))_{[Q]},$$

where $\text{Map}_{C^0}(X, BO(n))_{[Q]}$ denotes the connected component of $\text{Map}_{C^0}(X, BO(n))$ corresponding to the isomorphism class of Q.

Proof. Let $U_P = (P \times \mathcal{A}_P)/\mathcal{G}_P \to X \times \mathcal{B}_P$ denote the universal principal $U(n)$-bundle. This pulls back to a principal $U(n)$-bundle over $X \times \mathcal{B}_P^{\text{cla}}$, classified by a map

[8] For further background on topological stacks see Metzler [167] or Noohi [176], [177].

$f_P : X \times \mathcal{B}_P^{\mathrm{cla}} \to BU(n)$. Exponentiating this map gives a map

$$\mathcal{B}_P^{\mathrm{cla}} \longrightarrow \mathrm{Map}_{C^0}(X, BU(n))_{[P]} \subset \mathrm{Map}_{C^0}(X, BU(n))$$

which is shown to be a homotopy equivalence by Atiyah–Bott [16, Prop. 2.4] and Donaldson–Kronheimer [62, Prop. 5.1.4]. The proof of the $O(n)$ case is identical. □

Given principal $U(n)$- and $U(m)$-bundles P and Q on X, direct sum of connections gives a map $\Psi_{P,Q} : \mathcal{B}_P \times \mathcal{B}_Q \to \mathcal{B}_{P\oplus Q}$. Passing to classifying spaces makes $(\mathcal{B}^U, \Psi^{\mathrm{cla}})$ into an H-space such that the homotopy equivalence

$$\textstyle\coprod_{\substack{\text{iso. classes of } U(n)\text{-bundles } P\to X,\\ n\geq 0}} \Sigma_P : \mathcal{B}^U \xrightarrow{\ \sim\ } \mathrm{Map}_{C^0}(X, \coprod_{n\geq 0} BU(n))$$

is an equivalence of H-spaces.

Definition 3.4.3 (see [115, Def. 1.2])**.** Let $X, G, \pi : P \to X$ be as in Definition 3.4.1. Let $E_0, E_1 \to X$ be real vector bundles of rank r and let $D : C^\infty(E_0) \to C^\infty(E_1)$ be a linear partial differential operator of degree d. Given connections $\nabla^i_{E_0}$ on $E_0 \otimes \bigotimes^i T^*X$ we can express

$$D = \sum_{i=0}^{d} a_i \cdot \nabla^i_{E_0},$$

for some $a_i \in C^\infty(E_0^* \otimes E_1 \otimes S^i TX)$. For $\nabla_P \in \mathcal{A}_P$ let $\nabla_{\mathrm{ad}(P)}$ denote the induced connection on the adjoint bundle $\mathrm{ad}(P) \to X$. We define a twisted operator $D^{\nabla_{\mathrm{ad}(P)}} : C^\infty(\mathrm{ad}(P) \otimes E_0) \to C^\infty(\mathrm{ad}(P) \otimes E_1)$ by

$$e \mapsto \sum_{i=0}^{d} (\mathrm{id}_{\mathrm{ad}(P)} \otimes a_i) \nabla^i_{\mathrm{ad}(P)\otimes E_0} e,$$

where $\nabla_{\mathrm{ad}(P)\otimes E_0}$ denotes the induced connection on $\mathrm{ad}(P) \otimes E_0 \otimes \bigotimes^i T^*X$. If D is elliptic, then so is $D^{\nabla_{\mathrm{ad(P)}}}$.

Definition 3.4.4. Let $E_0, E_1 \to X$ be real rank r vector bundles and let $D : C^\infty(E_0) \to C^\infty(E_1)$ be a linear partial differential operator. For a pair $(P, Q) \in$

$\mathcal{B}^U \times \mathcal{B}^U$ consisting of a principal $U(n)$-bundle and a principal $U(m)$-bundle, choose connections ∇_P, ∇_Q on P and Q. There is a representation $\rho_{n,m} : U(n) \times U(m) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n \otimes_{\mathbb{C}} \overline{\mathbb{C}}^m)$ which is the tensor product of the ordinary representation of $U(n)$ on $\mathbb{C}^n$ and the conjugate representation of $U(m)$ on $\mathbb{C}^m$. The induced bundle $\rho_{n,m}(P \times_X Q)$ with fiber $\mathbb{C}^n \otimes_{\mathbb{C}} \overline{\mathbb{C}}^m$ carries an induced connection $\nabla_{\rho_{n,m}(P \times_X Q)}$. This gives a family of elliptic operators over $\mathcal{B}^U \times \mathcal{B}^U$, hence an *index bundle* $D^\nabla \in K^0_{\mathrm{top}}(\mathcal{B}^U \times \mathcal{B}^U)$ (cf. Atiyah–Singer [21])[9].

There is a homotopy-theoretic group completion $\mathcal{B}^U \times \mathcal{B}^U \to \overline{\mathcal{B}}^U \times \overline{\mathcal{B}}^U$. The index bundle D^∇ is classified by the homotopy class of a map $D^\nabla : \mathcal{B}^U \times \mathcal{B}^U \to BU \times \mathbb{Z}$. Because $BU \times \mathbb{Z}$ is group-like, the universal property of homotopy-theoretic group completions gives a weak H-map $D^\nabla : \overline{\mathcal{B}}^U \times \overline{\mathcal{B}}^U \to BU \times \mathbb{Z}$ well-defined up to weak homotopy. Note that the Chern classes of $\overline{D}^\nabla$ are all well-defined because the i^{th} Chern class depends only on the $2i$-skeletal truncation of a $\overline{\mathcal{B}}^U \times \overline{\mathcal{B}}^U$.

A similar construction works for moduli spaces of orthogonal connections.

Definition 3.4.5. Let $D : C^\infty(E_0) \to C^\infty(E_1)$ a real linear elliptic partial differential operator on X with $E_0, E_1 \to X$ real vector bundles of rank r. Let

$$\mathcal{B}^O = \coprod_{\text{isomorphism classes } [P] \text{ of } O(n)\text{-bundles}} \mathcal{B}_P^{\mathrm{cla}} \simeq \coprod_{n \geq 0} \mathrm{Map}_{C^0}(X, BO(n)).$$

For a pair $(P, Q) \in \mathcal{B}^O \times \mathcal{B}^O$ consisting of a principal $O(n)$-bundle and a principal $O(m)$-bundle, choose connections ∇_P, ∇_Q on P, Q. The bundle $\sigma_{n,m}(P \times_X Q)$ with fiber $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m$ induced by the natural representation $\sigma_{nm} : O(n) \times O(m) \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m)$ carries an induced connection $\nabla_{\sigma_{n,m}(P \times_X Q)}$. For an infinite-dimensional separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$, there is a homotopy equivalence $\mathrm{Fred}(\mathcal{H}_{\mathbb{R}}) \simeq BO \times \mathbb{Z}$.

[9] Atiyah and Singer assume that the parameter space is compact. Our parameter space is an infinite CW complex. In this case we use the equivalence $BU \times \mathbb{Z} \simeq \mathrm{Fred}(\mathcal{H})$, where $\mathrm{Fred}(\mathcal{H})$ denotes the space of Fredholm operators on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$, to define the index bundle.

Therefore the continuous assignment $(P, Q) \mapsto D^{\nabla_{\sigma_{n,m}(P \times_X Q)}}$ determines an index bundle $D^\nabla \in KO^0(\mathcal{B}^O \times \mathcal{B}^O)$.

We now move on to the topic of orientability of moduli spaces of connections. Recall that an *orientation* of a manifold X is a trivialization of the top power of its cotangent bundle. Spaces such as $\mathcal{B}^U$ or $\overline{\mathcal{B}}^U$ do not, however, have cotangent bundles; so a new definition is required.

Definition 3.4.6 (see [115, §. 1])**.** Let $E_0, E_1 \to X$ be two real rank r vector bundles on a compact manifold X and let $D : C^\infty(E_0) \to C^\infty(E_1)$ be an elliptic operator. For each $\nabla_P \in \mathcal{A}_P$ the twisted operator $D^{\nabla_{\mathrm{ad}(P)}}$ has finite-dimensional kernel and cokernel, giving a determinant line

$$\det(D^{\nabla_{\mathrm{ad}(P)}}) = \det(\mathrm{Ker}(D^{\nabla_{\mathrm{ad}(P)}})) \otimes \det(\mathrm{Coker}(D^{\nabla_{\mathrm{ad}(P)}}))^\vee.$$

These lines vary continuously with $\nabla_P \in \mathcal{A}_P$, forming a real line bundle $\check{L}_P^D \to \mathcal{A}_P$ as in Atiyah–Singer [20] and Quillen [183]. This line bundle is $\mathcal{G}_P$-equivariant. Therefore $\check{L}_D$ descends to a real line bundle $L_P^D \to \mathcal{B}_P = [\mathcal{A}_P/\mathcal{G}_P]$.

We define the *orientation bundle* O_P^D of L_P^D by

$$O_P^D := (L_P^D \setminus 0(\mathcal{B}_P))/(0, \infty),$$

where $0(\mathcal{B}_P)$ denotes the zero section of L_P^D and $(0, \infty)$ acts by scaling fibers. The orientation bundle is a principal $\mathbb{Z}_2$-bundle. We say that the pair $(\mathcal{B}_P, D)$ is *orientable* if O_P^D is isomorphic to the trivial principal $\mathbb{Z}_2$-bundle $\mathcal{B}_P \times \mathbb{Z}_2 \to \mathcal{B}_P$. An *orientation* of $(\mathcal{B}_P, D)$ is a choice of trivialization $O_P^D \cong \mathcal{B}_P \times \mathbb{Z}_2$. We may often just speak of orientations of $\mathcal{B}_P$ if the operator D can be inferred from context.

The line bundle $\check{L}_P^D$ is moreover $\mathcal{G}_P/Z(G)$-equivariant so that it descends to a line bundle $\hat{L}_P^D \to \hat{\mathcal{B}}_P$. By the same construction, $\hat{L}_P^D$ has an *orientation bundle*

$O_P^D \to \hat{\mathcal{B}}_P$ and an *orientation* of $\hat{\mathcal{B}}_P$ is a trivialization $O_P^D \cong \mathcal{B}_P \times \mathbb{Z}_2$. The projection $\pi_P : \mathcal{B}_P \to \hat{\mathcal{B}}_P$ has fiber $[*/Z(G)]$ which is connected and simply-connected giving an isomorphism $O_P^D \cong \pi_P^*(\hat{O}_P^D)$. This implies that orientations of $\mathcal{B}_P$ are in natural correspondence with orientations of $\hat{\mathcal{B}}_P$. The orientation bundle on $\hat{\mathcal{B}}_P^{\text{irr}} \hookrightarrow \hat{\mathcal{B}}_P$ is the restriction of the orientation bundle $\hat{O}_D^P \to \hat{\mathcal{B}}_P$. So, to construct orientations on $\hat{\mathcal{B}}_P^{\text{irr}}$ it suffices to construct trivializations of $O_P^D \to \mathcal{B}_P$.

We may be interested in moduli spaces $\mathcal{M}_P^{\text{inst}}$ of irreducible instanton connections on P whose curvature satisfies some condition. Typically, $\mathcal{M}_P^{\text{inst}}$ will be a smooth manifold for a sufficiently generic choice of Riemannian structure on X and otherwise will be a derived smooth manifold. There is a natural inclusion $\mathcal{M}_P^{\text{inst}} \hookrightarrow \hat{\mathcal{B}}_P^{\text{irr}}$ and trivializations of $\hat{O}_P^D \to \hat{\mathcal{B}}_P^{\text{irr}}$ give orientations of $\mathcal{M}_P^{\text{inst}}$ in the usual sense of ordinary (or derived) manifolds where D is an elliptic operator related to the curvature condition. So, to construct orientations on some $\mathcal{M}_P^{\text{inst}}$ it suffices to construct orientations on $\mathcal{B}_P$. We give an example from Donaldson theory.

Example 3.4.7. Let (X, g) be a compact, connected, and simply-connected Riemannian 4-manifold. Let $P \to X$ be a principal $SU(2)$-bundle. The Hodge star

$$\star : \Omega^2(X) \longrightarrow \Omega^2(X)$$

is an isomorphism with $\star^2 = -1$. This splits $\Omega^2(X)$ into ± 1-eigenbundles

$$\Omega^2(X) \cong \Omega^2_+(X) \oplus \Omega^2_-(X).$$

Let $\pi_+ : \Omega^2(X, \mathrm{ad}(P)) \to \Omega^2_+(X, \mathrm{ad}(P))$ denote the natural projection. A connection $\nabla_P \in \mathcal{A}_P$ is said to be an *anti-self-dual (asd) instanton* if the curvature tensor $F^{\nabla_P} \in \Omega^2(X, \mathrm{ad}(P))$ of ∇_P satisfies the *asd equation*: $\pi_+(F^{\nabla_P}) = 0$. We write $\mathcal{M}_P^{\text{asd}} \subset \hat{\mathcal{B}}_P^{\text{irr}}$ for the moduli space of gauge equivalence classes of irreducible asd instantons.

Orientability of moduli spaces of irreducible asd instantons is established in [61].

Consider the elliptic operator

$$d + d_+^* : C^\infty(\Omega^0(X) \oplus \Omega^2_+(X)) \longrightarrow C^\infty(\Omega^1(X)),$$

where $d_+^* := \pi_+ \circ d^*$. By the results of Atiyah–Hitchin–Singer [19] on the deformation theory of asd instantons, the fiber of the determinant line bundle $L_P^{d+d_+^*}$ at some gauge equivalence class $[\nabla_P]$ is isomorphic to $\det(T^*_{[\nabla_P]}\mathcal{M}_P^{\mathrm{asd}})$. For sufficiently generic metric, $\mathcal{M}_P^{\mathrm{asd}}$ is a smooth manifold. In this case, orientations of $(\mathcal{B}_P, d + d_+^*)$ give orientations of $\mathcal{M}_P^{\mathrm{asd}}$, in the usual sense, as a manifold.

We will also be interested in comparing orientations on different connected components of $\mathcal{B}^U$ under the H-map $\Psi : \mathcal{B}^U \times \mathcal{B}^U \to \mathcal{B}^U$ induced by direct sum of connections. In [115, Def. 2.22] an isomorphism

$$\psi : O^D \boxtimes_{\mathbb{Z}_2} O^D \longrightarrow \Psi^*(O^D)$$

is defined. So given $\alpha, \beta \in \pi_0(\mathcal{B}^U)$ and orientations $o^D_\alpha, o^D_\beta, o^D_{\alpha+\beta}$ of O^D_α, O^D_β, and $O^D_{\alpha+\beta}$ there is some sign $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ such that

$$\psi(o^D_\alpha \boxtimes o^D_\beta) = \epsilon_{\alpha,\beta} \cdot \Psi^*(o^D_{\alpha+\beta}).$$

The following definition appears in [45].

Definition 3.4.8 (see [45, Def. 3.3])**.** Let $(X, \mu_X, 1_X)$ be an H-space.

1. A *weak H-principal* $\mathbb{Z}_2$*-bundle* is a principal $\mathbb{Z}_2$-bundle $P \to X$ such that the classifying map $X \to B\mathbb{Z}_2$ of P, which is defined up to homotopy, is an H-map. Equivalently, there is an isomorphism $P \boxtimes_{\mathbb{Z}_2} P \xrightarrow{\sim} \mu_X^*(P)$ of principal $\mathbb{Z}_2$-bundles.

2. A *strong H-principal $\mathbb{Z}_2$-bundle* is a trivializable weak H-principal $\mathbb{Z}_2$-bundle $P \to X$ along with a choice of isomorphism $p : P \boxtimes_{\mathbb{Z}_2} P \xrightarrow{\sim} P$ such that

$$(\mu_X \times \mathrm{id})^*(p) \circ (\mathrm{id} \boxtimes p) \simeq (\mathrm{id} \times \mu_X)^*(p) \circ (p \boxtimes \mathrm{id}).$$

A *morphism of strong H-principal $\mathbb{Z}_2$-bundles* $\iota : (P, p) \to (Q, q)$ is a morphism $\iota : P \to Q$ of principal $\mathbb{Z}_2$-bundles such that $\iota \circ p = q \circ (\iota \boxtimes \iota)$.

Remark 3.4.9. The reason for introducing the notions of weak and strong H-principal $\mathbb{Z}_2$-bundle is as follows. Let $[f] : X \to Y$ be the homotopy class of a continuous map of topological spaces and let $\pi : P \to Y$ be a principal $\mathbb{Z}_2$-bundle. Then, the pullback $[f]^*P \to X$ is well-defined up to isomorphism but not, in general, up to canonical isomorphism. Suppose $f_0, f_1 : X \to Y$ are continuous maps and $H : I \times X \to Y$ is a homotopy $H : f_0 \simeq f_1$. Then pullback of P along H gives a continuous family $f_t^*(P) \to X$ of principal $\mathbb{Z}_2$-bundles over X. Parallel transport gives an isomorphism $f_0^*(P) \cong f_1^*(P)$. However, unless P is trivializable, this isomorphism may depend on the choice of homotopy H (see, for example [115, Def. 2.18]).

Fortunately, we only care about the compatibility of principal $\mathbb{Z}_2$-bundles with H-maps in the case of a trivializable principal $\mathbb{Z}_2$-bundle. If this were not the case, we may try to remedy the issue by considering H-spaces X that can be enhanced to E_∞-spaces and considering principal $\mathbb{Z}_2$-bundles whose classifying maps $X \to B\mathbb{Z}_2$ are morphisms of E_∞-spaces.

Proposition 3.4.10 (see [45, Prop. 3.5])**.** *Let X, Y be H-spaces and let $f : X \to Y$ be a homotopy-theoretic group completion. Then given a weak H-principal $\mathbb{Z}_2$-bundle $P \to X$, there exists a weak H-principal $\mathbb{Z}_2$-bundle $Q \to Y$, unique up to isomorphism, such that $P \cong f^*Q$.*

Further, if (P, p) is a strong H-principal $\mathbb{Z}_2$-bundle on X then there exists a strong H-principal $\mathbb{Z}_2$-bundle (Q, q) on Y, unique up to canonical isomorphism, and an iso-

morphism $(P,p) \xrightarrow{\sim} f^*(Q,q)$ *of strong H-principal* $\mathbb{Z}_2$*-bundles.*

Definition 3.4.11. As per the isomorphism [115, Def. 2.22] $\psi : O^D \boxtimes_{\mathbb{Z}_2} O^D \xrightarrow{\sim} \Psi^*(O^D)$ of principal $\mathbb{Z}_2$-bundles over $\mathcal{B}^U \times \mathcal{B}^U$ [115, Def. 2.22], $O^D \to \mathcal{B}^U$ is a weak H-principal bundle. Let $\overline{\mathcal{B}}^U$ denote the homotopy-theoretic completion of $\mathcal{B}^U$. By Proposition 3.4.10, there exists a unique extension of O^D over $\overline{\mathcal{B}}^U$ which is a weak H-principal $\mathbb{Z}_2$-bundle and which we will also denote by O^D.

Lemma 3.4.12 (see [115, Lem. 3.10])**.** *Let* $O^D \to \overline{\mathcal{B}}^U$ *be as above. As* $O^D \to \overline{\mathcal{B}}^U$ *is a weak H-bundle extending* $O^D \to \mathcal{B}^U$ *there is an isomorphism* $\psi : O^D \boxtimes_{\mathbb{Z}_2} O^D \xrightarrow{\sim} \Psi^*(O^D)$ *extending [115, Def. 2.22]. If* $O^D \to \overline{\mathcal{B}}^U$ *is trivializable, then* (O^D, ψ) *is a strong H-principal* $\mathbb{Z}_2$*-bundle over* $\overline{\mathcal{B}}^U$*.*

3.5 Calabi–Yau manifolds and Spin(7)-manifolds

Let (X, g) be a Riemannian manifold. Then the *holonomy group* $\mathrm{Hol}(g)$ of g at a point $p \in X$ is the sub-group of $\mathrm{GL}(T_pX)$ generated parallel transport maps around loops at x. For further background on Riemannian holonomy groups, we refer the reader to Joyce [107] [108].

We call a Riemannian manifold *irreducible* if it is not locally a Riemannian product. We say that Riemannian manifold (X, g) is a *symmetric space* if for all $p \in X$ the geodesic reflection s_p is an isometry. For simply-connected irreducible Riemannian manifolds which are not symmetric spaces, Marcel Berger composed a list of all possible holonomy groups.

Theorem 3.5.1 (Berger [27])**.** *Let* (X, g) *be a simply-connected irreducible Riemannian manifold of dimension* n*. Suppose that* (X, g) *is not a symmetric space. Then the holonomy group Hol*(g) *is either*

- $SO(n)$,

- *$U(m)$ with $n = 2m$ and $m \geq 2$,*
- *$SU(m)$ with $n = 2m$ and $m \geq 2$,*
- *$Sp(m)$ with $n = 4m$ and $m \geq 2$,*
- *$Sp(m)Sp(1)$ with $n = 4m$ and $m \geq 2$,*
- *G_2 with $n = 7$, or*
- *$Spin(7)$ with $n = 8$.*

The first examples of manifolds with holonomy Spin(7) were non-compact and constructed by Bryant [41]. The first complete non-compact examples of manifolds holonomy Spin(7) were constructed by Bryant–Salamon [42]. The first examples of compact manifolds with holonomy Spin(7) were constructed by Joyce [106]. We now discuss the holonomy groups $SU(m)$ and Spin(7) in more detail.

First, we will mention that Riemannian manifolds with holonomy $SU(m)$ or $Spin(7)$ are spin [224].[10] Recall that the spin groups Spin(n) are double covers of $SO(n)$.

Definition 3.5.2. A *spin structure* on an oriented Riemannian manifold is a principal Spin(n)-bundle covering the natural $SO(n)$-bundle of frames induced by the orientation and Riemannian structure.

The group Spin(n) has a natural *spin representation* Δ^n of complex dimension 2^m. When $n = 2m$ is even the spin representation splits

$$\Delta^n \cong \Delta^n_+ \oplus \Delta^n_-$$

into irreducible representations $\Delta^n_\pm$ of complex dimension 2^{n-1}. For $n = 8k-1, 8k$, or $8k+1$ the spin representation is the complexification of a *real spin representation*

[10]This is actually the same reason that special holonomy manifolds are so pervasive in the high-energy physics literature. G_2-manifolds are also spin.

$\Delta^n_{\mathbb{R}}$. A spin structure $\tilde{P} \to X$ on a Riemannian manifold X of dimension n yields a *spin bundle* $\mathbb{S} \to X$ with fiber Δ^n.

For further reading on spinors and Clifford algebras, the reader is advised to consult Atiyah–Bott–Shapiro [17], Lawson–Michelson [141], or Harvey [97].

Definition 3.5.3. For a Riemannian spin manifold (X, g) the Levi-Civita connection ∇_g lifts to a *spin connection* $\nabla^{\mathbb{S}}$ on $\mathbb{S}$. The *Dirac operator* $\not{D} : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is the composition

$$C^\infty(\mathbb{S}) \xrightarrow{\nabla^{\mathbb{S}}} C^\infty(T^*X \otimes \mathbb{S}) \longrightarrow C^\infty(\mathbb{S}),$$

where $C^\infty(T^*X \otimes \mathbb{S}) \to C^\infty(\mathbb{S})$ is Clifford multiplication.

In particular, 8-dimensional Riemannian spin manifolds have splittings of their real spin bundles $\mathbb{S} \cong \mathbb{S}_+ \oplus \mathbb{S}_-$ and *positive Dirac operators* $\not{D}_+ : C^\infty(\mathbb{S}_+) \to C^\infty(\mathbb{S}_-)$.

Definition 3.5.4. A *Calabi–Yau manifold* of complex dimension n is an $2n$-dimensional Kähler manifold X together with a holomorphic $(n, 0)$-form θ such that

$$\omega^n = (-1)^{\frac{n(n-1)}{2}} i^n 2^{-n} n! \cdot \theta \wedge \overline{\theta},$$

where ω denotes the Kähler form of X.

Calabi–Yau manifolds X of complex dimension n have $\mathrm{Hol}(g) \subset SU(n)$ and $c_1(X) = 0$.

Definition 3.5.5. A *Calabi–Yau variety* is a smooth complex variety X with trivial canonical sheaf $K_X \cong \mathcal{O}_X$.

The underlying analytic space of a Calabi–Yau variety is a Calabi–Yau manifold. A Calabi–Yau manifold or variety of complex dimension n is called a *Calabi–Yau n-fold*.

The holomorphic volume form θ of a Calabi–Yau $4m$-fold X can be used to define generalized Hodge star operators $\star_q : \Omega^{0,q}(X) \to \Omega^{0,4-q}(X)$ for $q = 0, \ldots, 4m$ characterized by the equation

$$\alpha \wedge (\star_q \beta) = \langle \alpha, \beta \rangle \overline{\theta}$$

where $\langle \cdot, \cdot \rangle$ denotes the Hermitian metric determined by the Kähler structure on X. We call these *Calabi–Yau Hodge star operators.* The Calabi–Yau Hodge star operators satisfy the following identities

$$\begin{aligned} \star_{4m-q} \circ \star_q &= (-1)^q \mathrm{id}, \\ \overline{\partial}^* \circ \star_q &= (-1)^{q+1} \star_{q+1} \circ \overline{\partial}, \text{ and} \\ \overline{\partial} \circ \star_q &= (-1)^q \star_{q-1} \circ \overline{\partial}^* \end{aligned}$$

for $q = 0, \ldots, 4m$.

Consider the elliptic operator

$$D_{\mathbb{C}} := \overline{\partial} + \overline{\partial}^* : C^\infty(\bigoplus_q \Omega^{0,2q}(X)) \longrightarrow C^\infty(\bigoplus_q \Omega^{0,2q+1}(X)).$$

For brevity, we write $E_0 := \bigoplus_q \Omega^{0,2q}(X)$ and $E_1 := \bigoplus_q \Omega^{0,2q+1}(X)$. Using the Calabi–Yau Hodge star, one can define $\mathbb{C}$-antilinear isometric vector bundle automorphisms $\heartsuit_0 : E_0 \xrightarrow{\sim} E_0$, $\heartsuit_1 : E_1 \xrightarrow{\sim} E_1$ by $\heartsuit_0 := (-1)^q \star_{2q}$ and $\heartsuit_1 := (-1)^{q+1} \star_{2q+1}$ such that

$$\heartsuit_i^2 = \mathrm{Id} \tag{3.5.1}$$

and

$$D_{\mathbb{C}} \circ \heartsuit_0 = \heartsuit_1 \circ D_{\mathbb{C}}. \tag{3.5.2}$$

Equation (3.5.1) implies that $\heartsuit_i$ is a real structure, in the sense of Atiyah [14], on E_i. We write $E_i^{\mathbb{R}}$ for the real sub-bundle of E_i fixed by $\heartsuit_i$. Then there are isomorphisms $E_0^{\mathbb{R}} \cong \mathbb{S}_+ \to X, E_1^{\mathbb{R}} \cong \mathbb{S}_- \to X$ of real vector bundles over X and an equality

$$D_{\mathbb{R}} := D_{\mathbb{C}}|_{E_0^{\mathbb{R}}} = \not{D}_+ \tag{3.5.3}$$

of operators $E_0^{\mathbb{R}} \to E_1^{\mathbb{R}}$. Equation (3.5.3) will be useful in Section 4.4.

We now discuss Spin(7)-manifolds.

Definition 3.5.6. Let $(x_1, \ldots, x_8)$ be coordinates on $\mathbb{R}^8$. We use the notation $dx_{ijk\ell} := dx_i \wedge dx_j \wedge dx_k \wedge dx_\ell$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned}\Omega_0 := dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} - dx_{2358} - dx_{2367} \\ - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678}.\end{aligned}$$

An interesting description of the Lie group Spin(7) is the subgroup of $\mathrm{GL}(8, \mathbb{R})$ preserving Ω_0.

A *Spin*(7)*-structure* (Ω, g) on an 8-manifold X is a 4-form Ω and Riemannian metric g such that there exist isomorphisms $T_xX \xrightarrow{\sim} \mathbb{R}^8$ for all $x \in X$ that identify $\Omega|_x \cong \Omega_0$ and $g|_x \cong g_0$, where

$$g_0 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2$$

is the standard Euclidean metric on $\mathbb{R}^8$. A Spin(7)-structure is said to be *torsion-free* if Ω is closed. A *Spin*(7)*-manifold* is manifold with a torsion-free Spin(7)-structure.

Spin(7)-manifolds (X, Ω, g) have the property that $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$. Examples of compact Spin(7)-manifolds that do not have full holonomy Spin(7) include Calabi–Yau 4-folds and flat 8-tori.

