C^{∞}-Algebraic Geometry

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Abstract

C^∞-rings are $\mathbb{R}$-algebras with additional structure: instead of just addition, multiplication, and scalar multiplication, they have all smooth functions as operations. They have been used in synthetic differential geometry and in derived differential geometry. We study the category of C^∞-rings – ideals, quotients, localizations, and small colimits – as well as the important subcategories of finitely generated and fair C^∞-rings. We define strongly fair ideals and C^∞-rings. We explain the basics of algebraic geometry for C^∞-rings, mirroring the basics of ordinary algebraic geometry, as elaborated in [15]. We also discuss open and closed embeddings of C^∞-schemes. We then repeat much of this procedure with C^∞-rings with corners, which are generalized from C^∞-rings in the same way that manifolds with corners are from manifolds.

Contents

Introduction

Smooth manifolds, and the smooth functions from a manifold M to $\mathbb{R}$, denoted $C^\infty(M)$, are of fundamental importance in differential geometry. Functions $M \to \mathbb{R}$ can be added and multiplied point-wise, and in this way $C^\infty(M)$ is not just a set, it is an $\mathbb{R}$-algebra. The starting point of C^∞-algebraic geometry is the observation that $C^\infty(M)$ is not just an $\mathbb{R}$-algebra: in fact, any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ defines an n-'operation' C_f on $C^\infty(M)$:

$$C_f(g_1, \ldots, g_n) = f(g_1(-), \ldots, g_n(-)).$$

Sets which have all such operations (along with some other axioms) are called C^∞-rings, and of these $C^\infty(M)$ is only the most important example. The definition of C^∞-rings suggests C^∞-schemes, much as in ordinary algebraic geometry, and *C^∞-algebraic geometry* will be the focus of this dissertation.

C^∞-rings were first an object of interest in synthetic differential geometry. They were developed first by Lawvere, then by Dubuc [1], followed by Moerdijk and Reyes [11], and others. As a manifold M is characterized by $C^\infty(M)$, the category of manifolds can be embedded into the category of C^∞-rings, and thus into the category of C^∞-schemes. The category of C^∞-schemes carries certain advantages over the category of manifolds. The category of manifolds is not cartesian closed (the space of smooth functions between two manifolds is not a manifold), lacks pullbacks, and there is no language for 'infinitesimals', all of which are important in synthetic differential geometry. The relevance of C^∞-algebraic geometry to synthetic differential geometry is that the category of C^∞-schemes has finite inverse limits and infinitesimals (and the question of spaces of smooth functions can also be addressed using slightly more structure). González and Salas [4] have developed the basic tools one would expect from considering ordinary algebraic geometry and the theory of analytic spaces, but only consider special types of C^∞-schemes which they call differentiable spaces. They call the special type of C^∞-rings they consider differentiable algebras.

More recently, C^∞-rings have become objects of interest in derived differential geometry. With the motivation of understanding the geometric structure of certain moduli spaces arising in symplectic geometry, Joyce has defined d-manifolds and d-orbifolds [7]. C^∞-rings and C^∞-schemes are used in this definition. Joyce has written a book on algebraic geometry over C^∞-rings [5], addressing some of the same questions as González and Salas, especially with regard to modules and quasicoherent sheaves, but in a more general context.

The work of Joyce [5], Moerdijk and Reyes [11], and González and Salas [4] are the primary sources for this dissertation. Joyce's fair C^∞-rings and locally fair C^∞-schemes are the main objects of consideration. In the first chapter, after a quick review of smooth manifolds, and key facts about smooth functions, we focus on the category of C^∞-rings. Most of this originates from Moerdijk and Reyes' book [11]. We show that C^∞-rings are well-behaved with respect to small colimits, quotients and localization. There are two types of localization for C^∞-ring – localization as an $\mathbb{R}$-algebra, and localization as a C^∞-ring – and we will show that C^∞-localization can be expressed using localization of the underlying $\mathbb{R}$-algebra. We also discuss an important subcategory of C^∞-rings, fair C^∞-rings, which are characterized by a sort of 'sheafiness'. The idea behind fairness originated in Dubuc's article [1], and was developed under the name of germ-determinedness by Moerdijk and Reyes [11]. Joyce uses the term fair for the finitely generated case. Motivated by the localization theorem for the differentiable algebras of González and Salas [4], we define strongly fair C^∞-rings as those for which, loosely speaking, the localization theorem is true.

The second chapter defines C^∞-schemes. There are two ways of defining the Spec functor from C^∞-rings to C^∞-schemes. Joyce, González and Salas use the real (or archimedean) spectrum. As the necessity of the concept of fairness or germ-determined-ness suggests, unlike for ordinary affine schemes, the elements of a C^∞-ring are not even determined by their value on *neighborhoods* of points of the real spectrum. Joyce uses sheafification to address this issue, and the importance of the localization theorem for differentiable spaces is that in their special case, everything *is* germ-determined. Moerdijk and Reyes [10, 11], however, use a larger set to define their spectrum. We follow Joyce's approach. The issue of sheafification can usually be avoided because we mainly deal with questions that can be reduced to the level of stalks. Lastly, we discuss modules over C^∞-schemes, including cotangent sheaves over C^∞-schemes, again following Joyce [5].

The third chapter works to generalize the definitions and results on embeddings of differentiable spaces to C^∞-schemes, proving the embedding theorem. In the fourth chapter, we turn our attention to C^∞-rings with corners. C^∞-rings with corners have the same relationship to manifolds with corners as C^∞-rings to manifolds. First, we define manifolds with corners and their boundary, following Joyce's article [6] (except in the definition of smooth maps). Gillam and Molcho [3] have used log geometry and the work of González and Salas on differentiable spaces to study manifolds with corners. While we do not follow this approach, we briefly discuss their work on monoids and log geometry in [3], as they will provide a helpful context for C^∞-rings with corners. We introduce C^∞-rings with corners, and study their quotients, small colimits, and localizations, and then define and briefly discuss C^∞-schemes with corners. This last section is new, as C^∞-rings with corners have not been defined previously in the literature.

1. C^∞-rings

The basic building block of algebraic geometry is a ring and its spectrum. The topology on the spectrum is the Zariski topology. Open subschemes correspond to certain localizations, and closed subschemes to certain quotients. The same will be true of C^∞-algebraic geometry, and thus our focus in this chapter will be understanding what it means to quotient and localize in C^∞-rings, and how this relates to $C^\infty(V), C^\infty(U)$ for open and closed sets V, U respectively. First, we state some facts about smooth manifolds that will be very useful when applied to C^∞-rings. In the second section, we give the formal definition of C^∞-rings, and then introduce quotients. In the third section, we work towards understanding localization, requiring us to considering smooth functions on closed and open subsets, small colimits of C^∞-rings, and a version of Hilbert's Nullstallensatz. We then prove that $C^\infty(U) = C^\infty(\mathbb{R}^n)\{f\}^{-1}$, where f is the characteristic function of an open set $U \subset \mathbb{R}^n$. Finally, following Joyce [5], we discuss fair C^∞-rings and modules of C^∞-rings.

1.1 Smooth manifolds

Algebraic geometry over C^∞-rings will in some senses prove nicer than ordinary algebraic geometry, and this is largely because we are working with smooth functions. In particular, we will have partitions of unity, which allow for gluing arguments. The next section recalls the definition of a smooth manifold, and states some basic results of differential geometry which will play an important role in C^∞-rings.

Definition 1.1. A pair (M, A) where M is a topological space which is separable and second countable, and A is a collection of continuous maps $\{\phi_\alpha : U_\alpha \to M | \alpha \in I\}$ for open sets $U_\alpha \subset \mathbb{R}^n$, is a *smooth n-manifold* if the following conditions hold:

1. $\phi : U_\alpha \to \phi(U_\alpha)$ is a homeomorphism, and $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$.
2. The charts (ϕ_α, U_α) are smoothly compatible. That is, for any $\alpha, \beta \in I$, with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$,
$$\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)) \to \phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta))$$
is a diffeomorphism. The above conditions make A a smooth atlas.
3. A is a maximal smooth atlas: there is no strictly larger smooth atlas which contains A.

For the first three chapters, all manifolds will be smooth manifolds, so we will simply call them manifolds. The reason we will be interested in sheaves of C^∞-rings is suggested by the structure of smooth manifolds. In fact, one can alternatively look at a manifold as a sheaf (see [4, p.1]).

Lemma 1.2. *Let M be a smooth manifold, and $U \subset M$ an open set. Then there exists a characteristic function for U; that is, there is a smooth $f : M \to \mathbb{R}$ with $f^{-1}(\mathbb{R} - \{0\}) = U$.*

Lemma 1.3. *Lee [8, p.37] Let M be a smooth manifold and $\{U_i : i \in I\}$ an open covering of M. Then there exists a partition of unity subordinate to this covering; that is, there exist $\{f_i \in C^\infty(M) | i \in I\}$, such that $f_i(M) \subset [0,1]$, and the following conditions hold:*

1. $\operatorname{supp} f_i \subset U_i$.

2. *The set $\{f_i : M \to [0,1] : i \in I\}$ is locally finite. That is, for every $p \in M$ there exists a neighborhood V of p such that only finitely many of the f_i are nonzero on V.*

3. $\sum_{i \in I} f_i = 1$, *where the sum is well defined because of the previous condition.*

If M is a manifold, and $V \subset M$ is a closed set, then we say a map $f : V \to \mathbb{R}$ is *smooth* if it can be extended to a smooth function $f : U \to \mathbb{R}$ for some open neighborhood U of V.

Lemma 1.4 (The Extension Lemma, Lee [8, Lemma 2.27])**.** *Let $V \subset M$ be a closed subset of a manifold, and $f : V \to \mathbb{R}$ a smooth function. Then for every open set U such that $V \subset U$, there is a smooth function $g : M \to \mathbb{R}$ such that $g|_V = f$ and $\operatorname{supp} g \subset U$.*

Every smooth manifold M can be embedded as a closed subspace of $\mathbb{R}^N$ for some sufficiently large N. If $M \subset \mathbb{R}^N$, then there is an open neighborhood U of M such that M is a smooth retract of U.

Lemma 1.5 (Hadamard's Lemma)**.** *Let U be an open set in $\mathbb{R}^n$,and $f : U \to \mathbb{R}$ a smooth function. Then there exist smooth functions $g_i : U \times U \to \mathbb{R}$, $i = 1, \dots, n$, such that for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in U$,*

$$f(x) - f(y) = \sum_{i=1}^{n} (x_i - y_i) g_i(x, y).$$

Proof. Suppose U is convex, and fix a $y \in U$. For $x \in U$, define $q : [0,1] \to U, q(\lambda) = f(y + \lambda(x - y))$, which is smooth, and maps to U because U is convex. Then

$$q'(\lambda) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(y + \lambda(x - y))(x_i - y_i).$$

So setting $g_i(x, y) = \int_0^1 \frac{\partial f}{\partial x_i}(y + \lambda(x - y)) \mathrm{d}\lambda$, we have

$$f(x) - f(y) = q(1) - q(0) = \int_0^1 q'(\lambda) \mathrm{d}\lambda = \sum_{i=1}^{n} g_i(x, y)(x_i - y_i).$$

The general case, where U is not necessarily convex, follows from a partition of unity argument by taking a convex open covering of U. $\square$

The Shrinking Lemma will be useful in using gluing arguments.

Lemma 1.6. *[14, p. 82] Let X be a Hausdorff paracompact space, and $\mathcal{U} = \{U_a | a \in A\}$ an open cover of X. Then there is a locally finite refinement of $\mathcal{U}$, $\{V_a | a \in A\}$ such that $\overline{V_a} \subset U_a$.*

Proof. For any $x \in U_a$, as X is normal, there is an open neighborhood $W'_{x,a}$ of x such that $\overline{W'_{x,a}} \subset U_a$. Let $\{W_b | b \in B\}$ be a locally finite refinement of the cover $\{W'_{x,a}\}$. Let $V_a = \bigcup\{W_b | \overline{W_b} \subset U_a\}$, which is a locally finite open refinement of $\{U_a\}$. Moreover, because the $\{W_b\}$ are locally finite, and X is Hausdorff,

$$\overline{V_a} = \overline{\bigcup_{\overline{W_b} \subset U_a} W_b} = \bigcup_{\overline{W_b} \subset U_a} \overline{W_b} \subset U_a.$$

So this is the open cover we need. □

1.2 Basic definitions

In this section, we mostly follow the first chapter of the book by Moerdijk and Reyes [11]. Most material is also included in the first chapters of Joyce's book [5], and sometimes we follow his presentation.

Definition 1.7. A C^∞*-ring* is a set C together with operations $C_f : C^n \to C$ for every $f \in C^\infty(\mathbb{R}^n)$ such that:

1. If $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i^{th} coordinate, then $C_{\pi_i}(c_1, \ldots, c_n) = c_i$ for any $(c_1, \ldots, c_n) \in C^n$.

2. If $f \in C^\infty(\mathbb{R}^n)$, $g_i \in C^\infty(\mathbb{R}^m), i = 1, \ldots, n$, and $h = f(g_1(-), \ldots, g_n(-)) : \mathbb{R}^m \to \mathbb{R}$, then for all $c_1, \ldots, c_m \in C$,

$$C_h(c_1, \ldots, c_m) = C_f(C_{g_1}(c_1, \ldots, c_m), \ldots, C_{g_n}(c_1, \ldots, c_m)).$$

A morphism of C^∞-rings C, D is a map $\phi : C \to D$ such that for all $f \in C^\infty(\mathbb{R}^n)$, and $c_1, \ldots, c_n \in C$,

$$\phi(C_f(c_1, \ldots, c_n)) = D_f(\phi(c_1), \ldots, \phi(c_n)).$$

A C^∞-ring has an underlying $\mathbb{R}$-algebra structure. An *ideal* of a C^∞-ring is an ideal of the underlying $\mathbb{R}$-algebra. There is an equivalent definition of a C^∞-ring as a functor, which we call a *categorical C^∞-ring.*

Definition 1.8. Let *Euc* be the category with objects $\mathbb{R}^n, n \in \mathbb{N} = \{0, 1, \dots\}$ and morphisms smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. A *categorical C^∞-ring* is a finite product preserving functor from *Euc* to the category of sets. A morphism of categorical C^∞-rings is a natural transformation.

Lemma 1.9. *There is a one to one correspondence between C^∞-rings and categorical C^∞-rings.*

Proof. Suppose C is a C^∞-ring. Define a categorical C^∞-ring $F : \mathrm{Euc} \to \mathrm{Sets}$ to be the finite product preserving functor determined by $F(\mathbb{R}) = C$, and given $f : \mathbb{R}^n \to \mathbb{R}$, $F(f) = C_f : C^n \to C$. If we are given a categorical C^∞-ring F, then set $C = F(\mathbb{R})$, and $C_f = F(f)$ for $f : \mathbb{R}^n \to \mathbb{R}$. □

The following proposition shows that even though ideals of C^∞-rings depend only on the $\mathbb{R}$-algebra structure, we can quotient by them to obtain a C^∞-ring.

Proposition 1.10. *[5, p.8] Let C be a C^∞-ring, and $I \subset C$ an ideal of C. Then there is a unique C^∞-ring structure on C/I such that the projection $\pi : C \to C/I$ is a C^∞-ring morphism.*

Proof. The C^∞-ring structure on C/I is determined by the condition that the projection is a C^∞-ring morphism, as then

$$(C/I)_f(g_1 + I, \dots, g_n + I) = (C/I)_f(\pi(g_1), \dots, \pi(g_n)) = \pi(C_f(g_1, \dots, g_n)).$$

So it suffices to show that given any $f \in C^\infty(\mathbb{R}^n)$, $g_i, h_i \in C, i = 1, \dots, n$ with $g_i - h_i \in I$,

$$C_f(g_1, \dots, g_n) - C_f(h_1, \dots, h_n) \in I.$$

By Hadamard's Lemma, there exist smooth functions $q_i : \mathbb{R}^{2n} \to \mathbb{R}, i = 1, .., n$, such that for all $x, y \in \mathbb{R}^n$,

$$f(x) - f(y) = \sum_{i=1}^{n} (x_i - y_i) q_i(x, y).$$

Using this equation,

$$C_f(g_1, \dots, g_n) - C_f(h_1, \dots, h_n) = \sum_{i=1}^{n} (g_i - h_i) C_{q_i}(g_1, \dots, g_n, h_1, \dots, h_n) \in I.$$

$\square$

Proposition 1.11. *[11, p.17] The C^∞-ring $C^\infty(\mathbb{R}^n)$ is the free C^∞-ring on n generators.*

Proof. $C^\infty(\mathbb{R}^n)$ is generated by $\{x_1, \dots, x_n\}$, where x_i is the i^{th} projection map. To show that it is free, we need that if A is any C^∞-ring, and $a_1, \dots, a_n \in A$ any n elements in A, there exists a unique morphism $C^\infty(\mathbb{R}^n) \to A$ with $x_i \mapsto a_i$. The morphism of C^∞-rings $\psi : C^\infty(\mathbb{R}^n) \to A, \psi(f) = A_f(a_1, \dots, a_n)$ satisfies this condition. Since ψ is a morphism of C^∞-rings, for any $f \in C^\infty(\mathbb{R}^n)$,

$$\psi(f) = \psi(C^\infty(\mathbb{R}^n)_f(x_1, \dots, x_n)) = A_f(\psi(x_1), \dots, \psi(x_n)).$$

So ψ is determined by its values on the projections, which shows uniqueness. $\square$

Example 1.12 (Ring of germs)**.** Let $p \in \mathbb{R}^n$. Let $C_p^\infty(\mathbb{R}^n)$ be the C^∞-ring of germs at p. It is the set of equivalence classes of pairs $[(g, U)]$ for U an open neighborhood of p and $g \in C^\infty(U)$, with the natural C^∞-operations. The equivalence relation is given by $(g, U) \sim (f, V)$ if there exists an open $p \in W \subset U \cap V$ such that $g|_W = f|_W$. Let $\psi : C^\infty(\mathbb{R}^n) \to C_p^\infty(\mathbb{R}^n); f \mapsto [(f, \mathbb{R}^n)]$. This map is surjective because by Lemma 1.4, if $[(g, U)] \in C_p^\infty(\mathbb{R}^n)$, we can find a smooth extension G of $g|_{\overline{V}}$ to all of $\mathbb{R}^n$, where V is an open neighborhood of p such that $\overline{V} \subset U$. Then $\psi(G) = [(g, U)]$. The kernel of ψ is the set of smooth functions on $\mathbb{R}^n$ which are zero on a neighborhood of p. Let n_p denote this ideal. Then

$$C_p^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)/n_p.$$

Let $\pi_p : C^\infty(\mathbb{R}^n) \to C_p^\infty(\mathbb{R}^n)$ denote the projection. Note that because π_p is surjective, for an ideal $J \subset C^\infty(\mathbb{R}^n)$, $\pi_p(J)$ is an ideal in $C_p^\infty(\mathbb{R}^n)$.

Definition 1.13. A C^∞-ring C is *finitely generated* if it is finitely generated as a C^∞-ring under the operations of all smooth functions. An ideal of C is *finitely* *generated* if it is finitely generated as an ideal of an $\mathbb{R}$-algebra. By the previous proposition, for any finitely generated C^∞-ring C, there exists an $n \in \mathbb{N}$ and ideal $I \subset C^\infty(\mathbb{R}^n)$ such that $C \cong C^\infty(\mathbb{R}^n)/I$. C is called *finitely presented* if this I is finitely generated.

Example 1.14 (Morphisms of finitely generated C^∞-rings, [11, p.21])**.** Let $C = C^\infty(\mathbb{R}^n)/I$ and $D = C^\infty(\mathbb{R}^m)/J$ be two finitely generated C^∞-rings. Let K be the set of smooth maps $\psi : \mathbb{R}^m \to \mathbb{R}^n$ satisfying $f \circ \psi \in J$ for all $f \in I$. Each $\psi \in K$ determines a morphism $C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$, $g + I \mapsto g \circ \psi + J$. Let $\phi : C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$ be a morphism. Then from the proof of Proposition 1.11, ϕ is determined by its values on $x_1 + I, \dots, x_n + I$. Let $y_i \in C^\infty(\mathbb{R}^m)$ be a representative of $\phi(x_i + I)$. We can define a smooth map $\psi : \mathbb{R}^m \to \mathbb{R}^n, \psi(a) = (y_1(a), \dots, y_n(a))$, and ψ is in K. Since ϕ and the projection $C^\infty(\mathbb{R}^m) \to J$ are morphisms of C^∞-rings, for any $f \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\phi(f + I) &= (C^\infty(\mathbb{R}^m)/J)_f(\phi(x_1 + I), \dots, \phi(x_n + I)) \\ &= (C^\infty(\mathbb{R}^m)/J)_f(y_1 + J, \dots, y_1 + J) = f \circ \psi + J.\end{aligned}$$

The morphism ϕ defines a $\psi \in K$ for each choice of representatives y_i.

Let $K/\sim$ be the set of these morphisms up to this equivalence, that is, $\psi_1 \sim \psi_2$ if $x_i \circ \psi_1 - x_i \circ \psi_2 \in J$. Thus we have that morphisms $C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$ are in one to one correspondence with the equivalence classes of morphisms in $K/\sim$.

1.3 Localization

Localization in C^∞-rings is more complicated than for $\mathbb{R}$-algebras. This is because to add an inverse, it does not (always) suffice to just add the fractions involving the inverse, as you would for an $\mathbb{R}$-algebra.

Definition 1.15. The C^∞-localization of a C^∞-ring C at a set S is a C^∞-ring $C\{S\}^{-1}$ and a morphism $i : C \to C\{S\}^{-1}$ such that $i(s)$ is invertible for all $s \in S$, and if $D, j : C \to D$ is another C^∞-ring and morphism such that $j(s)$ is invertible for all $s \in S$, there is a unique morphism $\phi : C\{S\}^{-1} \to D$ such that $\phi \circ i = j$.

To show that a C^∞-localization always exists, we first need to show the existence of colimits. We briefly review the definitions of inverse limits, directed colimits, colimits, and pushouts. Inverse limits are a way of gluing objects together under specific morphisms. More precisely, let X be a category, and $A_i, i \in I$ an inverse system of objects in X. That is, I is a directed poset (I has a partial order, and every finite set in I has a maximal element), and if $i, j, k \in J$, $i \leq j \leq k$, we have morphisms $f_{ij} : A_j \to A_i$, such that $f_{ii} = \mathrm{id}_{A_i}$ and $f_{ij} \circ f_{jk} = f_{ik}$. Then an *inverse limit* is an object A together with projections $\pi_i : A \to A_i$ such that $\pi_i = f_{ij} \circ \pi_j$ for all $i \leq j$. Moreover, (A, π_i) satisfies the universal property which you would expect (and which we don't need, so we don't write down).

A directed colimit is the dual of an inverse limit. That is, we are given a *direct* system of objects $A_i, i \in I$ instead: I is a directed poset, and if $i \leq j \in I$, we have morphisms in the other direction, $f_{ij} : A_i \to A_j$. Again, we require $f_{ii} = \mathrm{id}_{A_i}$, and if $i \leq j \leq k$, $f_{ij} \circ f_{jk} = f_{ik}$. A *directed colimit* is an object A together with morphisms $\phi_i : A_i \to A$ such that for all $i \leq j$, $\phi_i = \phi_j \circ f_{ij}$. Moreover, A is universal with respect to this property.

Given objects C, D, E and morphisms $\alpha : E \to C$, $\beta : E \to D$, a pushout is an object F, with morphisms $\delta : C \to F, \gamma : D \to F$, so that $\delta \circ \alpha = \gamma \circ \beta$, which is universal with respect to this property.

Directed colimits and pushouts are both special cases of *colimits*. Instead of starting with a direct system of objects, we start with a functor $F : J \to C$, where we think of J as a type of index (and its morphisms take the place of a directed partial order). A small colimit is when J is a small category. The colimit of $F : J \to C$ is an object A of C, with morphisms $\phi_X : F(X) \to C$ for all $X \in J$, such that for every $f : X \to Y$, $\phi_X = \phi_Y \circ F(f)$, which is universal with respect to this property.

The following proposition from Moerdijk and Reyes [11, p.22] gives us the construction of inverse limits and directed colimits, using the underlying sets of categorical C^∞-rings. We have to restrict ourselves to directed colimits to use underlying sets, because we want the resulting categorical C^∞-ring to be finite product preserving, and hence we need that directed colimits commute with finite limits.

Proposition 1.16. *Inverse limits and directed colimits of C^∞-rings exist, and can be constructed from the inverse limit and directed colimit of the underlying sets.*

That is, let $C_i, i \in I$ be an inverse system of categorical C^∞-rings . Then the functor

$$C : Euc \to Sets, C(\mathbb{R}^n) = \varprojlim_i (C_i(\mathbb{R}^n))$$

is a categorical C^∞-ring, and it is the inverse limit of the diagram. If $C_i, i \in I$ is a direct system, then $D : Euc \to Sets, D(\mathbb{R}^n) = \varinjlim_i (C_i(\mathbb{R}^n))$ is a categorical C^∞-ring and it is the directed colimit of the diagram.

We need the following example, however, to show that pushouts exist, as they are not directed colimits.

Example 1.17 (Explicit Construction of Pushouts for Finitely Generated C^∞-Rings). [5, Example 2.22] Let C, D, E be finitely generated C^∞-rings, and $\alpha : E \to C, \beta : E \to D$ morphisms. We will show that they fit into a pushout square:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & C \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ D & \xrightarrow{\gamma} & F. \end{array}$$

We can assume that $C = C^\infty(\mathbb{R}^l)/I, D = C^\infty(\mathbb{R}^m)/J$, and $E = C^\infty(\mathbb{R}^n)/K$. Let $x_1, \ldots, x_l$, $y_1, \ldots, y_m$, and $z_1, \ldots, z_n$ generate $C^\infty(\mathbb{R}^l), C^\infty(\mathbb{R}^m)$ and $C^\infty(\mathbb{R}^n)$ respectively, and hence $x_1 + I, \ldots, x_l + I$, $y_1 + J, \ldots, y_m + J$, and $z_1 + K, \ldots, z_n + K$ generate C, D and E respectively. Then $\alpha(z_i + K) = f_i(x_1, \ldots, x_l) + I$ for some $f_i \in C^\infty(\mathbb{R}^l)$, and $\beta(z_i + K) = g_i(y_1, \ldots, y_m) + J$ for some $g_i \in C^\infty(\mathbb{R}^m)$. Necessarily, F is generated by $\delta(x_1), \ldots, \delta(x_n), \gamma(y_1), \ldots, \gamma(y_m)$, so set $F = C^\infty(\mathbb{R}^{l+m})/L$, where L is generated by I, J and elements of the form $g_i - f_i$. The morphisms γ and δ are constructed by noting that the maps $C^\infty(\mathbb{R}^l) \to F, x_i \mapsto [x_i]$ and $C^\infty(\mathbb{R}^m) \to F, y_i \mapsto [y_i]$ factor through C and D respectively.

In particular, the coproduct over C^∞-rings, $C \otimes_\infty D$, is just the pushout square

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\alpha} & C \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ D & \xrightarrow{\gamma} & F, \end{array}$$

where the C^∞-ring $\mathbb{R}$ is the C^∞-ring $C^\infty(\mathbb{R}^0)$. So if C, D are as above, we have $C \otimes_\infty D \cong C^\infty(\mathbb{R}^{l+m})/(I, J)$.

Corollary 1.18. *[11, p.22] All small colimits exist in the category of C^∞-rings.*

Proof. Using generators and relations, one can show that every C^∞-ring is the directed colimit of finitely generated rings. The above example showed that pushouts exist in the finitely generated case, and thus pushouts exist for all C^∞-rings, as colimits commute with colimits. Moerdijk and Reyes state that in the category of C^∞-rings, all small colimits can be constructed using pushouts and directed colimits. So all small colimits exist. $\square$

Lemma 1.19. *(Joyce [5, p. 15]) The subcategories of finitely generated and finitely presented C^∞-rings are closed under finite colimits in the category of C^∞-rings.*

Proof. It suffices to show that they are closed under pushouts, as all finite colimits can be constructed using repeated pushouts [5, p. 15]. It is clear from Example 1.17 that this is true for the category of finitely generated C^∞-rings. Let C, D, E be finitely presented C^∞-rings which fit into a pushout square:

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "\delta"] \\ D \arrow[r, "\gamma"] & F. \end{tikzcd}$$

By assumption (using the notation of Example 1.17), for some $l, m, n \in \mathbb{N}$ and finitely generated $I \subset C^\infty(\mathbb{R}^l)$, $J \subset C^\infty(\mathbb{R}^m)$, $K \subset C^\infty(\mathbb{R}^n)$, we have $C \cong C^\infty(\mathbb{R}^l)/I$, $D \cong C^\infty(\mathbb{R}^m)/J$ and $E \cong C^\infty(\mathbb{R}^n)/K$. Then $F \cong C^\infty(\mathbb{R}^{l+m})/L$, where L is generated by (the finitely generated) I, J and the finite set of elements of the form $g_i - f_i$. Hence L is finitely generated, and F is finitely presented. $\square$

Now let us return to the questions with which we started this section.

Lemma 1.20. *[11, p.23] Let C be a C^∞-ring, and $S \subset C$ a set. Then $C\{S\}^{-1}$ exists.*

Proof. Suppose localizations at finite sets exist. Then because, loosely speaking, localization commutes with colimits, $C\{S\}^{-1}$ is the directed colimit of $C\{T\}^{-1}$ for finite subsets T of S. So we can assume S is finite. Since $f, g \in C$ are invertible if and only if fg is invertible, $C\{f, g\}^{-1} = C\{fg\}^{-1}$. Thus we have reduced to the case $S = \{f\}$. Since C is the colimit of finitely generated C^∞-rings containing S, we can, for the same reason, assume C is finitely generated, say $C = C^\infty(\mathbb{R}^n)/I$.

Let $S = \{f + I\}$ for some $f \in C^\infty(\mathbb{R}^n)$. Using the universal property, if we have defined $C^\infty(\mathbb{R}^n)\{f\}^{-1}$, then letting $\widetilde{I}$ be the image of I (under the map given by localization) in $C^\infty(\mathbb{R}^n)\{f\}^{-1}$, we have $(C^\infty(\mathbb{R}^n)/I)\{f+I\}^{-1} \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}/(\widetilde{I})$. So it is enough to consider $C^\infty(\mathbb{R}^n)$. Let y be the projection on the $n+1^{\text{th}}$ coordinate in $C^\infty(\mathbb{R}^{n+1})$. Now we prove that setting $C^\infty(\mathbb{R}^n)\{f\}^{-1} = C^\infty(\mathbb{R}^{n+1})/(yf-1)$ satisfies the universal property. The localization map is the composition $\pi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})/(yf-1)$. The element f is invertible by construction. Suppose D is another C^∞-ring, and $\phi : C^\infty(\mathbb{R}^n) \to D$ a morphism satisfying $\phi(f)$ is invertible. Then define $\psi : C^\infty(\mathbb{R}^{n+1}) \to D$ as $\psi(x_i) = \phi(x_i)$, and $\psi(y) = \phi(f)^{-1}$. Then ψ factors through to $\tilde{\psi} : C^\infty(\mathbb{R}^{n+1})/(yf-1) \to D$, $\tilde{\psi} \circ \pi = \phi$, and $\tilde{\psi}$ is unique with this property. □

In Corollary 1.26 we will state a result that shows a relationship between $\mathbb{R}$-algebra localization (adding elements of the form f/g) and C^∞-localization. However, we can see that there should be a connection by considering the ring of germs, as in the following example.

Example 1.21. Let $p \in \mathbb{R}^n$, and $S = \{f \in C^\infty(\mathbb{R}^n) | f(p) \neq 0\}$. Then for $f \in C^\infty(\mathbb{R}^n), g \in S$, by assumption g is nonzero on a neighborhood U of p, and so $\frac{f}{g} \in C^\infty(U)$. So $[(\frac{f}{g}, U)]$ is an element of $C^\infty_p(\mathbb{R}^n)$. In fact, $C^\infty_p(\mathbb{R}^n)$ is precisely the $\mathbb{R}$-algebra localization of $C^\infty(\mathbb{R}^n)$ at S. To see this, let D denote this $\mathbb{R}$-algebra localization, and $\phi : D \to C^\infty_p(\mathbb{R}^n), \phi(\frac{f}{g}) = [(f/g, U)]$ as just described. The map ϕ is surjective as every element of $C^\infty_p(\mathbb{R}^n)$ can be written in the form $[(f, \mathbb{R}^n)], f \in C^\infty(\mathbb{R}^n)$. The map ϕ is injective because if $\phi(\frac{f}{g}) = [(\frac{f}{g}, U)] = 0$, there is an open set $p \in W \subset U$ such that $\frac{f}{g}|_W = 0$. As g is never 0 in W, this means $f|_W = 0$. Let h be a characteristic function for W, and so $h \in S$, and $fh = 0$. So by definition of $\mathbb{R}$-algebra localization, $\frac{f}{g} = 0$.

Now that we have shown localization exists, as ordinary algebraic geometry suggests we ought to, we want to relate it to the restriction of functions to open sets. Similarly, we want to relate restriction to closed sets to quotienting. Having not yet introduced a functor that will correspond to the Spec functor in ordinary algebraic geometry, we can't discuss 'open' and 'closed' sets of the underlying topological space for an arbitrary C^∞-ring. However, for a C^∞-ring arising from a manifold M, we can understand how $C^\infty(U)$ and $C^\infty(V)$ relate to $C^\infty(M)$, for any open and closed sets U, V respectively.

Remark 1.22. Let $V \subset M$ be a closed set in a manifold M. Recall that $C^\infty(V)$ is the set of functions $f : V \to \mathbb{R}$ which can be extended to a smooth function on a neighborhood of V. More precisely, it is equivalence classes of pairs (f, U), where U is an open neighborhood of V, and $f : U \to \mathbb{R}$ is smooth, and $(f, U) \sim (g, W)$ if $f|_V = g|_V$. So by the Extension Lemma, $C^\infty(V) \cong C^\infty(M)/I(V)$, where $I(V) = \{f \in C^\infty(M) | f|_V = 0\}$. This also shows how $C^\infty(V)$ is a C^∞-ring.

For an ideal $J \subset C^\infty(M)$, let $Z(J) = \{x \in M | f(x) = 0 \text{ for all } f \in J\}$. Recall that we say $g_1, \ldots, g_n \in C^\infty(M)$ are *independent* if for each $x \in Z(g_1, \ldots, g_n)$ the linear map $(dg_{1x}, \ldots, dg_{nx}) : T_xM \to \mathbb{R}^n$ is surjective. For a set $X \subset M$, let $I(X) = \{f \in C^\infty(M) | f|_X = 0\}$. In ordinary algebraic geometry, we have $I(Z(J)) \cong J$ if J is a radical ideal of $\mathbb{K}[X_1, \ldots, X_n]$. The following proposition from Moerdijk and Reyes [11, 2.1 Lemma] gives a condition for this to be true for an ideal in a C^∞-ring – a sort of variation on the Nullstallensatz.

Proposition 1.23. *Let M be a manifold and $\{g_1, \ldots, g_n\}$ a set of independent functions in $C^\infty(M)$, and set $J = (g_1, \ldots, g_n)$. Then $J = I(Z(J))$, and so $C^\infty(Z(g_1, \ldots, g_n)) \cong C^\infty(M)/I(Z(g_1, \ldots, g_n)) \cong C^\infty(M)/(g_1, \ldots, g_n)$.*

Proof. It is clear that $(g_1, \ldots, g_n) \subset I(Z(J))$. Now suppose $f \in I(Z(J))$. Let $x \in M$. First, if $x \notin Z(J)$, then there exists $g \in J$ and a neighbourhood U of x such that g is never zero on U. So $\pi_x(g)$ is invertible, and using that $\pi_x(J)$ is an ideal, $\pi_x(J) = C^\infty_x(M)$, and so in particular $\pi_x(f) \in \pi_x(J)$. For $x \in Z(J)$, because $g_1, \ldots, g_n$ are independent, the map $G = (g_1, \ldots, g_n) : M \to \mathbb{R}^n$ is a submersion at x. By the local submersion theorem, there are open sets $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n, m \geq n$, each containing the origin, and local coordinates $\phi : U \to M, \psi : V \to \mathbb{R}^n$ such that

1. $\phi(0) = x$.

2. $\psi(0) = G(x)$.

3. Let π be the projection onto the first n coordinates. The diagram below commutes:

$$\begin{tikzcd} M \arrow[r, "G"] & \mathbb{R}^n \\ U \arrow[u, "\phi"] \arrow[r, "\pi"] & V. \arrow[u, "\psi"'] \end{tikzcd}$$

Note that for any $(0,\dots,0,a_{n+1},\dots,a_m) \in U$, $G\circ\phi((0,\dots,0,a_{n+1},\dots,a_m)) = \psi\circ\pi((0,\dots,0,a_{n+1},\dots,a_m)) = \psi(0) = G(x) = (0,\dots,0)$. So $\phi((0,\dots,0,a_{n+1},\dots,a_m)) \in Z(J)$. As $f \in I(Z(J))$, we have $f\circ\phi((0,\dots,0,a_{n+1},\dots,a_m)) = 0$. By Hadamard's Lemma, setting $x = (x_1,\dots,x_n), a = (a_{n+1},\dots,a_m)$, there exist $v_i \in C^\infty(U), i = 1,\dots,m$ such that

$$f\circ\phi(x,a) = f\circ\phi(0,\dots,0,a) + \sum_{i=1}^{n} x_i v_i(x,a) + \sum_{i=n+1}^{m} (a_i - a_i) v_i(x,a)$$

$$= \sum_{i=1}^{n} x_i v_i(x,a) \in (x_1,\dots,x_n).$$

As $\phi : U \to \phi(U)$ is a homeomorphism, this implies $f|_{\phi(U)} \in (J|_{\phi(U)})$, and thus $\pi_x(f) \in \pi_x(J)$. So for every $x \in \mathbb{R}^n$, $\pi_x(f) \in \pi_x(J)$. Because J is finitely generated, we can use a partition of unity argument to show that this implies that $f \in J$. The details of this argument are in the proof that finitely generated ideals are germ-determined or fair, in Lemma 1.29. □

Corollary 1.24. *Let $U \subset \mathbb{R}^n$ be an open set and f a characteristic function for U. Then $C^\infty(U) \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}$.*

Proof. Let y be the projection onto the $n+1^{\text{th}}$ coordinate, $y \in C^\infty(\mathbb{R}^{n+1})$. Let $\tilde{U} = Z(yf-1)$, which is a closed subset of $\mathbb{R}^{n+1}$. The set U is diffeomorphic to $\tilde{U}$ via the maps $\phi : U \to \tilde{U}, x \mapsto (f(x), f(x)^{-1})$ and $\psi : \tilde{U} \to U, (x,y) \mapsto x$. By the previous proposition, $C^\infty(\tilde{U}) \cong C^\infty(\mathbb{R}^n)/(yf-1) \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}$. So we have

$$C^\infty(U) \cong C^\infty(\tilde{U}) \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}.$$

□

While this is an explicit construction of localization, it is still more complicated than in the $\mathbb{R}$-algebra case. The issue is rooted in the fact that ideals in a C^∞-ring C are the ideals of C as a ring, but C is generated with all smooth operations. When we localize at f, we add an extra generator to the C^∞-ring, which involves adding all $C_g(1/f, f_1, \ldots, f_n)$, for all $g \in C^\infty(\mathbb{R}^n), f_i \in C$, which is very different from adding a generator to an $\mathbb{R}$-algebra. For example, $C^\infty(\mathbb{R}^{n+1}) = C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R})$, which is much larger than the $\mathbb{R}$-subalgebra in $C^\infty(\mathbb{R}^{n+1})$ generated by $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R})$. That is, C^∞-tensoring is more like a completion of the tensor product. However,the following proposition from Moerdijk and Reyes [10] gives us some information about $C^\infty(U)$. To prove it, we would need to diverge to introduce the Fréchet topology, so we simply state it.

Proposition 1.25. *[10, Theorem 1.3] Let $U \subset C^\infty(\mathbb{R}^n)$ be an open set. Then for every $h \in C^\infty(U)$, there exist $f, g \in C^\infty(\mathbb{R}^n)$ such that f is a characteristic function for U and*

$$hf|_U = g|_U.$$

As a corollary, we have the following key result, which shows when C^∞-ring and $\mathbb{R}$-algebra localization coincide.

Corollary 1.26. *Let $f \in C^\infty(\mathbb{R}^n)$, and $U = \{x|f(x) \neq 0\}$. Let $S = \{g \in C^\infty(\mathbb{R}^n)|g(x) \neq 0$ for all $x \in U\}$. Then the localization of $C^\infty(\mathbb{R}^n)$ at S as an $\mathbb{R}$-algebra is the C^∞-localization of $C^\infty(\mathbb{R}^n)$ at f.*

Proof. Let D denote the $\mathbb{R}$-algebra localization of $C^\infty(\mathbb{R}^n)$ at S. If $g \in C^\infty(\mathbb{R}^n), h \in S$, then g/h is smooth on U. Let $\phi : D \to C^\infty(U) \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}$ be the $\mathbb{R}$-algebra morphism which takes $\frac{g}{h} \in D$ to $\frac{g}{h} \in C^\infty(U)$. It is easy to check that ϕ is well-defined. By Proposition 1.25, ϕ is surjective. If $\phi(\frac{g}{h}) = 0$, then $g|_U = 0$. As f is a characteristic function for U, $f \in S$ and $fg = 0$. So $\frac{g}{h} = 0$ in D. Thus, ϕ is an isomorphism. □

This implies that a localization at $[f] \in C$ for any finitely generated C can be expressed as a $\mathbb{R}$-algebra localization. To see this, suppose $C = C^\infty(\mathbb{R}^n)/I$. Let $f \in C^\infty(\mathbb{R}^n)$, and set $U = D(f)$, $S' = \{g \in C^\infty(\mathbb{R}^n)|U \subset D(g)\}$. Then by Corollary 1.26, the $\mathbb{R}$-algebra localization of $C^\infty(\mathbb{R}^n)$ at S' is precisely the C^∞-localization of $C^\infty(\mathbb{R}^n)$ at f. As $C^\infty(\mathbb{R}^n)/I\{f + I\}^{-1} \cong C^\infty(\mathbb{R}^n)\{f\}^{-1}/(\tilde{I})$, where

$\tilde{I}$ is the image of I under the localization map, we have that $C\{[f]\}^{-1}$ is the $\mathbb{R}$-algebra localization at $[S']$.

We can also use characteristic functions to show the following result about coproducts:

Lemma 1.27. *[11, Lemma 2.4] Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open sets. Then*

$$C^\infty(U) \otimes_\infty C^\infty(V) \cong C^\infty(U \times V).$$

Proof. Let $f \in C^\infty(\mathbb{R}^n), g \in C^\infty(\mathbb{R}^m)$ be characteristic functions for U, V respectively. Then we have

$$\begin{aligned} C^\infty(U) \otimes_\infty C^\infty(V) &\cong C^\infty(\mathbb{R}^{n+1})/(y_1 f - 1) \otimes_\infty C^\infty(\mathbb{R}^{m+1})/(y_2 g - 1) \\ &\cong C^\infty(\mathbb{R}^{n+m+2})/(y_1 f - 1, y_2 g - 1), \\ C^\infty(\mathbb{R}^{n+m+2})/(y_1 f - 1, y_2 g - 1) &\cong C^\infty(\mathbb{R}^{n+m})\{f, g\}^{-1} \cong C^\infty(\mathbb{R}^{n+m})\{fg\}^{-1}. \end{aligned}$$

As fg is a characteristic function for $U \times V$, this proves the lemma. □

1.4 Fair C^∞-rings

The term *fair* was invented by Joyce in [5], but more as a short hand for germ-determined and finitely generated, than a new concept. The concept of germ-determined originates from Dubuc in [1] and is developed by Moerdijk and Reyes in [11]. The original ideas of this section are from Dubuc's article [1], but we follow the presentation of Joyce in [5]. To motivate the definition of fair, we need to look ahead to schemes. In regular algebraic geometry, the Spec functor takes rings to ringed spaces, with adjoint the global sections functor. This works because elements of a ring, while not determined by their values on points of spectrum, are determined by their values on stalks. This is not always true for C^∞-rings, but it is for fair C^∞-rings.

Definition 1.28. Let I be an ideal of $C^\infty(\mathbb{R}^n)$. I is called a *fair* ideal if I is *germ-determined*, that is

$$f \in I \iff \pi_p(f) \in \pi_p(I) \text{ for all } p \in \mathbb{R}^n.$$

A C^∞-ring C is called *fair* if there exists a fair ideal $I \subset C^\infty(\mathbb{R}^n)$ such that $C \cong C^\infty(\mathbb{R}^n)/I$.

In [5, Proposition 2.16], Joyce shows that fairness (being finitely presented) is independent of presentation: that is, if $C^\infty(\mathbb{R}^n)/I \cong C^\infty(\mathbb{R}^m)/J$, then I is fair (finitely presented) if and only if J is fair (finitely presented).

Lemma 1.29. *Dubuc [1]*

1. *An ideal $I \subset C^\infty(\mathbb{R}^n)$ is fair if only if*

$$f \in I \Leftrightarrow \text{ there is an open covering } \{U_i\} \text{ of } \mathbb{R}^n \text{ such that } f|_{U_i} \in I|_{U_i} \text{ for all } i.$$

2. *If $I \subset C^\infty(\mathbb{R}^n)$ is a finitely generated ideal, then I is fair.*

Proof. For 1., first assume I is fair. Suppose $f \in C^\infty(\mathbb{R}^n)$ and there exists an open covering $\{U_i\}$ such that $f|_{U_i} \in I|_{U_i}$ for all i. Given $p \in \mathbb{R}^n$, there exists an i such that $p \in U_i$ and a $g \in I$ such that $f|_{U_i} = g|_{U_i}$. So $\pi_p(f) = \pi_p(g) \in \pi_p(I)$. So as I is fair, $f \in I$.

Now assume

$$f \in I \Leftrightarrow \text{ there is an open covering } \{U_i\} \text{ such that } f|_{U_i} \in I|_{U_i} \text{ for all } i.$$

To show that I is fair, suppose that $f \in C^\infty(\mathbb{R}^n)$ such that $\pi_p(f) \in \pi_p(I)$ for all $p \in \mathbb{R}^n$. Hence, for every p there exists an open $U_p \subset \mathbb{R}^n$ and $g_p \in I$ such that $f|_{U_p} = g_p|_{U_p}$. The open covering $\{U_p | p \in \mathbb{R}^n\}$ is the required one, and so $f \in I$.

For 2., by assumption $I = (f_1, \ldots, f_m)$ for some $f_i \in C^\infty(\mathbb{R}^n)$. We will show that I is fair using 1. Suppose $f \in C^\infty(\mathbb{R}^n)$, and $\{U_i\}$ is an open cover of $C^\infty(\mathbb{R}^n)$ such that $f|_{U_i} \in I|_{U_i}$ for all i. Then there are $g_{1,i}, \ldots, g_{m,i} \in C^\infty(\mathbb{R}^n)$ such that $f|_{U_i} = (g_{1,i}f_1 + \cdots + g_{m,i}f_m)|_{U_i}$. Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Since ϕ_i is supported on U_i, $\phi_i f = \phi_i(g_{1,i}f_1 + \cdots + g_{m,i}f_m)$. Then

$$f = (\sum_i \phi_i)f = \sum_i \phi_i(g_{1,i}f_1 + \cdots + g_{m,i}f_m) = \sum_{j=1}^m \sum\nolimits_i \phi_i g_{j,i} f_j \in I.$$

So $f \in I \iff$ there is an open covering $\{U_i\}$ such that $f|_{U_i} \in I|_{U_i}$ for all i, and hence I is fair. $\square$

The second part of the lemma shows that if $I \subset C^\infty(\mathbb{R}^n)$ is fair, so is $(I, f_1, \ldots, f_n)$ for any $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^n)$. Let $J \subset C^\infty(\mathbb{R}^n)$ be an ideal. The *fairification* of J is $J^{fa} = \{f \in C^\infty(\mathbb{R}^n) | \pi_x(f) \in \pi_x(J) \text{ for all } x \in \mathbb{R}^n\}$.

A problem that arises is that the localization of a fair ring is not necessarily fair (for example, see [5, p. 12]), and the Spec functor, both in ordinary algebraic geometry and C^∞-algebraic geometry, is based on localization. So, unlike for ordinary rings, we will need sheafification. González and Salas [4] introduce *closed* C^∞-ideals, which are ideals which are closed under the Fréchet topology. All closed ideals are fair. A *differentiable algebra* is a C^∞-ring which is isomorphic to $C^\infty(\mathbb{R}^n)/I$ for some closed ideal $I \subset C^\infty(\mathbb{R}^n)$. The primary advantage of differentiable algebras is that a localization of a differentiable algebra is always a differentiable algebra, a result which they call the Localization Theorem [4, p. 41]. Motivated by this, let us define a subset of fair ideals, which is new.

Definition 1.30. Let $I \subset C^\infty(\mathbb{R}^n)$ be an ideal. Then I is strongly fair if for any open set $U \subset \mathbb{R}^n$, the ideal generated by the image (under restriction) of I in $C^\infty(U)$, which we denote $(I|_U)$, is fair. A C^∞-ring C is *strongly fair* if $C \cong C^\infty(\mathbb{R}^n)/I$ for some strongly fair ideal $I \subset \mathbb{R}^n$.

The localization of a strongly fair C^∞-ring is fair, because if $C = C^\infty(\mathbb{R}^n)/I$ for I strongly fair, and $f \in C^\infty(\mathbb{R}^n)$, then $C\{[f]\}^{-1} = C^\infty(U)/(I|_U)$, where $U = \{x \in \mathbb{R}^n | f(x) \neq 0\}$. So $C\{[f]\}^{-1}$ is fair.

Proposition 1.31. *Finitely generated ideals are strongly fair.*

Proof. Let $I \subset C^\infty(\mathbb{R}^n)$ be a finitely generated ideal. Let U be an open subset, with characteristic function f. Then $C^\infty(U) \cong C^\infty(\mathbb{R}^n)\{f\}^{-1} \cong C^\infty(\mathbb{R}^{n+1})/(yf - 1)$. Then the ideal $(I, yf - 1) \subset C^\infty(\mathbb{R}^{n+1})$ is finitely generated, and so it is fair. Because $C^\infty(U)/(I|_U) \cong C^\infty(\mathbb{R}^{n+1})/(I, yf - 1)$, and fairness is independent of presentation, $(I|_U)$ is fair. □

As mentioned, the closed ideals of [4] are strongly fair, and all strongly fair ideals are fair. I do not know if all strongly fair ideals are closed. We will show that we do not need sheafification to define the spectrum for strongly fair ideals. In Proposition 1.23 we showed that for an ideal J which is finitely generated by independent functions, we have a sort of Nullstallensatz: $I(Z(J)) = J$. The following proposition gives a weak Nullstallensatz for fair C^∞-rings.

Proposition 1.32. *Dubuc [1] Let J be a fair ideal in $C^\infty(\mathbb{R}^n)$. Then $Z(J) = \emptyset$ if and only if $1 \in J$.*

Proof. Suppose $Z(J) = \emptyset$. Then for every $x \in \mathbb{R}$, there is an open neighborhood U of x, and $g \in J$ with g never zero on U. So g is invertible on U, and hence $\pi_x(1) \in \pi_x(J)$. Since J is fair, $1 \in J$. □

We prove a stronger version of the Nullstallensatz for strongly fair ideals.

Lemma 1.33. *Let $J \subset C^\infty(\mathbb{R}^n)$ be a strongly fair ideal. Then $I(Z(J)) = \{f \in C^\infty(\mathbb{R}^n)|$ there exists $g \in J$ such that $Z(g) = Z(f)\}$.*

Proof. Let $f \in C^\infty(\mathbb{R}^n)$, and suppose there exists $g \in J$ such that $Z(g) = Z(f)$. Then $Z(J) \subset Z(g) = Z(f)$, so $f \in I(Z(J))$. For the other direction, let $f \in I(Z(J))$. Then consider $(J, fy - 1) \subset C^\infty(\mathbb{R}^{n+1})$. As $Z(J, fy - 1) = \emptyset$, by the Nullstallensatz for fair ideals, $1 \in (J, fy - 1)$. Let $U = D(f)$. So $1 \in (J|_U)$. That is, there exist $g_1, \ldots, g_k \in C^\infty(U)$ and $f_1, \ldots, f_k \in J$ such that

$$1 = g_1(f_1|_U) + \cdots + g_k(f_k|_U).$$

Using Proposition 1.25, for each g_j, there is $a_j \in C^\infty(\mathbb{R}^n)$ such that $D(a_j) = D(f)$, and $a_j|_U g_j = b_j|_U$ for some $b_j \in C^\infty(\mathbb{R}^n)$. Let $a = a_1 \cdots a_k$ and $\hat{a}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_k$. Then

$$a|_U = g_1(af_1)|_U + \cdots + g_k(af_k)|_U,$$

$$a|_U = (\hat{a}_1 b_1 f_1 + \cdots + \hat{a}_k b_k f_k)|_U \in J|_U.$$

By assumption, $D(a) = U = D(f)$. Let $a|_U = g|_U, g \in J$. Then $a^2 = ag \in J$, and $D(a^2) = D(a) = D(f)$ as needed. □

1.5 Modules over C^∞-rings

This section follows the chapter on modules over C^∞-rings in Joyce's paper [5, Chapter 5].

Definition 1.34. A *module* over a C^∞-ring C is a module over C as an $\mathbb{R}$-algebra. Let C-mod denote the category of C-modules.

If V is any $\mathbb{R}$-vector space, we can form a C-module $C \otimes V$, where for $c \in C, d \otimes v \in C \otimes V$, $c(d \otimes v) = (cd) \otimes v$.

Definition 1.35. A C-module M is *finitely generated* if there is an exact sequence

$$C \otimes \mathbb{R}^n \to M \to 0.$$

A C-module M is *finitely presented* if there is an exact sequence

$$C \otimes \mathbb{R}^n \to C \otimes \mathbb{R}^m \to M \to 0.$$

Let C-mod^{fp} denote the full subcategory of finitely presented C-modules.

Joyce [5, p.33] proves that C-mod^{fp} is closed under cokernels and extensions in C-mod.

The most important example of a C^∞-ring was a manifold. The most important example of a module over a C^∞-ring is the cotangent space of a manifold. Let M be a smooth manifold, $p \in M$. Let m_p be the maximal ideal of ring of germs $\mathcal{O}_{M,p}$ at p of M ($m_p = \{f | f(p) = 0\}, \mathcal{O}_{M,p}/m_p \cong \mathbb{R}$). The cotangent space of M at p is m_p/m_p^2. Equivalently, the cotangent space is the dual of the real vector space of maps $\mathcal{O}_{M,p} \to \mathbb{R}$, $f \mapsto f'$ such that $f'g + fg' = (fg)'$, the Leibniz rule. These two definitions are equivalent. The second definition is straightforward to generalize to C^∞-rings, as in [5].

Definition 1.36. Let C be a C^∞-ring, and M a C-module. A map $d : C \to M$ is a C^∞*-derivation* if for all $f \in C^\infty(\mathbb{R}^n), c_1, \ldots, c_n \in C$,

$$d(C_f(c_1, \ldots, c_n)) = \sum_{i=1}^{n} C_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) dc_i.$$

Note that this implies that d is $\mathbb{R}$-linear. We call this the C^∞*-Leibniz rule.* A pair (M, d) is the *cotangent module* of C if d is a C^∞-derivation and (M, d) is universal. That is, if (M', d') is another such pair, then there is a unique f such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "d'"] \arrow[d, "d"] & M' \\ M. \arrow[ur, dotted, "f"] & \end{tikzcd}$$

Cotangent modules exist, and are unique up to unique isomorphism. In order to define the equivalent of fair for C-modules, we need C^∞-ringed spaces, so we delay this until section 2.4.

2. C^∞-schemes

This chapter follows Joyce's chapter on C^∞-schemes [5, Chapter 4] in the main, but sometimes we use equivalent definitions or show more detail in order to more clearly portray the close connection with ordinary algebraic geometry. For a quick review of the ordinary algebraic geometry that we will parallel here, see Appendix A.

The first goal in this chapter will be to define the C^∞-ring equivalent to the Spec functor in ordinary algebraic geometry. There are two definitions of the spectrum in the literature. Dubuc [1], Joyce [5], González and Salas [4], all consider real maximal ideals, while Moerdijk and Reyes [11] consider the 'radical prime ideals'. We cannot take the topological space to be all prime ideals, as then stalks would not be local. Germs at real maximal ideals are local. However, in C^∞-algebraic geometry, knowing a germ of an element at every real maximal ideal only determines it globally if we are in a fair C^∞-ring, and moreover, since we need localization, it only determines it on open sets it if we are in a strongly fair C^∞-ring. But this is a very restrictive class. One can avoid this issue by considering sheafification, but then some of the connection between ordinary and C^∞-algebraic geometry is lost. So Moerdijk and Reyes [11] consider a larger topological space. However, we follow the first approach.

2.1 The real spectrum of a C^∞-ring

First we define the real spectrum of a C^∞-ring, following Joyce [5], and prove that real points coincide with evaluation maps.

Definition 2.1. Let C be a C^∞-ring. An ideal $m \subset C$ is a *real maximal ideal* if the inclusion map $\mathbb{R} \to C/m$ is an isomorphism. This implies that m is maximal.

We call a morphism from a C^∞-ring C to $\mathbb{R}$ an $\mathbb{R}$-*point*. As every morphism $\phi : C \to \mathbb{R}$ is surjective, there is a one to one correspondence between real maximal ideals and $\mathbb{R}$-points. To see why the set of $\mathbb{R}$-points is the natural underlying topological space to consider, let us look at the free C^∞-ring on n generators, $C^\infty(\mathbb{R}^n)$. Each point $p \in \mathbb{R}^n$ gives us an $\mathbb{R}$-point $\mathrm{ev}_p : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, the evaluation at p map. If $C = C^\infty(\mathbb{R}^n)/I$, and $p \in Z(I)$, then the evaluation map at p factors through to a map $C \to \mathbb{R}$, which we also call ev_p. As the next lemma shows, all $\mathbb{R}$-points of a finitely generated C^∞-ring are in fact evaluation maps.

Lemma 2.2. *[11, p. 33] Let $C \cong C^\infty(\mathbb{R}^n)/I$ be a finitely generated C^∞-ring, and $\phi : C \to \mathbb{R}$ an $\mathbb{R}$-algebra morphism which is not the zero map. Then ϕ is a morphism of C^∞-rings, and there is a unique $p \in Z(I)$ such that $\phi = \mathrm{ev}_p$.*

Proof. Let $\{K_m : m \in \mathbb{N}\}$ be a set of compact subsets of $\mathbb{R}^n$ such that $K_m \subset K^\circ_{m+1}$ and $\cup_{m\in\mathbb{N}} K_m = \mathbb{R}^n$. For each K_m there is a smooth function $f_m : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq f \leq 1$, $f_m|_{K_m} = 0$, and f_m is 1 outside of the interior of K_{m+1}. Then $\{f_m\}$ is locally finite, so $f = \sum_{m\in\mathbb{N}} f_m$ is a smooth function. For $x \notin K_m$, $f(x) \geq m-1$. There exists $r \in \mathbb{R}$ such that $\phi(f + I) = r$. The set $\{x \in C^\infty(\mathbb{R}^n) | f(x) = r\}$ is contained in K_l for $l > r + 1$, and so it is compact. Suppose

$$\bigcap_{\{g\in C^\infty(\mathbb{R}^n)|g+I\in\ker\phi\}} g^{-1}(0) = \emptyset.$$

As $\{f = r\}$ is compact, there are $g_1, \ldots, g_m \in C^\infty(\mathbb{R}^n), g + I \in \ker\phi$, such that

$$\bigcap_{j=1}^{m} g_j^{-1}(0) \cap \{f = r\} = \emptyset.$$

Then

$$g_1^2 + \cdots + g_m^2 + (f - r)^2 \in \ker\phi$$

and $g_1^2+\cdots+g_m^2+(f-r)^2$ is invertible, and hence $\phi=0$, which is a contradiction. So there exists a $p \in \bigcap_{\{g\in C^\infty(\mathbb{R}^n)|\phi(g+I)=0\}} g^{-1}(0) \subset Z(I)$. Then $\ker\phi \subset \ker \mathrm{ev}_p$, and as $\ker\phi$ is a maximal ideal, $\ker\phi = \ker\mathrm{ev}_p$. For any $g \in C^\infty(\mathbb{R}^n)$, $\phi(g+I)=s\in\mathbb{R}$, then $g-s+I \in \ker\phi$, so $g(p)-s=0$. So $\phi=\mathrm{ev}_p$, and in particular it is a morphism of C^∞-rings. $\square$

So for $C^\infty(\mathbb{R}^n)/I$ we have a one to one correspondence between $\mathbb{R}$-points and points in $Z(I)$.