Definition 3.5.7. Let (X, Ω, g) be a Spin(7)-manifold. The operation $\alpha \mapsto \star(\alpha \wedge \Omega)$

splits $\Omega^2(X)$ into eigenbundles

$$\Omega^2(X) \cong \Omega^2_7(X) \oplus \Omega^2_{21}(X)$$

of ranks 7 and 21. Let $\pi_7 : \Omega^2(X) \to \Omega^2_7(X)$ denote the natural projection onto $\Omega^2_7(X)$.

Let $P \to X$ be a principal G bundle on X. A *Spin*(7)*-instanton* is a connection ∇_P on P such that $\pi_7(F^{\nabla_P}) = 0$. We write $\mathcal{M}_P^{\mathrm{Spin}(7)}$ for the moduli space of gauge equivalence classes of irreducible Spin(7)-instantons on X.

Explicit examples of Spin(7)-instantons have been constructed by Lewis [144], Tanaka [210], and Walpuski [223]. Let $d_7 := \pi_7 \circ d$. The operator

$$d + d_7^* : C^\infty(\Omega^0(X) \oplus \Omega^2_7(X)) \longrightarrow \Omega^1(X)$$

is elliptic, giving an orientation bundle $O_P^{d+d_7^*} \to \mathcal{B}_P$ which pulls back to one on $\mathcal{M}_P^{\mathrm{Spin}(7)}$. Orientation bundles actually depend only on the symbol of the elliptic operator. Note that the symbol of $d + d_+^*$ equals the symbol of the positive Dirac operator $\not{D}_+$ of X.

In [63] Donaldson–Thomas outline a program for extending Donaldson's invariants of 4-manifolds [59] to higher-dimensional manifolds in the presence of a special holonomy structure. Constructing higher-dimensional gauge theories requires results on compactness, smoothness, and orientability of the instanton moduli spaces. Analytically, constructing compactifications is fiendishly difficult. However, Kobayshi–Hitchin type results give isomorphisms between moduli spaces of instantons and moduli spaces of semistable bundles [60] [127] [219]. In algebraic geometry, compactifications are actually easy to construct. Moduli spaces of bundles are compactified by moduli spaces of torsion-free coherent sheaves (see Gieseker [87], Maruyama [156] [157], and Simpson [197]).

By passing to algebraic geometry, however, one can no longer perturb metrics to obtain smooth moduli spaces. Smoothness issues are gotten around by using the theory of virtual fundamental classes obtained through perfect obstruction theories (see Behrend–Fantechi [26] and Li–Tian [149]) or through Kuranishi spaces (see Fukaya–Oh–Oha–Ono [84], Joyce [113] [111], and McDuff–Wehrheim [164] [165]). Moduli spaces of sheaves or instantons on Calabi–Yau 3-folds have canonical orientations. Orientability results for G_2-instanton moduli spaces have been established by Joyce–Upmeier [116]. Orientability of moduli spaces of Spin(7)-instantons was proved by Cao, Joyce, and the author and is stated below. This theorem has applications towards defining the aforementioned conjecturally existing gauge-theoretic enumerative invariants of Spin(7)-manifolds. The proof uses a generalization of Donaldson's excision principle [59] [61] [62]. Excision arguments require understanding phenomena along of codimension 3 submanifolds in X. When X is a compact 4-manifold, codimension 3 submanifolds are just circles. However, in the following theorem X is an 8-manifolds and its proof invokes Crowley's classification of compact simply-connected 5-manifolds [54].

Theorem 3.5.8 (Cao–Gross–Joyce [45, Thm. 1.11])**.** *Let (X, g) be a compact oriented Riemannian manifold of dimension 8. Suppose X is spin and let $\not{D}_+ : C^\infty(\mathbb{S}_+) \to C^\infty(\mathbb{S}_-)$ denote the positive Dirac operator. Let $G = U(m)$ or $SU(m)$ and let $P \to X$ be a principal G-bundle. Then $(\mathcal{B}_P, \not{D}_+)$ is orientable. In particular $(\mathcal{B}^U, \not{D}_+)$ is orientable i.e. $O^{\not{D}_+} \to \mathcal{B}^U$ is trivializable. The extension $O^{\not{D}_+} \to \overline{\mathcal{B}}^U$ is also trivializable.*

Theorem 3.5.8 also implies that given a compact Spin(7)-manifold X and principal $U(m)$ or $SU(m)$ bundle $P \to X$, the moduli space $\mathcal{M}_P^{\mathrm{Spin}(7)}$ of irreducible Spin(7)-instantons on P is orientable.

Theorem 3.5.8 was also used in [45] to establish the existence of orientations for Donaldson–Thomas type invariants of Calabi–Yau 4-folds (see Borisov–Joyce [34] and Cao-Leung [47]), generalizing the results of Cao–Leung [48] and Munoz–Shahbazi [174]. This is discussed in Section 4.4.

Chapter 4

Topology of stabilized moduli spaces

In this chapter we study the topology of moduli stacks of complexes of coherent sheaves on smooth complex projective varieties.

Section 4.1 contains background material on derived stacks and shifted symplectic structures. Section 4.2 discusses homology theories of higher stacks and the Betti homology of higher $\mathbb{C}$-stacks. Section 4.3 contains material from [92], wherein we compute the rational homology of the moduli stack of objects in the derived category of a smooth complex projective variety in class D. Finally, in Section 4.4, we prove orientability of moduli stacks of perfect complexes of coherent sheaves on projective Calabi–Yau 4-folds. Section 4.4 is based on joint work with Yalong Cao and Dominic Joyce [45]. Section 4.3 is based on a paper of the author [92].

4.1 Derived stacks and orientability of shifted symplectic stacks

This section begins with the definition of dg-categories. As far as the author is aware, the first appearance of dg-categories in the literature is Kelly [123]. However, the modern idea of using dg-categories to enhance triangulated categories first appeared in Bondal–Kapranov [32]. For surveys of dg-category theory the reader may also wish to consult Drinfeld [64], Keller [122], and Toën [215].

The problem of constructing moduli stacks parameterizing objects in (dg-enhanced) triangulated categories was solved by Toën–Vaquié using the theory of higher and derived stacks [217]. A derived stack can be thought of as a scheme where points have cotangent *complexes* rather than mere cotangent spaces. Just as complexes replace bundles in the derived world, cohomology classes replace sections. This leads to the notion of a differential p-form of cohomological degree q and in particular to the shifted symplectic and Poisson geometry of Pantev–Toën–Vaquié–Vezzosi [179].

In [34] Borisov–Joyce define a notion of orientation of -2-shifted symplectic $\mathbb{C}$-schemes[1,2] $\underline{X}$. They show that the underlying complex analytic space X^{an} of the classical truncation X of $\underline{X}$ can be made into a derived manifold $\underline{X}_{\mathrm{dm}}$. It is moreover shown that there is a natural bijection between orientations of the -2-shifted symplectic derived $\mathbb{C}$-scheme $\underline{X}$ and orientations of the derived manifold $\underline{X}_{\mathrm{dm}}$. Compact oriented derived manifolds have virtual classes and these virtual classes can be used to define a holomorphic analogue of Donaldson theory which 'counts' semistable coherent sheaves on Calabi–Yau 4-folds. This is the main application of the orientability results in Section 4.4.

Definition 4.1.1. Let k be a field. A *k-dg-category* is a k-linear category $\mathcal{C}$ whose

[1] This notion also makes sense for -2-shifted symplectic $\mathbb{C}$-stacks.

[2] It is known that moduli stacks of (complexes of) coherent sheaves on Calabi–Yau 4-folds are -2-shifted symplectic. It is hoped that moduli schemes of coherent sheaves on Calabi–Yau 4-folds are also -2-shifted symplectic.

Hom spaces are k-dg-modules such that for all objects a, b, c in $\mathcal{C}$ composition

$$\mathrm{Hom}_{\mathcal{C}}(a, b) \otimes \mathrm{Hom}_{\mathcal{C}}(b, c) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(a, c)$$

is a morphism of k-dg-modules.

Remark 4.1.2. A k-dg-category with one object is equivalent to a k-dg-algebra.

Definition 4.1.3. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{C}$ be a dg-category. We say that $\mathcal{C}$ is an *enhancement* of $\mathcal{T}$ is there is a triangulated equivalence $H^0(\mathcal{C}) \xrightarrow{\sim} \mathcal{T}$.

Examples of triangulated categories admitting dg-enhancements are $\mathrm{Perf}(X)$ where X is a complex variety and $\mathrm{Rep}(Q)$ where Q is a quiver. There is a class of dg-categories satisfying a finiteness condition called *saturated dg-categories* (see [122, § 4.7], [217, Def. 1.4]). Examples of saturated dg-categories are $\mathrm{Perf}(X)$ where X is a smooth projective complex variety and $\mathrm{Rep}(Q)$ where Q is a finite quiver without loops.

Recall that k-schemes and algebraic k-spaces are certain functors

$$F : \mathrm{Aff}_k^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

which are required, for example, to be étale sheaves (see [137]). Similarly, k-stacks can be regarded as certain functors

$$F : \mathrm{Aff}_k^{\mathrm{op}} \longrightarrow \mathrm{Groupoid}.$$

There are several versions of the theory of *higher k-stacks* or ∞-k-stacks (see Lurie [151], Pridham [182], Simpson [198], and Toën–Vezzosi [218]). Higher k-stacks are modeled as simplicial presheaves

$$F : \mathrm{Aff}_k^{\mathrm{op}} \longrightarrow \mathrm{sSet}.$$

The local model structures on simplicial presheaves (see Blander [29], Dugger–Hollander–Isaksen [65], and Jardine [105]) model the homotopy theory of higher stacks. Higher *Artin* k-stacks are a particularly nice class of ∞-k-stacks which form a sub-(∞-)category $\mathrm{HArt}_k \subset \mathrm{HSt}_k$.

The highest level of generality in this context is provided by the theory of *derived stacks* (see Lurie [151], Toën–Vezzosi [218], and Toën [214] [3]). A *derived k-stack* is a kind of homotopy sheaf

$$F : \mathrm{scalg}_k \longrightarrow \mathrm{sSet},$$

where scalg_k denotes the (∞-)category of simplicial commutative k-algebras. Again, there is a nice sub-(∞-)category $\mathrm{dASt}_k \subset \mathrm{dSt}_k$ of derived *Artin* k-stacks. Many derived k-stacks $\mathcal{M}$ admit *cotangent complexes* $\mathbb{L}_{\mathcal{M}}$ (see [150, § 7.3] and [218, § 1.4]) which generalize the usual cotangent sheaves of schemes. The inclusion functor $\mathrm{Aff}_k^{\mathrm{op}} \hookrightarrow \mathrm{scalg}_k$ induces an inclusion (∞-)functor $i : \mathrm{HSt_k} \hookrightarrow \mathrm{dSt_k}$ which admits a left adjoint $\tau_0 : \mathrm{dSt}_k \to \mathrm{HSt_k}$ called the *classical truncation*.

Remark 4.1.4. When k is a field of characteristic zero there is an (∞-)equivalence $\mathrm{scalg}_k \xrightarrow{\sim} \mathrm{cdga}_k^{\leq 0}$, where $\mathrm{cdga}_k^{\leq 0}$ denotes the (∞-)category of commutative differential graded algebras concentrated in non-positive degrees [150, § 8.1.4]. This can be a pleasant fact as dg-algebras are generally easier to work with than simplicial algebras.

For a small k-dg-category $\mathcal{T}$, Toën–Vaquié define a derived k-stack $\mathcal{M}_T : \mathrm{scalg}_k \to \mathrm{sSet}$ of $\mathcal{T}^{\mathrm{op}}$-dg-modules [217, § 3.1].

Proposition 4.1.5 (see [217, Thm. 3.6, Cor. 3.17]). *Let $\mathcal{T}$ be a saturated k-dg-category. Let $E : \mathrm{Spec}(k) \to \mathcal{M}_T$ be a k-point of $\mathcal{M}_T$. Then*

$$\mathbb{T}_{\mathcal{M}_{\mathcal{T}}}|_E \simeq \mathbb{R}Hom(E, E)[1].$$

[3] We also find Gaitsgory–Rozenblyum to be a helpful resource on derived algebraic geometry [86].

If $\mathcal{X}$ is a derived k-stack there exist exterior powers $\Lambda^p\mathbb{L}_{\mathcal{X}}$ for all $p \geq 0$ and a morphism of complexes $d_{dR} : \Lambda^p\mathbb{L}_{\mathcal{X}} \to \Lambda^{p+1}\mathbb{L}_{\mathcal{X}}$ called the *de Rham differential*. Each $\Lambda^p\mathbb{L}_{\mathcal{X}}$ being a complex has itself an internal differential d. These two differentials are compatible in that

$$d^2 = d_{\mathrm{dR}}^2 = d \circ d_{\mathrm{dR}} + d_{\mathrm{dR}} \circ d = 0.$$

Definition 4.1.6. Let $\mathcal{X}$ be a derived Artin k-stack with cotangent complex $\mathbb{L}_{\mathcal{X}}$. A *p-form of degree q* is a cohomology class $[\omega] \in H^q(\Lambda^p\mathbb{L}_{\mathcal{X}})$. A *closed p-form of degree q* is an element $([\omega^0, \omega^1, \dots)] \in H^q(\oplus_{i\geq 0}\Lambda^{p+i}\mathbb{L}_{\mathcal{X}}[i])$, where $d + d_{dR}$ gives the differential on $\oplus_{i\geq 0}\Lambda^{p+i}\mathbb{L}_{\mathcal{X}}[i]$. If $[(\omega^0, \omega^1, \dots)]$ is a closed p-form of degree q then $[\omega^0]$ is a p-form of degree q.

Definition 4.1.7. Let $\mathcal{X}$ be a derived k-stack with cotangent complex $\mathbb{L}_{\mathcal{X}}$. A p-form $[\omega]$ induces a map $\omega : \mathbb{T}_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}}[p]$. We say that ω is *non-degenerate* if $\omega : \mathbb{T}_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}}[p]$ is a quasi-isomorphism. A *q-shifted symplectic structure* on $\mathcal{X}$ is a non-degenerate closed 2-form of degree q on $\mathcal{X}$.

Example 4.1.8. If G is an affine algebraic group scheme, then the quotient stack $[*/G]$ has a natural 2-shifted symplectic structure. The moduli stack Perf_k of perfect complexes of k-vector spaces has a 2-shifted symplectic structure compatible with the inclusion $[*/\mathrm{GL}(n, k)] \hookrightarrow \mathrm{Perf}_k$ for all $n \geq 0$ [179, Thm. 0.3]. If X is a smooth projective Calabi–Yau n-fold then the mapping stack

$$\mathrm{Map}_{\mathrm{dSt}}(X, \mathrm{Perf}) \simeq \mathcal{M}_{\mathrm{Perf}(X)}$$

has a natural $(2 - n)$-shifted symplectic structure [179, Thm. 0.4].

In particular, if X is a Calabi–Yau 3-fold then $\mathcal{M}_{\mathrm{Perf}(X)}$ is -1-shifted symplectic. There is a notion of orientation data for -1-shifted symplectic k-stacks for which much interesting work has been done [117] [118] [129] [175]. In [34] Borisov–Joyce introduce a notion of orientation of $(2 - 4m)$-shifted symplectic $\mathbb{C}$-stacks.

Definition 4.1.9. Let $(\underline{\mathcal{X}}, \omega)$ be a k-shifted symplectic derived Artin $\mathbb{C}$-stack with $k < 0$ and $k \equiv 2 \mod 4$. Let $\mathcal{X} := \tau_0(\underline{\mathcal{X}})$ denote the classical truncation of $\underline{\mathcal{X}}$. The k-shifted symplectic structure ω gives a quasi-isomorphism $\mathbb{T}_{\underline{\mathcal{X}}} \xrightarrow{\sim} \mathbb{L}_{\underline{\mathcal{X}}}[k]$. Restricting to the classical truncation and taking determinants gives an isomorphism $\iota_\omega : \det(\mathbb{L}_{\underline{\mathcal{X}}|\mathcal{X}}) \xrightarrow{\sim} \det(\mathbb{L}_{\underline{\mathcal{X}}|\mathcal{X}})^{-1}$. An *orientation* of $(\underline{\mathcal{X}}, \omega)$ is a choice of isomorphism

$$o^\omega : \det(\mathbb{L}_{\underline{\mathcal{X}}|\mathcal{X}}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$$

such that $(o^\omega)^\vee \circ o^\omega = \iota^\omega$. The *orientation bundle* $O^\omega \to \mathcal{X}$ of $(\underline{\mathcal{X}}, \omega)$ is the algebraic principal $\mathbb{Z}_2$-bundle parameterizing étale local choices of o^ω. A global trivialization of O^ω is equivalent to a global orientation of $(\mathcal{X}, \omega)$.

4.2 The homology of higher stacks

We explain a notion of homology theories for higher stacks as in [112, §. 3.2.1].

Definition 4.2.1 (see [112])**.** Let k be a field. A *(co)homology theory of higher k-stacks over R* is a collection of covariant functors $H_i(-) : \mathrm{Ho}(\mathrm{HArt}_k) \to R$-mod and contravariant functor $H^i(-) : \mathrm{Ho}(\mathrm{HArt}_k) \to R$-mod for $i = 0, 1, \ldots$ such that for all higher Artin k-stacks $\mathcal{X}, \mathcal{Y}$

1. there exist R-linear functorial graded *cup products*

$$\cup : H^*(\mathcal{X}) \otimes_R H^*(\mathcal{X}) \longrightarrow H^*(\mathcal{X}),$$

and *cap products*

$$\cap : H_*(\mathcal{X}) \otimes_R H^*(\mathcal{X}) \longrightarrow H_*(\mathcal{X})$$

making $(H^*(\mathcal{X}), \cup)$ into a graded unital ring and making $H_*(\mathcal{X})$ into a graded $H^*(\mathcal{X})$-module such that for any morphism of higher k-stacks $f : \mathcal{X} \to \mathcal{Y}$

$$H_*(f)(a \cap H^*(f)(\beta)) = H_*(f)(a) \cap \beta$$

for all $a \in H_*(\mathcal{X}), \beta \in H^*(\mathcal{X})$,

2. there exist R-linear, functorial, associative, and supercommutative *external products*

$$\boxtimes : H^*(\mathcal{X}) \otimes_R H^*(\mathcal{Y}) \longrightarrow H^*(\mathcal{X} \times \mathcal{Y}),$$

3. for a family $\{\mathcal{X}_i\}_{i\in I}$ of higher k-stacks the inclusion $\mathcal{X}_i \hookrightarrow \coprod_{i\in I} \mathcal{X}_i$ induce isomorphisms

$$H_*(\coprod_{i\in I} \mathcal{X}_i) \cong \bigoplus_{i\in I} H_*(\mathcal{X}_i)$$

and

$$H^*(\coprod_{i\in I} \mathcal{X}_i) \cong \prod_{i\in I} H^*(\mathcal{X}_i),$$

4. the functors $H^*(-), H_*(-)$ are $\mathbb{A}^1$-*homotopy invariant* in the sense that the projection $\pi : \mathbb{A}^1 \times \mathcal{X} \to \mathcal{X}$ induces isomorphisms

$$H_*(\pi) : H_*(\mathbb{A}^1 \times \mathcal{X}) \cong H_*(\mathcal{X})$$

and

$$H^*(\pi) : H^*(\mathbb{A}^1 \times \mathcal{X}) \cong H^*(\mathcal{X}),$$

5. there exists a notion of R-linear functorial *Chern classes* of elements of $K_0(\mathrm{Perf}(\mathcal{X}))$ satisfying the usual Chern class identities (see, for example, Hirzebruch [101] or Milnor–Stasheff [171]),

6. there are canonical isomorphisms $H_*([*/\mathbb{G}_m]) \cong R[x]$ and $H^*([*/\mathbb{G}_m]) \cong R[[\tau]]$ of graded R-modules where $\tau = c_1(E_1)$ and $E_1 \to [*/\mathbb{G}_m]$ corresponds to the weight 1 representation of $\mathbb{G}_m$ and $\tau^n \cdot x^n = 1$, and

7. there are canonical isomorphisms

$$H_i(\{\mathrm{pt}\}) = \begin{cases} R, & i = 0 \\ 0, & i \neq 0 \end{cases} \qquad H^i(\{\mathrm{pt}\}) = \begin{cases} R, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Definition 4.2.2. If a collection of functors $E_i(-), E^i(-) : \mathrm{Ho}(\mathrm{HArt}_k) \to R\text{-mod}$ satisfies conditions 1.-6. of Definition 4.2.1 but not 7., we say that $E_i(-), E^i(-) : \mathrm{Ho}(\mathrm{HArt}_k) \to R\text{-mod}$ is a *generalized (co)homology theory of higher k-stacks.*

Remark 4.2.3. Note that we require our homology theories of higher k-stacks to have pushforwards along *all* stack morphisms, not just proper ones. In particular, this excludes algebraic K-theory, Borel–Moore homology, and algebraic cobordism. We are, in fact, only aware of one example of a (co)homology theory of higher k-stacks in the sense of Definition 4.2.1. This is the Betti homology, which is defined for $\mathbb{C}$-stacks only. The author believes that Lurie's étale cohomology of stacks [152] should also be an example but he has not proved this.

Remark 4.2.4. For us, the (co)homology of a derived $\mathbb{C}$-stack $\mathcal{X}$ is defined to be the (co)homology of its classical truncation $\tau_0(\mathcal{X})$. That is, we make no real distinction between (co)homology theories of derived stacks and (co)homology theories of higher stacks.

Given a higher $\mathbb{C}$-stack, there is an associated simplicial set called its *Betti realization* or *topological realization* and we can construct a (co)homology theory of higher $\mathbb{C}$-stacks, in the sense of Definition 4.2.1, by applying ordinary singular (co)homology to that associated simplicial set. The Betti realization was first defined by Simpson [199] (see also Blanc [28, §. 3.1], Morel–Voevodsky [173], and Dugger–Isaksen [66]).

Definition 4.2.5. Given a finite type affine $\mathbb{C}$-scheme U, there is a simplicial set U^{an} which is the singular complex of the underlying complex analytic space of U.

Taking the left Kan extension along the simplicial Yoneda embedding yields a functor $(-)^{\mathrm{Betti}} : sPr(\mathrm{Aff}_{\mathbb{C}}) \to \mathrm{sSet}$. This functor is called the *Betti realization* of simplicial presheaves.

Example 4.2.6. Let G be a complex algebraic group acting on a $\mathbb{C}$-scheme X. Then

$$[X/G]^{\mathrm{Betti}} \simeq (EG^{\mathrm{an}} \times X^{\mathrm{an}})/G^{\mathrm{an}}$$

[199, §8].

Definition 4.2.7. Let $\mathcal{X}$ be a higher $\mathbb{C}$-stack. The *Betti cohomology of $\mathcal{X}$ with coefficients in R* is defined to be

$$H^*(\mathcal{X}, R) := H^*(\mathcal{X}^{\mathrm{Betti}}, R),$$

where $H^*(-) : Ho(\mathrm{sSet}) \to R\text{-mod}$ denotes the R-coefficient singular cohomology of simplicial sets. The Betti homology of $\mathcal{X}$ with coefficients in R is defined similarly, as is the (Betti) E-(co)homology of $\mathcal{X}$ for a spectrum E.

An important fact about the Betti realization is that it sends $\mathbb{A}^1$-étale equivalences of higher $\mathbb{C}$-stacks to homotopy equivalences of topological spaces [66, Thm. 5.2].

Remark 4.2.8. The Betti homology of higher $\mathbb{C}$-stacks does not depend on, for example, whether the $\mathbb{A}^1$-étale model structure or the global model structure is used (see [28, Def. 3.6]).

Properties 1.-2. (and 7.) in Definition 4.2.1 of the Betti homology are automatically inherited from singular (or E-) (co)homology of simplicial sets. $\mathbb{A}^1$-homotopy invariance follows from contractibility of $\mathbb{C} \cong (\mathbb{A}^1)^{\mathrm{an}}$. Property 6. of Definition 4.2.1 is a consequence of the fact that $[*/\mathbb{G}_m]^{\mathrm{Betti}} \simeq BU(1)$.

For the existence of Chern classes we use the equivalence $(\mathrm{Perf}_{\mathbb{C}})^{\mathrm{Betti}} \simeq BU \times \mathbb{Z}$. A perfect complex $\mathcal{E}^\bullet \to \mathcal{X}$ of rank $r \in \mathbb{Z}$ on a higher $\mathbb{C}$-stack $\mathcal{X}$ is classified by a higher $\mathbb{C}$-stack morphism

$$f_{\mathcal{E}^\bullet} : \mathcal{X} \longrightarrow \mathrm{Perf}^r_{\mathbb{C}}.$$

Taking E-cohomology of $f_{\mathcal{E}^\bullet}$ gives a map

$$E^*(f_{\mathcal{E}^\bullet}) : E^*(BU) \cong R[[c_1^E, c_2^E, \dots]] \longrightarrow E^*(\mathcal{X}^{\mathrm{Betti}})$$

and we can define

$$c_k^E(\mathcal{E}^\bullet) := E^*(f_{\mathcal{E}^\bullet})(c_k^E).$$

4.3 Blanc's theorem and the homology of moduli stacks of complexes

In this section we compute the rational Betti homology of the higher $\mathbb{C}$-stack $\mathcal{M}$ of objects in $\mathrm{Perf}(X)$, where X is a smooth complex projective variety in class D.

Note that the Betti realization of $\mathcal{M}$ is an H-space. The Milnor–Moore theorem therefore implies that, if the identity component $\mathcal{M}_0$ is finite type, the Hopf algebra $H_*(\mathcal{M}_0, \mathbb{Q})$ is a free commutative-graded algebra on its primitive elements. This essentially reduces the problem to identifying the primitive elements of $H_*(\mathcal{M}_0, \mathbb{Q})$ more explicitly. To this end, we begin with a model $\tilde{\mathbf{K}}^{\mathrm{sst}}(\mathrm{Perf}(X))$, due to Blanc [28], of the homotopy type of the connective spectrum determined by the group-like E_∞-space $\mathcal{M}^{\mathrm{Betti}}$. Antieau–Heller identify the homotopy type of Blanc's model with the connective semi-topological K-theory spectrum $K^{\mathrm{sst}}(X)$ of X itself [8]. This implies that the 0^{th} space of the spectrum $K^{\mathrm{sst}}(X)$ is equivalent, as an infinite loop space, to $\mathcal{M}^{\mathrm{Betti}}$. The reader will notice that we care only about the H-space structure on $\Omega^\infty K^{\mathrm{sst}}(X)$, not the full infinite loop space structure.

More explicitly, in [28] Blanc introduces a functor $\tilde{\mathbf{K}}^{\mathrm{sst}} : \mathrm{dgCat}_{\mathbb{C}} \to \mathrm{Sp}$ called the *connective semi-topological K-theory of complex non-commutative spaces* [28, Def. 4.1].

This functor is defined as follows: for a $\mathbb{C}$-dg-category $\mathcal{A}$ there is a spectral presheaf $\tilde{\mathbf{K}}(\mathcal{A}) : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sp}$ such that $\tilde{\mathbf{K}}(\mathcal{A})(\mathrm{Spec}(B)) \simeq \tilde{K}(\mathcal{A} \otimes_{\mathbb{C}}^{\mathbb{L}} B)$, where $\tilde{K}(\mathcal{A} \otimes_{\mathbb{C}}^{\mathbb{L}} B)$ denotes the connective K-theory spectrum of the $\mathbb{C}$-dg-category $\mathcal{A} \otimes_{\mathbb{C}}^{\mathbb{L}} B$. The *connective semi-topological K-theory* of $\mathcal{A}$ is then defined to be the spectral realization (see Blanc [28, §. 3.4]) of $\tilde{K}(\mathcal{A})$.

Proposition 4.3.1 (see Antieau–Heller [8, Thm. 2.3], Blanc [28, Thm. 4.21])**.** *Let $\mathcal{A}$ be a saturated $\mathbb{C}$-linear dg-category and let $\mathcal{M}_{\mathcal{A}}$ denote the moduli stack of objects in $\mathcal{A}$. Then there is an equivalence*

$$\Omega^\infty \tilde{\mathbf{K}}^{sst}(\mathcal{A}) \simeq \mathcal{M}_{\mathcal{A}}^{\mathrm{Betti}}$$

of infinite loop spaces which is canonical up to homotopy. When $\mathcal{A} = \mathrm{Perf}(X)$ there is a further equivalence

$$\Omega^\infty \tilde{\boldsymbol{K}}^{sst}(Perf(X)) \simeq \Omega^\infty K^{sst}(X).$$

In particular, the set of connected components of $\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{Betti}}$ is given by $K_{sst}^0(X)$.

It is also worth pointing out that the semi-topological K-theory of saturated dg-categories was previously defined by Bertrand Toën [216]. Toën actually *defines* the semi-topological K-theory space of a saturated dg-category $\mathcal{A}$ to be the Betti realization the moduli stack of objects in $\mathcal{A}$ (see also Kaledin [121, §. 8]).