We are now ready to define the spectrum of C^∞-ring as a topological space. For a C^∞-ring C, the *real spectrum* as a topological space is

$$\operatorname{Spec}^{\mathrm{r}} C = \{m | m \text{ is a real maximal ideal of } C\},$$

which is a subset of the spectrum for C as an algebra. The topology is the Zariski topology: the closed subsets are $Z(I)$ where I is an ideal of C.

Remark 2.3. Let C be a C^∞-ring. For every $f \in C$, define a function $\tilde{f} : \operatorname{Spec}^{\mathrm{r}} C \to \mathbb{R}, m \mapsto f+m \in \mathbb{R}$. Then $\tilde{f}$ is continuous. To see this, let $V \subset \mathbb{R}$ be a closed subset of $\mathbb{R}$. Let $g \in C^\infty(\mathbb{R})$ be the characteristic function of $\mathbb{R}-V$. Then $f^{-1}(V) = Z(C_g(f))$, as an $\mathbb{R}$-point ϕ is in $f^{-1}(V)$ if and only if $\phi(f) \in V$ if and only $C_g(\phi(f)) = 0$, and $C_g(\phi(f)) = \phi(C_g(f))$. The topology defined by requiring all functions of the form $\tilde{f}$ to be continuous is called the *Gelfand topology.*

For a C^∞-ring C, $X = \operatorname{Spec}^{\mathrm{r}} C$ as a topological space, with elements $m \in X$ understood as maximal ideals, define the following maps

$$Z(-) : \{\text{ideals in } C\} \to \{\text{sets in } X\}, Z(S) = \{m \in X | S \subset m\},$$

$$I(-) : \{\text{sets in } X\} \to \{\text{ideals in } C\},$$
$$I(K) = \{f \in C | f \in m \text{ for all } m \in K\} = \cap_{m\in K} m.$$

For $f \in C$, $D(f) = \{m \in X : f \notin m\}$. That is, $I(-)$, $D(-)$, and $Z(-)$ for C^∞-rings are just the usual $I(-), D(-)$ and $Z(-)$ for rings, but restricted to the real spectrum as a topological space, rather than all prime ideals.

Definition 2.4. Let C be a C^∞-ring. Let X denote $\operatorname{Spec}^{\mathrm{r}} C$ as a topological space. Define a functor

$$C^\infty\text{-rings} \to \text{presheaves of } C^\infty\text{-rings on X}, C \mapsto \tilde{\mathcal{O}}_C$$

in the following way. X is the real spectrum with the Zariski/Gelfand topology, discussed above. The topology has as base sets of the form $D(f), f \in C$, and we associate to $D(f)$ the C^∞-ring $\tilde{\mathcal{O}}_C(D(f)) = C\{f\}^{-1}$, or using $\mathbb{R}$-algebra localization, $C\{S\}^{-1}, S = \{g \in C | D(f) \subset D(g)\}$. The $\mathbb{R}$-algebra localization definition shows that $\tilde{\mathcal{O}}_C(D(f)) = C\{f\}^{-1}$ is well-defined. The restriction maps are given by further localization. That is, if $D(f) \subset D(g) \in C$, then $D(fg) = D(f)$, and so $\tilde{\mathcal{O}}_C(D(g)) = \tilde{\mathcal{O}}_C(D(fg)) = C\{fg\}^{-1} = C\{f,g\}^{-1}$, so the restriction map is the natural map

$$\tilde{\mathcal{O}}_C(D(g)) = C\{g\}^{-1} \to C\{f,g\}^{-1}.$$

Let $\mathcal{O}_C$ be the sheafification of this presheaf. Then we define $\operatorname{Spec}^{\mathrm{r}} C = (X, \mathcal{O}_C)$.

Morphisms of C^∞-ringed spaces are analogous to those of ringed spaces. That is, a morphism $(\pi, \pi_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\pi : X \to Y$ and a morphism of sheaves of C^∞-rings $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)$. Sometimes we will just denote a morphism of C^∞-ringed spaces as $\pi : X \to Y$.

2.2 Fairification

To define the Spec functor, we used sheafification, but we will show that it coincides with fairification, which we define as:

Definition 2.5. The fairification of a finitely generated C^∞-ring C is a fair C^∞-ring D and a morphism $\phi : C \to D$ such that if D', ϕ' is another such pair, there is a unique morphism $f : D \to D'$ making the following diagram commute:

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\phi'"] & D \arrow[dl, "f"] \\ D' & \end{tikzcd} .$$

Lemma 2.6. *Let C be a finitely generated C^∞-ring. The fairification of C is C/J, where $J = \{f \in C | \pi_x(f) = 0$ for all $x \in \operatorname{Spec}^r C\}$.*

Proof. Clearly C/J is fair. Let $\pi : C \to C/J$ be the projection, and suppose D is a fair C^∞-ring, and $\phi : C \to D$ a morphism. We just need to show that $J \subset \ker \phi$, as then there is a unique morphism $\tilde{\phi} : C/J \to D$ making the diagram below commute:

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[d, "\phi"] & C/J \arrow[dl, "\tilde{\phi}"] \\ D & \end{tikzcd} .$$

To see that $J \subset \ker \phi$, suppose $f \in J$. If $y \in \operatorname{Spec}^{\mathrm{r}} D$, $\phi^{-1}(y) \in \operatorname{Spec}^{\mathrm{r}} C$. $\pi_y(\phi(f)) = 0$ since $\pi_{\phi^{-1}(y)}(f) = 0$, so as D is fair, $\phi(f) = 0$, and $f \in \ker \phi$. $\square$

As an ideal is contained in its fairification, any maximal ideal in $C^\infty(\mathbb{R}^n)$ is fair. If C, D are fair rings, and $\phi : C \to D$ is a morphism, then $C/\ker \phi$ is fair.

Lemma 2.7. *Let C, D be finitely generated C^∞-rings, and $\phi : C \to D$ a morphism. Then there is a unique morphism $\psi : C^{fa} \to D^{fa}$ such that the following diagram commutes:*

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\pi_C"] & D \arrow[d, "\pi_D"] \\ C^{fa} \arrow[r, "\psi"] & D^{fa} \end{tikzcd} .$$

Proof. Suppose we have $\phi : C \to D$. Then $\pi_D \circ \phi : C \to D^{fa}$, so by the universal property, there is a unique map $\psi : C^{fa} \to D^{fa}$ such that $\psi \circ \pi_C = \pi_D \circ \phi$. $\square$

2.3 Affine C^∞-schemes

For the first part of this section, we parallel Vakil's proofs in ordinary algebraic geometry [15] to prove that the Spec functor is full and faithful for fair C^∞-rings. In the second part of the section, we follow Joyce [5], and prove that open C^∞-subschemes of fair affine C^∞-schemes are fair affine, and prove the existence of partitions of unity.

Proposition 2.8. *Let C be a strongly fair C^∞-ring. Then the presheaf $\tilde{\mathcal{O}}_C$ on X is a sheaf, not just a presheaf.*

Proof. We can assume $C = C^\infty(\mathbb{R}^n)/J$, J strongly fair. Then $X = Z(J)$. We prove the identity axiom first for the special case $\bigcup_{i\in S} D(f_i + J) = \operatorname{Spec} C$. Suppose for all $i \in S$, $f + J|_{D(f_i+J)} = 0$. That is, under localization at $f_i + J$, $f + J$ is mapped to 0. Using that $C\{f_i + J\}^{-1} = C^\infty(D(f_i))/(J|_{D(f_i)})$, this implies $f \in (J|_{D(f_i)})$. Then $\pi_x(f) \in \pi_x(J)$ for all $x \in Z(J)$, so as J is fair, $f \in J$. So far we have only used that C is fair. The general case where $\cup_{i\in S} D(f_i + J) = D(g + J)$ for some g, reduces to the above, because $(D(g), \tilde{\mathcal{O}}_C|_{D(g)}) \cong (D(g), \tilde{\mathcal{O}}_{C\{g\}^{-1}})$, and $C\{g\}^{-1}$ is still fair because C is a strongly fair C^∞-ring.

It is enough to show gluability for the special case $X = \cup_{i\in S} D(f_i + J)$, because we will only use that C is finitely generated. Let $J_i = (J|_{D(f_i)})$, $J_{ij} = (J|_{D(f_i)\cap D(f_j)})$. Suppose we have $g_i + J_i \in \tilde{\mathcal{O}}_C(D(f_i + J)) = (C^\infty(\mathbb{R}^n)/J)\{f_i + J\}^{-1} \cong C^\infty(D(f_i))/J_i$ which agree on overlaps. The set $\{D(f_i)\}$ are an open covering of $U = \cup\{D(f_i)\}$. Because U is normal, by the Shrinking Lemma, there is a locally finite refinement $\{V_i\}$ of $\{D(f_i)\}$ such that $\overline{V_i} \subset D(f_i)$. Let $\{\phi_i\}$ be a partition of unity subordinate to the cover $\{V_i\}$. Then $\phi_i g_i \in C^\infty(D(f_i))$, but $\phi_i g_i|_{D(f_i)-V_i} = 0$, and $\overline{V_i} \subset D(f_i)$. So we can extend $\phi_i g_i$ by 0 to an element of $C^\infty(\mathbb{R}^n)$. Call this h_i, and note that $h_i + J_i = \phi_i g_i + J_i$ as $h_i = \phi_i g_i$ in $C^\infty(D(f_i))$. Also, in $C^\infty(D(f_l))$, $h_i + J_l = \phi_i g_l + J_l$. Let $g = \sum_i h_i$. Now we claim that $g|_{D_{U_i}} + J_i = g_i + J_i$.

$$g|_{D_{U_i}} + J_i = \sum_l h_l + J_i = \sum_l \phi_l g_i + J_i = (\sum \phi_l)(g_i + J_i) = g_i + J_i$$

So $g + J \in \tilde{\mathcal{O}}_C(X)$, and $g|_{D_{U_i}} + J_i = g_i + J_i$. Notice that in the proof of the gluability axiom, we did not use fairness. □

Lemma 2.9. *Let $\Phi : C \to D$ be a morphism, and $X_D = \operatorname{Spec}^r D, X_C = \operatorname{Spec}^r C$. Then Φ induces a map of C^∞-ringed spaces $(\Psi, \Psi_\#) : (X_D, \mathcal{O}_D) \to (X_C, \mathcal{O}_C)$.*

Proof. Let $m \in X_D$, corresponding to $\phi : D \to \mathbb{R}$. Let $\Psi(m) = \Phi^{-1}(m) = \ker(\phi \circ \Phi)$, which is a maximal real ideal of C since $\phi \circ \Phi : C \to \mathbb{R}$ is an $\mathbb{R}$-point. To see that Ψ is continuous, let $Z(f), f \in C$ be a closed subset of X_C. Then $\Psi^{-1}(Z(f)) = \{m \in X_D | f \in \Phi^{-1}(m)\} = \{m \in X_D | \Phi(f) \in m\} = Z(\Phi(f))$, which is closed in X_D. Now we can define a morphism of locally C^∞-ringed spaces. For $D(f) \subset X_C$, define

$$\Psi_\#(D(f)) : \mathcal{O}_C(D(f)) \to \mathcal{O}_D(\Psi^{-1}(D(f))) = \mathcal{O}_D(D(\Phi(f)))$$

as the sheafification of the natural map $C\{f\}^{-1} \to D\{\Phi(f)\}^{-1}$. For this to be a well-defined map, we need that the definition does not depend on f (it is enough to check this prior to sheafification), but this is just because these are morphisms of C^∞-rings. To see this, suppose $D(f) = D(g)$. Then there exist $f_i \in C, h \in C^\infty(\mathbb{R}^{n+1})$ such that $1/g = (C\{f\}^{-1})_h(1/f, f_1, \ldots, f_n)$. So the definition of $\Psi_\#(D(f))$ results in $\Psi_\#(D(f))(1/g) = (D\{\Phi(f)\}^{-1})_h(\Phi(1/f), \Phi(f_1), \ldots, \Phi(f_n)) = 1/\Phi(g) = \Psi_\#(D(g))(1/g)$. This is a map of ringed spaces, because it commutes with restriction, as restriction is just further localization. It is also a morphism of *locally* ringed spaces, automatically, as morphisms of local C^∞-rings are local morphisms: Suppose $\phi : C_1 \to C_2$ is a morphism, and C_i is local with maximal ideal m_i. Then the composition of ϕ with projection $\pi \circ \phi : C_1 \to C_2/m_2 \cong \mathbb{R}$ is surjective, and so its kernel is a maximal ideal, so the kernel must be m_1. So $\phi(m_1) \subset m_2$. □

Definition 2.10. A C^∞-ringed space $(X, \mathcal{O}_X)$ is an *affine C^∞-scheme* if there is a C^∞-ring C such that $(X, \mathcal{O}_X) \cong \mathrm{Spec}^{\mathrm{r}}\, C$. An affine C^∞-scheme is fair or finitely presented if C is. Let $\mathbb{L}, \mathbb{L}^{fa}, \mathbb{L}^{fp}$ denote the categories of affine C^∞-schemes, fair affine C^∞-schemes, and finitely presented affine C^∞-schemes. A C^∞-scheme is a C^∞-ringed space $(X, \mathcal{O}_X)$ with an open covering U of X such that $(U, \mathcal{O}_X|_U)$ is an affine C^∞-scheme. C^∞-schemes are locally C^∞-ringed spaces. A C^∞-scheme is locally fair or locally finitely presented if it can be covered by fair or finitely presented affine C^∞-schemes respectively. We call a C^∞-scheme separated, second countable, compact, etc. if the underlying topological space is.

Morphisms of C^∞-schemes are just morphisms of C^∞-ringed spaces. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are C^∞-schemes, if it is clear in the context, we sometimes refer to them just as X and Y, and write a morphism between them as just $\Psi : X \to Y$.

As mentioned in the introduction, from the perspective of synthetic differential geometry, the category of C^∞-rings is primarily important because the opposite category of smooth manifolds can be embedded into it. In fact, the category of manifolds can be fully and faithfully embedded into the opposite category of finitely presented C^∞-rings (and hence in the category of C^∞-schemes) in such a way that preserves transversal fibre products. A full proof of this can be found in Appendix B.

Proposition 2.11. *Let C be a finitely generated C^∞-ring, say $C \cong C^\infty(\mathbb{R}^n)/J$. Let $D = C^\infty(\mathbb{R}^n)/J^{fa}$. Then $\operatorname{Spec}^r C \cong \operatorname{Spec}^r D$, and for $f + J \in C$,*

$$\mathcal{O}_C(D(f+J)) = (C\{f+J\}^{-1})^{fa}.$$

That is, sheafification coincides with fairification.

Proof. As topological spaces, $\operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^n)/J$ is $Z(J)$ and $\operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^n)/J^{fa}$ is $Z(J^{fa})$. Since $J \subset J^{fa}$, $Z(J^{fa}) \subset Z(J)$. If $x \in Z(J)$, and $f \in J^{fa}$, then $f(x) = 0$ (there is an open neighborhood U of x such that $f|_U \in J|_U$). So $x \in Z(J^{fa})$. Thus $Z(J) = Z(J^{fa})$, and for $f \in C^\infty(\mathbb{R}^n)$, $D(f+J) = D(f) \cap Z(J) = D(f) \cap Z(J^{fa}) = D(f + J^{fa})$. Let $(Z(J), \mathcal{O}_X)$ be the sheaf on X defined by $\mathcal{O}_X(D(f+J)) = (C\{f+J\}^{-1})^{fa} = C^\infty(D(f))/(J|_{D(f)})^{fa}$. The restriction maps are given by Lemma 2.7. That is, given $f, g \in C^\infty(\mathbb{R}^n)$ where $D(g) \subset D(f)$, so $\mathcal{O}_X(D(g+J)) = C\{f+J, g+J\}^{-1}$, the restriction map is the bottom map of the following commutative diagram given by the lemma (where π is the respective fairification map):

$$\begin{array}{ccc}
C\{f+J\}^{-1} & \longrightarrow & C\{f+J, g+J\}^{-1} \\
\Big\downarrow \pi & & \Big\downarrow \pi \\
(C\{f+J\}^{-1})^{fa} & \xrightarrow{\psi} & (C\{f+J, g+J\}^{-1})^{fa}.
\end{array}$$

We can use the same arguments as in the case of a strongly fair C^∞-ring to see that this is not just a presheaf, but a sheaf. Let $\pi : (Z(J), \tilde{\mathcal{O}}_C) \to (Z(J), \mathcal{O}_X)$ be the morphism of presheaves given by fairification:

$$\tilde{\mathcal{O}}_C(D(f+J)) = C\{f+J\}^{-1} \to (C\{f+J\}^{-1})^{fa}.$$

By the construction of the restriction maps of $(Z(J), \mathcal{O}_X)$ this is a morphism of presheaves. Now suppose $\mathcal{F}$ is another sheaf of C^∞-rings on $Z(J)$, and $\phi : \operatorname{Spec}^{\mathrm{r}} C \to \mathcal{F}$ a morphism of presheaves. Let $D(f+J)$ be an open set, and $[g] \in \tilde{\mathcal{O}}_C(D(f+J))$ such that $g \in (J|_{D(f)})^{fa}$. Then there is an open covering $\{U_i\}$ of $D(f+J)$ such that $[g]|_{U_i} = 0$. So $\phi(U_i))([g]) = 0$, and as $\mathcal{F}$ is a sheaf, $\phi(D(f+J))([g]) = 0$. So ϕ factors uniquely through $\pi : (X, \tilde{\mathcal{O}}_C) \to (X, \mathcal{O}_X)$. This is the universal property of sheafification, so $(X, \mathcal{O}_X)$ is the sheafification of C. In particular, $\operatorname{Spec}^{\mathrm{r}} C \cong \operatorname{Spec}^{\mathrm{r}} D$. $\square$

This proposition shows that for a finitely generated C^∞-ring $C \cong C^\infty(\mathbb{R}^n)/J$, $\Gamma(\operatorname{Spec}^{\mathrm{r}} C)$ is the fairification of C, so in fact $\Gamma(\operatorname{Spec}^{\mathrm{r}} C) \cong C$ if and only if C is fair. The advantage of restricting our attention to fair rings, as in [5], is that $\operatorname{Spec}^{\mathrm{r}} C$ of any finitely generated C is a *fair* affine C^∞-scheme. Following Vakil [15, p. 179], we prove:

Proposition 2.12. *The functor* Spec^r *from the category of fair rings (finitey presented rings) to fair affine schemes (finitely presented affine schemes) is full and faithful.*

Proof. The only thing left to prove is that $\operatorname{Spec}^{\mathrm{r}}$ is full and faithful on morphisms. For a morphism $\Phi : C \to D$, we can define $\operatorname{Spec}^{\mathrm{r}}(\Phi) : \operatorname{Spec}^{\mathrm{r}} D = (X_D, \mathcal{O}_D) \to \operatorname{Spec}^{\mathrm{r}} C = (X_C, \mathcal{O}_C)$ as in Lemma 2.9. So all we need is that if $(\Psi, \Psi_\#) : \operatorname{Spec}^{\mathrm{r}} D \to \operatorname{Spec}^{\mathrm{r}} C$ is C^∞-ringed space morphism, $\operatorname{Spec}^{\mathrm{r}} \Gamma(\Psi, \Psi_\#)$ is $(\Psi, \Psi_\#)$. First, we need that $\Psi : X_D \to X_C$ is determined by $\operatorname{Spec}^{\mathrm{r}} \Gamma(\Psi, \Psi_\#)$. Note that for $m \in X_D$, $m = \{f \in \Gamma(\mathcal{O}_D) | f \in m\}$, that is, the set of functions that vanish at m. Because by assumption, $(\Psi, \Psi_\#)$ is a map of locally ringed spaces, the image of m under Ψ is precisely the unique point of X_C of functions $\{g \in C | \Psi_\#(X_C)(g) \text{ vanishes at } m\}$. As this coincides with our definition of the $\operatorname{Spec}^{\mathrm{r}}$ of a morphism, $\operatorname{Spec}^{\mathrm{r}} \Gamma(\Psi, \Psi_\#)$ coincides with $(\Psi, \Psi_\#)$ on the topological spaces. Now we look at whether $\operatorname{Spec}^{\mathrm{r}} \Gamma(\Psi, \Psi_\#)$ agrees with $(\Psi, \Psi_\#)$ as morphisms of sheaves. That is, for an open set $D(f)$, we want that $\Psi_\#(D(f)) : \mathcal{O}_C(D(f)) \to \mathcal{O}_D(\Psi^{-1}(D(f))) = \mathcal{O}_D(D(\Psi_\#(X_C)(f)))$ to be determined by the map of global sections (where the last equality is because $(\Psi, \Psi_\#)$ is a local morphism). Let $g = \Psi_\#(X_C)(f)$. First we consider C, D strongly fair. We have the following commutative diagram, because $\Psi_\#$ is a morphism of sheaves:

$$\begin{tikzcd}
C \cong \mathcal{O}_C(X_C) \arrow[r, "\Psi_\#(X_C)"] \arrow[d, "\rho_{X_C, D(f)}"] & \mathcal{O}_D(X_D) \cong D \arrow[d, "\rho_{X_D, D(g)}"] \\
C\{f\}^{-1} \cong \mathcal{O}_C(D(f)) \arrow[r, "\Psi_\#(D(f))"] & \mathcal{O}_D(D(\Phi(g)) \cong D\{g\}^{-1}
\end{tikzcd}.$$

Since C, D are strongly fair, restrictions are just localizations. As this diagram commutes, $\Psi_\#(D(f))$ is just the localization of $\Psi_\#(X_C)$. To generalize to the case when C, D are fair, we just consider the fairification of the above diagram. $\square$

In ordinary algebraic geometry, the Spec functor is full and faithful for all rings.

Remark 2.13. Using Spec, we can generalize the definition of *fair ideal* to ideals in any finitely generated C^∞-ring. Let C be a finitely generated C^∞-ring. Let $\mathrm{Spec^r}\, C = (X, \mathcal{O}_X)$. Let $x \in \mathrm{Spec^r}\, C = (X, \mathcal{O}_X)$. Let $\mathcal{O}_{X,x}$ be the ring of germs, and $\pi_x : C \to \mathcal{O}_{X,x}$ the projection. An ideal J in C is *fair* if $f \in J$ if and only if for every $x \in X$, $\pi_x(f) \in \pi_x(J)$. An ideal $J \subset C$, where C is finitely generated, is *strongly fair* if for every $f \in C$, the ideal generated by the image of J in $C\{f\}^{-1}$ is fair. Just for ordinary rings, if K is a closed subset of $\mathrm{Spec^r}\, C$, then $Z(I(K)) = K$. Then just as before, if $J \subset C$ is a strongly fair ideal, $I(Z(J)) = \{f \in C^\infty(\mathbb{R}^n) |$ there exists $g \in J$ such that $Z(g) = Z(f)\}$. To see this, assume $C = C^\infty(\mathbb{R}^n)/I$ for some ideal $I \subset C^\infty(\mathbb{R}^n)$. Points of C correspond to points of $Z(I)$. Then J corresponds to an ideal $\tilde{J}$ in $C^\infty(\mathbb{R}^n)$ containing I, and similarly for $I(Z(J))$. As $Z(J) = Z(\tilde{J}) \cap Z(I) = Z(\tilde{J})$, J is strongly fair if and only if $\tilde{J}$ is strongly fair, and the ideal corresponding to $I(Z(J))$ in $C^\infty(\mathbb{R}^n)$ is $I(Z(\tilde{J}))$. So we just need to find $I(Z(\tilde{J}))$, and we can assume $C = C^\infty(\mathbb{R}^n)$, and apply our previous result.

Using the fact that $\mathrm{Spec^r}$ is full and faithful, one can prove the following proposition from [5, p. 26].

Proposition 2.14. $\mathbb{L}, \mathbb{L}^f, \mathbb{L}^{fp}$ *are each closed under finite limits. In particular, fibre products and finite limits exist in each.*

By using open covers, as in [5, p.31], a corollary of this is that the subcategories of locally finitely generated and locally fair C^∞-schemes are also closed under fibre products and finite limits in the category of C^∞-ringed spaces. An example of a fibre product that will be of use to us in considering finite maps is the fibre of a morphism.

Example 2.15. Let $\Phi : Y = \mathrm{Spec^r}\, B \to X = \mathrm{Spec^r}\, A$ be a morphism of fair affine C^∞-schemes. There is a unique $\phi : A \to B$ corresponding to Φ, since $\mathrm{Spec^r}$ is full and faithful. Let m_x be the maximal ideal of $\mathcal{O}_{X,x}$. We also write m_x for its lift in A and for the image of this lift in B under ϕ. The fibre $\Phi^{-1}(x) = x \times_X Y = \mathrm{Spec^r}\, A/m_x \times_X Y$ in the category of locally fair C^∞-schemes is $\mathrm{Spec^r}\, A/m_x \coprod_A B = \mathrm{Spec^r}\, B/m_x B$. The ideal m_x is finitely generated, so it is fair, and so B/m_xB is also fair.

Lemma 2.16. *[5, p. 28] Let $(X, \mathcal{O}_X)$ be a fair affine (finitely presented) C^∞-scheme. Then if U is an open set, $(U, \mathcal{O}_X|_U)$ is a fair (finitely presented) affine C^∞-scheme.*

Proof. $(\Phi, \Phi_\#) : (X, \mathcal{O}_X) \to \operatorname{Spec}^{\mathrm{r}} C$ is an isomorphism for some $C = C^\infty(\mathbb{R}^n)/J$, so in particular, $\Phi : X \to Z(J)$ is a homeomorphism. Let $U \subset X$ be open. Then there is an open $V \subset \mathbb{R}^n$ with $V \cap Z(J) = \Phi(U) \cong U$. Let $f \in C^\infty(\mathbb{R}^n)$ be a characteristic function for V. Then $D(f + J) = Z(J) \cap V$, so $f + J$ is a characteristic function for $\Phi(U)$. As proved above, $(U, \mathcal{O}_X|_U) \cong (D(f + J), \mathcal{O}_C|_{D(f+J)}) \cong \operatorname{Spec}^{\mathrm{r}}(C\{f + J\}^{-1})$. As C is finitely generated, $\operatorname{Spec}^{\mathrm{r}} C\{f + J\}^{-1}$ is a fair affine C^∞-scheme. If C is finitely presented, so is $C\{f + J\}^{-1} \cong C^\infty(\mathbb{R}^{n+1})/(J, yf - 1)$, as $(J, yf - 1)$ is finitely generated. $\square$

This clearly generalizes to show that open C^∞-subschemes of locally fair (finitely presented) C^∞-schemes are locally fair (finitely presented).

In ordinary algebraic geometry, only some points are closed. For a locally fair C^∞-scheme $(X, \mathcal{O}_X)$, however, all points are closed, because points coincide with maximal ideals. Also, X is locally compact, because it is locally homeomorphic to a closed subset of $\mathbb{R}^n$, which is always locally compact.

Definition 2.17. Let $(X, \mathcal{O}_X)$ be a locally fair C^∞-scheme. A *locally finite* sum on $(X, \mathcal{O}_X)$ is a formal sum $\sum_{i \in I} c_i$ where I is an indexing set, $c_i \in \mathcal{O}_X(X)$, and there is an open covering $\{U_a\}$ of X such that all but finitely many of the c_i have $\rho_{XU_a}(c_i) = 0$. Since $\sum_{i \in I} \rho_{XU_a}(c_i)$ is a well-defined element of $\mathcal{O}_X(U_a)$, by gluability, there is a unique *limit* of this locally finite sum. That is, there is a $c \in \mathcal{O}_X(X)$ such that for all a, $\rho_{XU_a}(c) = \sum_{i \in I} \rho_{XU_a}(c_i)$. A *partition of unity* subordinate to an open cover $\{U_i | i \in I\}$ is a locally finite sum $\sum_{i \in I} c_i$ such that c_i is supported in U_i, and $\sum_{i \in I} c_i = 1$.

Theorem 2.18. *[5, p. 32] Let $(X, \mathcal{O}_X)$ be a separated, second countable, locally fair C^∞-scheme. Then every open cover of X has a partition of unity subordinate to it.*

Proof. Since X is Hausdorff, second countable, and locally compact, it is paracompact. Let $\{U_i | i \in I\}$ be the open cover. We can cover each U_i with fair affine

C^∞-schemes $U_{ij}, j \in I_i$, since it is locally fair, to get a subcover $\{U_{ij} | i \in I, j \in J\}$. We can then take a locally finite refinement of this subcover $\{V_{ij}\}$, and applying the Shrinking Lemma we can in fact assume that the closure of V_{ij} is also contained in U_{ij}. Characteristic functions exist for open subsets of fair affine C^∞-schemes (we can use part of the proof of the previous lemma), so let f_{ij} be a characteristic function for V_{ij}, with $f_{ij}(x) = 0$ for $x \in U_{ij} - \overline{V_{ij}}$. So the support of f_{ij} is $\overline{V_{ij}}$. We can extend f_{ij} by 0 to an element of $\mathcal{O}_X(X)$, and by squaring it if necessary, we can assume it is nonnegative. Then $\sum f_{ij}$ is a locally finite sum, as f_{ij} is supported in $\overline{V_{ij}}$, so it has a unique limit c, with $c(x)$ invertible in $\mathbb{R}$. So c is invertible in $\mathcal{O}_X(X)$. Defining $d_i = c^{-1} \sum_{j \in I_i} f_{ij}$ gives us $\sum_{i \in I} d_i$, a partition of unity subordinate to U_i as required. □

2.4 Sheaves of modules on C^∞-schemes

Both Joyce [5] and González and Salas [4] have developed concepts of sheaves of modules over C^∞-schemes or differentiable spaces. Joyce's is the more general construction, and this section is based on his chapter on sheaves of modules on C^∞-schemes [5, Chapter 6]. Because of space considerations, we only can only briefly discuss modules over C^∞-rings. In this section, we define the concepts for modules corresponding to fairness and the Spec functor, and quasicoherent sheaves of modules.

Let M a module over a C^∞-ring C, and $x \in \operatorname{Spec}^{\mathrm{r}} C$. Then the module of germs at x is just $\mathcal{O}_{C,x} \otimes_{\mathbb{R}} M$. Let $\pi_{M,x} = \pi_x \otimes \mathrm{id}_M : M \cong C \otimes M \to \mathcal{O}_{C,x} \otimes_{\mathbb{R}} M$. If $c \in C$, and M is a module, then we can define a $C\{c\}^{-1}$-module $M\{c\}^{-1} = M \otimes C\{c\}^{-1}$.