We now discuss the 0^{th} semi-topological K-group in a bit more detail.

Definition 4.3.2. Let $E, F \to X$ be algebraic vector bundles on X. They are said to be *algebraically equivalent* if there exists a connected algebraic curve C and an algebraic vector bundle $V \to X \times C$ specializing to E at a point $p \in C$ and to F at a point $q \in C$. One says that E and F are *rationally equivalent* if they are algebraically equivalent over $\mathbb{A}^1$.

The 0^{th} algebraic K-group of X is isomorphic to the group completion of the additive monoid of algebraic vector bundles modulo rational equivalence

$$K^0_{\text{alg}}(X) \cong \left(\frac{\{\text{algebraic vector bundles on } X\}}{\text{rational equivalence}} \right)^+.$$

Proposition 4.3.3 (see Friedlander–Walker [83, Prop. 2.10-11])**.** *There is a monoid isomorphism*

$$\pi_0(\text{Map}_{\text{IndSch}_{\mathbb{C}}}(X, \text{Gr}(\mathbb{C}^\infty))) \cong \frac{\{\textit{globally generated algebraic vector bundles on } X\}}{\textit{algebraic equivalence}}.$$

Group completion gives an isomorphism

$$K^0_{\text{sst}}(X) \cong \left(\frac{\{\textit{globally generated algebraic vector bundles on } X\}}{\textit{algebraic equivalence}} \right)^+$$

and hence a surjection $K^0_{\text{alg}}(X) \twoheadrightarrow K^0_{\text{sst}}(X)$ *(see [82, Prop. 2.12]).*

We now further analyze the topology of $\mathcal{M}$.

Lemma 4.3.4. *Let* $\alpha \in K^0_{sst}(X)$ *and let* $\mathcal{M}_\alpha \subset \mathcal{M}$ *denote the substack of perfect complexes of coherent sheaves on* X *of class* α*. Then there is an* $\mathbb{A}^1$*-homotopy equivalence* $\mathcal{M}_\alpha \simeq \mathcal{M}_0$.

Proof. Let $\mathcal{E}^\bullet$ be a perfect complex with $\alpha = [\mathcal{E}^\bullet] \in K^0_{\text{sst}}(X)$. Define a morphism $\Phi_{\mathcal{E}^\bullet} : \mathcal{M}_0 \to \mathcal{M}_\alpha$ by $\mathcal{F}^\bullet \mapsto \mathcal{E}^\bullet \oplus \mathcal{F}^\bullet$. Similarly, define a morphism $\Psi_{\mathcal{E}^\bullet} : \mathcal{M}_\alpha \to \mathcal{M}_0$ by $\mathcal{F}^\bullet \mapsto \mathcal{E}^\bullet[1] \oplus \mathcal{F}^\bullet$. We claim that $\Psi_{\mathcal{E}^\bullet} \circ \Phi_{\mathcal{E}^\bullet} \cong \text{Id}_{\mathcal{M}_0}$. Define $H : \mathbb{A}^1 \times \mathcal{M}_0 \to \mathcal{M}_0$ by

$$(t, \mathcal{F}^\bullet) \mapsto \mathcal{F}^\bullet \oplus \text{Cone}(t \cdot \text{Id}_{\mathcal{E}^\bullet}). \tag{4.3.1}$$

Then, for $\mathcal{F}^\bullet \in \text{ob}(\text{Perf}(X))$,

$$H(0, \mathcal{F}^\bullet) = \mathcal{F}^\bullet \oplus \mathcal{E}^\bullet \oplus \mathcal{E}^\bullet[1] = \Psi_{\mathcal{E}^\bullet} \circ \Phi_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet)$$

and, for $t \in \mathbb{A}^1/\{0\}$,

$$H(t, \mathcal{F}^\bullet) = \mathcal{F}^\bullet = \mathrm{Id}_{\mathcal{M}_0}(\mathcal{F}^\bullet).$$

Thus $\Psi_{\mathcal{E}^\bullet}$ is a left homotopy inverse of $\Phi_{\mathcal{E}^\bullet}$. For the other direction, one may define a homotopy $H' : \mathbb{A}^1 \times \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ between $\Phi_{\mathcal{E}^\bullet} \circ \Psi_{\mathcal{E}^\bullet}$ and $\mathrm{Id}_{\mathcal{M}_\alpha}$ again by the formula (4.3.1). $\square$

Theorem 4.3.5. *Suppose that the identity component $\Omega^\infty K^{\mathrm{sst}}(X)^0$ of $\Omega^\infty K^{\mathrm{sst}}(X)$ has finite Betti numbers. Then there is a natural isomorphism of graded $\mathbb{Q}$-Hopf algebras*

$$H_*(\mathcal{M}, \mathbb{Q}) \cong \mathbb{Q}[K^0_{sst}(X)] \otimes \mathrm{SSym}_{\mathbb{Q}}[\bigoplus_{i>0} K^i_{\mathrm{sst}}(X)]. \tag{4.3.2}$$

Proof. By Proposition 4.3.1, there is an H-equivalence $\mathcal{M}^{\mathrm{Betti}} \simeq \Omega^\infty K^{\mathrm{sst}}(X)$. This and the Milnor–Moore theorem then give an isomorphism

$$H_*(\mathcal{M}_0, \mathbb{Q}) \cong \mathrm{SSym}_{\mathbb{Q}}[\bigoplus_{i>0} K^i_{\mathrm{sst}}(X)]$$

of graded $\mathbb{Q}$-Hopf algebras. Künneth then gives

$$\begin{aligned} H_*(\mathcal{M}, \mathbb{Q}) &\cong H_*(\pi_0(\mathcal{M}) \times \mathcal{M}_0, \mathbb{Q}) \\ &\cong \mathbb{Q}[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{SSym}_{\mathbb{Q}}[\bigoplus_{i>0} K^i_{\mathrm{sst}}(X)]. \end{aligned}$$ $\square$

Computing semi-topological K-theory is, in general, not an easy task. There is however a certain class of varieties for which computing semi-topological K-theory is not hard.

Definition 4.3.6. A smooth complex variety V is said to be *in class D* if the natural map $\Omega^\infty K^{\mathrm{sst}}(V) \to \Omega^\infty K^{\mathrm{top}}(V^{\mathrm{an}})$ induces an isomorphism $K^i_{\mathrm{sst}}(V) \to K^i_{\mathrm{top}}(V^{\mathrm{an}})$ for all $i \geq 1$ and a monomorphism $K^0_{\mathrm{sst}}(V) \hookrightarrow K^0_{\mathrm{top}}(V^{\mathrm{an}})$.

Remark 4.3.7. Our terminology is motivated as follows: Friedlander–Haesemeyer–Walker say that a variety V is *in class C* if the refined cycle maps $L_tH_n(X) \to \tilde{W}_{-2t}H_n^{\mathrm{BM}}(X)$ [79, Def. 5.8] are isomorphisms for all t and n. The condition of being in class C is much stronger than what we want. For example, a smooth algebraic surface S will not be in class C unless all of $H^2(S^{\mathrm{an}})$ is algebraic. This would exclude many interesting surfaces, such as K3 surfaces. If a variety is in class C then $K^i_{\mathrm{sst}}(V) \to K^i_{\mathrm{top}}(V^{\mathrm{an}})$ is an isomorphism for $i \geq \dim_{\mathbb{C}} V - 1$ and injective for $i = \dim_{\mathbb{C}} V - 2$; one could call this property being in class E. If so, then *all* surfaces are in class E [79, Thm. 3.7] and class C $\subset$ class D $\subset$ class E.

Fortunately, many varieties are in class D. Examples of projective varieties in class D are curves, surfaces, toric varieties, flag varieties, and rational 3- and 4-folds [79, Thm. 6.18, Prop. 6.19]. This is a non-exhaustive list. For example, some more examples of degenerate 3- and 4-folds in class D were computed recently by Voineagu [221]. The kernel of $K^0_{\mathrm{sst}}(X) \otimes \mathbb{Q} \to K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes \mathbb{Q}$ is isomorphic to the rational Griffiths group [82, Ex. 1.5]. The Griffiths group of an algebraic variety measures the difference between homological and algebraic equivalence of its algebraic cycles. No variety with a non-vanishing rational Griffiths group can be in class D. Examples of varieties with non-vanishing rational Griffiths groups are quintic 3-folds, general Calabi–Yau 3-folds, cubic 7-folds, and certain 5-folds (see Albano–Collino [5], Clemens [51], Favero–Iliev–Katzarkov [69], Griffiths [91], and Voisin [222]). The following lemma justifies our choice to focus on varieties in class D.

Lemma 4.3.8. *If X is in class D then for all $\alpha \in K^0_{\mathrm{sst}}(X)$ the K-theory comparison map induces a homotopy equivalence $\mathcal{M}^{\mathrm{Betti}}_\alpha \simeq \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU \times \mathbb{Z})_\alpha$.*

Proof. By definition of class D, each connected component $\Gamma_\alpha : \Omega^\infty K^{\mathrm{sst}}(X)_\alpha \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})_\alpha$ of the natural K-theory comparison map is a weak homotopy equivalence. Proposition 4.3.1 then gives a weak homotopy equivalence

$$\mathcal{M}_\alpha^{\mathrm{Betti}} \simeq \Omega^\infty K^{\mathrm{sst}}(X)_\alpha \xrightarrow{\Gamma} \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})_\alpha \simeq \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU \times \mathbb{Z})_\alpha. \qquad (4.3.3)$$

As X^{an} is a compact metric space and $BU \times \mathbb{Z}$ has the homotopy type of a countable CW complex, $\mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU \times \mathbb{Z})_\alpha$ has the homotopy type of a CW complex [168, Cor. 2]. Because $\mathcal{M}_\alpha^{\mathrm{Betti}}$ is the realization of a simplicial set it is a CW complex too. Whitehead's theorem then lifts the weak homotopy equivalence (4.3.3) to a homotopy equivalence. □

Remark 4.3.9. We know that there is a graded vertex algebra structure on $\hat{H}_*(\mathcal{M})$ whether or not X is in class D. Although, when X is not in class D it may be very difficult to identity this vertex algebra structure explicitly. However, there will always be a vertex algebra morphism from $\hat{H}_*(\mathcal{M})$ into a generalized super-lattice vertex algebra on the complex topological K-theory of X^{an}.

Remark 4.3.10. That algebraic curves are in class D is a stabilization of Kirwan's result on the cohomology of spaces of maps from Riemann surfaces to Grassmannians [125].

The following proposition makes it possible to compute the Chern classes of the universal complex over $X \times \mathcal{M}$ when X is in class D.

Proposition 4.3.11. *Let X be in class D and $\alpha \in K^0_{\mathrm{sst}}(X)$. Let $\mathbb{E}_\alpha^\bullet$ be the universal perfect complex over $X \times \mathcal{M}_\alpha$ and let $\mathcal{E}_\alpha : X^{\mathrm{an}} \times \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU)_\alpha \to BU$ denote the evaluation map. Then there is a homotopy*

$$(\mathbb{E}_\alpha^\bullet)^{\mathrm{Betti}} \simeq \mathcal{E}_\alpha : X^{\mathrm{an}} \times \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU)_\alpha \to BU$$

In particular, the image of $c_i(\mathbb{E}_\alpha^\bullet)$ under the isomorphism $H^{2i}(X \times \mathcal{M}_\alpha) \cong H^{2i}(X^{\mathrm{an}} \times \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU)_\alpha)$ equals $c_i([\mathcal{E}_\alpha])$ for all $i \geq 0$.

Proof. The evaluation map $X \times \mathrm{Map}_{\mathrm{HSt}_{\mathbb{C}}}(X, \mathrm{Perf}_{\mathbb{C}})_\alpha \to \mathrm{Perf}_{\mathbb{C}}$ classifies $\mathbb{E}^\bullet{}_\alpha$. Taking Betti realizations gives a map

$$(ev_\alpha)^{\mathrm{Betti}} : X^{\mathrm{an}} \times \mathcal{M}_\alpha^{\mathrm{Betti}} \to BU \times \mathbb{Z}.$$

Exponentiating gives a continuous map $\Xi_\alpha : \Omega^\infty K^{\mathrm{sst}}(X)_\alpha \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})_\alpha$ by Proposition 4.3.1. Write

$$\Xi := \coprod_{\alpha \in K^0_{\mathrm{sst}}(X)} \Xi_\alpha : \Omega^\infty K^{\mathrm{sst}}(X) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}).$$

Note that both Ξ and the natural K-theory comparison map $\Gamma : \Omega^\infty K^{\mathrm{sst}}(X) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ make the homotopy-theoretic group completion diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C})^\infty)^{\mathrm{an}} \arrow[r] \arrow[d] & \mathrm{Map}_{C^0}(X^{\mathrm{an}}, \coprod_{n \geq 0} BU(n)) \arrow[d] \\
\Omega^\infty K^{\mathrm{sst}}(X) \arrow[r, "\Gamma, \Xi"'] & \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})
\end{tikzcd}$$

homotopy commute. By the weak universal property of homotopy-theoretic group completions, Γ is weakly homotopic to Ξ [50, Prop. 1.2]. Therefore, the restrictions of Γ, Ξ along any map $S^n \to \Omega^\infty K^{\mathrm{sst}}(X)$ are homotopic. In particular, Γ and Ξ induce the same maps on homotopy groups. Because X is in class D, Γ_α is a homotopy equivalence for all $\alpha \in K^0_{\mathrm{sst}}(X)$. This now implies that Ξ_α is a homotopy equivalence for all $\alpha \in K^0_{\mathrm{sst}}(X)$. Because Ξ_α makes the diagram

$$\begin{tikzcd}
X^{\mathrm{an}} \times \mathcal{M}_\alpha^{\mathrm{Betti}} \arrow[r, "(\mathbb{E}^\bullet_\alpha)^{\mathrm{Betti}}"] \arrow[d, "1_{X^{\mathrm{an}}} \times \Xi_\alpha"'] & BU \\
X^{\mathrm{an}} \times \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU)_\alpha \arrow[ur, "\mathcal{E}_\alpha"'] &
\end{tikzcd}$$

homotopy commute, $(\mathbb{E}^\bullet_\alpha)^{\mathrm{Betti}}$ is homotopic to $\mathcal{E}_\alpha$ for all $\alpha \in K^0_{\mathrm{sst}}(X)$. □

The following theorem can be seen as a stabilization of the results of Atiyah–Bott [16, Prop. 2.20] on Riemann surfaces, which applies to varieties in class D.

Theorem 4.3.12. *Let X be in class D and let $\alpha \in K^0_{\mathrm{sst}}(X)$. Then $H^*(\mathcal{M}_\alpha, \mathbb{Q})$ is freely generated as a commutative-graded $\mathbb{Q}$-algebra by the Künneth components of Chern classes of the universal complex $\mathbb{E}^\bullet_\alpha$ over $X \times \mathcal{M}_\alpha$.*

Proof. By Proposition 4.3.11, it suffices to show that $H^*(\mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU)_\alpha, \mathbb{Q})$ is freely generated by Künneth components of Chern classes of $[\mathcal{E}_\alpha]$. As X^{an} is a Kähler manifold, it is formal. The theorem then follows from Corollary 3.3.10. $\square$

Because the leading coefficients of the universal Chern character polynomials are non-zero, one can also regard the cohomology $H^*(\mathcal{M}_\alpha, \mathbb{Q})$ as being freely generated as a commutative-graded $\mathbb{Q}$-algebra by the Künneth components of Chern *characters* of $\mathbb{E}^\bullet_\alpha$. More generally, we can consider any complex-oriented cohomology theory whose coefficient ring is a $\mathbb{Q}$-algebra.

Corollary 4.3.13. *Let E be a complex-oriented spectrum. Write $R := E^*(\mathrm{pt})$ and suppose that R is a $\mathbb{Q}$-algebra. Then $E^*(\mathcal{M}_\alpha)$ is freely generated as a commutative-graded $\mathbb{Q}$-algebra by the Künneth components of the E-Chern characters of the universal complex $\mathbb{E}^\bullet_\alpha$ over $X \times \mathcal{M}_\alpha$.*

Proof. Because E is a rational spectrum, there is an E-Dold–Chern character isomorphism $E^*(\mathcal{M}_\alpha) \cong H^*(\mathcal{M}_\alpha, R)$. It is clear from the definition of E-Chern characters that the E-Dold-Chern character sends E-Chern characters to Chern characters. $\square$

4.4 Orientability of moduli spaces of sheaves on Calabi–Yau 4-folds

Let X be a smooth projective complex variety. Let $\underline{\mathcal{M}}$ denote the derived $\mathbb{C}$-stack of objects in $\mathrm{Perf}(X)$. If X is a Calabi–Yau 4-fold then, by [179, Thm. 0.4], $\underline{\mathcal{M}}$ has a natural -2-shifted symplectic structure ω. Let $\mathcal{M}$ denote the higher $\mathbb{C}$-stack which is the classical truncation of $\mathcal{M}$. Let $O^\omega \to \mathcal{M}$ denote the algebraic orientation bundle of $\mathcal{M}$ (see Definition 4.1.9).

Taking Betti realizations gives a topological principal $\mathbb{Z}_2$-bundle $O^{\omega,\mathrm{Betti}} \to \mathcal{M}^{\mathrm{Betti}}$. Let $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ denotes the $\mathbb{C}$-stack morphism induced by direct sum of perfect complexes. In [45] an isomorphism

$$\phi : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \xrightarrow{\sim} \Phi^*(O^\omega)$$

of algebraic principal $\mathbb{Z}_2$-bundles is constructed (see [45, Thm. 1.15.c]). This has the effect of making $O^{\omega,\mathrm{Betti}} : \mathcal{M}^{\mathrm{Betti}} \to B\mathbb{Z}_2$ into a weak H-principal $\mathbb{Z}_2$-bundle. The following Lemma is proved in [45].

Lemma 4.4.1 (see [45, Lem. 3.13]). *Let $O^{\omega,\mathrm{Betti}} \to \mathcal{M}^{\mathrm{Betti}}$ denote the weak H-principal $\mathbb{Z}_2$-bundle described above. Then if $O^{\omega,\mathrm{Betti}}$ is trivializable, $(\phi^{\mathrm{Betti}}, O^{\omega,\mathrm{Betti}})$ is a strong H-principal $\mathbb{Z}_2$-bundle.*

By Theorem 4.3.1, there is a homotopy equivalence $\mathcal{M}^{\mathrm{Betti}} \simeq \Omega^\infty K^{\mathrm{sst}}(X)$ of H-spaces. We will write $O^{\omega,\mathrm{Betti}} \to \Omega^\infty K^{\mathrm{sst}}(X)$ for the weak H-principal $\mathbb{Z}_2$-bundle pulled back from $\mathcal{M}^{\mathrm{Betti}}$ under this equivalence. Recall that there is a natural K-theory comparison map $\Gamma : \Omega^\infty K^{\mathrm{sst}}(X) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ (Definition 3.1.11). In this section, we will show that when X is a Calabi–Yau 4-fold, Γ pulls the differential-geometric orientation bundle $O^{\not{D}_+}$ on $\Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$ back to the algebro-geometric orientation bundle $O^{\omega,\mathrm{Betti}}$ on $\Omega^\infty K^{\mathrm{sst}}(X)$.

Theorem 4.4.2 (see Cao–Gross–Joyce [45, Thm. 1.15]). *Let X be a Calabi–Yau 4-fold and let $\underline{\mathcal{M}}$ denote the derived $\mathbb{C}$-stack of objects in Perf(X), which has a -2-shifted symplectic structure ω. Let $\mathcal{M}$ denote the classical truncation of $\underline{\mathcal{M}}$ and let Γ be the natural K-theory comparison map. Let $O^\omega \to \mathcal{M}$ denote the orientation bundle induced by ω. The underlying analytic space X^{an} of X is a spin 8-manifold with a positive Dirac operator $\not{D}_+ : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$. Let $O^{\not{D}_+} \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$*

denote the principal $\mathbb{Z}_2$-bundle induced by $\not{D}_+$ as in Definition 3.4.11. Then there is an isomorphism

$$O^{\omega,\mathrm{Betti}} \xrightarrow{\sim} \Gamma^*(O^{\not{D}_+})$$

of principal $\mathbb{Z}_2$-bundles.

Theorem 4.4.2 implies that if $O^{\not{D}_+}$ is trivializable, which it is by Theorems 3.5.8 and 3.4.10, then so is $O^{\omega,\mathrm{Betti}}$. This implies that $O^\omega \to \mathcal{M}$ is trivializable as an algebraic principal $\mathbb{Z}_2$-bundle so that $\mathcal{M}$ is orientable in the sense of Borsiov–Joyce. The higher $\mathbb{C}$-stack $\mathcal{M}$ also contains the $\mathbb{C}$-stacks $\mathcal{M}^{\mathrm{coh}}, \mathcal{M}^{\mathrm{coh,ss}}$, and $\mathcal{M}^{\mathrm{vect}}$ of coherent sheaves, semistable coherent sheaves, and (algebraic) vector bundles on X as substacks. The orientation bundle $O^\omega \to \mathcal{M}$ restricts to orientation bundles on $\mathcal{M}^{\mathrm{coh}}, \mathcal{M}^{\mathrm{coh,ss}}$, and $\mathcal{M}^{\mathrm{vect}}$ where it will be trivializable since it is trivializable over $\mathcal{M}$.

Note that by Lemmas 3.4.12 and 4.4.1, trivializability of both $O^{\not{D}_+}$ and $O^{\omega,\mathrm{Betti}}$ implies that the isomorphism $O^{\omega,\mathrm{Betti}} \cong \Gamma^*(O^{\not{D}_+})$ is actually canonical. Therefore, an orientation on $\overline{\mathcal{B}}^U$ determines an orientation on $\mathcal{M}^{\mathrm{Betti}}$. This establishes the existence of orientations needed to define Donaldson–Thomas type invariants of Calabi–Yau 4-folds as in [34, Thm. 1.1].

We can also compare the behavior of $O^{\not{D}_+}$ under direct sums to that of O^ω.

Theorem 4.4.3. *Let $X, \mathcal{M}, O^\omega$, and $O^{\not{D}_+}$ be as in Theorem 4.4.2. Given $\alpha \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$, there is an open and closed substack $\mathcal{M}_\alpha \subset \mathcal{M}$ of perfect complexes of X of class α. Let $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ denote the $\mathbb{C}$-stack morphism induced by direct sum of complexes and let $\Psi : \overline{\mathcal{B}}^U \times \overline{\mathcal{B}}^U \to \overline{\mathcal{B}}^U$ denote the H-map induced by direct sum of connections. Recall that there are isomorphisms*

$$\phi : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \xrightarrow{\sim} \Phi^*(O^\omega), \qquad \psi : O^{\not{D}_+} \boxtimes_{\mathbb{Z}_2} O^{\not{D}_+} \xrightarrow{\sim} \Psi^*(O^{\not{D}_+})$$

of principal $\mathbb{Z}_2$-bundles [45, Thm. 1.15.c] [115, Def. 2.22].

By Theorems 3.4.10, 3.5.8, and 4.4.2 an orientation $o_\alpha^{\not{D}_+}$ *of* $\overline{\mathcal{B}}_\alpha^U$ *for* $\alpha \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$ *induces an orientation* o^ω_α *of* $\mathcal{M}_\alpha$*. Then, for all* $\alpha, \beta \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$ *and* $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ *if*

$$\psi_*(o_\alpha^{\not{D}_+} \boxtimes o_\beta^{\not{D}_+}) = \epsilon_{\alpha,\beta} \cdot \Psi^*(o_{\alpha+\beta}^{\not{D}_+})$$

then

$$\phi_*(o^\omega_\alpha \boxtimes o^\omega_\beta) = \epsilon_{\alpha,\beta} \cdot \Phi^*(o^\omega_{\alpha+\beta}).$$

Theorem 4.4.3 implies that any relations which hold among the orientations on different connected components of $\overline{\mathcal{B}}^U$ also hold among orientations on different connected components of $\mathcal{M}$. The kinds of relations which may hold, for example, are as follows. Given two complex vector bundles $P, Q \to X^{\mathrm{an}}$ with K-theory classes α, β one can define an Euler form by

$$\chi^{\not{D}_+}(\alpha, \beta) = \mathrm{ind}_{\mathbb{C}}(\not{D}_+^{\nabla_{\overline{P}\otimes Q}}).$$

Then with $o_\alpha^{\not{D}_+}$, $o_\beta^{\not{D}_+}$ defined as in Theorem 4.4.3, there is some $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ such that

$$\psi(o_\alpha^{\not{D}_+} \boxtimes o_\beta^{\not{D}_+}) = \epsilon_{\alpha,\beta} \cdot \Psi^*(o_{\alpha+\beta}^{\not{D}_+}).$$

Then

$$\epsilon_{\beta,\alpha} = (-1)^{\chi^{\not{D}_+}(\alpha,\beta) + \chi^{\not{D}_+}(\alpha,\alpha)\chi^{\not{D}_+}(\beta,\beta)} \epsilon_{\alpha,\beta}$$

[115, Rem. 2.23.b].

Theorems 4.4.2 and 4.4.3 will be proved in several parts. We begin with some background. Throughout, we fix a smooth projective Calabi–Yau 4-fold X although some of the material would apply to arbitrary smooth complex projective varieties.

Definition 4.4.4. By an *algebraic vector bundle* we mean a locally free coherent sheaf. We say that an algebraic vector bundle is *globally generated* if there exists a surjective sheaf morphism $V \otimes_{\mathbb{C}} \mathcal{O}_X \to E$ where V is a finite-dimensional $\mathbb{C}$-vector space. We write $\mathcal{M}^{\mathrm{vb,gs}}$ for the Artin $\mathbb{C}$-stack of globally generated algebraic vector bundles on X. There in an injective $\mathbb{C}$-stack morphism $\mathcal{M}^{\mathrm{vb,gs}} \hookrightarrow \mathcal{M}$ that sends an algebraic vector bundle to a complex concentrated in degree 0.

For all $n \geq 0$, the Grassmannian $\mathrm{Gr}(\mathbb{C}^n) = \coprod_{k\geq 0} \mathrm{Gr}_k(\mathbb{C}^n)$ has a tautological bundle $F_n^{\mathrm{taut}} \to \mathrm{Gr}(\mathbb{C}^n)$ which has rank k over the component $\mathrm{Gr}_k(\mathbb{C}^n)$. Let $\underline{\mathbb{C}}^n$ denote the trivial rank n complex vector bundle on $\mathrm{Gr}(\mathbb{C}^n)$. Then $F_n^{\mathrm{taut}} \subset \underline{\mathbb{C}}^n$ and we write $F_n^{\mathrm{quot}} := \underline{\mathbb{C}}^n / F_n^{\mathrm{taut}}$ for the quotient bundle.

Consider the evaluation map

$$\mathrm{ev}_{X,n} : X \times \mathrm{Map}_{\mathrm{Sch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^n)) \longrightarrow \mathrm{Gr}(\mathbb{C}^n).$$

Pulling back F_n^{quot} along $\mathrm{ev}_{X,n}$ gives a family of algebraic vector bundles over $X \times \mathrm{Map}_{\mathrm{Sch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^n))$, globally generated because $\mathrm{ev}_{X,n}^*(F_n^{\mathrm{quot}}) \cong \underline{\mathbb{C}}^n / \mathrm{ev}_{X,n}^*(F_n^{\mathrm{taut}})$ is surjected onto by $\underline{\mathbb{C}}^n \to X$. This family can be expressed as a $\mathbb{C}$-stack morphism

$$\Delta^n : \mathrm{Map}_{\mathrm{Sch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^n)) \longrightarrow \mathcal{M}^{\mathrm{vb,ss}} \subset \mathcal{M}.$$

Taking a direct limit over n gives a $\mathbb{C}$-stack morphism

$$\Delta : \mathcal{T} \longrightarrow \mathcal{M}^{\mathrm{vb,gs}} \subset \mathcal{M},$$

where $\mathcal{T}$ denotes the $\mathbb{C}$-ind-scheme $\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))$. Taking a direct limit over the maps $\mathrm{Gr}(\mathbb{C}^N) \times \mathrm{Gr}(\mathbb{C}^M) \to \mathrm{Gr}(\mathbb{C}^{N+M})$ gives a $\mathbb{C}$-ind-scheme map $\mathrm{Gr}(\mathbb{C}^\infty) \times \mathrm{Gr}(\mathbb{C}^\infty) \to \mathrm{Gr}(\mathbb{C}^\infty)$ that makes $\mathcal{T}^{\mathrm{an}}$ into an H-space.[4] The Betti realization $\Delta^{\mathrm{Betti}} : \mathcal{T}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{Betti}}$ of Δ is a morphism of H-spaces. From Proposition 4.3.1 we can

[4]We already know that $\mathcal{T}^{\mathrm{an}}$ is an H-space because we mentioned in Chapter 3 that Friedlander–Walker proved $\mathrm{Map}_{\mathrm{IndSch}_{\mathbb{C}}}(X, \mathrm{Gr}(\mathbb{C}^\infty))$ is an E_∞-space. We chose to write out the H-space structure in this way so as to make it more clear that Δ^{Betti} is an H-map.

further show that the H-map $\Delta^{\mathrm{Betti}} : \mathcal{T}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{Betti}}$ is a homotopy-theoretic group completion.