Definition 2.19. Let M be a module over a fair C^∞-ring C. A locally finite sum in M, $\sum_{i \in I} m_i$ is a formal sum such that there is an open covering $\{U_j | j \in J\}$ of $\operatorname{Spec}^{\mathrm{r}} C$ such that on each U_j, for all but finitely many $i \in I$, $\pi_{M,x}(m_i) = 0$ for all $x \in U_j$. Two locally finite sums $\sum_{i \in I} m_i$ and $\sum_{i \in J} n_i$ are said to be equivalent if for all $x \in C, \pi_{M,x}(\sum_{i \in I} m_i) = \pi_{M,x}(\sum_{i \in J} n_i)$. An element $m \in M$ is a limit of a locally finite sum if it is equivalent to it. M is a *complete* module if every locally finite sum has a unique limit. The completion of a C-module M is formed by adding equivalence classes of all locally finite sums, with the C-operation being point-wise.

Let C be a C^∞-ring, and I fair ideal. Then I is a complete module. Joyce proves that the subcategory of complete C-modules is an abelian subcategory, and finitely presented C-modules are complete [5, p.36].

Definition 2.20. A $\mathcal{O}_X$*-module* (X, ξ) for a C^∞-ringed space $(X, \mathcal{O}_X)$ is a sheaf over X such that for an open $U \subset X$, $\xi(U)$ is an $\mathcal{O}_X(U)$-module, the restriction maps are linear maps, and the action of $\mathcal{O}_X(U)$ on $\xi(U)$ is compatible with restriction. Let M be a C-module. Define a presheaf over $\mathrm{Spec}^{\mathrm{r}}\, C$ by setting for open U, $\tilde{\xi}_M(U) = \mathcal{O}_C(U) \otimes_C M$. Write $\mathrm{MSpec}(M) = (\mathrm{Spec}^{\mathrm{r}}\, C, \xi_M)$ for the sheafification of this presheaf.

Just as for finitely generated C^∞-rings and fairification, the sheafification of this presheaf is precisely the completion at every open set. That is, if M is a complete module over a fair C^∞-ring C, $\xi_M(U)$ is the completion of $\mathcal{O}_C(U) \otimes_C M$. The proof can be found in [5, p.45]. Hence there is an equivalence of categories between complete modules over a fair C^∞-ring C, and $\mathcal{O}_C$-modules. For a C^∞-scheme $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module ξ_X is said to be *quasicoherent* if (just as in ordinary algebraic geometry) there is an affine open cover $\{U_i\}$ of X such that $\xi_X(U_i) \cong \mathrm{MSpec}(M_i)$ for an $\mathcal{O}_X(U_i)$-module M_i. Joyce has proved that every $\mathcal{O}_X$-module of a locally fair C^∞-scheme is quasicoherent.

Example 2.21. Let C be a C^∞-ring, and Ω_C its cotangent module. For any $f \in C$, the localization of Ω_C at f is isomorphic to the cotangent module of $C\{f\}^{-1}$ [5, p. 40]. If C is fair (finitely generated, finitely presented), then Ω_C is complete (finitely generated, finitely presented) [5, p.38]. Using sheafification, one can define the cotangent sheaf of a C^∞-scheme (see [5, p.49]). In the particular case of a fair affine C^∞-scheme, say $\mathrm{Spec}^{\mathrm{r}}\, C$, the cotangent sheaf is $\mathrm{MSpec}(\Omega_C)$.

3. Embeddings

In Joyce's book, fair affine C^∞-schemes and locally fair C^∞-schemes are the main objects of consideration. In González's and Salas' book [4], they only discuss differentiable spaces, which are locally fair C^∞-schemes, with the additional assumption that they look locally like $\operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^n)/I$ for some n and some I which is closed under the Fréchet topology. They develop good definitions of embeddings, dimension and smoothness for differentiable algebras. In this chapter, we will generalize their results on embeddings to locally fair C^∞-schemes. The reason that these results can largely be generalized is because they will depend on the stalk of a C^∞-scheme $\mathcal{O}_{X,x}$.

3.1 Open and closed C^∞-subschemes

Following [4, p.57], we give the following definitions.

Definition 3.1. Let $(X, \mathcal{O}_X)$ be C^∞-scheme. Let $V \subset X$ be locally closed. Let $\mathcal{I}$ be a sheaf of ideals of $\mathcal{O}_X|_V$. As this is a sheaf, $\mathcal{I}(U)$ must be fair. Define a sheaf $\mathcal{O}_X/\mathcal{I}$ to be $\mathcal{O}_X/\mathcal{I}(U) = \mathcal{O}_X|_V(U)/\mathcal{I}(U)$ for U open in V. This is in fact a sheaf, not merely a presheaf, because $\mathcal{I}(U)$ is a fair ideal by assumption, and so $\mathcal{O}_X|_V(U)/\mathcal{I}(U)$ is fair. When $(V, \mathcal{O}_X/\mathcal{I})$ is C^∞-scheme, we call it a C^∞-*subscheme* of $(X, \mathcal{O}_X)$. If $\mathcal{I} = 0$, and V is open, then it is called an *open* C^∞-*subscheme* of $(X, \mathcal{O}_X)$. If V is closed, we call it a *closed* C^∞*-subscheme.* A C^∞-subscheme $(V_1, \mathcal{O}_X/\mathcal{I}_1)$ is said to be contained in another, $(V_2, \mathcal{O}_X/\mathcal{I}_2)$ if $V_1 \subset V_2$ and $\mathcal{I}_2|_{V_1} \subset \mathcal{I}_1$.

Remark 3.2. Open C^∞-subschemes of $(X, \mathcal{O}_X)$ correspond to $(U, \mathcal{O}_X|_U)$, for U open, so if X is affine, so is any open subscheme of $(X, \mathcal{O}_X)$. Closed C^∞-subschemes correspond to sheaves of ideals on X. To see this, if $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_X$, let $V = \{x \in X | \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$, the support of the quotient sheaf $\mathcal{O}_X/\mathcal{I}$, and so V is closed. Then $(V, \mathcal{O}_X/\mathcal{I})$ is a closed subscheme, because we consider $\mathcal{O}_X/\mathcal{I}$ as a sheaf on V via restriction to V. If it is a C^∞-scheme, then it is a closed C^∞-subscheme of $(X, \mathcal{O}_X)$. In the other direction, if $(V, \mathcal{O}_X/\mathcal{I})$ is a closed C^∞-subscheme, we can extend $\mathcal{I}$ from a sheaf of ideals of $\mathcal{O}_X|_V$ to a sheaf of ideals of $\mathcal{O}_X$ by having it coincide with $\mathcal{O}_X$ on the open set $X - V$.

When working with embeddings, it is sometimes more convenient to consider morphisms of ringed space $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to be a pair (π, π^*), where $\pi : X \to Y$ is as before, but π^* is a map of sheaves on X, $\pi^*\mathcal{O}_Y \to \mathcal{O}_X$. Using universal properties, one can show that this is equivalent to our previous definition of morphisms of C^∞-ringed spaces.

C^∞-subschemes come with canonical inclusion morphisms. That is, if $(V, \mathcal{O}_X/\mathcal{I})$ is a C^∞-subscheme of $(X, \mathcal{O}_X)$, there is a morphism of C^∞-schemes

$$(i, i^*) : (V, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$$

where $i : V \to X$ is just the inclusion morphism. Then $i^*\mathcal{O}_X$ is just $\mathcal{O}_X|_V$, and so

$$i^*(U) : \mathcal{O}_X|_V(U) \to \mathcal{O}_X/\mathcal{I}(U) = \mathcal{O}_X|_V(U)/\mathcal{I}(U)$$

is the projection, for U open in V. So i^* is surjective. For $f \in \mathcal{O}_X(X)$, we call $i^*(X)(f)$ the restriction of f to V.

Proposition 3.3. *[4, p.59] Closed C^∞-subschemes of a fair affine C^∞-scheme* $\mathrm{Spec}^r C$ *correspond precisely to fair ideals of the fair ring C.*

Proof. Let I be a fair ideal of C. Then C/I is a fair ring, and as a topological space $\mathrm{Spec}^{\mathrm{r}}\, C/I = Z(I)$. Let $\pi : C \to C/I$ be the projection map. It induces a morphism $\mathrm{Spec}^{\mathrm{r}}\, C/I \to \mathrm{Spec}^{\mathrm{r}}\, C$. If $U = D(f)$ is an open set, then

$$\begin{aligned} \pi_\#(D(f)) : C\{f\}^{-1fa} &\to \mathcal{O}_{C/I}(V \cap D(f)) \\ &= C/I\{f + I\}^{-1fa} \cong (C\{f\}^{-1}/\tilde{I})^{fa} \cong C\{f\}^{-1}/(\tilde{I})^{fa}, \end{aligned}$$

where $\tilde{I}$ is the image of I in $C\{f\}^{-1}$, and $\tilde{I}^{fa}$ its fairification. Let $\mathcal{I}$ be sheaf of ideals given by I, so by definition $\mathcal{I}(D(f)) = (\tilde{I})^{fa}$. So the kernel of $\pi_\#(D(f))$ is $\tilde{I}^{fa} = \mathcal{I}(D(f))$. As $\mathcal{O}_C/\mathcal{I}$ has support $Z(I)$, $(\mathrm{Spec}^{\mathrm{r}}\, C/I, \mathcal{O}_{C/I}) \cong (Z(I), \mathcal{O}_C/\mathcal{I})$, which is a closed C^∞-subscheme.

If $(V, \mathcal{O}_C/\mathcal{I})$ is a closed C^∞-subscheme, then by extending $\mathcal{I}$ from a sheaf of ideals of $\mathcal{O}_X|_V$ to a sheaf of ideals of $\mathcal{O}_X$, $I = \mathcal{I}(V)$ is a ideal of C. It is complete as a module, so it is in fact a fair ideal of C. Joyce has proved that for any $\mathcal{O}_C$-module (X, ξ_X), $\mathrm{MSpec}(\xi_X(X))$ is canonically isomorphic to ξ_X [5, p.45]. So $(V, \mathcal{O}_C/\mathcal{I}) \cong (V, \mathcal{O}_{C/I}) \cong \mathrm{Spec}^{\mathrm{r}}\, C/I$. In particular, $(V, \mathcal{O}_C/\mathcal{I})$ is affine. □

This is again quite different from the situation in ordinary algebraic geometry, where sheaves of ideals do not necessarily define closed subschemes (they need to be quasicoherent). An immediate corollary of the above is that affine C^∞-schemes are isomorphic to closed C^∞-subschemes of $\mathbb{R}^n$ for some n. We can now generalize Joyce's result which we proved in Lemma 2.16.

Corollary 3.4. *C^∞-subschemes of affine C^∞-schemes are affine.*

Proof. We have proved that closed C^∞-subschemes and open C^∞-subschemes are affine. Given an arbitrary C^∞-subscheme, $(V, \mathcal{O}_X/\mathcal{I})$, we have by definition that V is locally closed, and so there is an open U and a closed Y, subsets of $\operatorname{Spec^r} C$ such that $V = Y \cap U$. Then $(V, \mathcal{O}_U/\mathcal{I})$ is a closed C^∞-subscheme of $(U, \mathcal{O}_U)$, which is an open C^∞-subscheme of $(X, \mathcal{O}_X)$. As $(X, \mathcal{O}_X)$ is assumed to be affine, $(U, \mathcal{O}_U)$ is affine, and so $(V, \mathcal{O}_V/\mathcal{I}) \cong (V, \mathcal{O}_U/\mathcal{I})$ is affine. □

Definition 3.5. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally fair C^∞-schemes. A morphism $\phi : X \to Y$ of C^∞-schemes is called an *embedding* if there is a C^∞-subscheme Y' of Y and an isomorphism $\phi' : X \to Y'$ such that $\phi = i \circ \phi'$, where i is the inclusion. Then ϕ is called an *open embedding* or a *closed embedding* when Y' is a open or closed C^∞-subscheme respectively. A morphism $\phi : X \to Y$ is called a *local embedding* at $x \in X$ if there is an open neighborhood U of x such that $\phi|_U : (U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ is an embedding.

By Proposition 3.3, if $\operatorname{Spec^r} C = (X, \mathcal{O}_X)$ is affine, then $Y \to X$ is an embedding if and only if $(Y, \mathcal{O}_Y) \cong \operatorname{Spec^r} C/J$ for $J \subset C$ a fair ideal, if and only if Y is affine and $C \to \mathcal{O}_Y Y$ is surjective.

3.2 Covering dimension

Understanding embeddings will require the notion of covering dimension.

Definition 3.6. Let X be a topological space, and $\{U_i | i \in I\}$ an open covering. The *order* of the open covering is the maximal $n \in \mathbb{N}$ such that there exist $i_1, \ldots, i_{n+1} \in I$ such that $\bigcap_{j=1}^{j=n+1} U_{i_j} \neq \emptyset$ if the maximum exists, and is ∞ otherwise. The *covering dimension* of a normal topological space is the minimal d, if it exists, such that if U_i is any finite open cover of X, then it has a finite refinement of order less than d, and ∞ if it doesn't exist.

Following [4, p. 53-55], we prove some results about covering dimension, which will we later apply to C^∞-schemes.

Lemma 3.7. *Let X be a topological space which is Hausdorff, and let $K \subset X$ be a compact subset of X such that dim$(K) \leq d$. Then given any open cover*

$\{U_a | a \in A\}$, there is a refinement $\{V_a : a \in A\}$ such that $V_a \subset U_a, V_a \cap (X - K) = U_a \cap (X - K)$, and $\{V_a \cap K : a \in A\}$ has order less than d.

Proof. Since K is compact, there is a finite subcover of $\{U_a \cap K : a \in A\}$, call it $\{U_b \cap K : b \in B\}$. K has dimension less than d, so $\{U_b \cap K : b \in B\}$ has a finite refinement of order less than d, which we call $\{U'_b | b \in B\}$. For $b \in B$, let $a_b \in A$ such that $U'_b \subset U_{a_b}$. U'_b is open in K, so there is an open neighborhood W'_b of K such that $U'_b = K \cap W'_b$. Set $W_b = W'_b \cap U_{a_b} \subset U_{a_b}$. Then the open cover that we need is

$$V_a = (U_a - K \cap U_a) \cup (\cup_{\{b | a_b = a\}} W_b)$$

Clearly $V_a \subset U_a$, $V_a \cap (X - K) = U_a \cap (X - K)$, and as $V_a \cap K = \cup_{b | a_b = a} U'_b$, $\{V_a \cap K\}$ has order less than d. $\square$

This lemma allows us to characterise the dimension of certain topological spaces in terms of the compact neighborhoods.

Theorem 3.8. *[4, p. 54] Let X be a separated, second countable space, and assume that for every $x \in X$, x has a compact neighborhood of dimension less than d. Then $dim X \leq d$.*

Proof. We will prove this theorem in two steps. First, we will show that we can find a countable set of compact sets $K_n, n \in \mathbb{N}$ of dimension less than or equal to d, which cover X, such that $K_n \subset K^\circ_{n+1}$. Taking the compact neighborhoods given by assumption, there is an open cover of $\{V_i\}$ of X such that the closure of V_i is compact and of dimension less than d. Since X is second countable, there is a countable subcover $\{V_n\}_{n \in \mathbb{N}}$. We define the K_n inductively: $K_1 = \overline{V_1}$. Assume we have $K_1, \dots, K_n$ with the needed property. K_n is compact, so let $V_{n_1}, \dots, V_{n_i}$ be a finite subcover of K_n. Then define $K_{n+1} = \cup_{j=1}^{i} \overline{V_{n_j}}$, which is compact, and its interior contains K_n. It has dimension less than d because it is the finite union of spaces which have dimension less than d (one can use the previous lemma to show this).

Next, we show that any open cover $\{U_i\}$ has a refinement of order less than the d, and so $\dim X \leq d$. Define $Q_n = K_n - K^\circ_{n-1}$, which is compact and of dimension less than d because the K_m are. Also,

$$X = \bigcup_{n \in \mathbb{N}} K_n = \bigcup_{n \in \mathbb{N}} Q_n.$$

By the previous lemma, given an open covering $\mathcal{U} = \{U_i | i \in I\}$, there is a refinement $\mathcal{U}^1 = \{U_i^1\}$ of order less than d on Q_1 such that $\mathcal{U}$ and $\mathcal{U}^1$ agree on $X - Q_1$. Define $\mathcal{U}^n$ inductively as the open refinement of $\mathcal{U}^{n-1}$ of order less than d on Q_n such that $\mathcal{U}^n$ and $\mathcal{U}^{n-1}$ agree on $X - Q_n$. Let $\{V_i\}$ be the open covering where $V_i = \cap_n U_i^n$. This covers X because if $x \in X$, $x \in Q_n$ for some n, and so $x \in U_i^n$ for all i. To show that it has order less than d, suppose that $V_{i_1} \cap \cdots \cap V_{i_{d+2}}$ is nonempty, and has nonempy intersection with Q_n. But then $U_{i_1}^n \cap \cdots \cap U_{i_{d+2}}^n \cap Q_n \neq \emptyset$, which is a contradiction because $\mathcal{U}^n$ has order less than d on Q_n. □

We can apply this to C^∞-schemes.

Corollary 3.9. *[4, p. 55] Let $(X, \mathcal{O}_X)$ be a separated, second countable locally fair C^∞-scheme, and assume that for every $x \in X$, x has a compact neighborhood of dimension less than d. Let $\mathcal{U} = \{U_i\}$ be an arbitrary open cover of X. Then there are $d+1$ families of disjoint open sets $\mathcal{U}^i, i = 0, \ldots, d$ such that their union $\mathcal{V} = \bigcup_{i=0}^d \mathcal{U}^i$ is a locally finite refinement of $\mathcal{U}$.*

Proof. Because of Theorem 3.8, we can assume that $\mathcal{U}$ has order less than d. We have proved that there is a partition of unity subordinate to this open covering, $\{\phi_i\}$. Since $\mathcal{U}$ has order less than d, $\{\mathrm{Supp}\phi_i\}$ also has order less than d, so for every $x \in X$, no more than d of the ϕ_i have $\phi_i(x) \neq 0$.

For $j_0, \ldots, j_n \in I$, define $U(j_0, \ldots, j_n) = \{x \in X | \phi_i(x) < \phi_{j_k}(x), \forall k = 0, \ldots, n, \forall i \notin \{j_0, \ldots, j_n\}\}$. Because at most d of the $\phi_i(x) \neq 0$, we can assume $n \leq d$. Let $\mathcal{W} = \{U(j_0, \ldots, j_n)\}$. $\mathcal{W}$ covers X, as $\phi_i(x) = 0$ for some i, so $x \in U(j), j \neq i$. As $U(i_0) \subset \mathrm{Supp}\phi_{i_0} \subset U_{i_0}$, $\mathcal{W}$ is a refinement of $\mathcal{U}$. As $\{\mathrm{Supp}\phi_i\}$ is locally finite, for any $x \in X$, there is a neighborhood U such that only finitely many ϕ_i are nonzero on U, say $\phi_{i_1}, \ldots, \phi_{i_k}$. Then $U(j_0, \ldots, j_n) \cap U \neq \emptyset$ if and only if $\{j_0, \ldots, j_n\} \subset \{i_1, \ldots, i_k\}$, so $\mathcal{W}$ is locally finite.

For $0 \leq n \leq d$, let

$$F_n = \{U(j_0, \ldots, j_n)\}.$$

This is a family of disjoint open sets: suppose $x \in U(j_0, \ldots, j_n) \cap U(i_0, \ldots, i_n)$. If $(j_0, \ldots, j_n) \neq (i_0, \ldots, i_n)$, then there is $i_k \notin \{j_0, \ldots, j_n\}, j_l \notin \{i_0, \ldots, i_n\}$. But then $\phi_{i_k}(x) < \phi_{j_l}(x)$, and $\phi_{i_k}(x) > \phi_{j_l}(x)$, which is a contradiction. $\square$

3.3 The embedding theorem

In this section, we follow González and Salas [4] in proving the embedding theorem (which is stated, but not proved in [5]). Their proof generalizes to locally fair C^∞-schemes because it relies on understanding T_x^*X, which is defined using only the ring of germs at x, so we can avoid the issue of sheafification/fairification.

Definition 3.10. Let $(X, \mathcal{O}_X)$ be a locally fair C^∞-scheme. For $x \in X$ define $\mathcal{O}_{X,x}$ to be the stalk at x with maximal ideal m_x. If $(X, \mathcal{O}_X) \cong \mathrm{Spec}^{\mathrm{r}}\,(C^\infty(\mathbb{R}^n)/I)$, then $X \cong Z(I)$ and $\mathcal{O}_x$ is the localization of $C^\infty(\mathbb{R}^n)/I$ at the set $\{f + I \in C^\infty(\mathbb{R}^n)/I, f(x) \neq 0\}$, that is, it is $\mathcal{O}_{C^\infty(\mathbb{R}^n),x}/\pi_x(I)$, which has maximal ideal $m_x = \{f | f(x) = 0\}$. If $(X, \mathcal{O}_X)$ is locally fair, and $x \in X$ has affine open neighborhood $(U, \mathcal{O}_X|_U)$, then $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$, so stalks can be computed as in the affine case. Define $T_x^*X = m_x/m_x^2$. Let $\mathrm{Der}_\infty(\mathcal{O}_{X,x}, \mathbb{R})$ be the set of C^∞-derivations.

Lemma 3.11. *Let C be a fair C^∞-ring, $X = \mathrm{Spec}^r C$. Let $x \in X$. Then the dual vector space of $Der_\infty(\mathcal{O}_{X,x}, \mathbb{R})$ is canonically isomorphic to T_x^*X.*

Proof. Let $d \in \mathrm{Der}_\infty(\mathcal{O}_{X,x}, \mathbb{R})$, and $gh \in m_x^2$. If $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = xy$, then $d(gh) = d(C_f(g, h)) = gd(h) + d(g)h \in m_x$ and so $d(gh) = 0$. So $\mathrm{Der}_\infty(\mathcal{O}_{X,x}, \mathbb{R}) = \mathrm{Der}_\infty(\mathcal{O}_{X,x}/m_x^2, \mathbb{R})$. So each $d : \mathcal{O}_{X,x} \to \mathbb{R}$ gives us a restricted $\mathbb{R}$-linear map $m_x/m_x^2 \to \mathbb{R}$. On the other hand, suppose $\phi : m_x/m_x^2 \to \mathbb{R}$ is an $\mathbb{R}$-linear map. There is an $\mathbb{R}$-linear map $\mathcal{O}_{X,x} \to m_x/m_x^2$ which takes $[(f, U)] \in \mathcal{O}_{X,x} \mapsto [(f - f(x), U)] + m_x^2$ (where $f(x)$ is understood as $f + m_x \in \mathcal{O}_{X,x}/m_x$). Composing this with ϕ we get an $\mathbb{R}$-linear map $\mathcal{O}_{X,x} \to \mathbb{R}$, which we call d. Now we show that it satisfies the C^∞-Leibniz condition. By assumption, $C \cong C^\infty(\mathbb{R}^n)/J$, for J fair. Then $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{R}^n,x}/\pi_x(J)$. So an $\mathbb{R}$-linear map $\mathcal{O}_{X,x} \to \mathbb{R}$ corresponds to an $\mathbb{R}$-linear map $d : \mathcal{O}_{\mathbb{R}^n,x} \to \mathbb{R}$ with $d(\pi_x(J)) = 0$. Denote $\pi_x(J) = J_x$. So we have reduced to the case $C = C^\infty(\mathbb{R}^n)$, because for $f \in C^\infty(\mathbb{R}^m), c_1, \ldots, c_n \in C_x^\infty(\mathbb{R}^n)$, if the lemma is true for $C^\infty(\mathbb{R}^n)$, then $d(C_f(c_1 + J_x, \ldots, c_m + J_x)) = d(C_f(c_1, \ldots, c_m) + J_x) = d(C_f(c_1, \ldots, c_m)) = \sum_{i=1}^m C_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_m) dc_i = \sum_{i=1}^m C_{\frac{\partial f}{\partial x_i}}(c_1 + J_x, \ldots, c_m + J) d(c_i + J_x)$.

So we assume that $C = C^\infty(\mathbb{R}^n)$, and we have an $\mathbb{R}$-linear map $d : \mathcal{O}_{X,x} \to \mathbb{R}$, $m_x^2 \subset \ker d$. For any $f, g \in C_x^\infty(\mathbb{R}^n)$,

$$0 = \phi((f - f(x)(g - g(x)) + m_x^2) = \phi(fg - f(x)g - g(x)f + f(x)g(x) + m_x^2),$$

$$0 = \phi(fg - f(x)g(x) - (f(x)g + g(x)f - 2f(x)g(x)) = d(fg) - d(f(x)g + g(x)f),$$

$$d(fg) = f(x)d(g) + g(x)d(f).$$

So d is an $\mathbb{R}$-derivation of $\mathbb{R}^n$ at x. The set of derivations of $\mathbb{R}^n$ at x is spanned by $\frac{\partial}{\partial x_i}, i = 1, \ldots, n$ (see Lee [8, p. 57]). So $d = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}, a_i \in \mathbb{R}$. By the composition rule, $\frac{\partial}{\partial x_i}$ satisfies the C^∞-Leibniz condition, and so we conclude that d does as well. □

Let $\phi : Y \to X$ be a morphism of C^∞-schemes, $\phi(y) = x$. Let ϕ^* denote the corresponding morphism of sheaves, $\mathcal{O}_X \to \phi_*\mathcal{O}_Y$. We have a (local) morphism of C^∞-rings $\phi^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. As $\phi^*(m_x) \subset m_y$, it induces linear maps $\phi^* : T_x^*X \to T_y^*Y$.

Let $(Y, \mathcal{O}_Y)$ be a C^∞-subscheme of $(X, \mathcal{O}_X)$, and $i : Y \to X$ the canonical inclusion map. Then $i^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,x}$ is surjective. Consider the induced map $\mathrm{Der}_\infty(\mathcal{O}_{Y,y}, \mathbb{R}) \to \mathrm{Der}_\infty(\mathcal{O}_{X,x}, \mathbb{R}), d \mapsto d \circ i^*$. If $d_1 \circ i^* = d_2 \circ i^*$, then because i is surjective, $d_1 = d_2$. So this map is injective. Hence its dual, the induced map $T_x^*X \to T_x^*Y$, is surjective.

Proposition 3.12. *[4, p. 64] Let $(X, \mathcal{O}_X)$ be a locally fair C^∞-scheme, $x \in X$. Then there is an open neighborhood U of x and $j : U \to \mathbb{R}^n$ a closed embedding such that $j^* : T_{j(p)}^*\mathbb{R}^n \to T_p^*U$ is an isomorphism.*

Proof. We can assume that $(X, \mathcal{O}_X) \cong \mathrm{Spec}^{\mathrm{r}}\, C^\infty(\mathbb{R}^n)/J$ for J fair (rings of germs can be computed in an affine neighborhood), and $x \in Z(J)$. Then we have an exact sequence

$$0 \to J \to C^\infty(\mathbb{R}^n) \to C \to 0,$$

which gives us an exact sequence

$$0 \to \pi_x(J) \to \mathcal{O}_{\mathbb{R}^n,x} \to \mathcal{O}_{X,x} \to 0,$$

and hence an exact sequence

$$0 \to \pi_x(J)/\pi_x(J) \cap m^2_{C^\infty(\mathbb{R}^n),x} \to m_{C^\infty(\mathbb{R}^n),x}/m^2_{C^\infty(\mathbb{R}^n),x} \to m_{X,x}/m^2_{X,x} \to 0$$

as vector spaces. So we can choose a basis of $m_{C^\infty(\mathbb{R}^n),x}/m^2_{C^\infty(\mathbb{R}^n),x}$, call it

$$\{d_x f_1, \ldots, d_x f_m, d_x g_{m+1}, \ldots, d_x g_n\},$$

$g_i, f_i \in C^\infty(\mathbb{R}^n)$, such that $d_x f_1, \ldots, d_x f_m$ is a basis for $T^*_x X$ and $g_{m+1}, \ldots, g_n \in J$. This means that $(f_1, \ldots, f_m, g_{m+1}, \ldots, g_n)$ is a coordinate system, and for some open neighborhood U' of x, $Y = Z(g_{m+1}, \ldots, g_n)$ cuts out a submanifold which is diffeomorphic to $\mathbb{R}^m$. Now by Proposition 1.23, $I(Z(g_{m+1}, \ldots, g_n)) = (g_{m+1}, \ldots, g_n) \subset J$, so

$$(U', C^\infty(\mathbb{R}^n)/(g_{m+1}, \ldots, g_n)|_{U'}) \cong (U', \mathcal{O}_Y|_{U'}).$$

As $(g_1, \ldots, g_n) \subset J$, J defines a closed C^∞-subscheme $(Z(J), \mathcal{O}_{C^\infty(\mathbb{R}^n)/(g_{m+1},\ldots,g_n)}/\mathcal{J})$ of $\operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^n)/(g_{m+1}, \ldots, g_n)$. So $(U = U' \cap X, \mathcal{O}_X|_U)$ is a closed C^∞-subscheme of $(U', \mathcal{O}_Y|_{U'})$. So the map induced by the inclusion $T^*_p Y \to T^*_p U$ is surjective because this is an embedding, but as these real vector spaces have the same dimension, it is an isomorphism. $\square$

Definition 3.13. The *embedding dimension* at $x \in X$, for $(X, \mathcal{O}_X)$ a locally fair C^∞-scheme, is the dimension of $T^*_x X$ as a $\mathbb{R}$ vector space.

We will use embedding dimension to characterize affine C^∞-schemes. To do this, we will give characterizations of local embeddings, and then embeddings, using $T^*_x X$.