Definition 4.4.5. Points of $\mathcal{T}^{\mathrm{an}}$ may be regarded as pairs $(F, (s_1, s_2, \dots))$ consisting of an algebraic vector bundle $F \to X$ and a sequence of sections $(s_1, s_2, \dots)$ generating F. Given such a pair choose $n \gg 0$ such that the map $(s_1, s_2, \dots, s_n) : \underline{\mathbb{C}}^n \twoheadrightarrow F$ is surjective. We split

$$\underline{\mathbb{C}}^n = F^{\mathrm{an}} \oplus \ker(s_1, \dots, s_n)^{\mathrm{an}} \tag{4.4.1}$$

orthogonally by choosing a Hermitian metric on $\underline{\mathbb{C}}^n$. The trivial connection on $\underline{\mathbb{C}}^n$ projects under (4.4.1) to a unitary connection $\nabla_{F^{\mathrm{an}}}$ on $F^{\mathrm{an}} \to X^{\mathrm{an}}$. We can then define a morphism topological stacks

$$\Lambda : \mathcal{T}^{\mathrm{an}} \longrightarrow \coprod_{\text{iso. classes } [P] \text{ of } U(n)\text{-bundles } , n \geq 0} \mathcal{B}_P.$$

by $F \mapsto \nabla_{F^{\mathrm{an}}}$. Taking classifying spaces and composing with the Σ_P maps (see Proposition 3.4.2) gives an H-map $\Lambda^{\mathrm{cla}} : \mathcal{T}^{\mathrm{an}} \to \mathcal{B}^U$.

Functoriality of homotopy-theoretic group completions gives a homotopy commutative diagram

$$\begin{tikzcd}
\mathcal{T}^{\mathrm{an}} \arrow[r, "\Lambda^{\mathrm{cla}}"] \arrow[d, "\Xi"] & \mathcal{B}^U \arrow[d] \\
\Omega^\infty K^{\mathrm{sst}}(X) \arrow[r, "\Gamma"] & \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}),
\end{tikzcd}$$

where $\Xi : \mathcal{T}^{\mathrm{an}} \to \Omega^\infty K^{\mathrm{sst}}(X)$ denotes the homotopy-theoretic group completion of $\mathcal{T}^{\mathrm{an}}$. Consider now the following diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{Betti}} \arrow[dr, "\sim"'] & \mathcal{T}^{\mathrm{an}} \arrow[l, "\Delta^{\mathrm{Betti}}"'] \arrow[r, "\Lambda^{\mathrm{cla}}"] \arrow[d, "\Xi"] & \mathcal{B}^U \arrow[d] \\
 & \Omega^\infty K^{\mathrm{sst}}(X) \arrow[r, "\Gamma"] \arrow[dr, "O^{\omega,\mathrm{Betti}}"'] & \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \arrow[d, "O^{\not{D}_+}"] \\
 & & B\mathbb{Z}_2,
\end{tikzcd}$$

where the equivalence $\mathcal{M}^{\mathrm{Betti}} \xrightarrow{\sim} \Omega^\infty K^{\mathrm{sst}}(X)$ denotes the equivalence existing uniquely up to weak[5] homotopy which is compatible with Δ^{Betti} and Ξ. If it can be shown that $\Xi^*(O^{\omega,\mathrm{Betti}})$ is isomorphic to $(\Lambda^{\mathrm{cla}})^*(O^{\not{D}_+})$ then, by Proposition 3.4.10, $O^{\omega,\mathrm{Betti}}$ will be isomorphic to $\Gamma^*(O^{\not{D}_+})$.

We are now in a position to prove Theorem 4.4.2.

Proof. (of Theorem 4.4.2) Given an isomorphism class $[P]$ of principal $U(n)$-bundles on X for some $n \geq 0$, we write $\mathcal{T}^{\mathrm{an}}_{[P]} := \Lambda^{-1}(\mathcal{B}_P)$. Restriction of Λ^{cla} to $\mathcal{T}^{\mathrm{an}}_{[P]}$ maps $\mathcal{T}^{\mathrm{an}}_{[P]} \to \mathcal{B}^{\mathrm{cla}}_P$ and Λ^{cla} can be written as the union of $\Lambda^{\mathrm{cla}}|_{\mathcal{T}^{\mathrm{an}}_{[P]}}$ over all isomorphism classes $[P]$ of principal $U(n)$-bundles on X for all $n \geq 0$. Therefore, it suffices to construct an isomorphism

$$\lambda_{[P]} : (\Delta^{\mathrm{Betti}})^*(\Xi^*(O^{\omega,\mathrm{Betti}})|_{\mathcal{T}^{\mathrm{an}}_{[P]}}) \xrightarrow{\ \sim\ } (\Lambda^{\mathrm{cla}})^*(O^{\not{D}_+}|_{\mathcal{B}^{\mathrm{cla}}_P})$$

for all such $[P]$. Fixing an isomorphism class $[P]$, we will define $\lambda_{[P]}$ pointwise on $\mathcal{T}^{\mathrm{an}}$ using only continuous operations so that our pointwise definition gives a continuous identification of fibers i.e. an isomorphism of principal $\mathbb{Z}_2$-bundles.

Let F be a globally generated algebraic vector bundle of isomorphism class $[P]$ on X and let $(s_1, s_2, \dots)$ be a sequence of generating sections so that the pair $(F, (s_1, s_2, \dots))$ can be regarded as a point of $\mathcal{T}^{\mathrm{an}}$. By definition

$$(\Lambda^{\mathrm{cla}})^*(O^{\not{D}_+}|_{\mathcal{B}^{\mathrm{cla}}_P})|_{(F,(s_1,s_2,\dots))} \cong \mathrm{Or}(\det(\not{D}_+^{\nabla_P})) \tag{4.4.2}$$

where we identify F^{an} with P and write ∇_P for the connection on P determined by $\nabla_{F^{\mathrm{an}}}$ under this identification. By definition, there is an isomorphism

$$(\Delta^{\mathrm{Betti}})^*(\Xi^*(O^{\omega,\mathrm{Betti}})|_{\mathcal{T}^{\mathrm{an}}_{[P]}})|_{(F,(s_1,s_2,\dots))} \cong \{o_F : \det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \xrightarrow{\sim} \mathbb{C} \mid o_F^\vee \circ o_F = i^\omega|_{[F]}\},$$

[5]For our purposes of proving that two principal $\mathbb{Z}_2$-bundles are isomorphic, it will not be important to distinguish between weak homotopy and homotopy. That is because a principal $\mathbb{Z}_2$-bundle $P : X \to B\mathbb{Z}_2$ on a CW complex X is determined by its *monodromy map* i.e. the induced map $\pi_1(P) : \pi_1(X) \to \pi_1(B\mathbb{Z}_2) \cong \mathbb{Z}_2$ on fundamental groups.

where $\iota^\omega|_{[F]}$ agrees with the isomorphism $\det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \xrightarrow{\sim} \det(\mathbb{L}_{\underline{\mathcal{M}}})^\vee|_{[F]}$ induced by Serre duality. Proposition 4.1.5 gives

$$\det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \cong \bigotimes_{k=0}^{4} \det(\mathrm{Ext}^k(F,F))^{(-1)^k}. \tag{4.4.3}$$

The Ext groups $\mathrm{Ext}^k(F,F)$ can be computed as cohomology groups of the elliptic complex

$$\dots \xrightarrow{\overline{\partial}^{\mathrm{ad}(P)}} \mathrm{ad}(P) \otimes_{\mathbb{R}} \Omega^{0,k} X^{\mathrm{an}} \xrightarrow{\overline{\partial}^{\mathrm{ad}(P)}} \mathrm{ad}(P) \otimes_{\mathbb{R}} \Omega^{0,k+1} X^{\mathrm{an}} \xrightarrow{\overline{\partial}^{\mathrm{ad}(P)}} \dots$$

which can be compressed into a single complex elliptic operator $D_{\mathbb{C}}^{\nabla_{\mathrm{ad}(P)}} := \overline{\partial}^{\mathrm{ad}(P)} + (\overline{\partial}^{\mathrm{ad}(P)})^* : \mathrm{ad}(P) \otimes_{\mathbb{R}} \Omega^{0,2*} X^{\mathrm{an}} \to \mathrm{ad}(P) \otimes_{\mathbb{R}} \Omega^{0,2*+1} X^{\mathrm{an}}$ by taking L^2-adjoints of the Dolbeault-type operators $\overline{\partial}^{\mathrm{ad}(P)}$ with respect to some choice of Hermitian metrics on the $\mathrm{ad}(P) \otimes_{\mathbb{R}} \Omega^{0,k} X^{\mathrm{an}}$ bundles. This gives an isomorphism

$$\det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \cong \det{}_{\mathbb{C}}(D_{\mathbb{C}}^{\nabla_{\mathrm{ad}(P)}}). \tag{4.4.4}$$

Moreover, the real structures $\heartsuit_0, \heartsuit_1$ induce an isomorphism

$$\heartsuit : \det{}_{\mathbb{C}}(D_{\mathbb{C}}^{\nabla_{\mathrm{ad}(P)}})|_{[F]} \xrightarrow{\sim} \overline{\det{}_{\mathbb{C}}(D_{\mathbb{C}}^{\nabla_{\mathrm{ad}(P)}})}|_{[F]}$$

hence an isomorphism

$$\heartsuit : \det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \xrightarrow{\sim} \overline{\det(\mathbb{L}_{\underline{\mathcal{M}}})}|_{[F]},$$

which we also call $\heartsuit$ by abuse of notation. As our definition of $D^{\nabla_{\mathrm{ad}(P)}}$ involved choosing Hermitian metrics, by (4.4.3), there is an induced Hermitian metric on $\det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]}$ which determines an isomorphism $\det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \xrightarrow{\sim} \det(\mathbb{L}_{\underline{\mathcal{M}}})^\vee|_{[F]}$. Combining this with $\heartsuit$ gives the Serre duality isomorphism $\iota^\omega : \det(\mathbb{L}_{\underline{\mathcal{M}}})|_{[F]} \xrightarrow{\sim} \det(\mathbb{L}_{\underline{\mathcal{M}}})^\vee|_{[F]}$. All together, we get a series of canonical isomorphisms

$$(\Delta^{\mathrm{Betti}})^*(\Xi^*(O^{\omega,\mathrm{Betti}})|_{\mathcal{T}^{\mathrm{an}}_{[P]}})|_{(F,(s_1,s_2,\dots))} \cong \mathrm{Or}(\det(\mathbb{L}_{\underline{\mathcal{M}}}|_{[F]})^{\mathbb{R}})$$

$$
\begin{aligned}
&\cong \mathrm{Or}(\det_{\mathbb{C}}(D_{\mathbb{C}}^{\nabla_{\mathrm{ad}(P)}})^{\mathbb{R}}) \\
&\cong \mathrm{Or}(\det_{\mathbb{R}}(D_{\mathbb{R}}^{\nabla_{\mathrm{ad}(P)}})) \\
&\cong \mathrm{Or}(\det_{\mathbb{R}}(\not{D}_{+}^{\nabla_{\mathrm{ad}(P)}})) \\
&\cong (\Lambda^{\mathrm{cla}})^*(O^{\not{D}_+}|_{\mathcal{B}_P^{\mathrm{cla}}})|_{(F,(s_1,s_2,\dots))}
\end{aligned}
$$

which we may take as the definition of $\lambda_{[P]}$. □

Proof. (of Theorem 4.4.3) Let $\gamma : O^{\omega,\mathrm{Betti}} \xrightarrow{\sim} \Gamma^*(O^{\not{D}_+})$ denote our canonical isomorphism of strong H-principal $\mathbb{Z}_2$-bundles. Let $\phi : O^{\omega} \boxtimes_{\mathbb{Z}_2} O^{\omega} \xrightarrow{\sim} \Phi^*(O^{\omega})$ and $\psi : O^{\not{D}_+} \boxtimes_{\mathbb{Z}_2} O^{\not{D}_+} \xrightarrow{\sim} \Psi^*(O^{\not{D}_+})$ be the isomorphisms constructed in [45, Thm. 1.15.c] and [115, Def. 2.22]. A consequence of γ being a strong H-principal bundle map is that

$$
(\Phi^{\mathrm{Betti}})^*(\gamma) \circ \phi^{\mathrm{Betti}} \cong (\Gamma \times \Gamma)^*(\psi) \circ (\gamma \boxtimes \gamma) \tag{4.4.5}
$$

as maps $O^{\omega,\mathrm{Betti}} \boxtimes_{\mathbb{Z}_2} O^{\omega,\mathrm{Betti}} \to (\Gamma \circ \Phi^{\mathrm{Betti}})^*(O^{\not{D}_+}) \cong (\Psi \circ (\Gamma \times \Gamma))^*(O^{\not{D}_+})$. The identification $(\Gamma \circ \Phi^{\mathrm{Betti}})^*(O^{\not{D}_+}) \cong (\Psi \circ (\Gamma \times \Gamma))^*(O^{\not{D}_+})$ is induced by a homotopy $h : \Gamma \circ \Phi^{\mathrm{Betti}} \simeq \Psi \circ (\Gamma \times \Gamma)$ witnessing the fact that Γ is an H-map. Because $O^{\not{D}_+}$ is trivializable, we may regard this identification as being independent of h.

Fix a trivialization $o^{\not{D}_+}$ of $O^{\not{D}_+}$. For $\alpha \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$, let $o_\alpha^{\not{D}_+}$ denote the trivialization of $O^{\not{D}_+}|_{\overline{\mathcal{B}}_\alpha^U}$ induced by $o^{\not{D}_+}$. Then $o_\alpha^{\omega,\mathrm{Betti}} := \Gamma^*(o_\alpha^{\not{D}_+}) \circ \gamma$ is a trivialization of $O^{\omega,\mathrm{Betti}}|_{\mathcal{M}_\alpha^{\mathrm{Betti}}}$. Let $\alpha, \beta \in K^0_{\mathrm{top}}(X^{\mathrm{an}})$ and $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ such that

$$
\psi_*(o_\alpha^{\not{D}_+} \boxtimes o_\beta^{\not{D}_+}) = \epsilon_{\alpha,\beta} \cdot \Psi^*(o_{\alpha+\beta}^{\not{D}_+}). \tag{4.4.6}
$$

Applying $(\Gamma \times \Gamma)^*$ to (4.4.6) gives

$$
\Gamma^*(o_\alpha^{\not{D}_+}) \boxtimes \Gamma^*(o_\beta^{\not{D}_+}) \circ (\Gamma \times \Gamma)^* \psi^{-1} = \epsilon_{\alpha,\beta} \cdot (\Gamma \times \Gamma)^* \Psi^*(o_{\alpha+\beta}^{\not{D}_+}). \tag{4.4.7}
$$

Pre-composing (4.4.7) with $\gamma \boxtimes \gamma$ gives

$$\Gamma^*(o_\alpha^{\not{D}_+}) \boxtimes \Gamma^*(o_\beta^{\not{D}_+}) \circ (\gamma \boxtimes \gamma) = \epsilon_{\alpha,\beta} \cdot (\Gamma \times \Gamma)^* \Psi^*(o_{\alpha+\beta}^{\not{D}_+}) \circ (\Gamma \times \Gamma)^*(\psi) \circ (\gamma \boxtimes \gamma) \tag{4.4.8}$$

$$= \epsilon_{\alpha,\beta} \cdot (\Phi^{\mathrm{Betti}})^* \Gamma^*(o_{\alpha+\beta}^{\not{D}_+}) \circ (\Phi^{\mathrm{Betti}})(\gamma) \circ \phi^{\mathrm{Betti}}. \tag{4.4.9}$$

Equations (4.4.8) and (4.4.9) then give

$$(\phi^{\mathrm{Betti}})_*(o_\alpha^{\omega,\mathrm{Betti}} \boxtimes o_\beta^{\omega,\mathrm{Betti}}) = \epsilon_{\alpha,\beta} \cdot (\Phi^{\mathrm{Betti}})^*(o_{\alpha+\beta}^{\omega,\mathrm{Betti}}). \qquad \square$$

Chapter 5

Geometric constructions of vertex algebras

In Section 5.1, we review background material from [112]. Namely, the construction of graded vertex algebras on the homology of moduli stacks of objects in certain dg-categories. We also review an example worked out in [112]: the homology of the moduli stack of representations of a finite quiver.

In Section 5.2 we present an explicit example of Joyce's construction and identify it with a known vertex algebra. Specifically, we consider the rational homology[1] of the moduli stack of objects in the derived category of a smooth projective complex variety X in class D. Using the fact that the rational Betti homology of such a space is freely generated by Künneth components of Chern characters of the universal complex (Theorem 4.3.12), we will be able to compute that Joyce's formula agrees with a generalized super-lattice vertex algebra associated to the rational cohomology of X^{an}.

In Section 5.3, we will consider H-spaces $\mathcal{X}$ with $BU(1)$-actions $BU(1) \times \mathcal{X} \to \mathcal{X}$

[1] In Section 5.2, we actually will consider general rational complex-oriented homology theories. However, this is a rather soft generalization because all rational spectra are canonically equivalent to graded Eilenberg–Maclane spectra [190, Thm. 7.11].

that are also H-maps. Given a complex-oriented spectrum E with associated formal group law F, it is easy to write down a holomorphic F-bicharacter on $E_*(\mathcal{X})$. To obtain a singular F-bicharacter we introduce an operator $(-) \cap C_z^E(V) : E_*(\mathcal{X}) \to E_*(\mathcal{X})((z))$, natural in $V \in K^0_{\mathrm{top}}(\mathcal{X})$, uniquely characterized by two axioms (Theorem 5.3.3). Our main example of an H-space $\mathcal{X}$ with $BU(1)$-action for which $(-) \cap C_z^E(-)$ can be used to construct an F-bicharacter on $E_*(\mathcal{X})$ is the moduli space of all unitary connections on a compact manifold, as well as its homotopy-theoretic group completion.

Given a Künneth isomorphism $E_*(\mathcal{X} \times \mathcal{X}) \cong E_*(\mathcal{X}) \otimes E_*(\mathcal{X})$ such an F-bicharacter will determine a vertex F-algebra on $E_*(\mathcal{X})$. Because $(-) \cap C_z^E(V) : E_*(\mathcal{X}) \to E_*(\mathcal{X})((z))$ is not, in general, degree-preserving it will be necessary to shift the grading of $E_*(\mathcal{X})$ by a quadratic form in order to obtain a graded vertex F-algebra. The state-to-field correspondence built from the F-bicharacter of Section 5.3 is actually independent of the coproduct on $E_*(\mathcal{X})$, so we do not believe that the Künneth isomorphism is truly necessary. Nonetheless, we think the F-bicharacter construction is interesting and serves as a model for a generalized bicharacter construction which could be performed in the symmetric monoidal category of spectra on (the suspension spectrum of) an H-space itself.

In Section 5.4, we consider H-spaces $\mathcal{Y}$ with $BO(1)$-actions that are also H-maps. Again, it is simple to build a holomorphic bicharacter (hence, holomorphic vertex algebra) on $H_*(\mathcal{Y}, \mathbb{Z}_2)$. To introduce singularities we will introduce an operator $(-) \cap W_u(V) : H_*(\mathcal{Y}, \mathbb{Z}_2) \to H_*(\mathcal{Y}, \mathbb{Z}_2)((u))$ natural in $V \in KO^0(\mathcal{Y})$. Our main example of an H-space $\mathcal{Y}$ for which a vertex algebra can be built on its $\mathbb{Z}_2$-coefficient homology in this way will be the moduli space of all orthogonal connections on a compact manifold, as well as its homotopy-theoretic group completion. To conclude, we sketch an idea to build graded vertex F-algebras using *Real-oriented spectra* in the sense of Araki [9].

Sections 5.3 and 5.4 are based on joint work with Markus Upmeier [94].

5.1 A geometric construction of vertex algebras

In this section, we review Joyce's geometric construction of graded vertex algebras [112, §4.2]. Let k be a field and let $\mathcal{A}$ be a k-linear dg-category such that there exists a higher stack $\mathcal{M}$ parameterizing objects in $\mathcal{A}$. The category $\mathcal{A}$ could be an abelian category like $\mathrm{Rep}(Q)$ or $\mathrm{Coh}(X)$ or a triangulated category like $D^b\mathrm{Rep}(Q)$ or $\mathrm{Perf}(X)$.

Direct sum of objects in $\mathcal{A}$ induces a stack morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The k-linearity of $\mathcal{A}$ induces a $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ which acts trivially on points and acts on stabilizers by scaling. Given a homology theory $H_*(-)$: $\mathrm{Ho}(\mathrm{HSt}_k) \to R\text{-mod}$ (Definition 4.2.1), inclusion of the zero object of $\mathcal{A}$ corresponds to a k-stack morphism $\mathrm{Spec}(k) \to \mathcal{M}$ which induces a map $|0\rangle : H_*(\{\mathrm{pt}\}) \cong \mathrm{R} \to \mathrm{H}_*(\mathcal{M})$ in homology. Further, as in Corollary 3.2.7, Ψ induces a shift operator $\mathcal{D}(z)$: $H_*(\mathcal{M}) \to H_*(\mathcal{M})[[z]]$ compatible with the graded (bi)algebra structure on $H_*(\mathcal{M})$.

After choosing some additional data, a graded vertex algebra structure with shift operator $\mathcal{D}(z)$ and vacuum vector $|0\rangle$ can be built on the (shifted) homology of $\mathcal{M}$.

Theorem 5.1.1 (see Joyce [112, Thm. 4.11]). *Let $\mathcal{A}, \mathcal{M}, \Phi, \Psi, H_*(-), \mathcal{D}(z)$, and $|0\rangle$ be as above. Given*

- *a quotient $K(\mathcal{A})$ of the Grothendieck group $K_0(\mathcal{A})$,*
- *signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for all $\alpha, \beta \in K(\mathcal{A})$, and*
- *a perfect complex $\Theta^\bullet \in \mathrm{Perf}(\mathcal{M} \times \mathcal{M})$*

such that

- *the map $\mathcal{M} \to K(\mathcal{A})$, $E \mapsto [E]$ is locally constant giving a decomposition*

$$\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha,$$

- *for all* $\alpha, \beta, \gamma \in K(\mathcal{A})$

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}, \tag{5.1.1}$$

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}, \text{ and} \tag{5.1.2}$$

$$\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 1, \tag{5.1.3}$$

where $\chi(\alpha,\beta) := \mathrm{rk}(\Theta^\bullet_{\alpha,\beta})$, *and*

- *there are isomorphisms*

$$(\Phi \times \mathrm{id})^*(\Theta^\bullet) \cong \pi_{13}^*(\Theta^\bullet) \oplus \pi_{23}^*(\Theta^\bullet), \tag{5.1.4}$$

$$(\mathrm{id} \times \Phi)^*(\Theta^\bullet) \cong \pi_{12}^*(\Theta^\bullet) \oplus \pi_{13}^*(\Theta^\bullet), \tag{5.1.5}$$

$$(\Psi \times \mathrm{id})^*(\Theta^\bullet) \cong \pi_{[*/\mathbb{G}_m]}^*(E_1) \otimes \pi_{23}^*(\Theta^\bullet), \tag{5.1.6}$$

$$(\mathrm{id}_{\mathcal{M}} \times \Psi)^*(\Theta^\bullet) \cong \pi_{[*/\mathbb{G}_m]}^*(E_1^*) \otimes \pi_{23}^*(\Theta^\bullet), \tag{5.1.7}$$

$$\sigma^*(\Theta^\bullet) \cong (\Theta^\bullet)^\vee[2n], \tag{5.1.8}$$

for some $n \in \mathbb{Z}$, *where* $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ *denotes the exchange of factors morphism.*

Note that (5.1.4), (5.1.5), and (5.1.8) make $\chi : K(\mathcal{A}) \otimes K(\mathcal{A}) \to \mathbb{Z}$ *a symmetric* $\mathbb{Z}$*-bilinear form. Let* $Q(\alpha) := \chi(\alpha,\alpha)$ *be the associated quadratic form and let* $\hat{H}_*(\mathcal{M})$ *denote the* Q*-shift (Definition 2.4.5) of* $H_*(\mathcal{M})$*. Define a graded* R*-linear map* $Y(-,z) : \hat{H}_*(\mathcal{M}) \to \mathcal{F}(\hat{H}_*(\mathcal{M}))$ *by*

$$Y(a,z)b = \epsilon_{\alpha,\beta}(-1)^{aQ(\beta)} z^{\chi(\alpha,\beta)} \Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \boxtimes b \cap \sum_{j \geq 0} c_j(\Theta^\bullet) z^{-j}), \tag{5.1.9}$$

for $a \in \hat{H}_*(\mathcal{M}_\alpha)$ *and* $b \in \hat{H}_*(\mathcal{M}_\beta)$*. Then* $(\hat{H}_*(\mathcal{M}), Y(-,z), |0\rangle, \mathcal{D}(z))$ *is a graded vertex algebra.*

Example 5.1.2. Let Q be a finite quiver, let $\mathcal{A} = \mathrm{Rep}(Q)$ or $\mathcal{A} = D^b(\mathrm{Rep}(Q))$, and let $k = \mathbb{C}$ so that $\mathcal{A}$ is $\mathbb{C}$-linear. One can take $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$, $\Theta^\bullet$ to be the symmetrized Ext complex

$$\Theta^\bullet := (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet),$$

and signs $\epsilon_{\alpha,\beta} = (-1)^{\chi^{\mathrm{eu}}(\alpha,\beta)}$ where $\chi^{\mathrm{eu}} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is the *Euler form* of Q

$$\chi^{\mathrm{eu}}([E],[F]) = \dim_{\mathbb{C}}(\mathrm{Hom}(E,F)) - \dim_{\mathbb{C}}(\mathrm{Ext}^1([E],[F])).$$

If Q happens to be a symmetric quiver then one could instead take $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$, $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, and there always exists a choice of signs $\epsilon_{\alpha,\beta}$ satisfying (5.1.1)-(5.1.3) by a theorem of Kac [120, § 7.8].

For a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, write

$$\mathrm{GL}_{\mathbf{d}} := \prod_{v \in Q_0} \mathrm{GL}(\mathbf{d}(v), \mathbb{C}).$$

Then

$$\mathcal{M}^{\mathrm{Betti}}_{\mathrm{Rep}(Q)} \simeq \coprod_{\mathbf{d} \in \mathbb{N}^{Q_0}} B\mathrm{GL}_{\mathbf{d}}$$

and

$$\mathcal{M}^{\mathrm{Betti}}_{D^b\mathrm{Rep}(Q)} \simeq \prod_{v \in Q_0} (BU \times \mathbb{Z})$$

[112, Prop. 7.13]. This gives

$$H_*(\mathcal{M}_{\mathrm{Rep}(Q)}, \mathbb{Q}) \cong \mathbb{Q}[b_{\mathbf{d},v,i} : \mathbf{d} \in \mathbb{Z}^{Q_0}, v \in Q_0, i = 1, \dots, \mathbf{d}(v)]$$

and

$$H_*(\mathcal{M}_{D^b\mathrm{Rep}(Q)}, \mathbb{Q}) \cong \mathbb{Q}[b_{\mathbf{d},v,i} : \mathbf{d} \in \mathbb{Z}^{Q_0}, v \in Q_0, i = 1, 2, \dots].$$

Moreover, $\hat{H}_*(\mathcal{M}_{D^b\mathrm{Rep}(Q)}, \mathbb{Q})$ is the graded lattice vertex algebra associated to the lattice $\mathbb{Z}^{Q_0}$ with form depending upon the initial choice of $\Theta^\bullet$ [112, Thm. 7.19].

5.2 Moduli spaces of complexes of coherent sheaves

In this section, we work out a further example of Theorem 5.1.1. Throughout, E will denote a rational complex-oriented spectrum, R will denote the coefficient ring of E i.e. $R = E^*(\{\mathrm{pt}\})$, X will denote a smooth complex projective variety, and $\mathcal{M}$ will denote the moduli stack of objects in the $\mathbb{C}$-dg-category $\mathrm{Perf}(X)$.