Theorem 3.14. *[4, p.65] Let X, Y be locally fair C^∞-schemes. A morphism $\phi : Y \to X$ is a local embedding at y, setting $x = \phi(y)$, if and only if the induced map $\phi^* : T^*_x X \to T^*_y Y$ is surjective.*

Proof. Assume that ϕ is a local embedding. Since the cotangent space is defined locally, we can assume that this is an embedding, and hence the induced map of cotangent spaces is surjective. Now suppose $\phi^* : T^*_x X \to T^*_y Y$ is surjective. Again, we can assume that X and Y are fair affine, and so that for some $n, m \in \mathbb{N}$ and ideals $I \subset C^\infty(\mathbb{R}^n), J \subset C^\infty(\mathbb{R}^m)$, $X \cong \operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^n)/I, Y \cong \operatorname{Spec}^{\mathrm{r}} C^\infty(\mathbb{R}^m)/J$. By the previous proposition (composing with a closed embedding if necessary), we can assume $T_p(Y) = T_p\mathbb{R}^m$. The morphism $\phi : Y \to X$ corresponds to a morphism $\phi^* : C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$. By our explicit construction of morphisms of finitely generated C^∞-rings, $\phi^* : C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$ determines a map $\psi : \mathbb{R}^m \to \mathbb{R}^n$, so that defining $\psi^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m), \psi^*(f) = f \circ \psi$, the following diagram commutes:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^n) \arrow[r] \arrow[d, "\psi^*"] & C^\infty(\mathbb{R}^n)/I \arrow[d, "\phi^*"] \\
C^\infty(\mathbb{R}^m) \arrow[r] & C^\infty(\mathbb{R}^m)/J.
\end{tikzcd}$$

(ψ, ψ^*) is a morphism of C^∞-schemes, so this induces a commutative diagram

$$\begin{tikzcd}
T_x^*\mathbb{R}^n \arrow[r] \arrow[d, "\psi^*"] & T_x^*X \arrow[d, "\phi^*"] \\
T_y^*\mathbb{R}^m \arrow[r] & T_x^*Y.
\end{tikzcd}$$

The bottom arrow is an isomorphism, and the top is surjective. By assumption, ϕ^* is also surjective, so by commutativity, ψ^* is surjective. Hence there are open neighborhoods U and V of x and y respectively such that $\psi : V \to U$ is a closed embedding. $V \cap Y \to V$ is also a closed embedding, so the composition $V \cap Y \to U$ is a closed embedding. As $V \cap Y \to U \cap X \to U$ is a closed embedding, and $U \cap X \to U$ is a closed embedding, $V \cap Y \to U \cap X$ is a closed embedding. Thus, ϕ is a local embedding. $\square$

Corollary 3.15. *Let X, Y be locally fair C^∞-schemes. A morphism $(\phi, \phi^*) : Y \to X$ is an embedding if and only if $\phi : Y \to \phi(Y)$ is a homeomorphism, and for every $y \in Y$, setting $x = \phi(y)$, the induced map $\phi^* : T_x^*X \to T_y^*Y$ is surjective.*

Proof. If ϕ is an embedding, then Y is homeomorphic to $\phi(Y)$, and we have shown for every $y \in Y$, setting $x = \phi(y)$, the induced map $\phi^* : T_x^*X \to T_y^*Y$ is surjective. Suppose we are given a morphism $(\phi, \phi^*) : Y \to X$ such that $\phi : Y \to \phi(Y)$ is a homeomorphism, and for every $y \in Y$, setting $x = \phi(y)$, the induced map $\phi^* : T_x^*X \to T_y^*Y$ is surjective. Then by the theorem, ϕ is a locally closed embedding at every point of Y. So as topological spaces, $\phi(Y)$ is a locally closed in X. Because ϕ is a locally closed embedding at every point, $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is surjective. Since surjectivity can be checked at the level of stalks, $\phi^* : \phi^*\mathcal{O}_X \to \mathcal{O}_Y$ is surjective. So by definition, ϕ is an embedding. $\square$

Theorem 3.16 (Embedding Theorem)**.** *[4, p.67] Let $(X, \mathcal{O}_X)$ be a locally fair C^∞-scheme. Then $(X, \mathcal{O}_X)$ is fair affine if and only if it is a separated space, its topology has a countable basis, and there is an $m \in \mathbb{N}$ such that the embedding dimension at x for all $x \in X$ is less than m.*

Proof. If $(X, \mathcal{O}_X)$ is affine, then X is isomorphic to a closed C^∞-subscheme of $\mathbb{R}^n$ for some n. So the topology has a countable basis, and it is separated, and n is the bound on the embedding dimension.

For the other direction, by Proposition 3.12, each $x \in X$ has an affine open neighborhood which is isomorphic to a C^∞-subscheme of $\mathbb{R}^n$ for some $n \leq m$, and so isomorphic to a C^∞-subscheme of $\mathbb{R}^m$, which gives us an open covering. By Corollary 3.9, there are n families of disjoint open sets $\mathcal{U}^i, i = 1, \ldots, n$ such that their union $\mathcal{V} = \bigcup \mathcal{U}^i$ is a locally finite refinement of this covering. As X is second countable, $\mathcal{U}^i$ is countable. Because $\mathcal{V}$ is a refinement of an affine covering with each open set isomorphic to a C^∞-subscheme of $C^\infty(\mathbb{R}^m)$, each open set of $\mathcal{V}$ is isomorphic to a C^∞-subscheme of $\mathbb{R}^m$. Since $\mathcal{U}^i$ is countable and disjoint, the union of its sets is isomorphic to a C^∞-subscheme of $\mathbb{R}^m$, and hence an affine C^∞-subscheme, so in fact we have a finite affine open cover $\{U_1, \ldots, U_n\}$.

To show that X is affine, we will construct an embedding from X to $\mathbb{R}^r$ for some r. First note that we can find smooth functions defining a closed embedding for U_i into $\mathbb{R}^s$ for some s that doesn't depend on i (we choose a maximum and use that the open cover is finite). That is, we have smooth functions $f_{i_1}, \ldots, f_{i_s} : U_i \to \mathbb{R}$ such that $F_i = (f_{i1}, \ldots, f_{i_s}) : U_i \to \mathbb{R}^s$ is a closed embedding: it is injective and the induced map $T_x^*\mathbb{R}^s \to T_x^*U_i$ is surjective for all $x \in U_i$. Applying the Shrinking

Lemma twice, and because open C^∞-subschemes of affine C^∞-schemes are affine, we have two affine open covers of X, $\{V_1, \ldots, V_n\}$, and $\{W_1, \ldots, W_n\}$, such that for all i,

$$\overline{V_i} \subset U_i, \overline{W_i} \subset V_i.$$

Because of the existence of characteristic functions, we can find $\phi_i \in \mathcal{O}_X(U_i)$ such that $\phi_i|_{W_i} = 1, \phi_i|_{U_i - \overline{V_i}} = 0$. Extending by 0, we have $\phi_i, \phi_i f_{ij} \in \mathcal{O}_X(X)$.

Next, we construct a proper smooth map $\phi : X \to \mathbb{R}$. As X is separated, and its topology has a countable basis, we can find a locally finite, countable open cover $\{A_n\}$ with compact closure, and a refinement $\{B_n\}$ with $\overline{B_n} \subset A_n$. Moreover, by the existence of partitions of unity, we can find $\phi_n : X \to [0, 1]$ which is supported in A_n, and equal to 1 on a neighborhood of $\overline{B_n}$. Then $\phi = \sum_{n \in \mathbb{N}} n\phi_n$ is well-defined. It is proper because given any $c \in \mathbb{R}$, the closed set $\{\phi \leq c\}$ is compact because it is a subset of the compact $\overline{B_1} \cup \cdots \cup \overline{B_n}$, $n \geq c$. So because proper maps from second countable paracompact Hausdorff spaces are closed, ϕ is closed.

Now let A be the finite set of all of these global functions:

$$A = \{\phi, \phi_i, \phi f_{ij} | i, j\}$$

Let $r = |A|$, and $g_1, \ldots, g_r$ be an ordering of A such that $g_1 = \phi$. We claim that $\psi = (g_1, \ldots, g_r) : X \to \mathbb{R}^r$ defines a closed embedding. First, we show that it is injective. Suppose we have $x, y \in X$ such that $\psi(x) = \psi(y)$. There is an i such that $x \in W_i$, and by assumption $\phi_i(y) = \phi_i(x) = 1$, and so $y \in U_i$. As $f_{ij}(x) = \phi_i f_{ij}(x) = \phi_i f_{ij}(y) = f_{ij}(y)$, $F_i(x) = F_i(y)$. But F_i is injective, so $x = y$.

Next, we need that the induced cotangent map $T_x^*\mathbb{R}^r \to T_x^*X$ is surjective. This is clear because $x \in U_i$ for some i, and by assumption $T_x^*\mathbb{R}^s \to T_x^*U_i$ is surjective. By Corollary 3.15, ψ is an embedding, and so X is isomorphic to a closed C^∞-subscheme of $\mathbb{R}^r$, which is affine. So X is affine. $\square$

In their book [4], González and Salas also define finite morphisms, relative dimension, and smooth morphisms for differentiable spaces. The notion of smoothness is particularly important, because it characterizes manifolds and morphisms of manifolds within the category of differentiable spaces. Their work is used in Gillam and Molcho's paper on log differentiable spaces, which discusses manifolds with corners. In Appendix C, we discuss the generalization of finite morphisms to C^∞-schemes.

4. C^∞-rings with corners

C^∞-rings allowed us to embed the category of smooth manifolds (in a way which preserved transversal products) into another category with desired properties. Smooth manifolds can be generalized to smooth manifolds with corners, which are spaces locally modeled on $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}$ instead of Euclidean space. In the last chapter of this paper, we will work towards a corresponding generalization to C^∞-rings with corners. First, we will define manifolds with corners and their boundaries, following Joyce's paper [6]. Second, we will summarize relevant results on monoids, which can be found in full in Gillam and Molcho's paper [3]. Finally, we will define C^∞-rings with corners, their quotients and localizations, and C^∞-schemes with corners. C^∞-rings with corners are new.

4.1 Manifolds with corners

We follow Joyce [6] entirely in this section, except we do not use his definition of smooth morphisms from that paper, but instead use what are called interior b-maps in the literature. Fiber products of b-transversal interior b-maps in the category of positive log differentiable spaces are not necessarily manifolds with corners, however, Gillam and Molcho [3] show that the fiber products can be resolved to a manifold with corners.

A continuous function $f : A \to B$, where A, B are subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, is said to be *weakly smooth* if it can be extended to a smooth map of open neighborhoods of A and B. The map f is a *weak diffeomorphism* if it is a homeomorphism and its inverse is weakly smooth.

Definition 4.1. A pair (M, A) is a *n-manifold with corners* if:

1. M is a topological space which is Hausdorff and second countable.

2. A is a collection of continuous maps $\{\phi_\alpha : U_\alpha \to M | \alpha \in I\}$ for open sets $U_\alpha \subset \mathbb{R}^n_k$, for some $0 \leq k \leq n$, such that $\phi_\alpha : U_\alpha \to \phi(U_\alpha)$ is a homeomorphism, and $\{\phi_\alpha(U_\alpha) | \alpha \in I\}$ is an open covering of M.

3. A is a smooth atlas: the charts (ϕ_α, U_α) are smoothly compatible. That is, for any $\alpha, \beta \in I$, with $\phi_\alpha(U_\alpha) \cap \phi_\alpha(U_\beta) \neq \emptyset$, $\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)) \to \phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta))$ is a weak diffeomorphism.

4. A is a maximal smooth atlas: there is no strictly larger smooth atlas which contains A.

For a manifold with corners M, we say a map $f : M \to \mathbb{R}$ is *weakly smooth* if given any chart (ϕ, U), $f \circ \phi : U \to \mathbb{R}$ is a weakly smooth map. Generalizing the definition of morphisms for manifolds without corners in a straightforward way, we can define weakly smooth maps between manifolds with corners.

Definition 4.2. Let M^m, N^n be manifolds with corners. A map $f : M \to N$ is *weakly smooth* if given any charts $(\phi, U), (\psi, V)$ on M, N respectively, $\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \to V$ is weakly smooth.

As with manifolds without corners, we denote the $\mathbb{R}$-algebra of smooth maps $M \to \mathbb{R}$ as $C^\infty(M)$. Analogously to the ordinary definition, the *tangent space* of $x \in M$ is

$$T_x M = \{v : C^\infty(M) \to \mathbb{R} | v \text{ is linear and } v(fg) = f(x)v(g) + g(x)v(f)\}.$$

The *cotangent space* is $T_x^* M = (T_x M)^*$.

Definition 4.3. Let M be a manifold with corners, and $x \in M$. Let (ϕ, U), be a chart, $0 \in U$ open in $\mathbb{R}^n_k, \phi(0) = x$. Then $d\phi|_0 : T_0(\mathbb{R}^n_k) \to T_x M, v \mapsto v(- \circ \phi)$ is an isomorphism, and $T_0(\mathbb{R}^n_k) \cong \mathbb{R}^n$. The *inward sector* of $T_x M$ is $IS(T_x M) = d\phi|_0(\mathbb{R}^n_k)$. Because of the compatibility condition on charts, this definition is independent of the choice of (ϕ, U).

We need to understand the boundary of a manifold with corners. This is a complicated construction that will take us several steps.

Definition 4.4. Let $U \subset \mathbb{R}^n_k$, and $u = (u_1, \dots, u_n) \in U$. The number of boundary faces of U which contain u is called the depth of u, that is

$$\operatorname{depth} u = \#\{i | 1 \leq i \leq k, u_i = 0\}.$$

The depth of $x \in M$ for an n-manifold with corners M is defined using charts. Given a chart $(\phi, U), x \in \phi(U)$, $\operatorname{depth} x = \operatorname{depth} \phi^{-1}(x)$.

For $k = 0, \dots, n$, the *depth k stratum* of M is

$$S^k(M) = \{x \in M : \operatorname{depth} x = k\}.$$

The depth k strata are disjoint, and cover M. By a projection onto the nonzero coordinates, $S^k(M)$ has the structure of an $n-k$ manifold without boundary. The closure of $S^k(M)$ is $\bigcup_{l=k}^n S^l(M)$, as if $x \in \overline{S^k(M)}$, then $x \in M$ and $k \leq \operatorname{depth} x$ (take a sequence in $S^k(M)$ converging to x).

Lemma 4.5. *Let $x \in S^k(M)$, for M a manifold with corners. Then $IS(T_xM) \cong \mathbb{R}^n_k$, and $IS(T_xM) \cap -IS(T_xM) \cong T_xS^k(M) \cong \mathbb{R}^{n-k}$.*

Proof. Since $x \in S^k(M)$, a chart (ϕ, U) for $x = \phi(0)$ has $U \subset \mathbb{R}^n_k$. As $d\phi_0$ is an isomorphism, $IS(T_xM) = d\phi_0(\mathbb{R}^n_k) \cong \mathbb{R}^n_k$. Using this isomorphism, we see that $IS(T_xM) \cap -IS(T_xM) \cong \mathbb{R}^{n-k} \cong T_xS^k(M)$, where the last isomorphism is because x has depth 0 in $S^k(M)$, which is a $n-k$ manifold without boundary, and so $T_xS^k(M) = \mathbb{R}^{n-k}$. $\square$

We need one more construction before we can define the boundary of a manifold with corners.

Definition 4.6. Let M be an n-manifold with corners, and $x \in M$. A *local boundary component β of M at x* assigns to each sufficiently small open neighborhood V of x a connected component W of $V \cap S^1(M), x \in \overline{W}$, such that the assignments are compatible. That is, given V, V' open neighborhoods, the assigned W, W' must have $x \in \overline{(W \cap W')}$.

Lemma 4.7. *Let M be an n-manifold with corners, and $x \in M$. Then there are* $\operatorname{depth} x$ *local boundary components of M at x.*

Proof. Let (ϕ, U) be a chart at x, $U \subset \mathbb{R}^n_k, \phi(u) = x$. Let $S^1_i(U)$ be the subset of $S^1(U)$ of $y \in U$ whose i^{th} coordinate is 0, $i = 1, \ldots, k$. Then $S^1(U)$ is the disjoint union of $S^1_i(U)$, as every $y \in S^1(U)$ has precisely one of its first k coordinates 0. Suppose $u_i = 0$. Then $u \in \overline{S^1_i(U)}$, and given any sufficiently small open ball $u \in B$, $B \cap S^1_i(U)$ is connected. So $S^1_i(U)$ determines a local boundary component of U at u for every i with $u_i = 0$. Distinct i, j give distinct boundary components. Because a local boundary component assigns connected components, which are maximal by definition, all local boundary components must be of this form. So there are precisely depth u local boundary components for U at u, and hence depth x local boundary components of M at x. □

Definition 4.8. Let M be a manifold. Define the *boundary* of M to be

$$\partial M = \{(x, \beta) : x \in M, \beta \text{ is a local boundary component of } M \text{ at } x\}.$$

Proposition 4.9. *If M is an n-manifold with corners for $n > 0$, then ∂M is an $(n-1)$-manifold with corners.*

Proof. Let (ϕ, U) be a chart for $x \in M$, $U \subset \mathbb{R}^n_k$. For $i = 1, \ldots, k$, define a chart (ϕ_i, U_i) as

$$U_i = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}_{k_1} : (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}) \in U\},$$

$$\phi_i : (x_1, \ldots, x_{n-1}) \mapsto (\phi(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}), \phi_*(S^1_i(U))).$$

These define compatible charts, so they induce an atlas on ∂M, which can be extended to a unique maximal atlas. □

Because ∂M is a manifold with corners, we can form $\partial\partial M$, which we call $\partial^2 M$. In general $\partial^k M$ is an $(n-k)$ manifold with corners. Here is a further characterization of $\partial^k M$.

Proposition 4.10. *Let M be an n manifold with corners. For $0 \leq k \leq n$,*

$$\partial^k M \cong \{(x, \beta_1, \ldots, \beta_k) | x \in M, \beta_i \text{ distinct local boundary components for } M \text{ at } x\}.$$

Proof. We use induction. The cases $k = 0, 1$ are by definition. For $k = 2$, we will show that local boundary components of ∂M at (x, β_1) are in one to one correspondence with local boundary components of x distinct from β_1. Points in $\partial^2 M$ are of the form $((x, \beta_1), \tilde{\beta}_2)$. Let (ϕ, U) be a chart for M, $\phi(u = (u_1, \ldots, u_n)) = x$, $U \subset \mathbb{R}^n_l, l \geq 2$. Suppose $u_{i_1} = 0$, and $\phi^{-1}(\beta_1)$ is the local boundary component induced by $S^1_{i_1}(U)$. By the definition of the smooth structure on $\partial^2 M$, we have a chart (U_{i_1}, ϕ_{i_1}) for ∂M, $\phi_{i_1}((u_1, \ldots, u_{i_1-1}, u_{i_1+1}, \ldots, u_n)) = (x, \beta_1)$. Then $\phi^{-1}_{i_1}(\tilde{\beta}_2)$ is a local boundary component for U_{i_1}, so it is induced by $S^1_j(U_{i_1})$ for $j = 1, \ldots, l-1$. Set $i_2 = j$ if $j < i_1$, and $i_2 = j + 1$ if $j \geq i_1$. So $u_{i_2} = 0$, as it is the j^{th} coordinate of $(u_1, \ldots, u_{i_1-1}, u_{i_1+1}, \ldots, u_n)$. The one to one correspondence (independent of the choice of charts) takes $\tilde{\beta}_2$ to β_2, the local boundary component induced by $S^1_{i_2}(U)$, which is distinct from β_1 as $i_1 \neq i_2$. This proves the claim for $k = 2$.

For the inductive step, suppose the claim is true for $2 \leq k < l < n$. A local boundary component of $\partial^l M$, can be identified with $(x, \beta_1, \ldots, \beta_{l-1}, \tilde{\beta}_l)$ where β_i are distinct local boundary components for M at x, and $\tilde{\beta}_l$ is a local boundary component for $\partial^{l-1} M$ at $(x, \beta_1, \ldots, \beta_{l-1})$. But using the same type of argument as for $k = 2$, $\tilde{\beta}_l$ can be uniquely associated to a local boundary component of M at x distinct from the β_i. □

Using this characterization, we see that there is a natural action of the symmetric group on k elements, S_k on $\partial^k M$, given by

$$\sigma : (x, \beta_1, \ldots, \beta_k) \mapsto (x, \beta_{\sigma(1)}, \ldots, \beta_{\sigma(k)}).$$

This action is free, so σ induces an isomorphism of $n - k$ manifolds with corners on $\partial^k M$. So we can form an $n - k$ manifold $C_k(M) = \partial^k M / S_k$. Explicitly,

$$C_k(M) = \{(x, \{\beta_1, \ldots, \beta_k\}) | x \in M, \beta_i \text{ distinct boundary components for } M \text{ at } x\}.$$

$C_k(M)$ is called the *k-corners* of M. Let $i_M : \partial M \to M, (x, \beta) \mapsto x$. Note that $|i^{-1}_M(x)| = \operatorname{depth} x$ by the previous lemma. In fact, i_M is continuous, finite, and proper, by considering local neighborhoods.

Definition 4.11. Let M be a manifold with corners, and $(x,\beta) \in \partial M$. A *boundary defining function for M at (x,β)* is a map $b : V \to [0,\infty)$, where V is an open neighborhood of x, such that:

1. b is weakly smooth in the sense that given any chart (ϕ, U), $b \circ \phi : U \to \mathbb{R}$ is a weakly smooth map.

2. $\mathrm{d}b : T_x V \to T_0[0,\infty)$ is nonzero.

3. There is an open neighborhood $\tilde{V}$ of (x,β) in $i_M^{-1}(V)$ such that $b \circ i_M|_{\tilde{V}} = 0$.

4. $i_M|_{\tilde{V}} : \tilde{V} \to \{x' \in V : b(x') = 0\}$ is a homeomorphism onto its image.

A boundary defining function exists for every (x,β). If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k$, and β is the boundary component corresponding to $x_i = 0, i \leq k$, then a boundary defining function at (x,β) is given by $b : V \to [0,\infty), b(y) = y_i$. As mentioned before, there have been multiple definitions of smooth maps between manifolds with corners. However, Melrose's interior b-maps work the best for our purposes, so we use this definition.

Definition 4.12. Let M, N be manifolds with corners, and $f : M \to N$ a map. f is *smooth* if for every $x \in M$, $y = f(x)$, with boundary components $\beta'_1, \ldots, \beta'_k$ at x and $\beta_1, \ldots, \beta_l$ at y, the following conditions hold. For boundary defining functions (V'_i, b'_i) at (x, β'_i), (V_j, b_j) at (y, β_j), we require that for every j, there exist $n_{1j}, \ldots, n_{kj} \in \mathbb{N}$ and a (weakly) smooth $G_j > 0$ such that $b_j \circ f = G_j b_1'^{n_{1j}} \cdots b_k'^{n_{kj}}$.

Locally, this gives a straightforward requirement. Let $f : \mathbb{R}^n_k \to \mathbb{R}^l_j, f = (f_1, \ldots, f_n)$. For $x \in \mathbb{R}^n_k$, each of its boundary components correspond to a 0 in the first k coordinates, and a boundary defining function for each is just the projection map onto that coordinate. Similarly for $y = f(x) = (y_1, \ldots, y_n)$. Suppose $x_{i_1}, \ldots, x_{i_s}$ are the zero coordinates of x. Then requiring f to be an interior map just means that for each j such that $y_j = 0, j \leq l$, $b_j \circ f = y_j \circ f = f_j$, there is $n_{1j}, \ldots, n_{sj} \in \mathbb{N}$ and a smooth $G_j > 0$ such that

$$f_j = G_j x_{i_1}^{n_{1j}} \cdots x_{i_s}^{n_{sj}}.$$

So in particular, if $f : \mathbb{R}^n_k \to [0,\infty)$, then $f = G x_1^{n_1} \cdots x_k^{n_k}$ for $G > 0, n_i \geq 0$.

4.2 Monoids

Gillam and Molcho [2, 3] use monoids and log structures to study manifolds with corners. They give differentiable spaces log structures, and maps between manifold with corners are then defined to be maps of log differentiable spaces, that is, maps which respect the log structure (these coincide with interior b-maps). So in a sense, the log structure keeps track of the corners. Instead of log differentiable spaces, we will want to consider C^∞-rings with corners. Monoids play an important role for C^∞-rings with corners, so we will need some basic definitions and results about monoids. The results of this section are from [2, 3], which give a thorough introduction to monoids.

Definition 4.13. A *monoid* is a set M with a commutative, associative operation $+$ with a unit. That is, a monoid is basically an abelian group without inverses. A morphism of monoids is a map which respects the operation and takes 0 to 0. M^* is the set of units in M.

One has to be careful with surjectivity and injectivity for monoid morphisms, because unlike for groups, for example, a monoid morphism can have a 0 kernel and not be injective. For example, the map $\mathbb{N}^2 \to \mathbb{N}, (n, m) \mapsto n + m$ is not injective but has kernel 0. Injectivity and surjectivity mean injectivity and surjectivity at the level of sets.

Definition 4.14. A *monoidal equivalence* on M is an equivalence relation $\sim$ satisfying $m_1 \sim m_2, n_1 \sim n_2 \Rightarrow m_1 + n_1 \sim m_2 + n_2$. Given a monoidal equivalence, we define the set of equivalence classes $M/\sim$ to have operation $[m] + [n] = [m + n]$. This is well defined, because if $[m_1] = [m_2], [n_1] = [n_2]$, then $[m_1 + n_1] = [m_2 + n_2]$.

Often, we identify a monoidal equivalence $\sim$ on M with the set $R = \{(m, n) | m \sim n\} \subset M \times M$.

Lemma 4.15. *[2, p.5] Let N, M be monoids, and $f : N \to M$ a morphism. Then the set $R \subset M \times M$ given by*

$$(m, n) \in R \Leftrightarrow \textit{there exist } p, q \in N \textit{ such that } f(p) + m = f(q) + n$$

defines a monoidal equivalence $\sim$. $M/\sim$ is the cokernel of f.

Proof. R is clearly symmetric and reflexive. It is transitive since if $(m,n) \in R, (n,l) \in R$, there exist $p,q,r,s \in N$ such that $f(p) + m = f(q) + n$ and $f(r)+n = f(s)+l$. As $f(p+r)+m = f(q+r)+n = f(q+s)+l$, $(m,l) \in R$. R is a monoidal equivalence since given $(m,n),(m',n') \in R$, there exist $p,q,p',q' \in N$ such that $f(p)+m = f(q)+n$ and $f(p')+m' = f(q')+n'$. As $f(p+p')+(m+m') = f(q+q') + (n+n')$, $(m+m', n+n') \in R$. To show that $M \to M/\sim$ is the cokernel of f, we use the universal property. Suppose $g : M \to P$ satisfies $g \circ f = 0$. Define $M/\sim \to P, [m] \mapsto g(m)$, which is well defined because if $[m] = [n]$, then there exist $p,q \in N$ such that $f(p) + m = f(q) + n$, and so $g(m) = g(m) + g(f(p)) = g(m + f(p)) = g(n + f(q)) = g(n)$. The composition $M \to M/\sim \to P$ is g. This map $M/\sim \to P$ is the unique such map because $M \to M/\sim$ is surjective. $\square$

In particular, using the inclusion map, if $N \subset M$ is a submonoid, then we can define M/N using the equivalence relation $m \sim m'$ if and only if there exist $n, n' \in N$ such that $m + n = m' + n'$.

Let $(M, \cdot)$ be a monoid. Every set in $R \subset M \times M$ is contained in a minimal monoidal equivalence, which can be constructed in the following way [2]. First, we take the reflexive, symmetric closure, $R_1 = R \cup \{(f,f) | f \in M\} \cup \{(f,g) | (g,f) \in R\}$. Second, we take the submonoid generated by R_1,

$$\begin{aligned} R_2 = &\{(f,g) | \text{ there exist } (f_i, g_i) \in R_1, i = 1, \ldots, n \\ &\text{such that } (f,g) = (f_1 \cdots f_n, g_1 \cdots g_n)\}. \end{aligned}$$

Finally, we take the transitive closure of R_2, which is

$$\begin{aligned} R_3 = &\{(f,g) | \text{ there exist } h_i \in M, i = 0, \ldots, n \\ &\text{such that } h_0 = f, h_n = g, \text{ and } (h_i, h_{i+1}) \in R_2, i = 0, \ldots, n-1\}. \end{aligned}$$

Gillam [2, p.3] proves that R_3 is in fact a monoidal equivalence relation.

A construction we will adapt to study C^∞-rings with corners is the following. Here M^* denotes the units in a monoid M.

Definition 4.16. A *prelog structure* on a ring A is a morphism of monoids $\alpha : M \to A$, where A is understood as a monoid under multiplication. A prelog structure is a *log structure* if $\alpha|_{\alpha^{-1}(A^*)} : \alpha^{-1}(A^*) \to A^*$ is an isomorphism. A prelog structure induces a log structure via the (monoid) pushout $M^a = M \oplus_{\alpha^{-1}(A^*)} A^*$. A *chart* for a log structure is a morphism $N \to M \to A$ such that $N^a \cong M$. A chart is a *characteristic chart* if $N \to M \to M/A^*$ is an isomorphism. For a log structure, $\overline{M} = M/A^*$ is called the *characteristic monoid*.

In fact, if $M \to A$ is a prelog structure, the characteristic monoid of M^a is naturally isomorphic to $M/\alpha^{-1}(A^*)$ [2, p. 41].

4.3 C^∞-rings with corners

The set of smooth functions $M \to \mathbb{R}$ on manifold with corners M is a C^∞-ring in the usual way. But we can also generalize the notion of a C^∞-ring to a C^∞-ring with corners. First, using the categorical definition:

Definition 4.17. Let Euc^c be the category with objects $\mathbb{R}^n_k, n, k \in \mathbb{N}, k \leq n$ and morphisms the smooth maps, that is, interior b-maps $\mathbb{R}^n_k \to \mathbb{R}^m_j$. A *categorical C^∞-ring with corners* is a finite product preserving functor from Euc^c to the category of sets.

As in the case of C^∞-rings without corners, we can see immediately that inverse limits and *directed* colimits of categorical C^∞-rings with corners exist, and can be constructed from inverse limits and directed colimits of the underlying sets. As before, this is because inverse limits and directed colimits commute with finite limits, and so the inverse limit or directed colimit of the underlying set should also be finite product preserving.

As categorical C^∞-rings with corners are finite product preserving, they are determined by their values on $\mathbb{R}$ and $[0,\infty)$, and on smooth functions $\mathbb{R}^n_k \to \mathbb{R}$ and $\mathbb{R}^n_k \to [0,\infty)$. This suggests the equivalence of the definition of categorical C^∞-rings with corners with the following definition, just as in the case of C^∞-rings without corners.