We can regard $E_*(-) : \mathrm{Ho}(\mathrm{HSt}_{\mathbb{C}}) \to R\text{-mod}$ as a generalized Betti homology theory over R. Recall that the $\mathbb{Q}$-localization of any ring spectrum E is naturally equivalent to a graded Eilenberg–Maclane ring spectrum (see Rudyak [190, Thm. 7.11]) and that all graded formal group laws over $\mathbb{Q}$-algebras are canonically isomorphic to the additive formal group law. In particular, the generalization from ordinary homology to rational complex-oriented homology is a mild one. However, there are refined $\mathbb{C}$-linear enumerative invariants which take values in rational K-theory and rational cobordism (see Göttsche–Kool [88], Okounkov [178], Thomas [213], and Shen [200]). This may mean one will want to consider such graded vertex algebras to prove wall-crossing formulae in refined $\mathbb{C}$-linear enumerative invariant theories, although we do not know how to formulate a precise conjecture at this time.

For homology with rational coefficients, one can re-express the state-to-field correspondence (5.1.9) in terms of Chern characters as

$$Y(a,z)b = \epsilon_{\alpha,\beta}(-1)^{aQ(\beta)} z^{\chi(\alpha,\beta)} \Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \boxtimes b \cap \exp(\sum_{i \geq 0} (-1)^{i-1}(i-1)! z^{-i} \mathrm{ch}_i([\Theta^\bullet_{\alpha,\beta}]))) \tag{5.2.1}$$

(see [112, Eqn. (4.30)]). Given a rational spectrum such as E, one can write a state-to-field correspondence on $E_*(\mathcal{M})$ in terms of E-Chern characters $\mathrm{ch}^E(-)$ (Definition

3.2.9)

$$Y(a,z)b = \epsilon_{\alpha,\beta}(-1)^{aQ(\beta)} z^{\chi(\alpha,\beta)} \Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \boxtimes b \cap \exp(\sum_{i\geq 0}(-1)^{i-1}(i-1)! z^{-i}$$
$$\mathrm{ch}_i^E([\Theta^\bullet_{\alpha,\beta}])))$$
(5.2.2)

(see [92, Eqn. (4.3)]). After shifting the grading on $E_*(\mathcal{M})$ by means of a quadratic form, the state-to-field correspondence (5.2.2) will make $E_*(\mathcal{M})$ into a graded vertex algebra.

We will show that we can can choose data on $\mathrm{Perf}(X)$, as in Theorem 5.1.1, to identify the (shifted) E-homology of $\mathcal{M}$ with state-to-field correspondence given by (5.2.2) with a generalized super-lattice vertex algebra on $K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}})$.

Notation 5.2.1. Let $(-)^\vee : H^*(X^{\mathrm{an}}, \mathbb{Q}) \to H^*(X^{\mathrm{an}}, \mathbb{Q})$ denote the involution

$$v^\vee = \begin{cases} (-1)^{v/2} v, & 2|\deg(v) \\ (-1)^{(v-1)/2} v, & 2 \nmid \deg(v). \end{cases}$$

The following theorem will be proved in several parts.

Theorem 5.2.2 (see Gross [92, Thm. 6.8])**.** *Let X be a smooth complex projective variety in class D (Definition 4.3.6). Let E be a complex-oriented spectrum such that $R := E^*(\mathrm{pt})$ is a $\mathbb{Q}$-algebra. Let $\mathcal{M}$ denote the moduli stack of objects in the $\mathbb{C}$-dg-category Perf(X). Let $\mathbb{E}^\bullet$ denote the universal complex over $X \times \mathcal{M}$, write $\mathcal{E}xt^\bullet := \mathbb{R}\pi_*(\pi^*(\mathbb{E}^\bullet)^\vee \otimes^{\mathbb{L}} \pi^*(\mathbb{E}^\bullet))$, let $\chi : K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}) \to \mathbb{Z}$ denote the Euler form*

$$\chi(v,w) = \int_{X^{\mathrm{an}}} \mathrm{ch}(v)^\vee \cdot \mathrm{ch}(w) \cdot \mathrm{Td}(X^{\mathrm{an}}),$$

and let $Q(v) := \chi(v,v)$ *be the associated quadratic form. Similarly, define*

$$\chi^{\mathrm{sym}}(v,w) := \chi(v,w) + \chi(w,v)$$

and $Q^{\mathrm{sym}}(v) := \chi^{\mathrm{sym}}(v,v)$*. Then*

1. *The* Q^{sym}*-shift* $\hat{E}_*(\mathcal{M})$ *of* $E_*(\mathcal{M})$ *can be made into a graded vertex algebra by taking* $K(\mathrm{Perf}(X)) = K^0_{\mathrm{sst}}(X)$*,* $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)$*, and* $\epsilon_{\alpha,\beta} = (-1)^{\chi(\alpha,\beta)}$ *in (5.2.2). In this case,* $\hat{E}_*(\mathcal{M})$ *is isomorphic, as a graded vertex algebra, to*

$$R[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{Sym}(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-1}R[t^{-1}]) \otimes \bigwedge(K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-\frac{1}{2}}R[t^{-1}]), \quad (5.2.3)$$

   where (5.2.3) is given the structure of a generalized super-lattice vertex algebra associated to $(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}), \chi^{\mathrm{sym}})$ *and the inclusion* $K^0_{\mathrm{sst}}(X) \hookrightarrow K^0_{\mathrm{top}}(X^{\mathrm{an}})$*.*

2. *Suppose* X *is* $2n$*-Calabi–Yau and for all* $\alpha, \beta \in K^0_{\mathrm{sst}}(X)$ *we are given signs* $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ *such that the collection* $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$ *is a solution of the equations (5.1.1)-(5.1.3). Then the* Q*-shift* $\hat{E}_*(\mathcal{M})$ *of* $E_*(\mathcal{M})$ *can be made into a graded vertex algebra by taking* $K(\mathrm{Perf}(X)) = K^0_{\mathrm{sst}}(X)$*,* $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$*, and signs* $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$*. In this case,* $\hat{E}_*(\mathcal{M})$ *is isomorphic, as a graded vertex algebra, to*

$$R[K^0_{\mathrm{sst}}(X)] \otimes \mathrm{Sym}(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-1}R[t^{-1}]) \otimes \bigwedge(K^1_{\mathrm{top}}(X^{\mathrm{an}}) \otimes t^{-\frac{1}{2}}R[t^{-1}]), \quad (5.2.4)$$

   where (5.2.4) is given the structure of a generalized super-lattice vertex algebra associated to $(K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}}), \chi)$ *and the inclusion* $K^0_{\mathrm{sst}}(X) \hookrightarrow K^0_{\mathrm{top}}(X^{\mathrm{an}})$*. Up to isomorphism this graded vertex algebra is independent of the representative of the group cohomology class* $[\epsilon] \in H^2(K^0_{\mathrm{sst}}(X), \mathbb{Z}_2)$ *that* $\{\epsilon_{\alpha,\beta}\}_{\alpha,\beta \in K^0_{\mathrm{sst}}(X)}$ *defines.*

Remark 5.2.3. It is a fact that there always exists, up to coboundary, at least one way to choose signs satisfying (5.1.1)-(5.1.3) (see Kac [120, Cor. 5.5] and Joyce [112, Lem. 4.5]). The ambiguity in choice is controlled by the 2-torsion in $K^0_{\mathrm{sst}}(X)$ (see Joyce–Tanaka–Upmeier [115, Thm. 2.27]).

Example 5.2.4. Let X be a K3 surface. Then $K^0_{\mathrm{top}}(X^{\mathrm{an}})$ is a lattice and $K^1_{\mathrm{top}}(X^{\mathrm{an}}) \cong 0$. In particular, the fermionic piece of (5.2.4) vanishes. We then get that $\hat{E}_*(\mathcal{M})$ is the graded lattice vertex algbera associated to the Mukai lattice (with restricted group algebra $R[K^0_{\mathrm{sst}}(X)] \subset R[K^0_{\mathrm{top}}(X^{\mathrm{an}})]$).

We now commence our proof of Theorem 5.2.2.

Fix, for the remainder of this section, a basis $Q = \{v_1, \dots, v_r\}$ of the rational complex topological K-theory $K^0_{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}} \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}}$ of X^{an} and a dual basis $Q^\vee = \{v_1^\vee, \dots, v_r^\vee\}$ of $(K^0_{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}} \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}})^\vee$. By Corollary 4.3.13, if X is in class D, then for all $\alpha \in K^0_{\mathrm{sst}}(X)$ there is a canonical isomorphism of graded R-algebras

$$E^*(\mathcal{M}_\alpha) \cong \mathrm{SSym}_R[[\mu_{\alpha,v,i} : v \in Q, i \geq 1]], \tag{5.2.5}$$

given by $\mathrm{ch}^E_i([\mathbb{E}^\bullet{}_\alpha]/v^\vee) \mapsto \mu_{\alpha,v,i}$.

One can define a super-symmetric bilinear form on $K^0_{\mathrm{top}}(X^{\mathrm{an}}) \oplus K^1_{\mathrm{top}}(X^{\mathrm{an}})$ by

$$\chi^E(v,w) = \int^E_{X^{\mathrm{an}}} \mathrm{ch}^E(v)^\vee \cdot \mathrm{ch}^E(w) \cdot \mathrm{Td}^E(X^{\mathrm{an}})$$

if X is $2n$-Calabi–Yau for some $n \geq 1$ and by

$$\chi^E_{\mathrm{sym}}(v,w) = \int^E_{X^{\mathrm{an}}} \left(\mathrm{ch}^E(v)^\vee \cdot \mathrm{ch}^E(w) + \mathrm{ch}^E(w)^\vee \cdot \mathrm{ch}^E(v)\right) \cdot \mathrm{Td}^E(X^{\mathrm{an}})$$

if X is not $2n$-Calabi–Yau for any $n \geq 1$, where $\mathrm{ch}^E(-)^\vee := \mathrm{ch}^E((-)^\vee)$. We will need to know that $\chi^E|_{K^0_{\mathrm{sst}}(X)}$ is an *integral* form.

Lemma 5.2.5. *For all $v, w \in K^0_{\mathrm{sst}}(X)$*

$$\chi^E(v, w) = \chi(v, w) := \int_{X^{\mathrm{an}}} \mathrm{ch}(v)^\vee \cdot \mathrm{ch}(w) \cdot \mathrm{Td}(X^{\mathrm{an}}).$$

Proof. For brevity let Φ^E denote the E-Dold–Chern character and let $\tau_X^E, \tau_X^H, \tau_{\mathrm{pt}}^E$, τ_{pt}^H, and τ_X^{KU} denote the Thom isomorphisms. By the Hirzebruch–Riemann–Roch theorem, it suffices to show that

$$\begin{tikzcd}
K^0_{\mathrm{top}}(X^{\mathrm{an}}) \times K^0_{\mathrm{top}}(X^{\mathrm{an}}) \arrow[d, "\mathrm{ch}^E \times \mathrm{ch}^E"'] \arrow[dr, "\mathrm{ch} \times \mathrm{ch}"] & \\
E^*(X^{\mathrm{an}}) \times E^*(X^{\mathrm{an}}) \arrow[r, "\Phi^E \times \Phi^E"'] \arrow[d, "\cdot"'] & H^*(X^{\mathrm{an}}) \times H^*(X^{\mathrm{an}}) \arrow[d, "\cdot"] \\
E^*(X^{\mathrm{an}}) \arrow[r, "\Phi^E"'] \arrow[d, "\int^E_{X^{\mathrm{an}}}(-)\cdot \mathrm{Td}^E(X^{\mathrm{an}})"'] & H^*(X^{\mathrm{an}}) \arrow[dl, "\int_{X^{\mathrm{an}}}(-)\cdot \mathrm{Td}(X^{\mathrm{an}})"] \\
R &
\end{tikzcd} \tag{5.2.6}$$

commutes. Commutativity of the upper triangle and middle square of (5.2.6) follow from the definition and multiplicativity of the E-Dold–Chern character. Because Φ^E is induced by a ring morphism of ring spectra, it commutes with Thom isomorphisms [191, Prop. V.1.6]. So, for all $v \in E^*(X^{\mathrm{an}})$, we can calculate

$$\begin{aligned}
\int_{X^{\mathrm{an}}} \Phi^E(v) \cdot \mathrm{Td}(X^{\mathrm{an}}) &= (\tau_{\mathrm{pt}}^H)^{-1} \circ \tilde{H}(p) \circ \tau_X^H \big\{ \Phi^E(v) \cdot (\tau_X^H)^{-1} \circ \mathrm{ch} \circ \tau_X^{KU}(1) \big\} \\
&= (\tau_{\mathrm{pt}}^H)^{-1} \circ \tilde{H}(p) \circ \tau_X^H \big\{ \Phi^E(v) \cdot (\tau_X^H)^{-1} \circ \Phi^E \circ \mathrm{ch}^E \circ \\
&\qquad \tau_X^{KU}(1) \big\} \\
&= (\tau_{\mathrm{pt}}^H)^{-1} \circ \tilde{H}(p) \circ \tau_X^H \big\{ \Phi^E(v \cdot (\tau_X^E)^{-1} \circ \mathrm{ch}^E \circ \tau_X^{KU}(1)) \big\} \\
&= (\tau_{\mathrm{pt}}^E)^{-1} \circ \tilde{E}(p) \circ \tau_X^E \big\{ v \cdot (\tau_X^E)^{-1} \circ \mathrm{ch}^E \circ \tau_X^{KU}(1) \big\} \\
&= \int^E_{X^{\mathrm{an}}} v \cdot \mathrm{Td}^E(X^{\mathrm{an}}),
\end{aligned}$$

where $p : X^{\mathrm{an}} \to \mathrm{pt}$. This establishes commutativity of the bottom triangle of

(5.2.6). $\square$

Let $\mathbb{E}^\bullet$ denote the universal complex over $X \times \mathcal{M}$ and, for $\alpha \in K^0_{\text{sst}}(X)$, let $\mathbb{E}^\bullet{}_\alpha$ denote the universal complex over $X \times \mathcal{M}_\alpha$. Let $\mathcal{E}$ denote the evaluation map for $\mathrm{Map}_{C^0}(X, BU \times \mathbb{Z})$ and let $\mathcal{E}_\alpha$ denote the evaluation map for $\mathrm{Map}_{C^0}(X^{\text{an}}, BU \times \mathbb{Z})_\alpha$. For $\alpha, \beta \in K^0_{\text{sst}}(X)$ define complexes $\mathcal{E}\text{xt}^\bullet_{\alpha,\beta} \in \mathrm{Perf}(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$ by

$$\mathcal{E}\text{xt}^\bullet_{\alpha,\beta} := \mathbb{R}\pi_*(\pi_1^*(\mathbb{E}^\bullet{}_\alpha)^\vee \otimes^{\mathbb{L}} \pi_2^*(\mathbb{E}^\bullet{}_\beta)).$$

Then for all $m \in \mathbb{Z}$

$$\sigma^*_{\alpha,\beta}((\mathcal{E}\text{xt}^\bullet_{\alpha,\beta})^\vee \oplus \sigma^*_{\alpha,\beta}\mathcal{E}\text{xt}^\bullet_{\beta,\alpha}[2m]) \cong ((\mathcal{E}\text{xt}^\bullet_{\alpha,\beta})^\vee \oplus \sigma^*_{\alpha,\beta}\mathcal{E}\text{xt}^\bullet_{\beta,\alpha}[2m])^\vee[2m].$$

and if X happens to be $2n$-Calabi–Yau then

$$\sigma^*_{\alpha,\beta}((\mathcal{E}\text{xt}^\bullet_{\beta,\alpha})^\vee) \cong \mathcal{E}\text{xt}^\bullet_{\alpha,\beta}[2n].$$

To get an explicit formula for $Y(-,z)$ one has to calculate the E-Chern classes of $\Theta^\bullet$. We only know how to do this when X is in class D. For brevity we write $\mathcal{U} = \pi^*(\mathcal{E})^\vee \otimes \pi(\mathcal{E})$.

Lemma 5.2.6. *Let X be in class D and let $\alpha, \beta \in K^0_{\text{sst}}(X)$. Then, for all $i \geq 0$*

$$c_i((\mathcal{E}\text{xt}^\bullet_{\alpha,\beta})^{\text{Betti}}) = c_i(\pi_!^{KU}(\mathcal{U}_{\alpha,\beta})).$$

Proof. The class $\pi_!^{KU}(\pi_\alpha^*(\mathcal{E}_\alpha)^\vee \otimes \pi_\beta^*(\mathcal{E}_\beta))$ can be described using twisted elliptic operators. Consider the elliptic operator

$$D := \overline{\partial} + \overline{\partial}^* : C^\infty(\Lambda^{0,2*}T^*X^{\text{an}}) \to C^\infty(\Lambda^{0,2*+1}T^*X^{\text{an}})$$

on X^{an}. Given complex vector bundles $P, Q \to X^{\mathrm{an}}$ we can choose connections ∇_P, ∇_Q on them and we can write down a Fredholm operator $D^{\nabla_{\overline{P}\otimes Q}} : C^\infty(\Lambda^{0,2*}T^*X^{\mathrm{an}} \otimes \overline{P} \otimes Q) \to C^\infty(\Lambda^{0,2*+1}T^*X^{\mathrm{an}} \otimes \overline{P} \otimes Q)$ as in [115, Def. 2.20]. Then the correspondence $(P,Q) \mapsto D^{\nabla_{\overline{P}\otimes Q}}$ can be expressed as a continuous map

$$D^U : \coprod_{n\geq 0} \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU(n)) \times \coprod_{n\geq 0} \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU(n)) \to \mathrm{Fred}(\mathcal{H}),$$

defined up to weak[2] homotopy, whose (weak) homotopy class is independent of the choices of connections ∇_P, ∇_Q. By group-likeness of $\mathrm{Fred}(\mathcal{H})$ and the weak universal property of homotopy-theoretic group completions there exists a weak H-map

$$\overline{D}^U : \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \times \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \to \mathrm{Fred}(\mathcal{H}) \simeq BU \times \mathbb{Z}$$

such that the restriction of $\overline{D}^U$ along the completion map

$$\coprod_{n\geq 0} \mathrm{Map}_{C^0}(X^{\mathrm{an}}, BU(n)) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$$

is weakly homotopic to D^U. By the families index theorem [21, Thm. 3.1], the weak homotopy class of $\overline{D}^U$ equals the weak homotopy class of $\pi_!^{KU}(\mathcal{U})$. As X is in class D, it then suffices to show that the restrictions

$$\mathcal{M}^{\mathrm{Betti}} \times \mathcal{M}^{\mathrm{Betti}} \hookrightarrow \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \times \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}}) \underset{(\mathcal{E}\mathrm{xt}^\bullet)^{\mathrm{Betti}}}{\overset{\overline{D}^U}{\rightrightarrows}} BU \times \mathbb{Z}$$

are weakly homotopic. If $C \subset \mathrm{Map}_{\mathrm{alg}}(X, \mathrm{Gr})^{\mathrm{an}} \times \mathrm{Map}_{\mathrm{alg}}(X, \mathrm{Gr})^{\mathrm{an}}$ is compact then $[(\mathcal{E}\mathrm{xt}^\bullet)^{\mathrm{Betti}}|_C] \in K^0_{\mathrm{top}}(C)$ equals the index bundle $[D^U|_C]$. In particular, the restrictions of $(\mathcal{E}\mathrm{xt}^\bullet)^{\mathrm{Betti}}$ and $\overline{D}^U$ to $\mathrm{Map}_{\mathrm{alg}}(X, \mathrm{Gr})^{\mathrm{an}} \times \mathrm{Map}_{\mathrm{alg}}(X, \mathrm{Gr})^{\mathrm{an}}$ are weakly homo-

[2]Note that $D^{\nabla_{\overline{P}\times Q}}$ is a Fredholm map between two different infinite-dimensional separable complex Hilbert spaces. All infinite-dimensional separable complex Hilbert spaces are non-canonically isometrically isomorphic, up to a weakly contractible choice. In other words, there is a Hilbert space bundle which is trivial upon restriction to any finite CW complex and we may regard D^∇ as a section.

topic. The claim then follows from Proposition 4.3.1 and the fact that $\mathrm{Fred}(\mathcal{H})$ is group-like. $\square$

Proposition 5.2.7. *Let X be in class D. Then for all $\alpha, \beta \in K^0_{\mathrm{sst}}(X)$, $i \geq 1$*

$$\mathrm{ch}^E_i((\mathcal{E}\mathrm{xt}^\bullet_{\alpha,\beta})^\vee) = \sum_{\substack{j,k\geq 0; i=j+k \\ v,w\in Q}} (-1)^k \chi(v,w)\mu_{\alpha,v,j} \boxtimes \mu_{\beta,w,k}.$$

and

$$\mathrm{ch}^E_i((\mathcal{E}\mathrm{xt}^\bullet_{\alpha,\beta})^\vee \oplus (\sigma^*\mathcal{E}\mathrm{xt}^\bullet_{\beta,\alpha})) = \sum_{\substack{j,k\geq 0; i=j+k \\ v,w\in Q}} (-1)^k \chi_{\mathrm{sym}}(v,w)\mu_{\alpha,v,j} \boxtimes \mu_{\beta,w,k}.$$

Proof. By Lemmas 3.2.1 and 5.2.6, we compute

$$\begin{aligned}\mathrm{ch}^E((\mathcal{E}\mathrm{xt}^\bullet_{\alpha,\beta})^\vee) &= (\int^E_{X^{\mathrm{an}}} \pi^*_\alpha(\mathrm{ch}^E(\mathcal{E}_\alpha))^\vee \cdot \pi^*_\beta(\mathrm{ch}^E(\mathcal{E}_\beta)) \cdot \pi^*_{\alpha,\beta}\mathrm{Td}^E(X^{\mathrm{an}}))^\vee \\ &= (\int^E_{X^{\mathrm{an}}} \pi^*_\alpha \mathrm{ch}^E(\sum_{v\in Q} v^\vee \boxtimes ([\mathcal{E}_\alpha]/v)^\vee) \cdot \pi^*_\beta \mathrm{ch}^E(\sum_{w\in Q} w \boxtimes [\mathcal{E}_\beta]/w) \cdot\end{aligned}$$

$$\pi^*_{\alpha,\beta}\mathrm{Td}^E(X^{\mathrm{an}}))^\vee$$

$$= (\sum_{v,w\in Q} \chi^E(v,w)\mathrm{ch}^E([\mathcal{E}_\alpha]/v)^\vee \mathrm{ch}^E([\mathcal{E}_\beta]/w))^\vee.$$

So, for $i \geq 0$,

$$\mathrm{ch}^E_i((\mathcal{E}\mathrm{xt}^\bullet_{\alpha,\beta})^\vee) = \sum_{\substack{v,w\in Q \\ i=j+k}} (-1)^k \chi(v,w)\mu_{\alpha,v,j} \boxtimes \mu_{\beta,w,k}.$$

The $(\mathcal{E}\mathrm{xt}^\bullet_{\alpha,\beta})^\vee \oplus \sigma^*\mathcal{E}\mathrm{xt}^\bullet_{\beta,\alpha}$ case is similar. $\square$

For $\alpha \in K^0_{\mathrm{sst}}(X)$, consider the R-algebra $\mathrm{SSym}_R[u_{\alpha,v,i} : v \in Q, i \geq 1]$. Define a pairing

$$\mathrm{SSym}_R[[\mu_{\alpha,v,i} : v \in Q, i \geq 1]] \times \mathrm{SSym}_R[u_{\alpha,w,i} : w \in Q, i \geq 1] \longrightarrow R$$

by

$$(\prod_{v\in Q, i\geq 1} \mu_{\alpha,v,i}^{m_{v,i}})\cdot(\prod_{v\in Q, i\geq 1} u_{\alpha,v,i}^{n_{v,i}}) = \begin{cases} \prod_{v\in Q,i\geq 1}\frac{m_{v,i}!}{((i-1)!)^{m_{v,i}}}, & m_{v_i,i}=n_{v_i,i}, \forall i \\ 0, & \text{otherwise.}\end{cases} \tag{5.2.7}$$

We identify $E_*(\mathcal{M}_\alpha)$ with $\mathrm{SSym}_R[u_{\alpha,w,i} : i\geq 1, w\in Q]$ using (5.2.7).

Lemma 5.2.8. *Let X in class D. Then for all $\alpha,\beta\in K^0_{\mathrm{sst}}(X)$*

$$E_*(\Phi_{\alpha,\beta})[(\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{m_{v,i}})\boxtimes(\prod_{v\in Q,i\geq 1} u_{\beta,v,i}^{n_{v,i}})] = \prod_{v\in Q,i\geq 1} u_{\alpha+\beta,v,i}^{m_{v,i}+n_{v,i}}.$$

Proof. Pulling $\mathbb{E}^{\bullet}{}_{\alpha+\beta}$ along the map $\mathrm{id}\times\Phi_{\alpha,\beta} : X\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\to X\times\mathcal{M}_{\alpha+\beta}$ gives

$$(\mathrm{id}_X\times\Phi_{\alpha,\beta})^*(\mathbb{E}^{\bullet}{}_{\alpha+\beta}) \cong \pi_1^*(\mathbb{E}^{\bullet}{}_\alpha)\oplus\pi_2^*(\mathbb{E}^{\bullet}{}_\beta).$$

Taking K-theory classes, slanting with $v^\vee\in Q^\vee$, and then taking ch_i^E gives

$$E^*(\Phi_{\alpha,\beta})(\mu_{\alpha+\beta,v,i}) = \mu_{\alpha,v,i}\boxtimes 1 + 1\boxtimes\mu_{\beta,v,i}$$

so that

$$E^*(\Phi_{\alpha,\beta})(\prod_{v\in Q,i\geq 1}\mu_{\alpha+\beta,v,i}^{n_{v,i}}) = \prod_{v\in Q,i\geq 1}\sum_{0\leq m_{v,i}\leq n_{v,i}}\binom{n_{v,i}}{m_{v,i}}\mu_{\alpha,v,i}^{m_{v,i}}\boxtimes\mu_{\beta,v,i}^{n_{v,i}-m_{v,i}}. \tag{5.2.8}$$

Under (5.2.7), $(\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{m_{v,i}})\boxtimes(\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{n_{v,i}})$ gets identified with the functional

$$(\prod_{v\in Q,i\geq 1}\mu_{\alpha,v,i}^{m'_{v,i}})\boxtimes(\prod_{v\in Q,i\geq 1}\mu_{\beta,v,i}^{n'_{v,i}}) \mapsto \begin{cases} \prod_{v\in Q,i\geq 1}\frac{m_{v,i}!n_{v,i}!}{((i-1)!)^{m_{v,i}+n_{v,i}}}, & m'_{v,i}=m_{v,i}, \\ & n'_{v,i}=n_{v,i} \\ 0, & \text{otherwise}\end{cases}$$

so that, by (5.2.8), $E_*(\Phi_{\alpha,\beta})$ acts as

$$\prod_{v\in Q, i\geq 1} \mu_{\alpha+\beta,v,i}^{\ell_{v,i}} \mapsto (\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{m_{v,i}}) \boxtimes (\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{n_{v,i}})(\prod_{v\in Q,i\geq 1} \sum_{0\leq r_{v,i}\leq \ell_{v,i}} \binom{\ell_{v,i}}{r_{v,i}}$$

$$\mu_{\alpha,v,i}^{r_{v,i}} \boxtimes \mu_{\beta,v,i}^{\ell_{v,i}-r_{v,i}})$$

$$= \begin{cases} \prod_{v\in Q,i\geq 1} \binom{n_{v,i}+m_{v,i}}{m_{v,i}} \frac{n_{v,i}!m_{v,i}!}{((i-1)!)^{n_{v,i}+m_{v,i}}}, \ \ell_{v,i} = n_{v,i}+ \\ \qquad m_{v,i}, \text{ for all } i \\ 0, \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} \prod_{v\in Q,i\geq 1} \frac{(m_{v,i}+n_{v,i})!}{((i-1)!)^{m_{v,i}+n_{v,i}}}, \ \ell_{v,i} = n_{v,i}+m_{v,i}, \\ \qquad \text{for all } i, \\ 0, \quad \text{otherwise.} \end{cases}$$

This functional is represented by $\prod_{v\in Q,i\geq 1} u_{\alpha+\beta,v,i}^{n_{v,i}+m_{v,i}}$. ☐

Lemma 5.2.9. *Let X be in class D and let $\alpha \in K^0_{\mathrm{sst}}(X)$. Then*

$$(\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{n_{v,i}}) \cap (\prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{m_{v,i}}) = \begin{cases} \prod_{v\in Q,i\geq 1} \frac{n_{v,i}!}{(n_{v,i}-m_{v,i})!((i-1)!)^{m_{v,i}}} u_{\alpha,v,i}^{n_{v,i}-m_{v,i}}, \\ \qquad n_{v,i} \geq m_{v,i} \textit{ for all } v,i, \\ 0, \qquad \textit{otherwise.} \end{cases}$$

Proof. The cap product is dual to the cup product under (5.2.7). Therefore, the cap product $(\prod_{v\in Q,i\geq 1} u_{\alpha,v,i}^{n_{v,i}}) \cap (\prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{m_{v,i}})$ acts as

$$\prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{\ell_{v,i}} \mapsto u_{\alpha,v,i}^{n_{v,i}}(\prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{m_{v,i}} \cdot \prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{\ell_{v,i}})$$

$$= \begin{cases} \prod_{v\in Q,i\geq 1} \frac{n_{v,i}!}{((i-1)!)^{n_{v,i}}}, & \ell_{v,i} = n_{v,i} - m_{v,i} \\ 0, & \text{otherwise} \end{cases}$$

and $\prod_{v\in Q,i\geq 1} \frac{n_{v,i}!}{(n_{v,i}-m_{v,i})!((i-1)!)^{m_{v,i}}} u_{\alpha,v,i}^{n_{v,i}-m_{v,i}} (\prod_{v\in Q,i\geq 1} \mu_{\alpha,v,i}^{\ell_{v,i}})$ equals $\prod_{v\in Q,i\geq 1}$ $\frac{n_{v,i}!}{(n_{v,i}-m_{v,i})!((i-1)!)^{m_{v,i}}} \frac{(n_{v,i}-m_{v,i})!}{((i-1)!)^{n_{v,i}-m_{v,i}}}$ if $\ell_{v,i} = n_{v,i}+m_{v,i}$ for all i and equals 0 otherwise. □

Lemma 5.2.10. *Let X be in class D, let $\alpha \in K^0_{\mathrm{sst}}(X)$, $k \geq 0$, and $v \in Q$. Then*

$$E_*(\Psi_\alpha)(t^i \boxtimes u_{\alpha,v,1}) = u_{\alpha,v,i+1}.$$

Proof. There is an isomorphism

$$(\Psi_\alpha \times \mathrm{id}_X)^*(\mathbb{E}^\bullet{}_\alpha) \cong \pi^*_{[*/\mathbb{G}_m]}(E_1) \otimes \pi^*_{\mathcal{M}_\alpha \times X}(\mathbb{E}^\bullet{}_\alpha), \tag{5.2.9}$$

where $E_1 \to [*/\mathbb{G}_m]$ is the one-dimensional weight 1 representation of $\mathbb{G}_m$. Taking K-theory classes of (5.2.9), slanting both sides by some $v^\vee \in Q^\vee$, and then taking ch_i^E gives

$$E^*(\Psi_\alpha)(\mu_{\alpha,v,i}) = \sum_{j=0}^{i} \frac{1}{j!}\tau^j \boxtimes \mu_{\alpha,v,i-j},$$

where $\mathrm{ch}_j^E(E_1) = \frac{1}{j!}\tau^j$ under the isomorphism $E^*([*/\mathbb{G}_m]) \cong R[\tau]$. Therefore

$$[E_*(\Psi_\alpha)(t \boxtimes u_{\alpha,v,1})](\mu_{\alpha,v_1,k_1} \dots \mu_{\alpha,v_N,k_N}) = (t \boxtimes u_{\alpha,v,1})(\sum_{0\leq j_i\leq k_i, i=1,\dots,N} \frac{1}{j_1!\dots j_N!}$$

$$\tau^{j_1+\dots+j_N} \boxtimes (\mu_{\alpha,v_1,k_1-j_1} \cdots \mu_{\alpha,v_N,k_N-j_N}))$$

$$= \begin{cases} 1, & \text{if } N = 1, v_N = v \text{ and } k_1 = 2 \\ 0, & \text{otherwise} \end{cases}$$

so that $E_*(\Psi_\alpha)(t \boxtimes u_{\alpha,v,1}) = u_{\alpha,v,2}$. Iteration gives $E_*(\Psi_\alpha)(t^k \boxtimes u_{\alpha,v,1}) = u_{\alpha,v,k+1}$. □

Lemma 5.2.11. *Let X be in class D. Then for all $\alpha \in K^0_{\mathrm{sst}}(X)$, $v \in Q$, and $\eta \in$*

$\hat{E}_*(\mathcal{M}_\alpha)$

$$Y(u_{0,v,1},z)\eta=(-1)^{v\chi(\alpha,\alpha)}\big\{\sum_{i\geq 0}z^i\cdot u_{\alpha,v,i+1}\eta+\sum_{w\in Q,k\geq 0}k!\chi(v,w)z^{-k-1}(\eta$$
$$\cap\mu_{\alpha,w,k})\big\}$$

if X is $2n$-Calabi–Yau and

$$Y(u_{0,v,1},z)\eta=(-1)^{v\chi(\alpha,\alpha)}\big\{\sum_{i\geq 0}z^i\cdot u_{\alpha,v,i+1}\eta+\sum_{w\in Q,k\geq 0}k!\chi_{\mathrm{sym}}(v,w)z^{-k-1}$$
$$(\eta\cap\mu_{\alpha,w,k})\big\}$$

otherwise.

Proof. Assume that X is $2n$-Calabi–Yau. When X is not $2n$-Calabi–Yau, the proof is essentially identical. By definition

$$Y(u_{0,v,1},z)\eta=\epsilon_{0,\alpha}(-1)^{v\chi(\alpha,\alpha)}z^{\chi(0,\alpha)}\hat{E}_*(\Phi_{0,\alpha})\circ\hat{E}_*(\Psi_0\times\mathrm{id}_{\mathcal{M}_\alpha})\big\{(\sum_{i\geq 0}t^iz^i)$$
$$\boxtimes(u_{0,v,1}\boxtimes\eta)\cap\exp(\sum_{v,w\in Q}\sum_{j,k\geq 0,j+k\geq 1}(-1)^{j-1}(j+k-1)!z^{-j-k}\chi(v,w)\mu_{0,v,j}$$
$$\boxtimes\mu_{\alpha,w,k}\big\}$$

which gives

$$Y(u_{0,v,1}z)\eta=(-1)^{v\chi(\alpha,\alpha)}\hat{E}_*(\Phi_{0,\alpha})\circ\hat{E}_*(\Psi_0\times\mathrm{id}_{\mathcal{M}_\alpha})\big\{(\sum_{i\geq 0}z^it^i)\boxtimes(u_{0,v,1}\boxtimes$$
$$\eta)\cap[1+\sum_{v,w\in Q,k\geq 0}k!\chi(v,w)z^{-k-1}\mu_{0,v,1}\boxtimes\mu_{\alpha,w,k}\big\} \tag{5.2.10}$$

because, by Lemma 5.2.9, $u_{0,v,1}\cap\mu_{0,v,j}=0$ unless $j=1$, in which case $u_{0,v,1}\cap\mu_{0,v,1}=1$

and $u_{0,v,1} \cap \mu_{0,v,1}^k = 0$ for all $k > 1$. So, the right hand side of (5.2.10) equals

$$(-1)^{v\chi(\alpha,\alpha)} \hat{E}_*(\Phi_{0,\alpha}) \big\{ \hat{E}_*(\Psi_0)((\sum_{i\geq 0} z^i t^i) \boxtimes u_{0,v,1}) \boxtimes \eta + \hat{E}_*(\Psi_0)(\sum_{i\geq 0} z^i t^i \boxtimes 1_0) \boxtimes (\sum_{w\in Q} k! \chi(v,w) z^{-k-1} \cdot (\eta \cap \mu_{\alpha,w,k})) \big\} \tag{5.2.11}$$

By Lemma 5.2.10,

$$\hat{E}_*(\Psi_0)(\sum_{i\geq 0} z^i t^i \boxtimes \mu_{0,v,1}) = \sum_{i\geq 0} z^i u_{0,v,i+1} \text{ and } \hat{E}_*(\Psi_0)(\sum_{i\geq 0} z^i t^i \boxtimes 1_0) = 1_0.$$

Substituting this into (5.2.11) completes the proof. $\square$

Proof. (of Theorem 5.2.2) By Proposition 2.2.12 it suffices to study fields which are of the form $Y(u_{0,v,1}, z)$ for $v \in Q$. Without loss of generality we will assume that there is a decomposition $Q = Q^+ \cup Q^-$ where Q^+ is a basis for $K^0_{\text{top}}(X^{\text{an}})$ and Q^- is a basis for $K^1_{\text{top}}(X^{\text{an}})$. We will assume that $v \in Q^+$. The $v \in Q^-$ case is nearly identical.

First, suppose $n < 0$. For $\eta \in \hat{E}_*(\mathcal{M}_\alpha)$, by Lemma 5.2.11,

$$Y(u_{0,v,1}, z)_n \eta = (-1)^{v\chi(\alpha,\alpha)} u_{\alpha,v,i} \cdot \eta,$$

which corresponds to $v_n(\eta)$. For $n = 0$,

$$Y(u_{0,v,1}, z)_0 \eta = (-1)^{v\chi(\alpha,\alpha)} \eta \cap \mu_{\alpha,v,0} = 0 = (-1)^{v\chi(\alpha,\alpha)} \chi(\alpha, 0) \cdot \eta.$$

For $n > 0$ and $w \in Q$

$$Y(u_{0,v,1}, z)_n u_{\alpha,w,i} = (-1)^{v\chi(\alpha,\alpha)} n! \chi(v,w)(u_{\alpha,v,i} \cap \mu_{\alpha,w,n})$$

$$= \begin{cases} (-1)^{v\chi(\alpha,\alpha)} n\chi(v,w), & i = n, \\ 0, & \text{otherwise.} \end{cases}$$

This identifies $Y(u_{0,v,1}, z)_n$ with $v_n(-)$. $\square$

5.3 A geometric construction of graded vertex F-algebras in gauge theory

In this section we give a gauge-theoretic construction similar to the algebraic construction reviewed in Section 5.1. We will allow (even) complex-oriented homology theories and build vertex F-algebras using an even F-bicharacter.

Let X be a compact manifold. Then recall that

$$\mathcal{M}^U := \mathrm{Map}_{C^0}(X, \coprod_{n \geq 0} BU(n))$$

is homotopy equivalent to the topological realization of the classifying stack of all unitary connections on X (Proposition 3.4.2). Let $\overline{\mathcal{M}}^U$ denote the homotopy-theoretic group completion of $\mathcal{M}^U$. One can regard $\overline{\mathcal{M}}^U$ as the moduli space of virtual complex vector bundles on X. To make an analogy with the material of Section 5.1: $\mathcal{M}^U$ is to the moduli stack of objects in an abelian category $\mathcal{A}$ as $\overline{\mathcal{M}}^U$ is to the moduli stack of objects in its bounded derived category $D^b(\mathcal{A})$.

For all $n \geq 1$ the natural scaling action $\mathbb{C}^* \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is a group homomorphism. Applying the classifying space functor $B(-)$ yields an H-space homomorphism

$$BU(1) \times \coprod_{n \geq 0} BU(n) \longrightarrow \coprod_{n \geq 0} BU(n). \tag{5.3.1}$$

By the functoriality of homotopy-theoretic group completions there exists an H-map

$$BU(1) \times BU \times \mathbb{Z} \longrightarrow BU \times \mathbb{Z}$$

extending (5.3.1). The mapping spaces $\mathcal{M}^U, \overline{\mathcal{M}}^U$ inherit $BU(1)$-actions

$$BU(1) \times \mathcal{M}^U \longrightarrow \mathcal{M}^U, \quad BU(1) \times \overline{\mathcal{M}}^U \longrightarrow \overline{\mathcal{M}}^U,$$

which are also H-maps, as follows: For $n \geq 0$, let $\mathcal{U}_X^n \to X \times \mathrm{Map}_{C^0}(X, BU(n))$ denote the universal rank n complex vector bundle over X. The external product $\mathcal{U}_X^n \boxtimes EU(1)$ is classified by an H-map

$$BU(1) \times X \times \mathrm{Map}_{C^0}(X, BU(n)) \longrightarrow BU(1) \times BU(n) \longrightarrow BU(n). \tag{5.3.2}$$

Exponentiating (5.3.2) gives an H-map

$$BU(1) \times \mathrm{Map}_{C^0}(X, BU(n)) \longrightarrow \mathrm{Map}_{C^0}(X, BU(n)). \tag{5.3.3}$$

Taking the union of (5.3.3) over all $n \geq 0$ gives the desired H-map $BU(1) \times \mathcal{M}^U \to \mathcal{M}^U$ and then taking a homotopy-theoretic group completion gives the H-map $BU(1) \times \overline{\mathcal{M}}^U \to \overline{\mathcal{M}}^U$.

Let E be a complex-oriented spectrum with associated formal group law F. Taking E-homology then gives F-shift operators

$$E_*(\mathcal{M}^U) \longrightarrow E_*(\mathcal{M}^U)[[c_1^E]] \ \text{ and } \ E_*(\overline{\mathcal{M}}^U) \longrightarrow E_*(\overline{\mathcal{M}}^U)[[c_1^E]]$$

compatible with the algebra structures on $E_*(\mathcal{M}^U)$ and $E_*(\overline{\mathcal{M}}^U)$ (see Corollary 3.2.7).

Notation 5.3.1. By $\mathcal{M}$ we will mean either $\mathcal{M}^U$ or $\overline{\mathcal{M}}^U$.

We will want to introduce an F-bicharacter compatible with the above F-shift operator on $E_*(\mathcal{M})$. Let $R := E^*(\mathrm{pt})$ be the coefficient ring of E. Given an H-space $(\mathcal{X}, \mu, 1_{\mathcal{X}})$, the simplest possible F-bicharacter $r : E_*(\mathcal{X}) \otimes E_*(\mathcal{X}) \to R$ is given by taking E-homology of the projection

$$\mathcal{X} \times \mathcal{X} \xrightarrow{\mu} \mathcal{X} \xrightarrow{\pi} \{\mathrm{pt}\}.$$

Writing η for the E-homology of $\mathcal{X} \to \{\mathrm{pt}\}$ a formula for this F-bicharacter is

$$r(a \otimes b) = \eta(a) \cdot \eta(b).$$

Supposing that $BU(1)$ acts on $\mathcal{X}$ via an H-space homomorphism $\rho : BU(1) \times \mathcal{X} \to \mathcal{X}$ the E-homology of the composition

$$BU(1) \times \mathcal{X} \times \mathcal{X} \xrightarrow{\rho \times \mathrm{id}_{\mathcal{X}}} \mathcal{X} \times \mathcal{X} \xrightarrow{\mu} \mathcal{X} \longrightarrow \{\mathrm{pt}\} \tag{5.3.4}$$

produces a holomorphic F-bicharacter $r : E_*(\mathcal{M}) \otimes E_*(\mathcal{M}) \to E^*(BU(1)) \cong R[[c_1^E]]$ which can be written explicitly[3] as

$$r_z(a \otimes b) = \eta(\mathcal{D}(z)(a)) \cdot \eta(b).$$

In the language of spectra, we can suspend (5.3.4) to produce a homomorphism of spectra

$$\Sigma^\infty(\mathcal{X}_+) \wedge \Sigma^\infty(\mathcal{X}_+) \longrightarrow \mathbb{D}(\Sigma^\infty(BU(1)_+)), \tag{5.3.5}$$

where $\mathbb{D}(-)$ denotes the Spanier–Whitehead dual. Taking wedge products with E followed by homotopy groups gives

[3]This is sort of a silly modification because compatibility of $\mathcal{D}(z)$ with η implies $\eta(\mathcal{D}(z)(a)) \cdot \eta(b) = \eta(a) \cdot \eta(b)$. Nonetheless, we find this instructive.

$$\begin{array}{ccc}
\pi_*(\Sigma^\infty(\mathcal{X}\times\mathcal{X})_+\wedge E) & \longrightarrow & \pi_*(\mathbb{D}(\Sigma^\infty(BU(1))_+)\wedge E)\cong\pi_{-*}([\Sigma^\infty(BU(1))_+,E]) \\
\big\downarrow\sim & & \big\downarrow\sim \\
E_*(\mathcal{X}\times\mathcal{X}) & \longrightarrow & E^{-*}(BU(1))\cong R[[z]]
\end{array}$$

with z in degree -2. This explains our convention in Chapter 2 to take the degree of z to be -2. Pre-composing with external products gives an F-bicharacter

$$E_*(\mathcal{X})\otimes E_*(\mathcal{X}) \longrightarrow E_*(\mathcal{X}\times\mathcal{X}) \longrightarrow R[[z]].$$

If we wish for a non-holomorphic vertex F-algebra, we need to introduce some singularities into the above picture. To this end, we will produce for any K-theory class $\Theta\in K^0_{\mathrm{top}}(\mathcal{X})$ an operator $(-)\cap C^E_z(\Theta): E_*(\mathcal{X})\to E_*(\mathcal{X})((z))$.

Remark 5.3.2. In general, the map $(-)\cap C^E_z(\Theta): E_*(\mathcal{X})\to E_*(\mathcal{X})((z))$ will not preserve degree. Therefore, in order to obtain graded vertex F-algebras we will need to shift the grading on $E_*(\mathcal{M})$ by a quadratic form (see Definition 2.4.5).

Theorem 5.3.3. *Let E be a connective complex-oriented spectrum with associated formal group law F. Then there is a unique natural transformation*

$$(-)\cap C^E_z(-): K^0_{\mathrm{top}}(-)\Longrightarrow \mathcal{F}(E_*(-)),$$

where $\mathcal{F}(E_*(-))$ denotes the module of fields on $E^*(-)$, satisfying the following two conditions for all CW complexes $\mathcal{X}$

1. *Whitney sum: for $V,W\in K^0_{\mathrm{top}}(\mathcal{X})$*

$$(-)\cap C^E_z(V+W)=((-)\cap C^E_z(V))\cap C^E_z(W),$$

   and

2. *normalization: for a complex line bundle $L\to\mathcal{X}$, the operation $(-)\cap C^E_z([L])$ is given by cap product with the class $F(z,c^E_1(L))\in E^*(\mathcal{X})[[z]]$.*

Moreover, for all $\alpha \in \pi_0(\mathcal{X})$, the map $(-) \cap C_z^E(\Theta|_{\mathcal{X}_\alpha}) : E_*(\mathcal{X}_\alpha) \to E_*(\mathcal{X}_\alpha)((z))$ is graded of degree $-2\mathrm{rk}(\Theta|_{\mathcal{X}_\alpha})$.

Theorem 5.3.3 will be proved in several parts. It is clear that $(-) \cap C_z^E(\Theta)$ is uniquely determined when Θ is represented by a complex line bundle. We will first show that $(-) \cap C_z^E(\Theta)$ is uniquely determined when Θ is represented by a complex vector bundle, then we will use this result to show $(-) \cap C_z^E(\Theta)$ is uniquely determined for arbitrary $\Theta \in K^0_{\mathrm{top}}(\mathcal{X})$ when $\mathcal{X}$ is a finite CW complex, and lastly we will show that $(-) \cap C_z^E(\Theta)$ is uniquely determined for arbitrary $\Theta \in K^0_{\mathrm{top}}(\mathcal{X})$ when $\mathcal{X}$ is an infinite CW complex by taking a direct limit along the finite sub-skeleta of $\mathcal{X}$.

Proposition 5.3.4. *Let E be a complex-oriented spectrum. Then there exists a unique natural transformation $C_z^E(-) : \mathrm{Vect}_{\mathbb{C}}(-) \implies E^*(-)[[z]]$ with the following properties*

1. *$C_z^E(V \oplus W) = C_z^E(V) \cup C_z^E(W)$, and*
2. *$C_z^E(\mathcal{L}) = F(z, c_1^E(\mathcal{L}))$ where $\mathcal{L} \to BU(1)$ is the universal complex line bundle.*

Proof. Condition 1. defines C_z^E for any complex line bundle $L \to X$

$$C_z^E(L) = F(z, c_1^E(L)).$$

Now let $V \to X$ be a rank n complex vector bundle, let $q : Y \to X$ be the corresponding flag bundle, and let $L_1, \ldots, L_n$ be the roots of V i.e. $L_1 \oplus \cdots \oplus L_n = q^*(V) \to Y$. It is clear that the expression $F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n))$ is symmetric in $c_1^E(L_1), \ldots, c_1^E(L_n)$. Therefore the coefficient of each z^k is a symmetric polynomial in then Chern roots $c_1^E(L_1), \ldots, c_1^E(L_n)$ i.e. a polynomial in the E-Conner-Floyd Chern classes of q^*V. This implies that

$$C_z^E(q^*V) = F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n))$$

lies in the image of the injection $E^*(X) \hookrightarrow E^*(Y)$. We define $C_z^E(V)$ to be the unique preimage of $F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n))$. Naturality follows from the fact that both Conner–Floyd Chern classes and cup products commute with pullbacks. $\square$

Corollary 5.3.5. *Let E be a complex-oriented spectrum. Then there is a unique natural transformation $(-) \cap C_z^E(-) : \mathrm{Vect}_{\mathbb{C}}(-) \Longrightarrow \mathcal{F}(E_*(-))$ satisfying the conditions of Theorem 5.3.3. Moreover, given a rank n complex vector bundle $V \to \mathcal{X}$ over a CW complex $\mathcal{X}$, $(-) \cap C_z^E(\mathcal{X}) : E_*(\mathcal{X}) \to E_*(\mathcal{X})((z))$ is graded of degree $-2n$.*

Proof. Let $\mathcal{X}$ be a CW complex and let $V \to \mathcal{X}$ be a rank n complex vector bundle with roots $L_1, \ldots, L_n$. Then, by Proposition 5.3.4, $(-) \cap C_z^E([V]) : E_*(\mathcal{X}) \to E_*(\mathcal{X})[[z]] \subset E_*(\mathcal{X})((z))$ is given by

$$a \mapsto a \cap (F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n))).$$

Naturality is equivalent to the fact that

$$f_*(a) \cap C_z^E([V]) = f_*(a \cap C_z^E(f^*[V]))$$

for all $a \in E_*(\mathcal{X})$ and continuous maps $f : \mathcal{X} \to \mathcal{Y}$. Capping with $F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n))$ will reduce degree by $-2n$ so that $(-) \cap C_z^E(V) : E_*(\mathcal{X}) \longrightarrow E_*(\mathcal{X})((z))$ is indeed graded of degree $-2n$. $\square$

We are now in a position to prove Theorem 5.3.3.

Proof. (of Theorem 5.3.3) Let $\mathcal{X}$ be a (possibly infinite) CW complex and let $\Theta : K^0_{\mathrm{top}}(\mathcal{X})$. For $n \geq 0$, let $\mathcal{X}^n \subset \mathcal{X}$ be the n-skeleton of $\mathcal{X}$. Then $\Theta|_{\mathcal{X}^n} \in K^0_{\mathrm{top}}(\mathcal{X}^n)$ can be written as difference $\Theta|_{\mathcal{X}^n} = [V] - [W] \in K^0_{\mathrm{top}}(\mathcal{X}^n)$ where V, W are complex vector bundles on $\mathcal{X}^n$. Choose a large rank complex vector bundle $U \to \mathcal{X}^n$ such that $U \oplus W \cong \underline{\mathbb{C}}^N$ for some $N \gg 0$, where $\underline{\mathbb{C}}^N \to \mathcal{X}^n$ denotes the trivial rank N complex vector bundle on $\mathcal{X}^n$. Observe that

$$C_z^E([\underline{\mathbb{C}}^N]) = F(z, c_1^E(\underline{\mathbb{C}})) \cup \cdots \cup F(z, c_1^E(\underline{\mathbb{C}})) = z^N.$$

This implies that $C_z^E([W])^{-1} := z^{-N} C_z^E([U])$ is a multiplicative inverse of $C_z^E([W])$ in $E^*(\mathcal{X}^n)((z))$, well-defined by uniqueness of multiplicative inverses. Moreover, as the unit of $E^*(\mathcal{X}^n)((z))$ is graded of degree 0 we must have that $(-) \cap C_z^E([W])^{-1}$ is graded of degree $2\mathrm{rk}(W)$. The Whitney sum axiom then forces

$$(-) \cap C_z^E(\Theta|_{\mathcal{X}^n}) = (-) \cap (C_z^E([V]) \cup C_z^E([W])^{-1}) \in E^*(\mathcal{X}^n)((z)).$$

One consequence of the above is that $(-) \cap C_z^E(\Theta|_{\mathcal{X}^n})$ is graded of degree $-2\mathrm{rk}(\Theta)$.

For $m \geq n \geq 0$ there is an R-linear map $j_*^{n,m} : E_*(\mathcal{X}^n) \otimes_R R((z)) \to E_*(\mathcal{X}^m) \otimes_R R((z))$, which is the identity on $R((z))$, making $\{E_*(\mathcal{X}^n)((z)), j_*^{n,m}\}$ into a direct system. Taking the direct limit of the maps $(-) \cap C_z^E(\Theta|_{\mathcal{X}^n})$ along this system gives a map

$$\begin{tikzcd}
\varinjlim_n E_*(\mathcal{X}^n) \arrow[r] \arrow[d, "\sim"] & \varinjlim_n E_*(\mathcal{X}^n)((z)) \arrow[d, "\sim"] \\
E_*(\mathcal{X}) \arrow[r] & E_*(\mathcal{X})((z)).
\end{tikzcd}$$

by [208, Prop. 7.53]. The isomorphism $\varinjlim_n (E_*(\mathcal{X}^n) \otimes_R R((z))) \cong (\varinjlim_n E_*(\mathcal{X}^n)) \otimes_R R((z))$ follows from the fact that left adjoints commute with direct limits and that the direct limit of an identity system on an object is isomorphic to that object. $\square$

We now prove a property of $(-) \cap C_z^E(-) : K^0_{\mathrm{top}}(-) \Longrightarrow \mathcal{F}(E_*(-))$ which makes it an ideal candidate from which to build a singular F-bicharacter.

Proposition 5.3.6. *Let $\mathcal{X}$ be a CW complex, E a complex-oriented connective spectrum with associated formal group law F, and $L \to \mathcal{X}$ a complex line bundle. Then* *for all* $\Theta \in K^0_{\mathrm{top}}(\mathcal{X})$

$$(-) \cap C^E_z([L] \cdot \Theta) = i_{z,c_1^E(L)}(-) \cap C^E_{F(z,c_1^E(L))}(\Theta)$$

where $i_{z,c_1^E(L)}$ *denotes the expansion map defined in Section 2.4 (see Definition 2.4.1).*

Proof. First suppose that $\mathcal{X}$ is a finite CW complex. Consider a rank n complex vector bundle $V \to \mathcal{X}$ with roots $L_1, \dots, L_n$. Then

$$\begin{aligned}
C^E_z([L] \cdot [V]) &= F(z, c_1^E(L \otimes L_1)) \cup \cdots \cup F(z, c_1^E(L \otimes L_n)) \\
&= F(z, F(c_1^E(L), c_1^E(L_1))) \cup \cdots \cup F(z, F(c_1^E(L), c_1^E(L_n))) \\
&= F(F(z, c_1^E(L)), c^E(L_1)) \cup \cdots \cup F(F(z, c_1^E(L)), c^E(L_n)) \\
&= C^E_{F(z,c_1^E(L))}(L_1) \cup \cdots \cup C^E_{F(z,c_1^E(L))}(L_n) \\
&= C^E_{F(z,c_1^E(L))}([V]).
\end{aligned}$$

Given a general $\Theta \in K^0_{\mathrm{top}}(\mathcal{X})$ we can write $\mathcal{X} = [V] - [W]$ where $V, W \to \mathcal{X}$ are complex vector bundles. We claim that the multiplicative inverse of $C^E_z([L \otimes W])$ is given by $i_{z,c_1^E(L)} C^E_{F(z,c_1^E(L))}([W])^{-1}$. To see this, consider a complex vector bundle $U \to \mathcal{X}$ such that $W \oplus U = \underline{\mathbb{C}}^N$ for $N \gg 0$. Then

$$C^E_z([L] \cdot [W \oplus U]) = C^E_z([L] \cdot [W]) \cup C^E_z([L] \cdot [U]) = F(z, c_1^E(L))^N.$$

The term $F(z, c_1^E(L))^N$ is not invertible in $E^*(\mathcal{X})[[z]]$, but $i_{z,c_1^E(L)} F(z, c_1^E(L))^{-N}$ is a multiplicative inverse in $R((z))[[c_1^E(L)]] \subset E^*(\mathcal{X})((z))[[c_1^E(L)]]$. We then get that

$$i_{z,c_1^E(L)}(F(z, c_1^E(L))^{-N}) C^E_{F(z,c_1^E(L))}([W]) = i_{z,c_1^E(L)} C^E_{F(z,c_1^E(L))}([V])$$

is the multiplicative inverse of $C_z^E([W])$ in $E^*(\mathcal{X})((z))[[c_1^E(L)]]$. Therefore

$$\begin{aligned}C_z^E([L]\cdot\Theta) &= C_z^E([L\otimes V])\cup C_z^E([L\otimes W])^{-1}\\ &= i_{z,c_1^E(L)}C^E_{F(z,w)}([V])\cup C^E_{F(z,w)}([W])^{-1}\\ &= i_{z,c_1^E(L)}C^E_{F(z,w)}(\Theta).\end{aligned}$$

Because $E_{-k}(\mathcal{X}) = 0$ for all $k > 0$, the operator $(-)\cap i_{z,c_1^E(L)}C^E_{F(z,c_1^E(L))}(\Theta)$ maps $E_*(\mathcal{X}) \to E_*(\mathcal{X})((z))$.