Definition 4.18. A *C^∞-ring with corners* is a pair of sets (C, C_∂), such that for every smooth $f : \mathbb{R}^n_k \to \mathbb{R}$, there is an operation $C_f : C^k_\partial \times C^{n-k} \to C$, and for every smooth $g : \mathbb{R}^n_k \to [0,\infty)$ there is an operation $C^\partial_g : C^k_\partial \times C^{n-k} \to C_\partial$, such that:

1. Given $f_1, \dots, f_k \in C_\partial$ and $g_k, \dots, g_{n-k} \in C$, if $\pi_i : \mathbb{R}^n_k \to \mathbb{R}$ is the projection on to the i^{th} coordinate, then

$$C^\partial_{\pi_i}(f_1, \dots, f_k, g_{k+1}, \dots, g_n) = f_i,\ 1 \leq i \leq k,$$

$$C_{\pi_i}(f_1, \dots, f_k, g_{k+1}, \dots, g_n) = g_i,\ k+1 \leq i \leq n.$$

2. Given smooth maps $f : \mathbb{R}^n_k \to \mathbb{R},\ g_i : \mathbb{R}^m_j \to [0, \infty), i = 1, \dots, k,\ h_i : \mathbb{R}^m_j \to \mathbb{R}, i = k+1, \dots, n$ set

$$h = f(g_1(-), \dots, g_k(-), h_{k+1}(-), \dots, h_n(-)) : \mathbb{R}^m \to \mathbb{R}.$$

Then we require that

$$C_f(C^\partial_{g_1}(-), \dots, C^\partial_{g_k}(-), C_{h_{k+1}}(-), \dots, C_{h_n}(-)) = C_h.$$

3. Given smooth maps $f : \mathbb{R}^n_k \to [0, \infty),\ g_i : \mathbb{R}^m_j \to [0, \infty), i = 1, \dots, k,\ h_i : \mathbb{R}^m_j \to \mathbb{R}, i = k+1, \dots, n$ set

$$h = f(g_1(-), \dots, g_k(-), h_{k+1}(-), \dots, h_n(-)) : \mathbb{R}^m \to [0, \infty).$$

Then we require that

$$C^\partial_f(C^\partial_{g_1}(-), \dots, C^\partial_{g_k}(-), C_{h_{k+1}}(-), \dots, C_{h_n}(-)) = C^\partial_h.$$

A *morphism of* C^∞*-rings with corners* is a pair of maps $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$ satisfying:

1. Given $f : \mathbb{R}^n_k \to [0, \infty), g_1, \dots, g_k \in C_\partial, h_{k+1}, \dots, h_n \in C$,

$$\phi_\partial(C^\partial_f(g_1, \dots, g_k, h_{k+1}, \dots, h_n)) = D^\partial_f(\phi_\partial(g_1), \dots, \phi_\partial(g_k), \phi(h_{k+1}), \dots, \phi(h_n)).$$

2. Given $f : \mathbb{R}^n_k \to \mathbb{R}, g_1, \dots, g_k \in C_\partial, h_{k+1}, \dots, h_n \in C$,

$$\phi(C_f(g_1, \dots, g_k, h_{k+1}, \dots, h_n)) = D_f(\phi_\partial(g_1), \dots, \phi_\partial(g_k), \phi(h_{k+1}), \dots, \phi(h_n)).$$

That is, a C^∞-ring with corners is determined by two sets (C, C_∂), and two operation types, C_- and C_-^∂. Notice that C is a C^∞-ring without corners, and in particular an $\mathbb{R}$-algebra. C_∂ is not an $\mathbb{R}$-algebra; however, it is a monoid under multiplication, since $f : [0,\infty)^2 \to [0,\infty), (x,y) \mapsto xy$ is a smooth map. The inclusion map $i : [0,\infty) \to \mathbb{R}$ induces a morphism $C_i : C_\partial \to C$, which we usually just call i. Similarly, exp defines a map $C \to C_\partial$ via $C^\partial_{\exp}$, which we often just write exp. Note that these maps are not necessarily monic. The *characteristic monoid* of a C^∞-ring with corners (C, C_∂) is the monoid $C_\partial / \exp C$.

Just as for C^∞-rings without corners, we can embed $\mathbb{R}$ in C, and for $r \in \mathbb{R}$, $\phi(r) = r$ for any morphism of C^∞-rings with corners $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$. Similarly, for $r \in (0,\infty)$, the constant map r is smooth. We write $r \in C_\partial$ for $C^\partial_r(1)$. Then $\phi(r) = D^\partial_r(\phi(1)) = D^\partial_r(1) = r$.

Using interior b-maps has the advantage of making C_∂ integral in some cases. The author originally thought this would prove more useful than it did. However, this restriction does result in simpler characteristic monoids, which we define below. However, it would not be difficult to change the definition to require the maps to only be b-maps.

Example 4.19. Let M be a manifold with corners. Then $C = C^\infty(M), C_\partial = C^\infty_{\geq 0}(M) = \{f : M \to [0,\infty) | f \text{ is smooth}\}$ is a C^∞-ring with corners, where given $f : \mathbb{R}^n_k \to \mathbb{R}$, $\phi_f(g_1, \ldots, g_k, h_1, \ldots, h_{n-k}) = f(g_1, \ldots, g_k, h_1, \ldots, h_{n-k})$. Consider the case $M = \mathbb{R}^n_k$. We write

$$C^{n,k} = C^\infty(\mathbb{R}^n_k), C^{n,k}_\partial = C^\infty_{\geq 0}(\mathbb{R}^n_k).$$

The exponential function $\exp : \mathbb{R} \to [0,\infty)$ induces a map $C^{n,k} \to C^{n,k}_\partial, f \mapsto \exp f$. We compute the characteristic monoid $C^{n,k}_\partial / \exp C^{n,k}$. Let $\psi : \mathbb{N}^k \to C^{n,k}_\partial / \exp C^{n,k}, (n_1, \ldots, n_k) \mapsto x_1^{n_1} \cdots x_k^{n_k}$. Then ψ is clearly injective. In fact, it is an isomorphism, because if $f \in C^{n,k}_\partial$, then $f : \mathbb{R}^n_k \to [0,\infty)$ is an interior b-map, so as we have shown, $f = G x_1^{n_1} \cdots x_k^{n_k}$ for $G > 0, n_i \geq 0$. Since $\exp(C^{n,k}) \cong C^\infty_{>0}(\mathbb{R}^n_k)$, this shows that ψ is surjective.

Example 4.20. In the case of C^∞-rings without corners, $\mathbb{R}$ was itself a C^∞-ring, with operation given by $f : \mathbb{R}^n \to \mathbb{R}, C_f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n), x_i \in \mathbb{R}$. This was the C^∞-ring $C^\infty(\{0\})$: the smooth functions from the manifold with one point to $\mathbb{R}$. $\{0\}$ also has the structure of a manifold with corners. The smooth functions from $\{0\}$ to $\mathbb{R}$ is again $\mathbb{R}$. The set of smooth functions from $\{0\}$ to $[0,\infty)$ is $(0,\infty)$, because the zero map is not an interior b-map. So $C^\infty(\{0\}) = (\mathbb{R},(0,\infty))$. Note that $(\mathbb{R},[0,\infty))$ is *also* a C^∞-ring with corners, with operations given by evaluation.

Lemma 4.21. *The C^∞-ring with corners $(C^{n,k}, C^{n,k}_\partial) = (C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k))$ is the 'free C^∞-ring with corners on $(n-k,k)$ generators': that is, there are $n-k$ elements $y_i \in C^{n,k}$ and k elements $x_i \in C^{n,k}_\partial$ which together generate $(C^{n,k}, C^{n,k}_\partial)$ under all C^∞-rings with corners operations, and given any C^∞-ring with corners (D, D_∂), any $n-k$ elements $b_j \in D$ and k elements $a_i \in D_\partial$, there is a unique C^∞-ring with corners morphism $(\phi,\phi_\partial) : (C^{n,k}, C^{n,k}_\partial) \to (D, D_\partial)$ such that $\phi(y_i) = b_i$ and $\phi_\partial(x_i) = a_i$.*

Proof. Let $x_1,\dots,x_k,y_{k+1},\dots,y_n$ be the projection maps. Then $x_i \in C^{n,k}_\partial$. For any $f \in C^{n,k}, g \in C^{n,k}_\partial$,

$$f = C_f(x_1,\dots,x_k,y_{k+1},\dots,y_n),$$

$$g = C^\partial_g(x_1,\dots,x_k,y_{k+1},\dots,y_n).$$

So the x_i and y_i generate $C^{n,k}$ and $C^{n,k}_\partial$. Let $a_1,\dots,a_k \in D_\partial, b_{k+1},\dots,b_n \in D$. For $f \in C^{n,k}, g \in C^{n,k}_\partial$, define $\phi(f) = D_f(a_1,\dots,a_k,b_{k+1},\dots,b_n)$, and $\phi_\partial(g) = D^\partial_g(a_1,\dots,a_k,b_{k+1},\dots,b_n)$. By the axioms, this is a morphism of C^∞-ring with corners, and $\phi_\partial(x_i) = a_i$ and $\phi(y_i) = b_i$. Again by the axioms, for $f \in C^{n,k}, g \in C^{n,k}_\partial$,

$$\phi(f) = \phi(C_f(x_1,\dots,x_k,y_{k+1},\dots,y_n)) = D_f(\phi_\partial(x_1),\dots,\phi_\partial(x_k),\phi(y_{k+1}),\dots,\phi(y_n)),$$

$$\begin{aligned}\phi_\partial(g) &= \phi_\partial(C^\partial_g(x_1,\dots,x_k,y_{k+1},\dots,y_n))\\ &= D^\partial_g(\phi_\partial(x_1),\dots,\phi_\partial(x_k),\phi(y_{k+1}),\dots,\phi(y_n)).\end{aligned}$$

As (ϕ,ϕ_∂) is determined by its values on $x_1,\dots,x_k,y_{k+1},\dots,y_n$, it is unique. $\square$

4.4 Quotients

There are a number of different ways in which we might want to quotient a C^∞-ring with corners (C, C_∂). We could start with only an ideal in C. Alternatively, if we are given an ideal I of C, and a submonoid I_∂ of C_∂, when can we construct a C^∞-ring with corners $(C/I, C/I_\partial)$? Recall that not all monoidal equivalence relations on a monoid are given by submonoids. If we are given an ideal I of C, and a monoidal equivalence relation $\sim$ of C_∂, when can we construct a C^∞-ring with corners $(C/I, C_\partial/\sim)$? The first two problems are special cases of the last.

Proposition 4.22. *Let (C, C_∂) be a C^∞-ring with corners. Let I be an ideal of C, and $\sim$ a monoidal equivalence relation on C_∂. Then there is a (necessarily unique) C^∞-ring with corners structure on $(C/I, C_\partial/\sim)$ such that the projection map (π, π_∂) is a morphism of C^∞-rings with corners, if and only if the following conditions hold:*

1. *For all $f, g \in C_\partial$ such that $f \sim g$, $i(f) - i(g) \in I$.*
2. $\exp(I) \sim 1$.

Proof. For the 'if' direction, suppose we have such a pair $(I, \sim)$. Requiring that the projection map is a morphism of C^∞-rings with corners determines the C^∞-ring with corners structure on $(C/I, C_\partial/\sim)$. So showing that the projection map (π, π_∂) is a morphism of C^∞-rings with corners is equivalent to showing the following for any given $f_1 \in C^{n,k}$, $f_2 \in C_\partial^{n,k}$. Suppose $g_i, g_i' \in C_\partial, i = 1, \ldots, k, h_j, h_j' \in C, j = 1, \ldots, n-k$, such that $g_i \sim g_i'$ and $h_j - h_j' \in I$. Then we need to show that

$$C_{f_1}(g_1, \ldots, g_k, h_1, \ldots, h_{n-k}) - C_{f_1}(g_1', \ldots, g_k', h_1', \ldots, h_{n-k}') \in I,$$

$$C^\partial_{f_2}(g_1, \ldots, g_k, h_1, \ldots, h_{n-k}) \sim C^\partial_{f_2}(g_1', \ldots, g_k', h_1', \ldots, h_{n-k}').$$

We start with $f_1 \in C^{n,k}$. Just as for C^∞-rings without corners, we show the equation using Hadamard's Lemma. There exist $q_i(x, y) : (\mathbb{R}^n_k)^2 \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}^n_k$,

$$f_1(x) - f_1(y) = \sum_{i=1}^{n} (x_i - y_i) q_i(x, y).$$

By assumption, $i(g_i) - i(g_i'), h_j - h_j' \in I$. Hence

$$\begin{aligned}
&C_{f_1}(g_1, \ldots, g_k, h_1, \ldots, h_{n-k}) - C_{f_1}(g_1', \ldots, g_k', h_1', \ldots, h_{n-k}') \\
&= \sum_{i=1}^{k} (i(g_i) - i(g_i')) C_{q_i}(g_1, \ldots, h_{n-k}') + \sum_{i=1}^{n-k} (h_i - h_i') C_{q_i}(g_1, \ldots, h_{n-k}') \in I.
\end{aligned}$$

For the second equation, when $f_2 \in C_\partial^{n,k}$, we first prove three special cases. First, let $f_2 = x_1^{i_1} \cdots x_k^{i_k} : \mathbb{R}_k^n \to [0, \infty)$ for $(i_1, \ldots, i_k) \in \mathbb{N}^k$. Because $\sim$ is a monoidal equivalence, it commutes with multiplication, that is

$$\begin{aligned}
&C_{f_2}^\partial(g_1, \ldots, g_k, h_1, \ldots, h_{n-k}) \\
&\quad = g_1^{i_1} \cdots g_k^{i_k} \sim g_1'^{i_1} \cdots g_k'^{i_k} = C_{f_2}^\partial(g_1', \ldots, g_k', h_1', \ldots, h_{n-k}')
\end{aligned}$$

So the claim holds in this case. For notational simplicity, let

$$(G, H) = (g_1, \ldots, g_k, h_1, \ldots, h_{n-k}), (G', H') = (g_1', \ldots, g_k', h_1', \ldots, h_{n-k}').$$

Secondly, suppose we have $f_2 = \exp(g) : \mathbb{R}_k^n \to [0, \infty)$ for some $g : \mathbb{R}_k^n \to \mathbb{R}$. We have already shown that $C_g(G, H) - C_g(G', H') \in I$, and so by assumption $C_{\exp}(C_g(G, H) - C_g(G', H')) \sim 1$. Then

$$\begin{aligned}
1 &\sim C_{\exp}^\partial(C_g(G, H) - C_g(G', H')) \\
&= C_{f_2}^\partial(G, H) C_{\exp(-g)}(G', H') = C_{f_2}^\partial(G, H) C_{f_2}^\partial(G', H')^{-1}.
\end{aligned}$$

So

$$C_{f_2}^\partial(G, H) \sim C_{f_2}^\partial(G', H').$$

Now let $f_2 : \mathbb{R}_k^n \to [0, \infty)$ be a general smooth map. There exists $g \in C^{n,k}$ and $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $f_2 = \exp g \, x_1^{n_1} \cdots x_k^{n_k}$. So, using the previous two calculations, and the fact that this is a monoidal equivalence, $C_{f_2}^\partial(G, H) \sim C_{f_2}^\partial(G', H')$.

For the 'only if' direction, the two conditions are necessary conditions for defining such a C^∞-ring with corners because (π, π_∂) commute with C^∞-operations. That is, given $f, g \in C_\partial$ such that $f \sim g$, then by assumption $\pi_\partial(f) = \pi_\partial(g)$. Thus $\pi(i(f)) = i(\pi_\partial(f)) = i(\pi_\partial(g)) = \pi(i(g))$, and hence $i(f) - i(g) \in I$. Similarly, given $g \in I$, $\pi_\partial(\exp(g)) = \exp(\pi(g)) = \exp(0) = 1$. So $\exp(g) \sim 1$. $\square$

Example 4.23. 1. Let (C, C_∂) be a C^∞-ring with corners. Every ideal I of C determines a quotient C^∞-ring with corners $(C/I, C_\partial/\sim)$, where $\sim$ is the monoidal equivalence relation given by $f \sim g$ iff $i(f) - i(g) \in I$.

2. Let $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$ be a morphism, and $(I, \sim_D)$ a pair satisfying the conditions of the proposition for (D, D_∂). Define a monoidal equivalence relation on C_∂ as $f \sim_C g$ if $\phi_\partial(f) \sim_D \phi_\partial(g)$. Then $(\phi^{-1}(J), \sim_C)$ also satisfies the conditions of the proposition. If $I = (0)$ and for $f, g \in D_\partial$, $f \sim_D g \Leftrightarrow f = g$, then we call $(\ker \phi, \sim_C)$ the *kernel* of (ϕ, ϕ_∂).

Corollary 4.24. *Let (C, C_∂) be a C^∞-ring with corners. Let I be an ideal of C, and I_∂ a submonoid of C_∂. Then there is a (necessarily unique) C^∞-ring with corners structure on $(C/I, C_\partial/I_\partial)$, such that the projection map (π, π_∂) is a morphism of C^∞-rings with corners, if and only if the following conditions hold:*

1. $i(I_\partial) \subset 1 + I$.

2. $\exp(I) \subset I_\partial$.

Proof. First, suppose $i(I_\partial) \subset 1 + I$ and $\exp(I) \subset I_\partial$. Recall that by definition, $C_\partial/I_\partial = C_\partial/\sim$, where $\sim$ is the equivalence relation given by

$$f \sim g \Leftrightarrow \text{ there exist } h_1, h_2 \in I_\partial \text{ such that } fh_1 = gh_2.$$

Using Proposition 4.22, we show that the two conditions are satisfied for $(I, \sim)$. The second condition is satisfied, because given $h \in I, \exp(h) \in \exp(I) \subset I_\partial$, and so $\exp h = g \cdot 1$ for some $g \in I_\partial$. Thus, $h \sim 1$. For the first condition, suppose $f \sim g$. Then there exist $\alpha, \beta \in I_\partial$ such that $\alpha f = \beta g$, and so $i(\alpha f) = i(\beta g) \Leftrightarrow i(\alpha)i(f) = i(\beta)i(g)$. However, as $i(\alpha) \sim i(\beta) \sim 1$, we have $i(f) - i(g) \in I$.

For the only if direction, it is again just because π, π_∂ must commute with C^∞-operations. So if $f \in I_\partial$, $\pi(i(f)) = i(\pi_\partial(f)) = i(1) = 1$. Similarly, if $g \in \exp(I)$, then $\pi_\partial(\exp(g)) = \exp(\pi(g)) = \exp(0) = 1$. $\square$

Example 4.25. Let (C, C_∂) be a C^∞-ring with corners, and $J \subset C$ an ideal. The set $\exp J$ is a submonoid of C_∂. Suppose $f \in J$. Then $\exp(f) - 1 = \exp(f) - \exp(0)$, and by Hadamard's Lemma (just as we argued for C^∞-rings without corners), as

$f - 0 \in J$, $\exp(f) - \exp(0) \in J$. Hence $i(\exp(J)) \subset J + 1$. So the pair $(J, \exp J)$ satisfies the conditions of Corollary 4.24, and $(C/J, C_\partial/\exp J)$ is a well-defined C^∞-ring with corners.

Definition 4.26. Let (C, C_∂) be a C^∞-ring with corners, $I \subset C$ an ideal, and $\sim$ a monoidal equivalence on (C, C_∂). If the pair $(I, \sim)$ satisfies the conditions of Proposition 4.22, it is called an *corner equivalence.* If $\sim$ arises from an I as in part 1 of Example 4.23, we call the pair an *ideal equivalence.* If $\sim$ arises from a submonoid of C_∂ as in Corollary 4.24 we call the pair a *submonoid equivalence.*

Let (C, C_∂) be a C^∞-ring with corners. Recall that a monoidal equivalence on C_∂ is determined by a set in $C_\partial \times C_\partial$. Thus, sometimes instead of writing a corner equivalence as $(I, \sim)$, we will write it as (I, P), where I is an ideal and P is a set in $C_\partial \times C_\partial$ corresponding to a monoidal equivalence relation $\sim$ on C_∂. As stated in Gillam's book [2, p.7], the intersection of a set of monoidal equivalences is a monoidal equivalence.

Lemma 4.27. *Let (C, C_∂) be a C^∞-ring with corners, and let $(I_i, P_i), i \in S$ be corner equivalences. Then $(\bigcap_{i\in S} I_i, \bigcap_{i\in S} P_i)$ is a corner equivalence.*

Proof. As $I = \bigcap_{i\in S} I_i$ is an ideal, and $P = \bigcap_{i\in S} P_i$ is a monoidal equivalence, it suffices to check that $\exp I \times \{1\} \subset P$, and given $(f, g) \in P$, $i(f) - i(g) \in I$. Let $h \in I$. Then $(\exp h, 1) \in P_i$ for all $i \in S$, so $h \in P$. If $(f, g) \in P$, then $i(f) - i(g) \in I_i$ for all $i \in S$, so $i(f) - i(g) \in I$. □

Let $I \subset C$ be a set, and $P \subset C_\partial \times C_\partial$ a set. A corner equivalence (J, Q) on (C, C_∂) *contains* the pair (I, P) if $I \subset J$ and $P \subset Q$. The trivial corner equivalence $(C, C_\partial \times C_\partial)$ is a corner equivalence containing (I, P). Hence taking the intersection of all corner equivalences containing the pair (I, P) gives a minimal corner equivalence containing (I, P).

Corollary 4.28. *Let $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$ be a C^∞-ring with corners morphism, and $(I, \sim)$ a corner equivalence satisfying $I \subset \ker \phi$ and if $f, g \in C_\partial$, $f \sim g \Rightarrow \phi_\partial(f) = \phi_\partial(g)$. Then there is a unique morphism*

$$(\tilde{\phi}, \tilde{\phi_\partial}) : (C/I, C_\partial/\sim) \to (D, D_\partial)$$

such that

$$(\tilde{\phi}, \tilde{\phi_\partial}) \circ (\pi, \pi_\partial) = (\phi, \phi_\partial).$$

Proof. We need to check that the map

$$(\tilde{\phi}, \tilde{\phi_\partial}) : (C/I, C_\partial / \sim) \to (D, D_\partial), ([g], [h]) \mapsto (\phi(g), \phi_\partial(h))$$

is well-defined and a morphism of C^∞-ring with corners. That it is well-defined is by construction. To see that it is a morphism of C^∞-ring with corners follows from the fact that (ϕ, ϕ_∂) and (π, π_∂) are. That is, if $f : \mathbb{R}^n_k \to \mathbb{R}$, $[g_i] \in C_\partial / \sim, [h_i] \in C/I$, and writing $G = (g_1, \ldots, g_k), [G] = [(g_1], \ldots, [g_k]), H = (h_{k+1}, \ldots, h_n), [H] = ([h_{k+1}], \ldots, [h_n])$,

$$\begin{aligned} D_f(\tilde{\phi_\partial}([G]), \tilde{\phi}([H])) &= D_f(\phi_\partial(G), \phi(H)) = \phi(D_f(G, H)) \\ &= \tilde{\phi}([D_f(G, H)]) = \tilde{\phi}(D_f([G], [H])). \end{aligned}$$

The case where $f : \mathbb{R}^n_k \to [0, \infty)$ is similar. $\square$

Another example of quotienting comes from closed sets.

Example 4.29. Let $V \subset \mathbb{R}^n_k$ be a closed set. Let $I(V) = \{f \in C^\infty(\mathbb{R}^n_k) | f|_V = 0\}$. $I(V)$ is clearly an ideal. Define an equivalence on $C^\infty_{\geq 0}(\mathbb{R}^n_k)$ as $f \sim_V g \Leftrightarrow f|_V = g|_V$. It is easy to see that this is a monoidal equivalence. For any $f \in I(V)$, $\exp(f)|_V = \exp(0)|_V = 1|_V$, and so $\exp f \sim_V 1$. If $f, g \in C^\infty_{\geq 0}(\mathbb{R}^n_k), f \sim_V g$, then $i(f) - i(g)|_V = 0|_V$. So $i(f) - i(g) \in I(V)$. So the pair $(I(V), \sim_V)$ is a corner equivalence, and so defines a C^∞-ring with corners

$$(C^\infty(\mathbb{R}^n_k)/I(V), C^\infty_{\geq 0}(\mathbb{R}^n_k) / \sim_V).$$

Unlike the case for C^∞-rings without corners, this is not the set of maps $V \to \mathbb{R}$ which can be smoothly extended to an open neighborhood of V. This is because we do not have characteristic functions in $C^\infty_{\geq 0}(\mathbb{R}^n_k)$. We do have characteristic functions in $C^\infty(\mathbb{R}^n_k)$, as it is a C^∞-ring without corners.

4.5 Fairness

As we are interested in sheaves, we want some definition of germ-determinedness. The next example studies the germ at a point $x \in \mathbb{R}^n_k$. We want there to be a surjective map from a C^∞-ring with corners to the ring of germs at x, and so as the previous example suggests, we can't use the same definition as for C^∞-rings without corners.

Example 4.30. Let $(C, C_\partial) = (C^{n,k}, C^{n,k}_\partial)$, and $x \in \mathbb{R}^n_k$. Let $(m_x, \sim_x)$ be the pair given by

$$m_x = \{f \in C^{n,k} | \text{ there exists an open } x \in U \text{ such that } f|_U = 0\},$$

$$h_1 \sim_x h_2 \Leftrightarrow \text{ there exists an open } x \in U \text{ such that } h_1|_U = h_2|_U.$$

It is easy to check that $(m_x, \sim_x)$ is a corner equivalence. Then the *ring of germs at* x is the pair $(C^{n,k}_x, C^{n,k}_{\partial,x}) = (C^{n,k}/m_x, C^{n,k}/\sim_x)$. Note that $C^{n,k}_x$ is just as for C^∞-rings without corners. Let

$$(\pi_x, \pi_{\partial,x}) : (C^{n,k}, C^{n,k}_\partial) \to (C^{n,k}_x, C^{n,k}_{\partial,x})$$

be the projection. If I is an ideal in $C^{n,k}$, and $f \in C^{n,k}, g \in C^{n,k}_\partial$, we write I_x, f_x, g_x for their images under $(\pi_x, \pi_{\partial,x})$. Note that I_x is an ideal, as π_x is surjective. If P is a monoidal equivalence on $C^{n,k}_\partial$, we write P_x for the *transitive closure* of the image of P under $\pi_{\partial,x} \times \pi_{\partial,x}$. It is clear that the image of P is already reflexive, symmetric, and a submonoid, so P_x is a monoidal equivalence.

Definition 4.31. A C^∞-ring with corners (C, C_∂) is called *finitely generated* if $(C, C_\partial) \cong (C^\infty(\mathbb{R}^n_k)/J, C^\infty_{\geq 0}(\mathbb{R}^n_k)/\sim)$ for some corner equivalence $(J, \sim)$. A pair $(J, \sim)$ is called *fair* if

1. Given $f \in C^\infty(\mathbb{R}^n_k)$, then $f \in J$ if and only if for every $x \in \mathbb{R}^n_k$, $f_x \in J_x$.

2. Let $P \subset C^{n,k}_\partial \times C^{n,k}_\partial$ be the set corresponding to $\sim$. Then we require $(f, g) \in P$ if and only if for all $x \in \mathbb{R}^n_k$, $(g_x, h_x) \in P_x$.

A C^∞-ring with corners is called *fair* if it is isomorphic to $(C^{n,k}/J, C^{n,k}_\partial/\sim)$ for a fair $(J, \sim)$.

If (C, C_∂) is a finitely generated C^∞-ring, we can construct its *fairification* $(C, C_\partial)^{fa}$. Suppose $(C, C_\partial) \cong (C^{n,k}, C_\partial^{n,k})/(J, \sim)$. Let

$$J^{fa} = \{f \in C | f_x \in J_x \text{ for all } x \in \mathbb{R}^n_k\}.$$

Let $P^{fa} = \{(f,g)|$ for every $x \in \mathbb{R}^n_k, (f_x, g_x) \in P_x\}$. Note that P^{fa} is the intersection of the monoidal relations $\{(f,g)|(f_x, g_x) \in P_x\}$, and hence is itself a monoidal equivalence. We call it $\sim^{fa}$. Next, we prove that $(J^{fa}, \sim^{fa})$ is a corner equivalence. Suppose $f \in J^{fa}$. Then for each $x \in \mathbb{R}^n_k$ there exists $h \in J$ such that $f_x = h_x$. Then $\exp f_x = \exp h_x$. So as $\exp J \times \{1\} \in P$, $(\exp f_x, 1_x) \in P_x$ for all $x \in \mathbb{R}^n_k$, and hence $(\exp f, 1) \in P^{fa}$. So the condition that $\exp J^{fa} \sim^{fa} 1$ is satisfied. Secondly, suppose $f \sim^{fa} g$. As P_x is the transitive closure of the image of P at the germ of x, for each $x \in \mathbb{R}^n_k$ there exist $h_0, \ldots, h_n \in C_\partial^{n,k}$ such that $f_x = (h_0)_x, g_x = (h_n)_x$, and for each i, there exists $(a,b) \in P$ such that $((h_i)_x, (h_{i+1})_x) = (a_x, b_x)$. Hence $(h_i - h_{i-1})_x = (a-b)_x \in J_x$, as $(J, \sim)$ is a corner equivalence. As J_x is an ideal,

$$(f-g)_x = (h_0)_x - (h_1)_x + (h_1)_x - (h_2)_x + \cdots + (h_{n-1})_x - (h_n)_x \in J_x.$$

So $i(f) - i(g) \in J^{fa}$. So $(J^{fa}, \sim^{fa})$ is a corner equivalence. As $(P^{fa})_x = P_x$, this is a fair corner equivalence. The *fairification* of (C, C_∂) is

$$(C, C_\partial)^{fa} = (C^{n,k}, C_\partial^{n,k})/(J^{fa}, \sim^{fa}).$$

Let J be a fair ideal of $C^\infty(\mathbb{R}^n_k)$. Then the ideal equivalence given by J is fair.

Lemma 4.32. *Let (C, C_∂) be a fair C^∞-ring with corners given by an ideal equivalence. That is, $(C, C_\partial) \cong (C^\infty(\mathbb{R}^n_k)/J, C^\infty_{\geq 0}(\mathbb{R}^n_k)/\sim)$ for a fair J. Then* $\exp(C) = C_\partial^*$.

Proof. For $[f] \in C$, $\exp([f]) = [\exp(f)] \in C_\partial^*$, as $\exp(f) \in C^\infty_{>0}(\mathbb{R}^n_K)$. So $\exp(C) \subset C_\partial^*$. Let $[f] \in C_\partial^*$. Then there is a $g \geq 0$ such that $fg \sim 1 \Rightarrow fg - 1 \in J$. So $fg|_{Z(J)} = 1|_{Z(J)}$, and hence there are open neighborhoods U, V of $Z(J)$ such that $\overline{U} \subset V$, and $f|_V > 0$. So we can define $\log f$ on V, and choose an extension $g \in C^\infty(\mathbb{R}^n_k)$ of $\log f|_{\overline{U}}$. If g' is another extension, $g - g'$ is zero on a neighborhood of $Z(J)$, and by the fairness of J, $g - g' \in J$. $i(f)|_{\overline{U}} = i(\exp(g))|_{\overline{U}}$, so again by the fairness of $J, f \sim \exp(g)$. So $f \in \exp(C)$ and hence $\exp(C) = C_\partial^*$. $\square$

In the 'nice' circumstances of the previous lemma, the characteristic monoid of (C, C_∂) is $C_\partial / C_\partial^*$, which looks similar to the definition given for the characteristic monoid of a log ring.