Finally, if $\mathcal{X}$ is an infinite CW complex then

$$\begin{aligned}(-)\cap C_z^E([L]\cdot\Theta) &= \varinjlim_n (-)\cap C_z^E([L]\cdot\Theta|_{\mathcal{X}^n})\\ &= i_{z,c_1^E(L)}\varinjlim_n (-)\cap C^E_{F(z,c_1^E(L))}(\Theta|_{\mathcal{X}^n})\\ &= i_{z,c_1^E(L)}C^E_{F(z,w)}(\Theta).\end{aligned}$$

□

Recall that a complex-oriented spectrum E is said to be *even* if $E^{2i+1}(\{\mathrm{pt}\}) = 0$ for all $i\in\mathbb{Z}$. Examples of cohomology theories represented by even complex-oriented spectra are ordinary cohomology, complex topological K-theory, elliptic cohomology, and complex cobordism. For further reading on even cohomology theories, we refer the reader to Lurie [153].

Given an even complex-oriented spectrum E we are able to write down a singular even F-bicharacter on the E-homology of H-spaces with suitable $BU(1)$-actions.

Theorem 5.3.7. *Let $\mathcal{X}$ be an H-space with H-map $\Phi : \mathcal{X}\times\mathcal{X}\to\mathcal{X}$. Let $\Psi : BU(1)\times\mathcal{X}\to\mathcal{X}$ be a $BU(1)$-action on $\mathcal{X}$ which is also a morphism of H-spaces. Suppose that we are given the following data*

1. *a quotient $K(\mathcal{X})$ of $(\pi_0(\mathcal{X}))^+$, and*

2. *a complex topological K-theory class $[D^\nabla]\in K^0_{\mathrm{top}}(\mathcal{X}\times\mathcal{X})$*

such that for all $\alpha, \beta, \gamma \in K(\mathcal{X})$ there are equalities

$$(\Phi_{\alpha,\beta} \times \mathrm{id})^*([D^\nabla_{\alpha+\beta,\gamma}]) = \pi^*_{1,3}([D^\nabla_{\alpha,\gamma}]) + \pi^*_{2,3}([D^\nabla_{\beta,\gamma}]), \tag{5.3.6}$$

$$(\mathrm{id} \times \Phi_{\beta,\gamma})^*([D^\nabla_{\alpha,\beta+\gamma}]) = \pi^*_{1,2}([D^\nabla_{\alpha,\beta}]) + \pi^*_{1,3}([D^\nabla_{\alpha,\gamma}]), \tag{5.3.7}$$

$$(\Psi_\alpha \times \mathrm{id})^*([D^\nabla_{\alpha,\beta}]) = \pi^*_1([\mathcal{L}]) \cdot \pi^*_{2,3}([D^\nabla_{\alpha,\beta}]), \tag{5.3.8}$$

$$(\pi_2, \Psi_\beta \circ \pi^*_{1,3})([D^\nabla_{\alpha,\beta}]) = \pi^*_1([\mathcal{L}^\vee]) \cdot \pi^*_{2,3}([D^\nabla_{\alpha,\beta}]) \tag{5.3.9}$$

of K-theory classes over any finite sub-complex $C \subset \mathcal{X}$. Define a symmetric $\mathbb{Z}$-bilinear form $\chi^D : K(\mathcal{X}) \otimes K(\mathcal{X}) \to \mathbb{Z}$ by

$$\chi^D(\alpha, \beta) := \mathrm{rk}([D^\nabla_{\alpha,\beta}]) + \mathrm{rk}([D^\nabla_{\beta,\alpha}])$$

and let $Q^D(\alpha) := \chi^D(\alpha, \alpha)$ be the associated quadratic form. Let E be a connective even complex-oriented spectrum with associated formal group law F, let $\mathcal{D}(z)$ denote the F-shift operator induced by Ψ and E, let $\hat{E}_*(\mathcal{X})$ be the Q^D-shift of $E_*(\mathcal{X})$, let $R := E^*(\mathrm{pt})$, and let

$$(-) \cap G^E_z([D^\nabla_{\alpha,\beta}]) := ((-) \cap (-1)^{\mathrm{rk}([D^\nabla_{\beta,\alpha}])} C^E_z([D^\nabla_{\alpha,\beta}])) \cap C^E_{\iota(z)}(\sigma^*[D^\nabla_{\beta,\alpha}]).$$

Then the R-linear map $r : \hat{E}_*(\mathcal{X}) \otimes \hat{E}_*(\mathcal{X}) \to R((z))$ defined by

$$r(a, b) := (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes b \cap G^E_z([D^\nabla])) \tag{5.3.10}$$

is a graded even symmetric F-bicharacter. Therefore, if there is a Künneth[4] isomorphism

$$E_*(\mathcal{X} \times \mathcal{X}) \cong E_*(\mathcal{X}) \otimes E_*(\mathcal{X})$$

[4] For example, X could be a point or $E^*(\{\mathrm{pt}\})$ could be a $\mathbb{Q}$-algebra. The case $X = \{\mathrm{pt}\}$ includes moduli spaces of representations of finite quivers.

(5.3.10) endows $\hat{E}_*(\mathcal{X})$ with the structure of a graded vertex F-algebra.

Proof. Note that (5.3.6)-(5.3.9) give

$$\begin{aligned}
&(-)\cap(\Phi\times\mathrm{id})^*(G_z^E([D^\nabla])) = ((-)\cap\pi_{1,3}^*(G_z^E([D^\nabla])))\cap\pi_{2,3}^*(G_z^E([D^\nabla])),\\
&(-)\cap(\mathrm{id}\times\Phi)^*(G_z^E([D^\nabla])) = ((-)\cap\pi_{1,2}^*(G_z^E([D^\nabla])))\cap\pi_{1,3}^*(G_z^E([D^\nabla])),\\
&(-)\cap(\Psi\times\mathrm{id})^*(G_z^E([D^\nabla])) = i_{z,w}(-)\cap G_{F(z,w)}^E([D^\nabla]), \text{ and}\\
&(-)\cap(\mathrm{id}\times\Psi)^*(G_z^E([D^\nabla])) = i_{z,w}(-)\cap G_{F(z,\iota(w))}^E([D^\nabla]).
\end{aligned}$$

First, we prove that

$$r_z(\mathcal{D}(w)(a)\otimes b) = i_{z,w}r_{F(z,w)}(a\otimes b).$$

We calculate

$$\begin{aligned}
r_z(\mathcal{D}(w)(a)\otimes b) &= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta\otimes\eta)\circ(\mathcal{D}(z)\otimes\mathrm{id})(\mathcal{D}(w)(a)\otimes b\cap G_z^E([D^\nabla]))\\
&= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta\otimes\eta)\circ(\mathcal{D}(z)\otimes\mathrm{id})\circ(\mathcal{D}(w)\otimes\mathrm{id})((a\otimes b)\cap\\
&\qquad (\Psi\times\mathrm{id})^*G_z^E([D^\nabla]))\\
&= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta\otimes\eta)\circ(\mathcal{D}(F(z,w))\otimes\mathrm{id})((a\otimes b)\cap i_{z,w}G^E_{F(z,w)}\\
&\qquad ([D^\nabla]))\\
&= i_{z,w}r_{F(z,w)}(a\otimes b).
\end{aligned}$$

For

$$r_z(a\cdot b\otimes c) = (-1)^{bc'}r_z(a\otimes c')r_z(b\otimes c'') \tag{5.3.11}$$

note first that

$$(-1)^{(a+b)Q^D(\gamma)+\mathrm{rk}([D^\nabla_{\alpha+\beta,\gamma}])} = (-1)^{aQ^D(\gamma)+\mathrm{rk}([D^\nabla_{\alpha,\gamma}])}(-1)^{bQ^D(\gamma)+\mathrm{rk}([D^\nabla_{\beta,\gamma}])}.$$

We then have that

$$(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \cdot b \otimes c \cap G^E_z([D^\nabla])) = (\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id}) \circ (\Phi \times \mathrm{id})_* \\ ((a \otimes b \otimes c \cap \pi^*_{1,3}(G^E_z([D^\nabla])))\cap \\ \pi^*_{2,3}(G^E_z([D^\nabla])))$$

and

$$(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes c' \cap G^E_z([D^\nabla])) \cdot (\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(b \otimes c'' \cap G^E_z([D^\nabla]))$$
$$= (\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id} \otimes \mathcal{D}(z) \otimes \mathrm{id})(a \otimes c' \otimes b \otimes c'' \cap G^E_z([D^\nabla])) \otimes G^E_z([D^\nabla]))$$
$$= (-1)^{bc'}(\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id} \otimes \mathcal{D}(z) \otimes \mathrm{id})((\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(a \otimes b \otimes \Delta(c)) \\ \cap G^E_z([D^\nabla])) \otimes G^E_z([D^\nabla]))$$
$$= (-1)^{bc'}(\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id} \otimes \mathcal{D}(z) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \Delta) \circ (\mathrm{id} \otimes \sigma_* \otimes \mathrm{id}) \\ ((a \otimes b \otimes c \cap \pi^*_{1,3}(G^E_z([D^\nabla]))) \cap \pi^*_{2,3}(G^E_z([D^\nabla]))).$$

This means that to establish equation (5.3.11) it suffices to prove that the diagram

$$\begin{tikzcd}
B \otimes B \otimes B \arrow[r, "\mathrm{id}\otimes\sigma\otimes\mathrm{id}\circ\mathrm{id}\otimes\mathrm{id}\otimes\Delta"] \arrow[d, "m\otimes\mathrm{id}"] & B \otimes B \otimes B \otimes B \arrow[d, "\mathcal{D}(z)\otimes\mathrm{id}\otimes\mathcal{D}(z)\otimes\mathrm{id}"] \\
B \otimes B \arrow[d, "\mathcal{D}(z)\otimes\mathrm{id}"] & B \otimes B \otimes B \otimes B((z)) \arrow[d, "m\otimes m"] \\
B \otimes B((z)) \arrow[d, "\eta\otimes\eta"] & B \otimes B((z)) \arrow[d, "\eta"] \\
R((z)) \arrow[r, equal] & R((z))
\end{tikzcd}$$

commutes for any bialgebra $(B, \mu, \Delta, \eta, 1_B)$ with compatible shift operator $\mathcal{D}(z) : B \to B((z))$. This is a straightforward computation:

$$(\eta \otimes \eta)(\mathcal{D}(z)(a \cdot b) \otimes c) = (\eta \otimes \eta)(\mathcal{D}(z) \cdot \mathcal{D}(z)(b) \otimes c) \\ = \eta(\mathcal{D}(z)(a)) \cdot \eta(\mathcal{D}(z)(b)) \cdot \eta(c)$$

$$= \eta(\mathcal{D}(z)(a)) \cdot \eta(\mathcal{D}(z)(b)) \cdot \eta(c'\eta(c''))$$

$$= \eta \circ \mu(\mathcal{D}(z)(a) \otimes c') \cdot \eta \circ \mu(\mathcal{D}(z)(b) \otimes c'').$$

Note $(-) \cap G_z^E([D^\nabla])$ has a symmetry property

$$\begin{aligned}
(-) \cap G_z^E(\sigma^*[D^\nabla_{\beta,\alpha}]) &= ((-1)^{\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(-) \cap C_z^E(\sigma^*[D^\nabla_{\beta,\alpha}])) \cap C^E_{\iota(z)}([D^\nabla_{\alpha,\beta}]) \\
&= (-1)^{\chi^D(\alpha,\beta)+\mathrm{rk}(\sigma^*[D^\nabla_{\alpha,\beta}])}((-) \cap C^E_{\iota(\iota(z))}(\sigma^*[D^\nabla_{\beta,\alpha}])) \cap C^E_{\iota(z)}(\\
&\quad [D^\nabla_{\alpha,\beta}]) \\
&= (-1)^{\chi^D(\alpha,\beta)} G^E_{\iota(z)}([D^\nabla_{\alpha,\beta}]).
\end{aligned}$$

From this we show that $r_z(a \otimes b) = (-1)^{\hat{a}\hat{b}} r_{\iota(z)}(b, a)$, where $(-1)^{\hat{x}} := (-1)^{\deg(x) - Q^D(\gamma)}$ for any $x \in E_*(\mathcal{X}_\gamma)$, by computation

$$\begin{aligned}
(-1)^{\hat{a}\hat{b}} r_{\iota(z)}(b \otimes a) &= (-1)^{aQ^D(\beta)+Q^D(\alpha)Q^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathcal{D}(\iota(z)) \otimes \mathrm{id})(\sigma_*(a \otimes \\
&\quad b) \cap \sigma^*\sigma^* G^E_{\iota(z)}([D^\nabla])) \\
&= (-1)^{a\chi^D(\beta,\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathcal{D}(\iota(z)) \otimes \mathrm{id}) \circ \sigma_*(a \otimes b \cap G_z^E(\\
&\quad [D^\nabla])) \\
&= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathrm{id} \otimes \mathcal{D}(\iota(z)))(a \otimes b \cap G_z^E(\\
&\quad [D^\nabla])) \\
&= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathrm{id} \otimes \mathcal{D}(\iota(z)))(a \otimes b \cap (\overline{\Psi} \times \overline{\Psi})^* \\
&\quad G_z^E([D^\nabla])) \\
&= (-1)^{aQ^D(\beta)+\mathrm{rk}([D^\nabla_{\alpha,\beta}])}(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id}) \circ (\mathcal{D}(\iota(z)) \otimes \mathrm{id})(a \otimes \\
&\quad \mathcal{D}(\iota(z))(b) \cap (\overline{\Psi} \times \mathrm{id})^* G_z^E([D^\nabla])) \\
&= r_z(\mathcal{D}(\iota(w))(a) \otimes \mathcal{D}(\iota(w))(b))|_{w=z} \\
&= i_{z,w} r_{F(F(z,\iota(w)),w)}(a \otimes b)|_{w=z} \\
&= r_z(a \otimes b),
\end{aligned}$$

where $\overline{\Psi}$ denotes the $BU(1)$-action on $\mathcal{X}$ which is conjugate to Ψ.

Next, we show that

$$r_z(a \otimes 1) = \eta(a)$$

by computing

$$r_z(a \otimes 1) = (\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes 1 \cap G_z^E([D_{\alpha,0}^{\nabla}])) = \eta(\mathcal{D}(z)(a)) = \eta(a).$$

It remains to show that r is even with respect to the usual grading on $E_*(\mathcal{M})$. Suppose $a, b \in E_*(\mathcal{M})$ have different parities so that $a \otimes b$ has odd degree. Then $\Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes b \cap G_z^E([D^{\nabla}]))$ also has odd degree. The counit $\eta : E_*(\mathcal{M}) \to E_*(\mathrm{pt}) \cong R$ is graded. Therefore $r(a \otimes b) = 0$ by evenness of the spectrum E. $\square$

The state-to-field correspondence corresponding to the bicharacter of Theorem 5.3.7 is actually independent of the coproduct on $E_*(\mathcal{M})$.

Proposition 5.3.8. *Let $\mathcal{X}, \Phi, \Psi, K(\mathcal{X}), \mathcal{D}(z), [D^{\nabla}], \chi^D, Q^D$, and $(-) \cap G_z^E(-)$ be as in Theorem 5.3.7. Then*

$$r_z(a \otimes b) = (-1)^{aQ^D(\beta)+\mathrm{rk}([D_{\alpha,\beta}^{\nabla}])}(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes b \cap G_z^E([D^{\nabla}]))$$

gives the state-to-field correspondence

$$Y(a, z)b = (-1)^{aQ^D(\beta)+\mathrm{rk}([D_{\alpha,\beta}^{\nabla}])}\Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes b \cap G_z^E([D^{\nabla}])). \tag{5.3.12}$$

via the construction of Theorem 2.4.8.

Proof. It suffices to show that

$$[\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}](a \otimes b \cap C_z([D^{\nabla}])) = (-1)^{a''b'}\mathcal{D}(z)(a') \cdot b' \cdot [\eta \otimes \eta \circ \mathcal{D}(z) \otimes \mathrm{id}]$$
$$(a'' \otimes b'' \cap C_z([D^{\nabla}])).$$

By counitality, it suffices to show

$$\begin{aligned}\Delta([\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}](a \otimes b \cap C_z([D^\nabla]))) = (-1)^{a''b'}[\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}](a' \otimes b')\\ \otimes [\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}](a'' \otimes b'' \cap C_z([D^\nabla])).\end{aligned}$$

From the identity

$$\Delta \circ (\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}) = [\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}] \otimes [\Phi_* \circ \mathcal{D}(z) \otimes \mathrm{id}] \circ \Delta$$

it suffices to show that

$$\Delta(a \otimes b \cap C_z([D^\nabla])) = (-1)^{a''b'} a' \otimes b' \otimes (a'' \otimes b'' \cap C_z([D^\nabla]))$$

which we do by calculation:

$$\begin{aligned}\Delta(a \otimes b \cap C_z([D^\nabla])) &= (-1)^{a''b'}\Delta(a' \otimes b')C_z([D^\nabla])(a'' \otimes b''))\\ &= (-1)^{a''b'+a^{(2)}b^{(1)}} a^{(1)} \otimes b^{(1)} \otimes a^{(2)} \otimes b^{(2)} \otimes C_z([D^\nabla])\\ &\quad (a^{(3)} \otimes b^{(3)})\\ &= (-1)^{a''b'+a^{(3)}b^{(2)}} a^{(1)} \otimes b^{(1)} \otimes a^{(2)} \otimes b^{(2)} \otimes C_z([D^\nabla])\\ &\quad (a^{(3)} \otimes b^{(3)})\\ &= (-1)^{a''b'} a' \otimes b' \otimes (a'' \otimes b'' \cap C_z([D^\nabla])).\end{aligned}$$

□

Remark 5.3.9. The author realizes that it is more than slightly silly to require a Künneth isomorphism for the sole purpose of having a coproduct Δ and then use Δ for the sole purpose of writing down a formula which is ultimately independent of Δ. We do not believe that the Künneth isomorphism hypothesis is actually necessary in order to build a graded vertex F-algebra on $\hat{E}_*(\mathcal{M})$. It should be possible to give a direct proof of the fact that the state-to-field correspondence (5.3.12) make $\hat{E}_*(\mathcal{M})$

into a graded vertex F-algebra.

However, we like bicharacters because we believe they provide a geometric explanation for the, perhaps otherwise mysterious, existence of a vertex algebra on the homology of a moduli space. Therefore, we would prefer to remove the Künneth isomorphism assumption instead by lifting the entire construction to spaces, stacks, or spectra so that the moduli space is itself a vertex algebra object, built from a generalized bicharacter construction.

In [35] Borcherds defines generalized vertex algebras called a (A, H, S)*-vertex algebras*. This notion makes sense in any symmetric monoidal category. Given a suitable moduli space $\mathcal{M}$ we believe the diagonal coproduct $\Sigma^\infty(\mathcal{M}_+) \to \Sigma^\infty(\mathcal{M}_+) \wedge \Sigma^\infty(\mathcal{M}_+)$, which always exists, can be used to build an (A, H, S)-vertex algebra structure on $\Sigma^\infty(\mathcal{M}_+)$. As lax monoidal functors F send (A, H, S)-vertex algebras to $(F(A), F(H), F(S))$-vertex algebras, the complex-oriented homology of $\Sigma^\infty(\mathcal{M}_+)$ will then be a generalized vertex algebra. We hope that the details of this will appear in forthcoming work of the author and Markus Upmeier [94]. Results along these lines have also been achieved independently in forthcoming work of Sven Meinhardt [166].

Remark 5.3.10. For the ordinary homology of the moduli stack of objects in an even Calabi–Yau category, the natural choice of $\Theta^\bullet$ does not involve forced symmetrization. The reason that we do not state a separate "even Calabi–Yau" version of Theorem 5.3.7 is that the author is not aware of an analogue of the identity

$$(-) \cap C_z^H(V^\vee) = (-) \cap (-1)^{\mathrm{rk}(V)} C_{-z}^H(V),$$

which can be shown to hold in the case of ordinary homology, for a general even complex-oriented homology theory. Interestingly, at least in the case of complex topological K-theory, there does appear to be a natural definition which was shown to the author by Dominic Joyce. The operator

$$J_z^K(V) = (z+1)^{\mathrm{rk}(V)/2}(\det(V))^{-\frac{1}{2}} \sum_{k=0}^{\mathrm{rk}(V)} (-1)^k (z+1)^k \Lambda^k[V]$$

defined on complex vector bundles whose determinants have square roots, satisfies

$$J_z^K(V^\vee) = (-1)^{\mathrm{rk}(V)} J_{\iota(z)}^K(V).$$

Note that this fits nicely with the literature on K-theoretic enumerative invariants, which require a square root of the canonical bundle of the moduli space [12], [46], [88], [89], [134], [178], and [213].

We are now in a position to build a graded vertex F-algebra on $\hat{E}_*(\mathcal{M})$. For Theorem 5.3.7 we will simply take $K(\mathcal{M}) = K^0_{\mathrm{top}}(X)$. It suffices to build a suitable K-theory class. We will be able to do this for any elliptic operator D on X. Recall that by the Atiyah–Jänich[5] theorem there is a homotopy equivalence $BU \times \mathbb{Z} \simeq \mathrm{Fred}(\mathcal{H})$ where $\mathrm{Fred}(\mathcal{H})$ is the space of Fredholm operators on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$.

Definition 5.3.11. Let $E_0, E_1 \to X$ be complex vector bundles of rank r and let $D : C^\infty(E_0) \to C^\infty(E_1)$ be an elliptic differential operator. For all $P, Q \in \mathcal{M}^U = \mathrm{Map}^0_C(X, \coprod_{n\geq 0} BU(n))$ there is a twisted elliptic operator $D^{\nabla_{P\otimes\overline{Q}}}$ (Definition 3.4.3). We can write this correspondence as a continuous map

$$\mathcal{M}^U \times \mathcal{M}^U \longrightarrow \mathrm{Fred}(\mathcal{H}) \cong BU \times \mathbb{Z}. \tag{5.3.13}$$

defined up to weak homotopy. Let $[D^\nabla]$ denote the (weak) homotopy class of (5.3.13).

[5]Actually the classical Atiyah–Jänich theorem [13, Thm. A1] states that there is a homotopy equivalence $BU \times \mathbb{Z} \simeq \mathrm{Fred}(\mathcal{H})$ upon restriction to any compact CW complex X. However this weak homotopy equivalence can be improved to genuine homotopy equivalence, for example, by the methods of Segal [196, Prop. 3.3].

Further, as $BU \times \mathbb{Z}$ is a group-like H-space there is an induced map $\overline{\mathcal{M}}^U \times \overline{\mathcal{M}}^U \longrightarrow BU \times \mathbb{Z}$ defined up to (weak) homotopy which gives a class written $[D^\nabla] \in K^0_{\mathrm{top}}(\overline{\mathcal{M}}^U \times \overline{\mathcal{M}}^U)$ by abuse of notation.

Proposition 5.3.12. *The K-theory class $[D^\nabla]$ of Definition 5.3.11 satisfies equations (5.4.4)-(5.4.7) over any finite sub-complex $\mathcal{M}$.*

Proof. By the universal property of homotopy-theoretic group completions, it suffices to prove this fact for $\mathcal{M}^U$. Let C be a finite CW complex with a continuous map $C \to \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta \times \mathcal{M}^U_\gamma$ for some $\alpha, \beta, \gamma \in K^0_{\mathrm{top}}(X)$. This map assigns to each point $c \in C$ a triple of complex vector bundles $P_c, Q_c, R_c \to X$. The composition

$$C \longrightarrow \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta \times \mathcal{M}^U_\gamma \xrightarrow{\Phi \times \mathrm{id}} \mathcal{M}^U_{\alpha+\beta} \times \mathcal{M}^U_\gamma \xrightarrow{D^\nabla} \mathrm{Fred}(\mathcal{H})$$

sends c to $D^{\nabla_{(P_c \oplus Q_c) \otimes \overline{R_c}}} = D^{\nabla_{P_c \otimes \overline{R_c}}} \oplus D^{\nabla_{Q_c \otimes \overline{R_c}}}$, which establishes (5.3.6). Equation (5.3.7) follows similarly.

For (5.3.8) and (5.3.9) consider a finite CW complex C and a continuous map $C \to BU(1) \times \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta$ for some $\alpha, \beta \in K^0_{\mathrm{top}}(X)$. This is equivalent to the data of a complex line bundle $L \to C$ and a continuously varying family of complex vector bundles $P_c, Q_c \to C$ of for $c \in C$. Then the compositions

$$C \longrightarrow BU(1) \times \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta \xrightarrow{\Psi \times \mathrm{id}} \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta$$

and

$$C \longrightarrow BU(1) \times \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta \xrightarrow{\mathrm{id} \times \Psi} \mathcal{M}^U_\alpha \times \mathcal{M}^U_\beta$$

send $c \mapsto D^{\nabla_{(L \otimes P_c) \otimes \overline{Q_c}}} = D^{\nabla_L} \otimes D^{\nabla_{P_c \otimes \overline{Q_c}}}$ and $c \mapsto D^{\nabla_{P_c \otimes \overline{L \otimes Q_c}}} = D^{\nabla_{\overline{L}}} \otimes D^{\nabla_{P_c \otimes \overline{Q_c}}}$, respectively. $\square$

In conclusion, the state-to-field correspondence

$$Y(a,z)b = (-1)^{Q^D(\alpha) + \mathrm{rk}([D^\nabla_{\alpha,\beta}])} \Phi_* \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes b \cap G^E_z([D^\nabla]))$$

endows $\hat{E}_*(\mathcal{M})$ with the structure of a graded vertex F-algebra under the assumption that $E_*(\mathcal{M}\times\mathcal{M}) \cong E_*(\mathcal{M})\otimes E_*(\mathcal{M})$. Although, again, we do not believe the Künneth isomorphism assumption is necessary.

5.4 Moduli spaces of orthogonal connections

In this section we build a graded vertex algebra on the $\mathbb{Z}_2$-coefficient ordinary homology of the moduli space of orthogonal connections on a compact manifold.

Write

$$\mathcal{N}^O := \mathrm{Map}_{C^0}(X, \coprod_{n\geq 0} BO(n)).$$

Then $\mathcal{N}^O$ has the homotopy type of the moduli space of all $O(n)$-connections on X. The union $\coprod_{n\geq 0} BO(n)$ of classifying spaces is a Γ-space and

$$\coprod_{n\geq 0} BO(n) \longrightarrow BO \times \mathbb{Z}$$

is a homotopy-theoretic group completion [195, p. 305, $R = \mathbb{R}$]. We write $\overline{\mathcal{N}}^O$ for the homotopy-theoretic group completion of $\mathcal{N}^O$. Let $\mathcal{N}$ denote either one of $\mathcal{N}^O$ or $\overline{\mathcal{N}}^O$. The natural actions $O(1) \times O(n) \to O(n)$ for all $n \geq 0$ yield an H-map

$$BO(1) \times \mathcal{N} \longrightarrow \mathcal{N}.$$

For all $n \geq 1$ one can compute

$$H^*(BO(n), \mathbb{Z}_2) \cong \mathbb{Z}_2[[w_1, w_2, \dots]] \tag{5.4.1}$$

where w_i is a generator of degree i (see Milnor–Stasheff [171, Thm. 7.1] and Steenrod–Epstein [205]). The classes w_i are called *universal Stiefel–Whitney classes.* A real rank n vector bundle $V \to X$ is classified by the homotopy class of a map $f_V : X \to BO(n)$

and the *ith Stiefel–Whitney $w_i(V)$ class of V* is defined to be $H^*(f_V, \mathbb{Z}_2)(w_i)$.

Proposition 5.4.1. *There is an isomorphism of graded Hopf algebras*

$$H_*(BO(1), \mathbb{Z}_2) \cong \mathbb{Z}_2[\mathbb{G}_a],$$

where $\mathbb{Z}_2[\mathbb{G}_a]$ denotes the formal group ring of the additive formal group law over $\mathbb{Z}_2$ with generators $D^{(k)}$ of degree k.