The characteristic monoid of a C^∞-ring with corners (C, C_∂) behaves well with respect to quotienting. Let $M_C = C_\partial / \exp(C)$. Let $(J, \sim)$ be a corner equivalence, with $(D, D_\partial) \cong (C/J, C_\partial/\sim)$. As the projection $\pi_\partial : C_\partial \to C_\partial/\sim$ is morphism of C^∞-rings with corners, the image of $\exp(C)$ under the projection is $\exp C/J$. So

$$M_D = D_\partial / \exp D \cong (C_\partial/\sim)/(\exp(C/J)) = (C_\partial/\sim)/(\exp(C)/\sim) \cong M_C/\sim_M,$$

where $\sim_M$ is the monoidal equivalence on M_C generated by the image of $\sim$ in M_C under the projection.

Example 4.33. We have shown that the characteristic monoid of $C^\infty_{\geq 0}(\mathbb{R}^n_k)$ is $\mathbb{N}^k$, via the map $\phi : \mathbb{N}^k \to C^\infty_{\geq}(\mathbb{R}^n_k), (n_1, \ldots, n_k) \mapsto x_1^{n_1} \cdots x_k^{n_k}$.

Consider $(C^\infty([0,\infty)^4), C^\infty_{\geq 0}([0,\infty)^4))$, with coordinates (w, x, y, z), and ideal $J = (wx - yz)$. Then the characteristic monoid of the quotient C^∞-ring with corners is $\mathbb{N}^4/((1,1,0,0) \sim (0,0,1,1))$.

4.6 Small colimits

Recall for a C^∞-ring with corners (C, C_∂), the smallest corner equivalence containing a pair (J, P) always exists, where J is an ideal and $P \subset C_\partial \times C_\partial$.

Lemma 4.34. *Let (C, C_∂) be a C^∞-ring with corners. Let $J \subset C$ be an ideal, and $P \subset C_\partial \times C_\partial$ a set, satisfying $\exp J \times \{1\} \subset P$, and for $(f, g) \in P$, $i(f) - i(g) \in J$. If Q is the smallest monoidal equivalence containing P, then (J, Q) is a corner equivalence.*

Proof. Gillam [2, p.3] constructs Q from P in four steps. First, one takes the symmetric, reflexive closure $P_1 = \overline{P}^{rs}$. If $(f, g) \in \overline{P}^{rs}$, then either $(f, g) \in P$, $f = g$ or $(g, f) \in P$. In each case, it is clear that $i(f) - i(g) \in J$. Second, one takes the submonoid generated by P_1, which we call P_2. To show that if $(f, g) \in P_2, i(f) - i(g) \in J$, it suffices to show that if $(a, b), (c, d)$ satisfy $i(a) - i(b), i(c) - i(d) \in J$, then $i(ac) - i(bd) \in J$. But $i(ac) - i(bd) = i(a)i(c) - i(a)i(d) + i(a)i(d) - i(b)i(d) = i(a)(i(c) - i(d)) + i(d)(i(a) - i(b)) \in J$. Finally, one takes the transitive closure of P_2, which is a monoidal equivalence, and hence Q. If $(f,g) \in Q$, then there exist $f = h_0, \ldots, h_n = g$ such that $(h_i, h_{i+1}) \subset P_2$. Then $i(f) - i(g) = i(f) - i(h_1) + i(h_1) - i(h_2) + \cdots + i(h_{n-1}) - i(g) \in J$. □

We now explicitly construct the 'smallest' corner equivalences containing (J,P), which we will use to construct pushouts and localization. The use of the lemma will be that it will allow us to relate the 'C' part of these constructions to their equivalents for C^∞-rings without corners on C in (C, C_∂).

First, suppose we are just given J. The pair $(J, \exp J)$ is the smallest corner equivalence containing J.

Secondly, suppose we are given a monoidal equivalence relation $\sim$, corresponding to a set P. Let $J = (i(f) - i(g) | (f,g) \in P)$, and let Q be monoidal equivalence generated by P and $\exp(J) \times \{1\}$. Then by the lemma, this is a corner equivalence, and it is clearly the smallest corner equivalence containing P.

Finally, suppose we are given a pair (I,P). Let $J = (I, f - g | (f,g) \in P)$. Let Q be the monoidal equivalence generated by $P \cup (\exp J \times \{1\})$. We can apply the lemma to this pair, because for $(f,g) \in P, (\exp h, 1) \in \exp J \times \{1\}$, we have $f - g \in I$ by construction and $\exp h - 1 \in J$, as $h \in J$. So (J,Q) is the smallest corner equivalence containing (I,P).

As mentioned, directed colimits and inverse limits for C^∞-rings with corners can be constructed using underlying sets, just as for C^∞-rings without corners. General (small) colimits, however, cannot. We construct pushouts for C^∞-rings with corners in a series of three examples.

First, we construct coproducts for free C^∞-rings with corners.

Example 4.35. The coproduct

$$(C^{n,k}, C_\partial^{n,k}) \otimes_\infty (C^{m,l}, C_\partial^{m,l})$$

is $(C^{n+m,k+l}, C_\partial^{n+m,k+l})$ because each of these are free C^∞-rings with corners.

Next, we construct coproducts of finitely generated C^∞-rings with corners.

Example 4.36. Let $\mathfrak{C}_i = (C^{n_i,k_i}, C_\partial^{n_i,k_i})/(J_i, \sim_i), i = 1,2$ be finitely generated C^∞-rings with corners. Let P_i be the set corresponding to $\sim_i$. Let $(J, \sim)$ be the smallest corner equivalence containing the ideal $(J_1, J_2) \subset C^{n_1+n_2,k_1+k_2}$ and subset

$\{P_1, P_2\} \subset C_\partial^{n_1+n_2,k_1+k_2} \times C_\partial^{n_1+n_2,k_1+k_2}$. Let P be the set corresponding to $\sim$. Then we claim

$$\mathfrak{C}_1 \otimes_\infty \mathfrak{C}_2 = (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2})/(J, \sim).$$

Essentially, this is because of the minimality of $(J, \sim)$ and the fact that $(C^{a,b}, C_\partial^{a,b})$ is free. We present the full detail, however. The inclusion maps are constructed by using that the composition of

$$(C^{n_i,k_i}, C_\partial^{n_i,k_i}) \to (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2}) \to (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2})/(J, \sim)$$

factors through $(J_i, \sim_i)$, because by construction (J_i, P_i) is contained in (J, P).

To show the universal property, suppose we have a C^∞-ring with corners $\mathfrak{D}$, and maps $\Phi_i : \mathfrak{C}_i \to \mathfrak{D}$. Then by composition, we have maps $(C^{n_i,k_i}, C_\partial^{n_i,k_i}) \to \mathfrak{D}$, so by the previous example and the universal property, we have a commutative diagram:

$$\begin{tikzcd}
(C^{n_1,k_1}, C_\partial^{n_1,k_1}) \arrow[r] \arrow[d] & (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2}) \arrow[dd, "(\tilde{\pi},\tilde{\pi}_\partial)"] & (C^{n_2,k_2}, C_\partial^{n_2,k_2}) \arrow[l] \arrow[d] \\
\mathfrak{C}_1 \arrow[dr, "\Phi_1"] & & \mathfrak{C}_2 \arrow[dl, "\Phi_2"] \\
& \mathfrak{D}. &
\end{tikzcd}$$

So (J_i, P_i) is in the kernel of $(\tilde{\pi}, \tilde{\pi}_\partial)$. The kernel is a corner equivalence, and so by the minimality of $(J, \sim)$ it also contains $(J, \sim)$. Thus, finally, $(\tilde{\pi}, \tilde{\pi}_\partial)$ factors through $(J, \sim)$, so we get a unique morphism

$$(\pi, \pi_\partial) : (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2})/(J, \sim) \to \mathfrak{D}$$

satisfying

$$\begin{tikzcd}
\mathfrak{C}_1 \arrow[r] \arrow[dr, "\Phi_1"'] & (C^{n_1+n_2,k_1+k_2}, C_\partial^{n_1+n_2,k_1+k_2})/(J, \sim) \arrow[d, "(\pi,\pi_\partial)"] & \mathfrak{C}_2 \arrow[l] \arrow[dl, "\Phi_2"] \\
& \mathfrak{D}. &
\end{tikzcd}$$

The next example explicitly constructs pushouts of finitely generated C^∞-rings with corners. It isn't complicated, but we require twice as much notation as for C^∞-rings with corners.

Example 4.37. Let $\mathfrak{C}_i, i = 1, 2, 3$ be finitely generated C^∞-rings with corners. We will construct $\mathfrak{C}_4$ so that we have a pushout square:

$$\begin{array}{ccc} \mathfrak{C}_3 & \xrightarrow{(\alpha,\alpha_\partial)} & \mathfrak{C}_1 \\ \Big\downarrow{\scriptstyle (\beta,\beta_\partial)} & & \Big\downarrow{\scriptstyle (\delta,\delta_\partial)} \\ \mathfrak{C}_2 & \xrightarrow{(\gamma,\gamma_\partial)} & \mathfrak{C}_4. \end{array}$$

Suppose $\mathfrak{C}_i = (C^{n_i,k_i}, C^{n_i,k_i}_\partial)/(J_i, P_i), i = 1, 2, 3$. Let $x_{1,i}, \ldots, x_{k,i}, y_{k_i+1,i}, \ldots, y_{n_i,i}$ denote the generators of $(C^{n_i,k_i}, C^{n_i,k_i}_\partial)$. So $[x_{1,i}], \ldots, [x_{k,i}], [y_{k_i+1,i}], \ldots, [y_{n_i,i}]$ generate $\mathfrak{C}_i$, $i = 1, 2, 3$. We have that $\alpha_\partial([x_{j,3}]) = [a_j], \alpha([y_{j,3}]) = [b_j]$ for some $a_j \in C^{n_1,k_1}_\partial, b_j \in C^{n_1,k_1}$, and $\beta_\partial([x_{j,3}]) = [c_j], \beta([y_{j,3}] = [d_j]$ for some $c_j \in C^{n_2,k_2}_\partial, d_j \in C^{n_2,k_2}$. We want $\mathfrak{C}_4$ to be generated by $\delta_\partial(x_{j,1}), \delta(y_{j,1}), \gamma_\partial(x_{j,2}), \gamma(y_{j,2})$. Set $\mathfrak{C}_4 = (C^{n_1+n_2,k_1+k_2}, C^{n_1+n_2,k_1+k_2}_\partial)/(J, P)$ where (J, P) is the corner equivalence generated by the ideal $(J_1, J_2, \{b_j - d_j\})$, and the subset $(P_1, P_2, (a_j, c_j))$. By the same argument as in the previous example, which turns on the minimality of (J, P), this is the pushout.

Lemma 4.38. *Let (C, C_∂) be a C^∞-ring with corners. Then it is the directed colimit of finitely generated C^∞-rings with corners.*

Proof. Let $(C^i, C^i_\partial), i \in I$ be the set of finitely generated sub-C^∞-ring with corners of (C, C_∂), with partial order given by inclusion. Every element of (C, C_∂) is contained in a finitely generated C^∞-ring with corners. Let F_i be the categorical C^∞-ring with corners corresponding to (C^i, C^i_∂). Directed colimits can be computed at the level of sets, so as $\varinjlim_i (F^i(\mathbb{R}^n_k)) = F(\mathbb{R}^n_k)$, we are done. $\square$

Corollary 4.39. *Small colimits of C^∞-rings with corners exist.*

4.7 Localization

Definition 4.40. Let $S \subset C, S_\partial \subset C_\partial$ be subsets of a C^∞-ring with corners (C, C_∂). A *localization* of (C, C_∂) at (S, S_∂) is a C^∞-ring with corners (D, D_∂) and a morphism $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$ such that $\phi(s)$ is a unit for all $s \in S$ and $\phi_\partial(t)$ is a unit for all $t \in S_\partial$, which is universal in the sense that if $(\phi', \phi'_\partial), (D', D'_\partial)$ is another such pair, there is a unique $(\pi, \pi_\partial) : (D, D_\partial) \to (D', D'_\partial)$ such that $(\phi', \phi'_\partial) = (\pi, \pi_\partial) \circ (\phi, \phi_\partial)$.

In the following lemma, we show the existence of a localization in the two simplest cases.

Lemma 4.41. *Let $(C, C_\partial) = (C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k))$, $f \in C_\partial, g \in C$. Then there exists a localization of (C, C_∂) at $(g, \emptyset)$ and at $(\emptyset, f)$.*

Proof. We start with the localization at $(\emptyset, f)$. We will add a generator $y \in C_\partial$. Let y be the projection on the $k+1^{\text{th}}$ copy of $[0, \infty)$ in $\mathbb{R}^{n+1}_{k+1}$. We quotient by the smallest corner equivalence such that $fy \sim 1$, call it (J, P). By the constructions at the start of the previous section, $J = (fy - 1)$, and $P, \sim$ is the smallest monoidal equivalence containing $(fy, 1)$ and $(\exp(h(fy-1)), 1)$ for any $h \in C^{n,k}$. Then let $D_\partial = C^\infty_{\geq 0}(\mathbb{R}^{n+1}_{k+1})/\sim$, $D = C^\infty(\mathbb{R}^{n+1}_{k+1})/J$. Let (ϕ, ϕ_∂) be the inclusion of (C, C_∂) into $(C^{n+1,k+1}, C^{n+1,k+1}_\partial)$ followed by the projection onto (D, D_∂). Now we show that (D, D_∂) satisfies the universal property of localization. First of all, $\phi_\partial(f)y = 1$. Now suppose that $(\phi', \phi'_\partial), (D', D'_\partial)$ is another such pair. Define $(\pi, \pi_\partial) : (C^\infty(\mathbb{R}^{n+1}_{k+1}), C^\infty_{\geq 0}(\mathbb{R}^{n+1}_{k+1})) \to (D', D'_\partial)$ as $\pi(y_i) = \phi'(y_i), \pi_\partial(x_i) = \phi'_\partial(x_i), \pi_\partial(y) = \phi'_\partial(f)^{-1}$. As $\pi_\partial(fy) = \phi'_\partial(f)\phi'_\partial(f)^{-1} = 1$, the kernel of (π, π_∂) is a corner equivalence containing $fy \sim 1$, so by minimality of $(J, \sim)$, it also contains $(J, \sim)$. Hence (π, π_∂) factors through $(J, \sim)$. So there is an induced map $(\tilde{\pi}, \tilde{\pi}_\partial) : (D, D_\partial) \to (D', D'_\partial)$. By construction, $(\tilde{\pi}, \tilde{\pi}_\partial) \circ (\phi, \phi_\partial) = (\phi', \phi'_\partial)$, and $\tilde{\pi}, \tilde{\pi}_\partial$ is determined by this property, and hence unique.

The localization at $(g, \emptyset)$ is essentially the same. We add a generator $y \in C$: let y be the projection on the $n+1^{\text{th}}$ copy of $\mathbb{R}$ in $\mathbb{R}^{n+1}_k$. Let $J = (yg - 1)$. Then $(J, \exp J)$ is the smallest corner equivalence containing J. Let $D_\partial = C^\infty_{\geq 0}(\mathbb{R}^{n+1}_k)/\exp J$, $D = C^\infty(\mathbb{R}^{n+1}_k)/J$. Let (ϕ, ϕ_∂) be the inclusion of (C, C_∂) into $(C^{n+1,k}, C^{n+1,k}_\partial)$ followed by the projection. Now we show that (D, D_∂) satisfies the universal property of localization. First of all, $\phi(g)y = 1$. Now suppose that $(\phi', \phi'_\partial), (D', D'_\partial)$ is another such pair. Define $\pi : C^\infty(\mathbb{R}^{n+1}_k) \to D'$ as $\pi(y_i) = \phi'(y_i), \pi(y) = \phi'(g)^{-1}$, and $\pi_\partial(x_i) = \phi'_\partial(x_i)$. As $\pi(gy) = \phi'(g)\phi'(g)^{-1} = 1$, the kernel of (ϕ', ϕ'_∂) is a corner equivalence containing $(gy - 1)$, so by minimality of $(J, \exp J)$, it also contains $(J, \exp J)$. Hence (π, π_∂) factors through $(J, \exp J)$. So there is an induced map $(\tilde{\pi}, \tilde{\pi_\partial}) : (D, D_\partial) \to (D', D'_\partial)$. By construction, $(\tilde{\pi}, \tilde{\pi_\partial}) \circ (\phi, \phi_\partial) = (\phi', \phi'_\partial)$, and $(\tilde{\pi}, \tilde{\pi_\partial})$ is determined by this property, and hence unique. $\square$

Theorem 4.42. *Let (C, C_∂) be a C^∞-ring with corners, and (S, S_∂) a pair of sets, $S \subset C, S_\partial \subset C_\partial$. Then there exists a localization of (C, C_∂) at (S, S_∂).*

Proof. It suffices to show the theorem in the case where either S or S_∂ is empty. Let T denote the nonempty one. The localization at T is the directed colimit of localizations at finite subsets of T, so it suffices to show the theorem for T finite. For $f, g \in C$, f, g is invertible if and only if fg is invertible. Similarly for $f, g \in C_\partial$, so it suffices to show the theorem for the case T is a singleton. We can write (C, C_∂) as the directed colimit of finitely generated C^∞-rings with corners which all contain $T = \{f\}$, say $(C, C_\partial) = \varinjlim_i (C^i, C^i_\partial)$. Then

$$(C, C_\partial)T^{-1} = \varinjlim_i ((C^i, C^i_\partial)T^{-1}),$$

so it suffices to show the theorem for (C, C_∂) is finitely generated.

Suppose $(C, C_\partial) \cong (C^\infty(\mathbb{R}^n_k)/J, C^\infty_{\geq 0}(\mathbb{R}^n_k)/\sim)$, and P is the set associated to $\sim$. To localize at $[f]$ in $C^\infty(\mathbb{R}^n_k)/J$ (or in $C^\infty_{\geq 0}(\mathbb{R}^n_k)/\sim$), we first localize $(C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k))$ at f using the previous lemma, denoting this (F, F_∂), with localization map $(\eta, \eta_\partial) : (C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k)) \to (F, F_\partial)$. We then quotient by the smallest corner equivalence containing $((\eta(J)), \eta_\partial \times \eta_\partial(P))$. Let (D, D_∂) be the C^∞-ring with corners constructed in this way. The morphism (η, η_∂) composed with the projection onto (D, D_∂) factors through $(J, \sim)$ by construction, so we get a C^∞-ring with corners morphism $(\phi, \phi_\partial) : (C, C_\partial) \to (D, D_\partial)$. Now $\phi([f])$ (or $\phi_\partial([f])$) is still a unit, because quotienting does not change whether an element is a unit. To show that (D, D_∂) is the localization, suppose $(\phi', \phi'_\partial), (D', D'_\partial)$ is another such pair. Let

$$(\psi, \psi_\partial) : (C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k)) \xrightarrow{\text{projection}} (C, C_\partial) \xrightarrow{(\phi', \phi'_\partial)} (D', D'_\partial).$$

Note that by the universal property of localization, we have a unique map $(\tilde{\pi}, \tilde{\pi_\partial}) : (F, F_\partial) \to (D', D'_\partial)$, such that $(\psi, \psi_\partial) = (\tilde{\pi}, \tilde{\pi_\partial}) \circ (\eta, \eta_\partial)$. Note that

$$\tilde{\pi}(\eta(J)) = \psi(J) = \phi'(0) = (0).$$

Let $(f_1, f_2) \in \eta_\partial \times \eta_\partial(P)$, say $(f_1, f_2) = (\eta_\partial(g_1), \eta_\partial(g_2)), (g_1, g_2) \in P$. By assumption, $\psi_\partial(g_1) = \psi_\partial(g_2)$, and so

$$\tilde{\pi}_\partial(f_1) = \tilde{\pi} \circ \eta_\partial(g_1) = \psi_\partial(g_1) = \psi_\partial(g_2) = \tilde{\pi}_\partial(f_2).$$

Hence the kernel of $(\tilde{\pi}, \tilde{\pi_\partial})$ contains $((\eta(J)), \eta_\partial \times \eta_\partial(P))$. As the kernel is a corner equivalence, it contains the smallest corner equivalence containing $(\eta(J), \eta_\partial \times \eta_\partial(P))$. Hence $(\tilde{\pi}, \tilde{\pi_\partial})$ factors through (D, D_∂), so we get a unique map $(\pi, \pi_\partial) : (D, D_\partial) \to (D', D'_\partial)$. By looking at the definition of (ψ, ψ_∂), it is clear that $(\phi', \phi'_\partial) = (\pi, \pi_\partial) \circ (\phi, \phi_\partial)$ as needed. $\square$

Note that if (C, C_∂) is a C^∞-ring with corners, $S \subset C, S_\partial \subset C_\partial$, and (D, D_∂) the localization of (C, C_∂) at (S, S_∂), then the explicit constructions of the localization in the previous lemma and the proof of the theorem show that D is the localization of C at $(S, i(S))$ as a C^∞-ring *without* corners.

Let $V \subset \mathbb{R}^n_k$ be an (relatively) open set. Let $f \in C^{n,k}$ be a characteristic function for V. Then let (D, D_∂) be the localization of $(C^{n,k}, C^{n,k}_\partial)$ at f. By the above, $D \cong C^\infty(V)$. Unfortunately, D_∂ is not necessarily $C^\infty_{\geq 0}(V)$. If the boundary of V is not connected, for example, then $C^\infty_{\geq 0}(V)$ will be larger than D_∂. There are not many ways to 'localize' $C^{n,k}_\partial$, because the elements of $C^{n,k}_\partial$ are maps of the form $\exp f\, x_1^{i_1} \cdots x_k^{i_k}$. Now $\exp f$ is already invertible. So in fact, any localization at $C^\infty_{\geq 0}(\mathbb{R}^n_k)$ is just a choice of localization at some subset of $\{x_1, \ldots, x_k\}$.

4.8 C^∞-schemes with corners

Let (C, C_∂) be a C^∞-ring with corners. An $\mathbb{R}$-*point* (ϕ, ϕ_∂) is a morphism of C^∞-rings with corners $(C, C_\partial) \to (\mathbb{R}, [0, \infty))$. Recall from Example 4.20 that $(\mathbb{R}, [0, \infty))$ is a C^∞-ring with corners via evaluation. If $p \in \mathbb{R}^n_k$, then ev_p is an $\mathbb{R}$-point of $(C^\infty(\mathbb{R}^n_k), C^\infty_{\geq 0}(\mathbb{R}^n_k))$. If $(C, C_\partial) = (C^{n,k}/J, C^{n,k}_\partial/\sim)$ for $(J, \sim)$ a corner equivalence, and $p \in Z(J) \cap \{x \in \mathbb{R}^n_k | f \sim g \Rightarrow f(x) = g(x)\} \subset \mathbb{R}^n_k$, then $(\mathrm{ev}_p, \mathrm{ev}_p) : (C^{n,k}, C^{n,k}_\partial) \to (\mathbb{R}, [0, \infty))$ factors through to an $\mathbb{R}$-point $(\mathrm{ev}_p, \mathrm{ev}_p) : (C, C_\partial) \to (\mathbb{R}, [0, \infty))$.

Lemma 4.43. *Let $(C, C_\partial) = (C^{n,k}/J, C^{n,k}_\partial/\sim)$ for $(J, \sim)$ a corner equivalence, and (ϕ, ϕ_∂) an $\mathbb{R}$-point of (C, C_∂). Then there exists $p \in Z(J) \cap \{x \in \mathbb{R}^n_k | f \sim g \Rightarrow f(x) = g(x)\} \subset \mathbb{R}^n_k$ such that $(\phi, \phi_\partial) = (\mathrm{ev}_p, \mathrm{ev}_p)$.*

Proof. As $\phi : C \to \mathbb{R}$ is an $\mathbb{R}$-point for C as a C^∞-ring without corners, there exists $p \in Z(J)$ such that $\phi = \mathrm{ev}_p$. For any $f \in C_\partial^{n,k}$, $\phi_\partial([f]) = i(\phi_\partial([f])) = \phi(i([f])) = i(f)(p) = f(p)$. So $\phi_\partial = \mathrm{ev}_p$. If $f \sim g$, then $\phi_\partial([f]) = \phi_\partial([g])$, and so $f(p) = g(p)$. □

Remark 4.44. As suggested by the proof of the lemma, $\mathbb{R}$-points (ϕ, ϕ_∂) are determined by ϕ, because the map $i : [0, \infty) \to \mathbb{R}$ is just the inclusion. So $\phi_\partial = i \circ \phi_\partial = \phi \circ i$.

Definition 4.45. Let $\mathfrak{C} = (C, C_\partial)$ be a C^∞-ring with corners. As a topological space, the *real spectrum* of (C, C_∂), denoted $\mathrm{Spec}^{\mathrm{r}}(\mathfrak{C})$, is the set of $\mathbb{R}$-points of C together with the Gelfand topology. The Gelfand topology is the smallest topology such that for all $f \in C, g \in C_\partial$, the maps

$$\mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C} \to \mathbb{R}, (\phi, \phi_\partial) \mapsto \phi(f),$$

$$\mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C} \to [0, \infty), (\phi, \phi_\partial) \mapsto \phi_\partial(g)$$

are continuous.

By the previous remark, the second condition is redundant, because the map $(\phi, \phi_\partial) \mapsto \phi_\partial(g)$ coincides with the map $(\phi, \phi_\partial) \mapsto \phi(i(g))$. So by the existence of characteristic functions $\mathbb{R}^n_k \to \mathbb{R}$, the Gelfand topology coincides with the Zariski topology, which declares $Z(f) = \{(\phi, \phi_\partial) | \phi(f) = 0\}$ closed for all $f \in C$.

Let $U \subset \mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C}$ be an open set. Let $S_U \subset C, T_U \subset C_\partial$ be the sets,

$$S_U = \{f \in C | \text{ for all } (\phi, \phi_\partial) \in U, \phi(f) \neq 0\},$$

$$T_U = \{f \in C_\partial | \text{ for all } (\phi, \phi_\partial) \in U, \phi_\partial(f) \neq 0\}.$$

Definition 4.46. Let $\mathfrak{C} = (C, C_\partial)$ be a C^∞-ring with corners. Let $X = \mathrm{Spec}^{\mathrm{r}}(\mathfrak{C})$. Define the *structure pre-sheaf* $(X, \tilde{\mathcal{O}}_X)$ to be the presheaf of C^∞-rings with corners on X that defines for $U \subset X$ open,

$$\tilde{\mathcal{O}}_X(U) = (C, C_\partial)(S_U, T_U)^{-1}.$$

The restriction maps are further localization. The *structure sheaf* associated to (C, C_∂), $(X, \mathcal{O}_X)$ is the sheafification of this presheaf. A C^∞-ringed with corners space is an *affine C^∞-scheme with corners* if it is isomorphic to $\mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C}$ for some C^∞-ring with corners $\mathfrak{C}$. A *C^∞-scheme with corners* is a C^∞-ringed with corners space that can be covered by affine C^∞-schemes with corners.

Let $(\psi, \psi_\partial) : \mathfrak{C} = (C, C_\partial) \to \mathfrak{D} = (D, D_\partial)$ be a morphism of C^∞-rings with corners. It defines a morphism $\mathrm{Spec}^{\mathrm{r}}\,\mathfrak{D} = (Y, \mathcal{O}_Y) \to \mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C} = (X, \mathcal{O}_X)$. Given an $\mathbb{R}$-point (ϕ, ϕ_∂) of (D, D_∂), $(\phi, \phi_\partial) \circ (\psi, \psi_\partial)$ is an $\mathbb{R}$-point of (C, C_∂), so we can define a map $\Psi : Y \to X$ which is continuous. Given an open set U of $\mathrm{Spec}^{\mathrm{r}}\,\mathfrak{C}$, note that $\psi_\partial(S_U) \subset S_{\Psi^{-1}(U)}$, and $\psi(T_U) \subset T_{\Psi^{-1}(U)}$, so there is a natural morphism of presheaves on X, $\Psi_\# : \tilde{\mathcal{O}}_X \to \Psi_*(\tilde{\mathcal{O}}_Y)$ induced by (ψ, ψ_∂),

$$\Psi_\# : \tilde{\mathcal{O}}_X(U) \to \Psi_*(\tilde{\mathcal{O}}_Y)(U) = \tilde{\mathcal{O}}_Y(\Psi^{-1}(U)).$$

This induces a morphism of presheaves of C^∞-rings with corners. Then $\mathrm{Spec}^{\mathrm{r}}(\psi, \psi_\partial) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is $\Psi : Y \to X$ and the sheafification of $\Psi_\#$. Morphisms of affine C^∞-schemes are defined to be morphisms which arise in this way.

4.9 Conclusion

For a finitely generated C^∞-ring C, the presheaf of C^∞-rings associated to C already satisfied the 'gluability' axiom for a sheaf. Fairness gave the identity axiom, at least globally. For C^∞-rings with corners, however, while our notion of fairness should again correspond to the identity axiom, it is not clear that there ought to be gluability. Sheafification will thus play a more important role for C^∞-schemes with corners. In general, most questions we discussed for C^∞-rings have analogues for C^∞-rings with corners. There is a cotangent space for manifolds with corners - is there a generalization for C^∞-rings with corners? This would perhaps provide insight on how to discuss modules over C^∞-rings with corners. One can also ask about the boundary of a C^∞-ring with corners. As mentioned before, it may be useful to loosen the definition of smooth maps to b-maps, rather than just interior b-maps.

A. Ordinary algebraic geometry

Let us briefly some basics of algebraic geometry. This appendix is based on Vakil's text [15]. This may be helpful in seeing the structural differences between ordinary algebraic geometry and C^∞-algebraic geometry.

Definition A.1. A *presheaf* $\mathcal{O}_X$ of sets (rings/modules) on a topological space X is a set (ring/module) $\mathcal{O}_X(U)$ for every open set $U \subset X$, together with the following data:

1. For every inclusion of open sets $V \subset U$ in X we have a *restriction map* $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which is a map of sets (rings/modules).
2. For open sets $W \subset V \subset U$ in X, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.
3. For all open sets $U \subset X$, the restriction map ρ_{UU} is the identity.

A presheaf is a *sheaf* if we have an additional two axioms:

1. Identity axiom: If $U \subset X$ is an open set, $f, g \in \mathcal{O}_X(U)$, and $\{V_i : i \in I\}$ is an open covering of U, such that for all $i \in I$,
$$\rho_{UV_i}(f) = \rho_{UV_i}(g),$$
then $f = g$.
2. Gluability axiom: If U is an open set in X, and $\{V_i : i \in I\}$ is an open covering of U, and for every V_i we have $f_i \in \mathcal{O}_X(V_i)$ such that for all $i, j \in I$,
$$\rho_{V_i V_i \cap V_j}(f_i) = \rho_{V_j V_i \cap V_j}(f_j),$$
then there exists an $f \in \mathcal{O}_X(U)$ such that $\rho_{UV_i}(f) = f_i$ for all $i \in I$.