Proof. The isomorphism $H^*(BO(1), \mathbb{Z}_2) \cong \mathbb{Z}_2[[w_1]]$ given by (5.4.1) is understood as an isomorphism of graded rings. By unicity of antipodes, if it can be shown that (5.4.1), for $n = 1$, is a graded coalgebra isomorphism then it will also be a graded Hopf algebra isomorphism. The coproduct $\mathbb{Z}_2[[w_1]] \to \mathbb{Z}_2[[w_1]]\hat{\otimes}\mathbb{Z}_2[[w_1]]$ is given by

$$w_1 \mapsto w_1 \otimes 1 + 1 \otimes w_1.$$

The coproduct on $H^*(BO(1), \mathbb{Z}_2)$ is induced by tensor product of real line bundles. Given two real line bundles $L, M \to X$ on a CW complex X, $w_1(L \otimes M) = w_1(L) + w_1(M)$. This implies that $H^*(BO(1), \mathbb{Z}_2) \cong \mathbb{Z}_2[[w_1]]$ is a graded coalgebra isomorphism. Taking restricted linear duals yields the desired isomorphism $H_*(BO(1), \mathbb{Z}_2) \cong \mathbb{Z}_2[\mathbb{G}_a]$ of graded Hopf algebras over $\mathbb{Z}_2$. $\square$

By Corollary 3.2.7, Proposition 5.4.1 implies that given an H-space $\mathcal{Y}$ equipped with a $BO(1)$-action $\rho : BO(1) \times \mathcal{Y} \to \mathcal{Y}$ which is also an H-map there is a shift operator

$$\mathcal{D}(w_1) : H_*(\mathcal{Y}, \mathbb{Z}_2) \longrightarrow H_*(\mathcal{Y}, \mathbb{Z}_2)[[w_1]].$$

As in Section 5.3, we can build a holomorphic bicharacter

$$r_{w_1}(a \otimes b) = (\eta \otimes \eta) \circ \Phi_* \circ (\mathcal{D}(w_1) \otimes \mathrm{id})(a \otimes b)$$

where $\Phi : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ denotes the H-map on $\mathcal{Y}$. We would like to modify the above by introducing singularities via a natural transformation $(-) \cap W_u(-) : KO^0(-) \Longrightarrow \mathcal{F}(H_*(-, \mathbb{Z}_2))$.

Theorem 5.4.2. *There exists a unique natural transformation*

$$(-) \cap W_u(-) : KO^0(-) \Longrightarrow \mathcal{F}(H_*(-, \mathbb{Z}_2))$$

such that for a CW complex $\mathcal{Y}$

1. *given $S, T \in KO^0(\mathcal{Y})$*

$$(-) \cap W_u(S + T) = (-) \cap (W_u(S) \cup W_u(T)),$$

   and

2. *given a real line bundle $L \to \mathcal{Y}$, $(-) \cap W_u([L])$ is given by cap product with $u + w_1(L)$.*

Moreover, for $\Theta \in KO^0(\mathcal{Y})$ the map $(-) \cap W_u(\Theta) : H_*(\mathcal{Y}, \mathbb{Z}_2) \to H_*(\mathcal{Y}, \mathbb{Z}_2)((u))$ is graded of degree $-\mathrm{rk}(\Theta)$ where the degree of u is taken to be -1.

Proof. The proof of this theorem is very similar to the proof of Theorem 5.3.3.

First suppose $\mathcal{X}$ is a finite CW complex. If L is a KO-theory class represented by a single real line bundle then $(-) \cap W_u(L)$ is uniquely determined by condition 2. Suppose $V \to \mathcal{Y}$ is a real rank n vector bundle. We may form a flag bundle via the pullback

$$\begin{array}{ccc} F_{\mathcal{Y}} & \longrightarrow & BO(1) \times \cdots \times BO(1) \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & BO(n) \end{array}$$

such that the columns are injections in $\mathbb{Z}_2$-coefficient cohomology (see May [161, Thm. 1.1]). This gives a splitting principle for Stiefel–Whitney classes. Therefore, if

$L_1, \dots, L_n$ denote the roots of V

$$(-) \cap W_u(V) = (-) \cap (u + w_1(L_1)) \dots (u + w_1(L_n)).$$

Note that this map reduces degree by n. To define $(-) \cap W_u(-)$ uniquely when $\mathcal{Y}$ is an infinite CW complex, we take a direct limit over its finite skeleta as in the proof of Theorem 5.3.3 $\square$

The operation $(-) \cap W_u(-) : KO^0(-) \Longrightarrow \mathcal{F}(H_*(-, \mathbb{Z}_2))$ enjoys the following properties.

Proposition 5.4.3. *Let $\mathcal{Y}$ be a CW complex and let $L \to \mathcal{Y}$ be a real line bundle. Then for all $\Theta \in KO^0(\mathcal{Y})$*

$$(-) \cap W_u(\Theta^\vee) = (-) \cap W_u(\Theta). \tag{5.4.2}$$

and

$$(-) \cap W_u([L] \cdot \Theta) = i_{u, w_1(L)}(-) \cap W_{u + w_1(L)}(\Theta). \tag{5.4.3}$$

Proof. As in Proposition 5.3.6, (5.4.2) and (5.4.3) are proved first on finite CW complexes using the splitting principle for Stiefel–Whitney classes and then on infinite CW complexes by passing through a direct limit.

Let $\mathcal{Y}$ be a finite CW complex. For (5.4.2) we first consider a real line bundle $L \to \mathcal{Y}$. Then

$$(-) \cap W_u([L]^\vee) = (-) \cap (u + w_1(L^\vee)) = (-) \cap (u - w_1(L)) = (-) \cap (u + w_1(L)).$$

Given a rank n real vector bundle $V \to \mathcal{Y}$ with roots $L_1, \dots, L_n$

$$(-) \cap W_u([V]^\vee) = (-) \cap (W_u([L_1]^\vee) \cup \dots \cup W_u([L_n]^\vee))$$

$$
\begin{aligned}
&= (-) \cap (-u + w_1(L_1)) \cup \cdots \cup (-u + w_1(L_n)) \\
&= (-) \cap W_u([V]).
\end{aligned}
$$

Then given $\Theta \in KO^0(\mathcal{Y})$ we can write $\Theta = [S] - [T]$ for real vector bundles $S, T \to \mathcal{Y}$ of respective ranks n and m, respectively. As in the proof of Theorem 5.3.3 we can build a multiplicative inverse for $W_u([T])$ in $H^*(\mathcal{Y}, \mathbb{Z}_2)((u))$ by choosing a large rank real vector bundle $U \to \mathcal{Y}$ with $T \oplus U = \mathbb{R}^N \times \mathcal{Y}$ for $N \gg 0$. Then

$$
W_u(\Theta^\vee) = W_u([S]^\vee) \cup W_u([T]^\vee)^{-1} = W_u([S]) \cup W_u([T])^{-1} = W_u(\Theta).
$$

If $\mathcal{Y}$ is an infinite CW complex then

$$
(-) \cap W_u(\Theta^\vee) = \varinjlim_n (-) \cap W_u(\Theta^\vee|_{\mathcal{Y}_n}) = \varinjlim_n (-) \cap W_u(\Theta|_{\mathcal{Y}_n}) = (-) \cap W_u(\Theta).
$$

For (5.4.3), let $L \to \mathcal{Y}$ be a real line bundle. Suppose again that $\mathcal{Y}$ is a finite CW complex. Given a real line bundle $M \to \mathcal{Y}$

$$
\begin{aligned}
(-) \cap W_u([L] \cdot [M]) &= (-) \cap (u + w_1(L \otimes M)) \\
&= (-) \cap (u + w_1(L) + w_1(M)) \\
&= (-) \cap W_{u + w_1(L)}(M).
\end{aligned}
$$

Given a real rank n vector bundle $V \to \mathcal{Y}$ with roots $L_1, \ldots, L_n$

$$
\begin{aligned}
(-) \cap W_u([L] \cdot [V]) &= (-) \cap \big((u + w_1(L) + w_1(L_1)) \cup \cdots \cup w_1(L) + w_1(L_n)\big) \\
&= (-) \cap W_{u + w_1(L)}([V]).
\end{aligned}
$$

Given arbitrary $\Theta \in KO^0(\mathcal{Y})$, one can write $\Theta = [S] - [T]$ for two real vector bundles

$S, T \to \mathcal{Y}$. Then

$$\begin{aligned}(-) \cap W_u([L] \cdot \Theta) &= (-) \cap W_u([L] \cdot [S]) \cup W_u([L] \cdot [T])^{-1} \\ &= i_{u,w_1(L)} W_{u+w_1(L)}([S]) \cup W_{u+w_1(L)}([T])^{-1} \\ &= i_{u,w_1(L)} W_{u+w_1(L)}(\Theta).\end{aligned}$$

Lastly, if $\mathcal{Y}$ is an infinite CW complex and $\Theta \in KO^0(\mathcal{Y})$ is arbitrary

$$\begin{aligned}(-) \cap W_u([L] \cdot \Theta) &= \varinjlim_n (-) \cap W_u([L] \cdot \Theta|_{\mathcal{Y}^n}) \\ &= i_{u,w_1(L)} \varinjlim_n (-) \cap W_{u+w_1(L)}(\Theta|_{\mathcal{Y}^n}) \\ &= i_{u,w_1(L)} (-) \cap W_{u+w_1(L)}(\Theta).\end{aligned}$$

□

Theorem 5.4.4. *Let $\mathcal{Y}$ be an H-space with H-map $\Phi : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$. Let $\Psi : BO(1) \times \mathcal{Y} \to \mathcal{Y}$ be a $BO(1)$-action on $\mathcal{Y}$ which is also a morphism of H-spaces. Suppose that we are given the following data*

1. *a quotient $K(\mathcal{Y})$ of $(\pi_0(\mathcal{Y}))^+$, and*
2. *a KO-theory class $[D^\nabla] \in KO^0(\mathcal{Y} \times \mathcal{Y})$*

such that for all $\alpha, \beta, \gamma \in K(\mathcal{Y})$ there are equalities in KO-theory

$$(\Phi_{\alpha,\beta} \times \mathrm{id})^*([D^\nabla_{\alpha+\beta,\gamma}]) = \pi^*_{1,2}([D^\nabla_{\alpha,\gamma}]) + \pi^*_{2,3}([D^\nabla_{\beta,\gamma}]), \tag{5.4.4}$$

$$(\mathrm{id} \times \Phi_{\beta,\gamma})^*([D^\nabla_{\alpha,\beta+\gamma}]) = \pi^*_{1,2}([D^\nabla_{\alpha,\beta}]) + \pi^*_{1,3}([D^\nabla_{\alpha,\gamma}]), \tag{5.4.5}$$

$$(\Psi_\alpha \times \mathrm{id})^*([D^\nabla_{\alpha,\beta}]) = \pi^*_1([\mathcal{L}]) \cdot \pi^*_{2,3}([D^\nabla_{\alpha,\beta}]), \tag{5.4.6}$$

$$(\pi_2, \Psi_\beta \circ \pi^*_{1,3})([D^\nabla_{\alpha,\beta}]) = \pi^*_1([\mathcal{L}^\vee]) \cdot \pi^*_{2,3}([D^\nabla_{\alpha,\beta}]) \tag{5.4.7}$$

over any finite sub-complex $C \subset \mathcal{Y}$, where $\mathcal{L} \to BO(1)$ denotes the universal real line bundle. Let $\chi^D : K(\mathcal{M}) \times K(\mathcal{M}) \to \mathbb{Z}$ be the symmetric $\mathbb{Z}$-bilienar form given by

$\chi^D(\alpha,\beta) := \mathrm{rk}([D^\nabla_{\alpha,\beta}])$, let $Q^D(\alpha) := \chi^D(\alpha,\alpha)$ be the associated quadratic form, and let $\hat{H}_*(\mathcal{Y},\mathbb{Z}_2)$ be the Q^D-shift of $H_*(\mathcal{Y},\mathbb{Z}_2)$. Then the $\mathbb{Z}_2$-linear map defined by

$$r_u(a\otimes b) := (\eta\otimes\eta)\circ(\mathcal{D}(z)\otimes\mathrm{id})(a\otimes b\cap W_u([D^\nabla])) \tag{5.4.8}$$

is a graded even symmetric bicharacter on $\hat{H}_*(\mathcal{Y},\mathbb{Z}_2)$ with respect to the shift operator $\mathcal{D}(u) : \hat{H}_*(\mathcal{Y},\mathbb{Z}_2) \to \hat{H}_*(\mathcal{Y},\mathbb{Z}_2)[[u]]$ induced by the action $BO(1)\times\mathcal{Y}\to\mathcal{Y}$. In particular, this makes $\hat{H}_*(\mathcal{Y},\mathbb{Z}_2)$ into a graded vertex algebra with state-to-field correspondence given by

$$Y(a,u)b = \Phi_*\circ(\mathcal{D}(u)\otimes\mathrm{id})(a\otimes b\cap W_u([D^\nabla])).$$

Proof. The proof of this theorem is highly similar to the proof of Theorem 5.3.7. However, we nonetheless include it for the sake of completeness.

Note that (5.4.4)-(5.4.7) give

$$\begin{aligned}
&(-)\cap(\Phi\times\mathrm{id})^*(W_u([D^\nabla])) = ((-)\cap\pi_{1,3}^*(W_u([D^\nabla])))\cap\pi_{2,3}^*(W_u([D^\nabla])),\\
&(-)\cap(\mathrm{id}\times\Phi)^*(W_u([D^\nabla])) = ((-)\cap\pi_{1,2}^*(W_u([D^\nabla])))\cap\pi_{1,3}^*(W_u([D^\nabla])),\\
&(-)\cap(\Psi\times\mathrm{id})^*(W_u([D^\nabla])) = (-)\cap i_{u,v}W_{u+v}([D^\nabla]),\ \text{and}\\
&(-)\cap(\mathrm{id}\times\Psi)^*(W_u([D^\nabla])) = (-)\cap i_{u,v}W_{u-v}([D^\nabla]).
\end{aligned}$$

First, we prove that

$$r_u(\mathcal{D}(v)(a)\otimes b) = i_{u,v}r_{u+v}(a\otimes b).$$

We calculate

$$\begin{aligned} r_u(\mathcal{D}(v)(a) \otimes b) &= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})((\mathcal{D}(v) \otimes \mathrm{id})(a \otimes b) \cap W_u([D^\nabla])) \\ &= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id}) \circ (\mathcal{D}(v) \otimes \mathrm{id})((a \otimes b) \cap (\Psi \times \mathrm{id})^* W_u([D^\nabla])) \\ &= (\eta \otimes \eta) \circ (\mathcal{D}(u+v) \otimes \mathrm{id})((a \otimes b) \cap (\Psi \times \mathrm{id})^* i_{u,v} W_{u+v}([D^\nabla])) \\ &= i_{u,v} r_{u+v}(a \otimes b). \end{aligned}$$

For

$$r_u(a \cdot b \otimes c) = r_u(a \otimes c') r_u(b \otimes c'') \tag{5.4.9}$$

we have

$$\begin{aligned} (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})(a \cdot b \otimes c \cap W_u([D^\nabla])) = (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id}) \circ (\Phi \times \mathrm{id})_* \\ ((a \otimes b \otimes c \cap \pi_{1,3}^*(W_u([D^\nabla])))\cap \\ \pi_{2,3}^*(W_u([D^\nabla]))) \end{aligned}$$

and

$$\begin{aligned} &(\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes c' \cap W_u([D^\nabla])) \cdot (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})(b \otimes c'' \cap W_u([D^\nabla])) \\ &= (\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id} \otimes \mathcal{D}(u) \otimes \mathrm{id})(a \otimes c' \otimes b \otimes c'' \cap W_u([D^\nabla])) \otimes W_u([D^\nabla])) \\ &= (\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id} \otimes \mathcal{D}(u) \otimes \mathrm{id})((\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(a \otimes b \otimes \Delta(c)) \\ &\qquad \cap W_u([D^\nabla])) \otimes W_u([D^\nabla])) \\ &= (\eta \otimes \eta \otimes \eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id} \otimes \mathcal{D}(u) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \Delta) \circ (\mathrm{id} \otimes \sigma_* \otimes \mathrm{id}) \\ &\qquad (a \otimes b \otimes c \cap \pi_{1,3}^*(W_u([D^\nabla]))) \cap \pi_{2,3}^*(W_u([D^\nabla]))). \end{aligned}$$

The above follows from the same bialgebraic identity used to establish bimultiplicativity in the proof of Theorem 5.3.7. Next, we show that

$$r_u(a \otimes b) = r_u(b \otimes a).$$

To this end first observe that Proposition 5.4.3 implies

$$W_u(\sigma^*[D^\nabla_{\alpha,\beta}]) = W_u([D^\nabla_{\alpha,\beta}]^\vee) = W_u([D^\nabla_{\alpha,\beta}]).$$

From this, we can compute

$$\begin{aligned}
r_u(b \otimes a) &= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})(\sigma_*(a \otimes b) \cap W_u([D^\nabla])) \\
&= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id})(\sigma_*(a \otimes b) \cap \sigma^*\sigma^* W_u([D^\nabla])) \\
&= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id}) \circ \sigma_*(a \otimes b \cap W_u([D^\nabla])) \\
&= (\eta \otimes \eta) \circ (\mathrm{id} \otimes \mathcal{D}(u))(a \otimes b \cap W_u([D^\nabla])) \\
&= (\eta \otimes \eta) \circ (\mathrm{id} \otimes \mathcal{D}(u))(a \otimes b \cap (\Psi \times \Psi)^* W_u([D^\nabla])) \\
&= (\eta \otimes \eta)(a \otimes \mathcal{D}(u)(b) \cap (\Psi \times \mathrm{id})^* W_u([D^\nabla])) \\
&= (\eta \otimes \eta) \circ (\mathcal{D}(u) \otimes \mathrm{id}) \circ (\mathcal{D}(u) \otimes \mathrm{id})(a \otimes \mathcal{D}(u)(b) \cap (\Psi \times \mathrm{id})^* W_u([D^\nabla])) \\
&= r_u(\mathcal{D}(u)(a) \otimes \mathcal{D}(u)(b)) \\
&= r_u(\mathcal{D}(v)(a) \otimes \mathcal{D}(v)(b))|_{v=u} \\
&= i_{u,v} r_{u+v}(a \otimes \mathcal{D}(v)(b))|_{v=u} \\
&= i_{u,v} r_{(u+v)+v}(a \otimes b)|_{v=u} \\
&= r_u(a \otimes b).
\end{aligned}$$

The identity

$$r_u(a \otimes 1) = \eta(a)$$

follows from the computation

$$r_u(a \otimes 1) = (\eta \otimes \eta) \circ (\mathcal{D}(z) \otimes \mathrm{id})(a \otimes 1 \cap W_u([D^\nabla_{\alpha,0}])) = \eta(\mathcal{D}(u)(a)) = \eta(a).$$

Evenness of r follows from that fact that $H\mathbb{Z}_2$ is an even spectrum.

That the bicharacter construction applied to r yields the state-to-field correspondence

$$Y(a,u)b = \Phi_* \circ (\mathcal{D}(u) \otimes \mathrm{id})(a \otimes b \cap W_u([D^\nabla]))$$

follows from the same argument used to prove Proposition 5.3.8. $\square$

We would like to use the above theorem to build a graded vertex algebra on $\hat{H}_*(\mathcal{N})$. We can take $K(\mathcal{N}) = (\pi_0(\mathcal{N}))^+$. To construct a suitable KO-theory class we will consider a real elliptic operator $D : C^\infty(E_0) \to C^\infty(E_1)$ where $E_0, E_1 \to X$ are rank r real vector bundles on X. In similar spirit to the case of complex topological K-theory there is a homotopy equivalence

$$BO \times \mathbb{Z} \simeq \mathrm{Fred}(\mathcal{H}_\mathbb{R}),$$

where $\mathrm{Fred}(\mathcal{H}_\mathbb{R})$ is the space of all Fredholm operators on an infinite-dimensional real separable Hilbert space (see [196, Prop. 3.3]).

Definition 5.4.5. Given $(P,Q) \in \mathcal{N}^O \times \mathcal{N}^O$ we can form the twisted real elliptic operator $D^{\nabla_{P\oplus Q^\vee}}$ as in Definition 3.4.3. This correspondence can be expressed as a continuous map

$$\mathcal{N}^O \times \mathcal{N}^O \longrightarrow \mathrm{Fred}(\mathcal{H}_\mathbb{R}) \simeq BO \times \mathbb{Z},$$

defined up to weak homotopy, whose homotopy class we call $[D^\nabla] \in KO^0(\mathcal{N}^O \times \mathcal{N}^O)$. As $BO \times \mathbb{Z}$ is group-like homotopy theoretic group completion extends $[D^\nabla]$ to a class in $KO^0(\overline{\mathcal{N}}^O \times \overline{\mathcal{N}}^O)$ with well-defined Stiefel–Whitney classes.

Proposition 5.4.6. *The KO-theory class $[D^\nabla]$ of Definition 5.4.5 satisfies equations (5.3.7)-(5.4.7) over any finite sub-complex of $\mathcal{N}$.*

Proof. The proof of this Proposition is quite similar to that of Proposition 5.3.12. Nonetheless, it is included for the sake of completeness.

By the universal property of homotopy-theoretic group completions, it suffices to prove this fact for $\mathcal{N}^O$. Let C be a finite CW complex with a continuous map

$C \to \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta \times \mathcal{N}^O_\gamma$ for some $\alpha, \beta, \gamma \in KO^0(X)$. This map assigns to each point $c \in C$ a triple of real vector bundles $P_c, Q_c, R_c \to X$. The composition

$$C \longrightarrow \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta \times \mathcal{N^O}_\gamma \xrightarrow{\Phi \times \mathrm{id}} \mathcal{N}^O_{\alpha+\beta} \times \mathcal{N}^O_\gamma \xrightarrow{D^\nabla} \mathrm{Fred}(\mathcal{H}_\mathbb{R})$$

sends c to $D^{\nabla_{(P_c \oplus Q_c) \otimes R_c^\vee}} = D^{\nabla_{P_c \otimes Q_c^\vee}} \oplus D^{\nabla_{Q_c \otimes R_c^\vee}}$.

For pullback against Ψ, consider a continuous map $C \to BO(1) \times \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta$ for some $\alpha, \beta \in KO^0(X)$. This is equivalent to the data of a real line bundle $L \to C$ and a continuously varying family of real vector bundles P_c, Q_c for $c \in C$. Then the compositions

$$C \longrightarrow BO(1) \times \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta \xrightarrow{\Psi \times \mathrm{id}} \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta$$

and

$$C \longrightarrow BO(1) \times \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta \xrightarrow{\mathrm{id} \times \Psi} \mathcal{N}^O_\alpha \times \mathcal{N}^O_\beta$$

send $c \mapsto D^{\nabla_{(L \otimes P_c) \otimes Q_c^\vee}} = D^{\nabla_L} \otimes D^{\nabla_{P_c \otimes Q_c^\vee}}$ and $c \mapsto D^{\nabla_{P_c \otimes (L \otimes Q_c^\vee)}} = D^{\nabla_{L^\vee}} \otimes D^{\nabla_{P_c \otimes Q_c^\vee}}$, respectively. $\square$

In sum, the state-to-field correspondence

$$Y(a, u)b = \Phi_* \circ (\mathcal{D}(u) \otimes \mathrm{id})(a \otimes b \cap W_u([D^\nabla]))$$

endows $\hat{H}_*(\mathcal{N}, \mathbb{Z}_2)$ with the structure of a graded vertex algebra.

To conclude, we will sketch an idea to bring together Theorem 5.3.7 and Theorem 5.4.4 using Araki's *Real-oriented spectra* (see Araki [9] [10] [11] and Hu–Kriz [104]). We appreciate that details are missing and we hope that full details of this construction may appear in forthcoming work of the author and Markus Upmeier [94].

Recall that a *Real space* is just a $\mathbb{Z}_2$-equivariant space. Let $\mathbb{S}^1$ denote the subspace $S^1 \subset \mathbb{C}^*$ with involution given by complex conjugation. Then $B\mathbb{S}^1$ is the Real space classifying complex lines in $\mathbb{C}^\infty$ with involution given by complex conjugation. Given a $\mathbb{Z}_2$-equivariant commutative ring spectrum E there is a notion of a

Real orientation on E (see [104, Def. 2.2]). Examples of Real-oriented cohomology theories are Atiyah's KR-theory [14], Landweber's Real cobordism [136] [135], Real Peterson–Brown cohomology [9], and Real Morava K-theory [104, §. 3].

Theorem 5.4.7 (see Araki [11])**.** *Let E be a Real-oriented spectrum. Then there is a canonical isomorphism*

$$E^*(B\mathbb{S}^1) \cong E^*(\mathrm{pt})[[v]]$$

such that the image of v under the Hopf algebra homomorphism induced by the H-product $B\mathbb{S}^1 \times B\mathbb{S}^1 \to B\mathbb{S}^1$ is a formal group law F. We call this the formal group law associated to E. In particular, $E_*(B\mathbb{S}^1)$ is the formal group ring of F.

Corollary 5.4.8. *Let $\mathcal{Z}$ be a Real H-space. Suppose there is a $B\mathbb{S}^1$-action $\rho : B\mathbb{S}^1 \times \mathcal{Z} \to \mathcal{Z}$ which is also a Real H-map. Let E be a Real-oriented spectrum with associated formal group law F. Then there is an F-shift operator*

$$E_*(\mathcal{Z}) \longrightarrow E_*(\mathcal{Z})((v)).$$

The above guarantees the existence of a holomorphic vertex F-algebra structure on the Real-oriented homology of a Real H-space with a suitable $B\mathbb{S}^1$-action. To introduce singularities, we will probably want an operator depending on a choice of KR-theory class $\Theta \in KR^0(\mathcal{Z})$. Note that forgetting involutions gives a natural transformation $KR^0(-) \implies K^0_{\mathrm{top}}(-)$ and taking $\mathbb{Z}_2$-fixed points gives a natural transformation $KR^0(-) \implies KO^0(-)$. Given a Real H-map $B\mathbb{S}^1 \times \mathcal{Z} \to \mathcal{Z}$ taking fixed points gives an H-map $BO(1) \times \mathcal{Z}_{\mathbb{R}} \to \mathcal{Z}_{\mathbb{R}}$ where $\mathcal{Z}_{\mathbb{R}} \subset \mathbb{Z}$ denotes the $\mathbb{Z}_2$-fixed points of $\mathcal{Z}$ and forgetting $\mathbb{Z}_2$-actions gives an H-map $BU(1) \times \mathcal{Z} \to \mathcal{Z}$.

Notation 5.4.9. Let X be a real algebraic variety. Then the underlying analytic space $X(\mathbb{C})^{\mathrm{an}}$ of the set of complex points X is a Real space via the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. This Real space is denoted $X_{\mathbb{R}}(\mathbb{C})$.

Example 5.4.10. Let X be a smooth real projective variety and let $\mathrm{Gr}(\mathbb{R}^\infty)$ denote the $\mathbb{R}$-indscheme parameterizing all real subspaces in $\mathbb{R}^\infty$. Friedlander–Walker prove that $\mathrm{Map}_{\mathbf{IndSch}_\mathbb{R}}(X, \mathrm{Gr}(\mathbb{R}^\infty))_\mathbb{R}(\mathbb{C})$ is a Real E_∞-space. The homotopy-theoretic group completion $\Omega^\infty K\mathbb{R}^{\mathrm{sst}}(X)$ of $\mathrm{Map}_{\mathbf{IndSch}_\mathbb{R}}(X, \mathrm{Gr}(\mathbb{R}^\infty))_\mathbb{R}(\mathbb{C})$ is called the *Real semi-topological K-theory space* of X [81, Def. 1.3].

The infinite complex Grassmannian $\mathrm{Gr}(\mathbb{C}^\infty)$ has a Real structure induced by complex conjugation–this real space coincides with $\mathrm{Gr}(\mathbb{R}^\infty)_\mathbb{R}(\mathbb{C})$. The mapping space $\mathrm{Map}_{C^0}(X_\mathbb{R}(\mathbb{C}), \mathrm{Gr}(\mathbb{R}^\infty)_\mathbb{R}(\mathbb{C}))$ is a Real E_∞-space. The homotopy-theoretic group completion of $\mathrm{Map}_{C^0}(X_\mathbb{R}(\mathbb{C}), \mathrm{Gr}(\mathbb{R}^\infty)_\mathbb{R}(\mathbb{C}))$ is homotopy equivalent to the KR-theory space $\Omega^\infty KR(X_\mathbb{R}(\mathbb{C}))$ of $X_\mathbb{R}(\mathbb{C})$. There is a K-theory comparison map

$$\Omega^\infty K\mathbb{R}^{sst}(X) \longrightarrow \Omega^\infty K^{\mathrm{sst}}(X_\mathbb{R}(\mathbb{C}))$$

which is a weak homotopy equivalence if X is a curve or a flag variety [81, Prop. 6,1 & Prop. 6.2].

The natural action of $BU(1)$ on $\mathrm{Gr}(\mathbb{C}^\infty)$ respects complex conjugation so that given a Real manifold X there is a (weak) Real H-map

$$B\mathbb{S}^1 \times \Omega^\infty KR(X) \to \Omega^\infty KR(X)$$

which supplies a shift operator on the Real-oriented homology of $\Omega^\infty KR(X)$.

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