A *morphism of sheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set $U \subset X$, such that for every inclusion of open sets $V \subset U$ in X the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{G}(V).
\end{tikzcd}$$

For $x \in X$, the *stalk* at x is the set of equivalence classes $\{[(f, U)] : f \in \mathcal{O}_X(U), x \in U\}$ under the equivalence relation $(f, U) \sim (g, V)$ if there exists an open $x \in W \subset U \cap V$ such that $\rho_{UW}(f) = \rho_{VW}(g)$. We denote the germ $[(f, U)]$ as f_x.

Presheaves can be made into sheaves by sheafification. This can be expressed by a universal property, and hence is unique up to unique isomorphism. The sheafification of the presheaf $\mathcal{F}$ is denoted $\mathcal{F}^{sh}$. For an open set U, $\mathcal{F}^{sh}(U)$ is the set of compatible germs:

$$\begin{aligned}\mathcal{F}^{sh}(U) = \{(f_x)_{x\in U} :\ & \text{for all } x \in U \\ & \text{there exists an open } V \subset U, g \in \mathcal{F}(V) \text{ such that for all } y \in V, g_y = f_y\}.\end{aligned}$$

Recall that given presheaves $\mathcal{F}, \mathcal{G}$ and a presheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism of sheaves $\psi : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$ such that, letting $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{sh}$ be the sheafification maps, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "\pi_{\mathcal{F}}"] & \mathcal{G} \arrow[d, "\pi_{\mathcal{G}}"] \\
\mathcal{F}^{sh} \arrow[r, "\psi"] & \mathcal{G}^{sh}.
\end{tikzcd}$$

One can describe the full data of a sheaf using only a base: that is, if we have a base of the topology of X, say $\mathcal{U}$, and the following data:

1. If $U, V \in \mathcal{U}$ and $V \subset U$, there is a restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which a map of sets (rings/modules).

2. For $U, V, W \in \mathcal{U}$ such that $W \subset V \subset U$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

3. For all $U \in \mathcal{U}$, the restriction map ρ_{UU} is the identity.

4. The gluability and identity axioms hold when restricted to $\mathcal{U}$.

Let $\pi : X \to Y$ be a continuous map of topological spaces, and $\mathcal{O}_X$ a sheaf on X. The *direct image* $\pi_*(\mathcal{O}_X)$ of $\mathcal{O}_X$ is the presheaf on Y given by, for $V \subset Y$ open, $\pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$. This is in fact a sheaf. There is also a pullback of a sheaf, $\pi^*\mathcal{O}_Y$, which is more difficult to define.

Definition A.2. A ringed space is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces $(\pi, \pi_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map of topological spaces $\pi : X \to Y$, and a morphism of sheaves on Y, $\pi_\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$. Equivalently, we could define $\pi_\#$ to be a morphism of sheaves on X, $\pi^*(\mathcal{O}_Y) \to \mathcal{O}_X$. A *locally ringed space* is a ringed space such that ring of germs at each point is local. A *morphism of locally ringed spaces* is a morphism of ringed spaces, with the additional requirement that it takes the maximal ideal of the germ in X to the maximal ideal of the germ in Y for every $x \in X$. Morphisms of locally ringed spaces induce maps of stalks. That is, if $x \in X, y = \pi(x)$, there is induced morphism of rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, $[(f, U)] \mapsto [(\pi_\#(U)(f), \pi^{-1}(U))]$ where $\pi_\#(U)(f) \in \pi_*\mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}(U))$ as needed.

A commutative ring with unity can be made into a locally ringed space using the Spec functor. Let R be a ring. As a topological space, let $\operatorname{Spec} R = \{p : p \text{ is a prime ideal of } R\}$. Define maps

$$Z(-) : \{\text{ideals in } R\} \longrightarrow \{\text{ sets in } \operatorname{Spec} R\}, Z(S) = \{p \in \operatorname{Spec} R | S \subset p\},$$

$$I(-) : \{\text{sets in } \operatorname{Spec} R\} \longrightarrow \{\text{ideals in } R\}, I(K) = \{f \in R | f \in p \text{ for all } p \in K\}.$$

The *Zariski topology* on $\operatorname{Spec} R$ is defined by saying $Z(S)$ is closed for every $S \subset R$. The following lemma lists some well-known facts about these maps.

Lemma A.3. *Let R be a ring, $J \subset R$ an ideal in R, and $K \subset \operatorname{Spec} R$ a closed set.*

1. $J \subset I(Z(J))$.

2. $K = Z(I(K))$.

3. $I(Z(J)) = \sqrt{J}$*, where* $\sqrt{J} = \{f|$ *there exists* $n \in \mathbb{N}$ *such that* $f^n \in J\}$.

The topology on $\operatorname{Spec} R$ has as an open basis $D(f) = \{p \in \operatorname{Spec} R | f \notin p\}$ for all $f \in R$. We think of elements in R as functions on $\operatorname{Spec} R$, where the value of f at p is the projection of f in R/p. However, because of nilpotents (which are precisely elements in $\bigcap_{p \in \operatorname{Spec} R} p$), functions may not be determined by their values at points. In the particular case of the ring $R = K[X_1, \ldots, X_n]$, where K is an algebraically closed field, functions (elements in R) are determined by their values on the spectrum, and moreover, they are determined by their value at the maximal ideals of R, which are in one to one correspondence with elements in K^n.

The ringed space $\operatorname{Spec} R = (X, \mathcal{O}_R)$ is $X = \operatorname{Spec} R$ as a topological space, together with the sheaf on the base of distinguished open set (sets of the form $D(f), f \in R$), where $\mathcal{O}_R(D(f))$ is the localization of R at the set of all elements $g \in R : D(f) \subset D(g)$. It is non-trivial that this in fact defines a sheaf on a base. The proof can be found in [15, p. 127]. We follow this proof to show the same result for C^∞-rings. An affine scheme is a ringed space which is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_R)$ for some ring R. A scheme is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. If $\phi : R \to S$ is a morphism of commutative rings, then it induces a morphisms of affine sheaves $\operatorname{Spec} S \to \operatorname{Spec} R$. We want morphisms of schemes to locally look like the morphisms that arise in this way. One can define morphisms of schemes like this, but equivalently, morphisms of locally ringed spaces coincide with them, which gives an alternative definition.

B. Manifolds as fair affine C^∞-schemes

The goal of this appendix (following Moerdijk and Reyes [11, p. 25-30]) will be to prove that the category of manifolds can be fully and faithfully embedded into the opposite category of finitely presented C^∞-rings (and hence in the category of C^∞-schemes) in a way that preserves transversal fibre products.

Theorem B.1. *Let M be a smooth manifold. Then $C^\infty(M)$ is a finitely presented C^∞-ring.*

Proof. We can embed M into $\mathbb{R}^n$ for some n. In fact, using the ϵ Neighborhood Theorem (see [13, p. 69]), M has an open neighborhood U in $\mathbb{R}^n$, and a smooth retract $r : U \to M$. That is, $r \circ i = id_M$, where $i : M \to U$ is the inclusion. We get morphisms $R : C^\infty(M) \to C^\infty(U), f \mapsto f \circ r$, and $I : C^\infty(U) \to C^\infty(M), g \mapsto g \circ i$, where $I \circ R = id_{C^\infty(M)}$. So $C^\infty(M)$ is a retract of $C^\infty(U)$. We have already proved $C^\infty(U)$ is a finitely presented C^∞-ring, and as $C^\infty(M)$ can be embedded as a subring of $C^\infty(U)$ by the injective R, it is also finitely presented. □

Corollary B.2. *Let M, N be smooth manifolds. Then $C^\infty(M) \otimes_\infty C^\infty(N) \cong C^\infty(M \times N)$.*

Proof. As in the proof of the above theorem, let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open neighborhoods of M and N respectively, such that $f : U \to M$ and $g : V \to N$ are retracts. $M \times N$ is a retract of $U \times V$, where the retraction is $r = (f, g) : U \times V \to M \times N$. Let i_U, i_V, and $i_{U\times V}$ be the inclusions, and $F, G, R, I_U, I_V, I_{U\times V}$ the induced morphisms of C^∞-rings of all of these. Recall that

$$C^\infty(U) \otimes_\infty C^\infty(V) \cong C^\infty(U \times V).$$

We have a commutative diagram

$$\begin{tikzcd}
C^\infty(U) \arrow[r] \arrow[d, shift left, "I_U"] & C^\infty(U \times V) \arrow[d, shift left, "I_{U\times V}"] & C^\infty(V) \arrow[l] \arrow[d, shift left, "I_V"] \\
C^\infty(M) \arrow[r] \arrow[u, shift left, "F"] & C^\infty(M \times N) \arrow[u, shift left, "R"] & C^\infty(N). \arrow[l] \arrow[u, shift left, "G"]
\end{tikzcd}$$

Using universal properties, the fact the top line is a coproduct requires that the bottom line is as well. □

Two maps $f_i : M_i \to N, i = 1, 2$ are transversal if given $x_i \in M_i$ such that $f_1(x_1) = y = f_2(x_2)$, $\mathrm{im}(df_{1_{x_1}})$ and $\mathrm{im}(df_{2_{x_2}})$ span T_yN. A map $f : M \to N$ is transversal to a submanifold Z of N if for all $x \in M$, $\mathrm{im}(df_x)$ and $T_{f(x)}Z$ span $T_{f(x)}M$. To complete the section, we show that transversal pullbacks of manifolds are taken to pushouts of C^∞-rings, under the contravariant functor $M \mapsto C^\infty(M)$.

Theorem B.3. *Let $f_i : M_i \to N, i = 1, 2$ be transversal maps. Let $\triangle \subset N \times N$ be the diagonal. Then the pullback diagram*

$$\begin{tikzcd} (f_1 \times f_2)^{-1}(\triangle) \arrow[r] \arrow[d] & M_1 \arrow[d, "f_1"] \\ M_2 \arrow[r, "f_1"] & N \end{tikzcd}$$

gives us a pushout diagram

$$\begin{tikzcd} C^\infty(N) \arrow[r] \arrow[d] & C^\infty(M_1) \arrow[d] \\ C^\infty(M_2) \arrow[r] & C^\infty((f_1 \times f_2)^{-1}(\triangle)). \end{tikzcd}$$

Proof. Following [11, p. 28], we prove the theorem in three steps.

1. Reduce to the case where we are given $f : M \to N$ transversal to a submanifold $Z \subset N$.

2. Reduce to showing the theorem for a particular open cover.

3. Show that such an open cover exists.

The first step is the easiest, as f_1, f_2 are transversal if and only if $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ is transversal to the diagonal $\triangle \subset N \times N$. The pullback of f_1 and f_2 is $(f_1 \times f_2)^{-1}(\triangle)$. So setting $f = f_1 \times f_2$ and $Z = \triangle$, $M = M_1 \times M_2$ and replacing

N with $N \times N$ it is enough to show the following statement: Let $f : M \to N$ be transversal to the submanifold $Z \subset N$. Then the pullback diagram

$$\begin{tikzcd} Z \arrow[r] & N \\ f^{-1}(Z) \arrow[u] \arrow[r] & M \arrow[u, "f"'] \end{tikzcd}$$

gives us a pushout diagram

$$\begin{tikzcd} C^\infty(N) \arrow[r] \arrow[d] & C^\infty(M) \arrow[d] \\ C^\infty(Z) \arrow[r] & C^\infty(f^{-1}(Z)). \end{tikzcd}$$

For the second step, the open cover we need is as follows. Suppose there exists an open cover $\{U_\alpha, \alpha \in I\}$ of N , such that for all finite subsets $A \subset I$, setting $U_A = \bigcap_{\alpha \in A} U_\alpha, V_A = f^{-1}(U_A)$, the diagram below is a pushout:

$$\begin{tikzcd} C^\infty(U_A) \arrow[r, "- \circ f"] \arrow[d] & C^\infty(V_A) \arrow[d] \\ C^\infty(U_A \cap Z) \arrow[r] & C^\infty(V_A \cap f^{-1}(Z)). \end{tikzcd}$$

We show that this implies the theorem. Suppose $\phi : C^\infty(M) \to B, \psi : C^\infty(Z) \to B$ are morphisms of C^∞-rings such that $\phi(g \circ f) = \psi(g|_Z)$ for all $g \in C^\infty(N)$. As M, Z are manifolds, $C^\infty(M), C^\infty(Z)$ are finitely presented, and so we can assume that B is finitely presented as well, say $B \cong C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_k)$. We get an induced map $Z(f_1, \ldots, f_k) \to Z$, so U_α induces an open cover W_α of $Z(f_1, \ldots, f_k)$. For every finite subset $A \subset I$, as restriction to an open subset (that is, localization) and quotienting commute, $C^\infty(W_A)/(f_1|_{W_A},\dots,f_k|_{W_A}) \cong C^\infty(\mathbb{R}^n)/(f_1,\dots,f_k)|_{W_A}$. So ϕ,ψ restrict to morphisms

$$\phi_A : C^\infty(V_A) \to C^\infty(W_A)/(f_1|_{W_A},\dots,f_k|_{W_A}),$$

$$\psi_A : C^\infty(U_A\cap Z) \to C^\infty(W_A)/(f_1|_{W_A},\dots,f_k|_{W_A}).$$

Because we have been given pushout diagrams for each A by assumption, the universal property of pushouts gives us unique factorizations

$$\epsilon_A : C^\infty(V_A\cap f^{-1}(Z)) \to C^\infty(W_A)/(f_1|_{W_A},\dots,f_k|_{W_A}).$$

To complete this step, we need to glue the ϵ_A together to the required, and unique $\epsilon : C^\infty(f^{-1}(Z)) \to B$. But B is finitely presented, so it is strongly fair, and so $\operatorname{Spec}^{\mathrm{r}} B$ is a C^∞-scheme without any sheafification required, and by the gluing and identity axioms ϵ exists and is unique.

For the second step, we show that such an open cover does indeed exist. Let Z have codimension q. The local immersion theorem (see, for example [13, p. 29]) states that there is an open covering U_α of N such that $Z\cap U_\alpha = Z(g_1,\dots,g_q)$ for independent $g_i : U_\alpha \to \mathbb{R}$. Since f is transversal to Z, the $g_i\circ f : V_\alpha = f^{-1}(U_\alpha) \to \mathbb{R}$ are also independent. By Proposition 1.23, $C^\infty(U_\alpha)/(g_1,\dots,g_q) \cong C^\infty(U_\alpha\cap Z)$, and $C^\infty(V_\alpha) \cong C^\infty(Z(g_1\circ f,\dots,g_q\circ f)) \cong C^\infty(V_\alpha)/(g_1\circ f,\dots,g_q\circ f)$. So the following two commutative diagrams are the same:

$$\begin{tikzcd}
C^\infty(U_\alpha) \arrow[r] \arrow[d] & C^\infty(V_\alpha) \arrow[d] \\
C^\infty(U_\alpha\cap Z) \arrow[r] & C^\infty(V_\alpha\cap f^{-1}(Z)),
\end{tikzcd}$$

$$\begin{tikzcd}
C^\infty(U_\alpha) \arrow[r, "-\circ f"] \arrow[d] & C^\infty(V_\alpha) \arrow[d] \\
C^\infty(U_\alpha)/(g_1,\dots,g_q) \arrow[r] & C^\infty(V_\alpha)/(g_1\circ f,\dots,g_q\circ f).
\end{tikzcd}$$

The second diagram is clearly a pushout diagram. For each U_A, A a finite subset of I, we can apply Proposition 1.23 in the same way to get the required pushout squares. So the cover U_α is the cover needed in the previous step. $\square$

C. Finiteness

In ordinary algebraic geometry, we can define Krull dimension for rings and schemes using chains of prime ideals (irreducible closed sets). Prime ideals are not a useful concept for C^∞-rings, however (for example, $Z(p)$ is always a singleton for a prime ideal p). However, there is a geometric interpretation of Noether's Normalization Lemma which relates the idea of a finite morphism to dimension: if k is a field, and X is a finitely generated affine k-scheme of Krull dimension n, then there is a surjective finite morphism $X \to \mathbb{A}^n_k$ [15, p.303]. The concept of finite morphisms can be generalized to C^∞-rings. This appendix, following González and Salas [4], proves basic results on this generalization.

Definition C.1. Let C be a C^∞-ring. C is *finite* if C is a finite dimensional $\mathbb{R}$-vector space. Its dimension is called the degree of C. C is *rational* if for every maximal ideal $m \subset C$, m is a real maximal ideal. Let C, D be C^∞-rings, and $\phi : C \to D$ a morphism. ϕ is *finite* if D is finitely generated over C.

In this section, we will occasionally use the Nakayama lemma, so we state a version of it below.

Lemma C.2. *Let C be a ring, M a finitely generated C-module, and $N \subset M$ a submodule. Let $J(C)$ be the Jacobson radical of C, the intersection of all maximal ideals of C. If $N + J(C)M = M$, then $N = M$.*

We slightly adapt the result from [4, p.100].

Proposition C.3. *A finite C^∞-ring C is fair if and only if it is rational.*

Proof. Suppose $C \cong C^\infty(\mathbb{R}^n)/I$ is a finite fair C^∞-ring. Let m be a maximal ideal of C, which corresponds to a maximal ideal of $C^\infty(\mathbb{R}^n)$ containing I, which we also call m . As C is a finite $\mathbb{R}$-algebra, $C/m \cong C^\infty(\mathbb{R}^n)/m$ is a finite field extension of $\mathbb{R}$, and hence it is either $\mathbb{R}$ or $\mathbb{C}$. Suppose that $C/m \cong \mathbb{C}$. Then there is an $f \in C$ such that $[f] = i$ under this isomorphism, and so $f^2 + 1 \in m$. But this is impossible because $f^2 + 1$ is never zero, and so is an invertible element of C. So C is rational.

Suppose C is a finite rational C^∞-ring. Since it is finitely generated as an $\mathbb{R}$-algebra, it is certainly a finitely generated C^∞-ring, so $C \cong C^\infty(\mathbb{R}^n)/I$. First, suppose C is local. Then $Z(I) = p$ for the unique maximal ideal m corresponding to $p \in \mathbb{R}^n$. Since C is finitely generated as an $\mathbb{R}$-algebra, $m^{k+1} = m^k$ for some integer k, and so by the Nakayama lemma applied to $m(m^k) = m^k$, $m^k = 0$. That is, for all g such that $g(p) = 0$, $g^k \in I$. In particular, if $f \in (x_i - p_i | i = 1, \ldots, n)$, then $f(p) = 0$, so $f^k \in I$. So I is an ideal of $D = C^\infty(\mathbb{R}^n)/(x_i - p_i | i = 1, \ldots, n)^k$, which is a finite $\mathbb{R}$-algebra. So I is finitely generated in D, and hence fair. So $C^\infty(\mathbb{R}^n)/I$ is fair. To generalize to the non-local case, recall that a finite $\mathbb{R}$-algebra can be written as the direct product of local finite algebras (see [4, p.100] for more details). $\square$

Definition C.4. A C^∞-scheme $(X, \mathcal{O}_X)$ is *finite* if it is affine, and $(X, \mathcal{O}_X) = \operatorname{Spec}^{\mathrm{r}} C$ for C a rational finite C^∞-ring.

Theorem C.5. *Let $(X, \mathcal{O}_X)$ be a C^∞-scheme. Then the following are equivalent conditions on $(X, \mathcal{O}_X)$:*

1. *$(X, \mathcal{O}_X)$ is finite.*
2. *X is finite, and $\mathcal{O}_X(X)$ is a finite C^∞-ring.*
3. *X is finite and for all $x \in X$, $\mathcal{O}_{X,x}$ is finite.*

Proof. $(1 \Rightarrow 2)$ Suppose that C is rational finite and $X = \operatorname{Spec}^{\mathrm{r}} C$. Then C is fair, so $\mathcal{O}_X X \cong C$, so $\mathcal{O}_X X$ is rational finite. If $X = \operatorname{Spec}^{\mathrm{r}} C$ was infinite, then we would have an infinite sequence $m_i, i \in \mathbb{N}$ of points in $\operatorname{Spec}^{\mathrm{r}} C$. But $m_1 \cap \cdots \cap m_k$ is a strict sub-vector space of $m_1 \cap \cdots \cap m_{k-1}$, so we get an strictly decreasing sequence. But as C is finite dimensional, this is a contradiction.

$(2 \Rightarrow 3)$ $\mathcal{O}_{X,x_i}$ is a quotient of $\mathcal{O}_X X$, so if $\mathcal{O}_X X$ is finite, so is $\mathcal{O}_{X,x_i}$.

$(3 \Rightarrow 1)$ Let $X = \{x_1, \ldots, x_k\}$. It is a discrete topological space, since points are closed, and so $\{x_i\}$ is also open. The finite disjoint union of affine C^∞-schemes is affine, so as $\mathcal{O}_{X,x_i} \cong \mathcal{O}_X\{x_i\}$ is affine, X is affine. Also, $\mathcal{O}_X X = \mathcal{O}_X\{x_1\} \oplus \cdots \oplus \mathcal{O}_X\{x_k\} = \mathcal{O}_{X,x_1} \oplus \cdots \oplus \mathcal{O}_{X,x_k}$. As $\mathcal{O}_{X,x_i}$ is finite, $\mathcal{O}_X X$ is finite and fair, and hence rational. □

Corollary C.6. *[4, p.102] Let $\phi : Y = \operatorname{Spec}^r D \to X = \operatorname{Spec}^r C$ be a morphism of fair affine C^∞-schemes. Recall from Example 2.15 that the fibre $\phi^{-1}(x) = x \times_X Y$ at $x \in X$ is $\operatorname{Spec}^r D/m_x D$. Then if $D/m_x D$ is finite, and $\phi^{-1}(x) = \{y_1, \ldots, y_k\}$, then*

$$D/m_x D \cong D_{y_1}/m_x D_{y_1} \oplus \cdots \oplus D_{y_k}/m_x D_{y_k}.$$

Proof. $Z = \operatorname{Spec}^{\mathrm{r}} D/m_x D = \phi^{-1}(x) = \{y_1, \ldots, y_k\}$ and by the proof of the previous theorem, $D/m_x D = \mathcal{O}_{Z,y_1} \oplus \cdots \oplus \mathcal{O}_{Z,y_k} = D_{y_1}/m_x D_{y_1} \oplus \cdots \oplus D_{y_k}/m_x D_{y_k}$. □

Finally, we can define a morphism of locally fair C^∞-schemes as *finite* if it is a closed separated morphism $\phi : Y \to X$ and $\phi^{-1}(x)$ is finite for every $x \in X$. The degree of $\phi^{-1}(x)$ is called the degree of ϕ at x. To prove the main theorem of this section, we will need Malgrange's Preparation Theorem, which is stated in [4] as: given differentiable spaces X, Y, and a morphism $\phi : Y \to X, y \in Y, x = \phi(y)$, the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is finite if and only if $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is a finite $\mathbb{R}$-algebra. So in particular this holds for $C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^m)$. The next lemma follows part of Malgrange's proof to show that this implies the Preparation Theorem for all locally fair C^∞-schemes.

Lemma C.7. *Let X, Y be locally fair C^∞-schemes, $\phi : Y \to X$ a morphism, $y \in Y, x = \phi(y)$. Then the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is finite if and only if $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is finite.*

Proof. Since the theorem is local, it suffices to show this for X, Y fair affine, say $X = \operatorname{Spec}^{\mathrm{r}} C, Y = \operatorname{Spec}^{\mathrm{r}} D$, $C \cong C^\infty(\mathbb{R}^n)/I, D \cong C^\infty(\mathbb{R}^m)/J$, I, J fair ideals, and $\phi : C \to D$. We use the argument of Malgrange's reduction to the free case of the preparation theorem for analytic algebras [9, p. 34] to reduce to the case $C = C^\infty(\mathbb{R}^n), D = C^\infty(\mathbb{R}^m)$.

If $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is finite, then as $\mathcal{O}_{X,x}/m_x \cong \mathbb{R}$, $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is a finite $\mathbb{R}$-algebra.

Conversely, assume $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is a finite $\mathbb{R}$-algebra. Note that m_x in C can be lifted to the maximal ideal in $C^\infty(\mathbb{R}^n)_x$, which we temporarily call $\tilde{m}_x$. This ideal contains the image of I in $\mathcal{O}_x$, which we call I_x, as $x \in Z(I)$. So using the composition $C^\infty(\mathbb{R}^n)_x \to (C^\infty(\mathbb{R}^n)/I)_x \to \mathcal{O}_{Y,y}$, we have that $\mathcal{O}_{Y,y}/\tilde{m}_x\mathcal{O}_{Y,y}$ is finite. If the composition is a finite morphism, or equivalently, D_y is finite over $C^\infty(\mathbb{R}^n)_x$, then D_y is finite over $(C^\infty(\mathbb{R}^n)/I)_x \cong C^\infty(\mathbb{R}^n)_x/I_x$. So we have reduced to the case $C \cong C^\infty(\mathbb{R}^n)$.

The next step is the to reduce to the case where J is finitely generated. Let ψ be a lift of ϕ, that is, $\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ and $\psi \circ \pi = \phi$. Let m_y denote the maximal ideal of $C^\infty_y(\mathbb{R}^m)$, and $\overline{m_y}$ its image in $D_y \cong \mathcal{O}_{Y,y}$. Since D_y/m_xD_y is a finite $\mathbb{R}$-algebra, there exists a nonnegative integer k such that $[\overline{m_y}^k] = 0$ in D_y/m_xD_y, and hence $\overline{m_y}^k \subset m_xD_y$. So $m_y^k \subset J_y + m_xC^\infty_y(\mathbb{R}^m)$. As m_y is finitely generated, there exists $J' \subset J$ finitely generated such that $m_y^k \subset J'_y + m_xC^\infty_y(\mathbb{R}^m)$. As $C^\infty_y(\mathbb{R}^m)/m_y \cong \mathbb{R}$, and m_y^l is finitely generated, $C^\infty_y(\mathbb{R}^m)/m_y^l$ is finite over $\mathbb{R}$. So as $m_y^k \subset J'_y + m_xC^\infty_y(\mathbb{R}^m)$, $(C^\infty(\mathbb{R}^m)/J')_y/m_x(C^\infty(\mathbb{R}^m)/J')_y$ is finite. If $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)/J'$ is finite, then $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)/J$ is finite, so we have reduced to the case when J is finitely generated.

So assume $J = (g_1, \ldots, g_p)$ is finitely generated. Without loss of generality, $x = 0$, and so $x_1, \ldots, x_n$ generate m_x. Let z_i denote the generators of $C^\infty(\mathbb{R}^{n+p})$. Define a map $\alpha : C^\infty(\mathbb{R}^{n+p}) \to C^\infty(\mathbb{R}^n)$ satisfying $\alpha(z_i) = \psi(x_i)$ for $1 \le i \le n$, and $\alpha(z_{n+j}) = g_j$, for $1 \le j \le p$. Let $m_{0,0}$ denote the image under α of the maximal ideal of $C^\infty_0(\mathbb{R}^{n+p})$. As $J \subset m_{0,0}$,

$$C^\infty_y(\mathbb{R}^m)/m_{0,0}C^\infty_y(\mathbb{R}^m) \cong D_y/m_xD,$$

and so $C^\infty_y(\mathbb{R}^m)/m_{0,0}C^\infty_y(\mathbb{R}^m)$ is finite. If α_0 is a finite morphism, then so is ϕ_x, and hence we have reduced to the case $J = 0$.

As we have reduced to the case $C \cong C^\infty(\mathbb{R}^n), D \cong C^\infty(\mathbb{R}^m)$, we can now apply Malgrange's preparation theorem as stated in [4] for *differentiable algebras* to complete the proof. $\square$

Theorem C.8. *[4, p. 103] Let C, D be fair C^∞-rings, and $\phi : C \to D$. Then ϕ is a finite morphism of $\mathbb{R}$-algebras if and only if $\psi = \operatorname{Spec}^r \phi : \operatorname{Spec}^r D \to \operatorname{Spec}^r C$ is a finite morphism of C^∞-schemes.*

Proof. Suppose $\phi : C \to D$ is finite. Then there are $b_1, \dots, b_d \in D$ such that $D = Cb_1 + \cdots + Cb_d$. Let $x \in \operatorname{Spec^r} C$. Then as $D/m_x D = (C/m_x)[b_1] + \cdots (C/m_x)[b_d]$, $D/m_x D$ is a finite algebra over $C/m_x \cong \mathbb{R}$ of degree less than d. So $\psi^{-1}(x) = D/m_x D$ is a finite C^∞-ring. The map ψ is separated because $\operatorname{Spec^r} C, \operatorname{Spec^r} D$ are Hausdorff. To show that ψ is closed, let $V \subset \operatorname{Spec^r} D$ be a closed set. Then $V = \operatorname{Spec^r} D/I(V) = \operatorname{Spec^r} D/I(V)^{fa} = \operatorname{Spec^r} D/(I(V)^{fa})$. Let $J = \phi^{-1}(I(V)^{fa})$. Note that J is fair because J is the kernel of $C \to D/I(V)^{fa}$, and $D/I(V)^{fa}$ is fair. So we have

$$\psi : V = Z(I(V)^{fa}) = \operatorname{Spec^r} D/(I(V)^{fa}) \to \operatorname{Spec^r} C/J = Z(J),$$

and it suffices to show that $\psi(V) = Z(J)$, which is equivalent to showing that the fiber $\phi^{-1}(x) \neq \emptyset$ for all $x \in \operatorname{Spec^r} C/J = Z(J)$. Let $A = C/J, B = D/I(V)^{fa}$. The induced map $\tilde{\phi} :: A \to B$ is finite and injective, and so $A_x \to B_x$ is injective. Since $A_x \neq 0$, $B_x \neq 0$, and by the Nakayama lemma (which we can apply because B_x is local and nonzero), $m_x B_x \neq B_x$, and hence $m_x B \neq B$. As $m_x B$ is finitely generated, it is fair, and so by the Nullstallensatz for fair rings, $\phi^{-1}(x) = \operatorname{Spec^r} B/m_x B \neq \emptyset$.

Conversely, suppose ψ is finite. Let $x \in \operatorname{Spec^r} C$, and $\psi^{-1}(x) = \{y_1, \dots, y_r\}$. By assumption, $D_{y_i}/m_x D_{y_i}$ is a finite algebra, so by the previous lemma, D_{y_i} is a finite over C_x. Let D_x denote the germ at x of the pushforward sheaf $\psi_*(\mathcal{O}_D)$. We can write D_x as a directed colimit over open neighborhoods of x, that is,

$$D_x = \varinjlim_{x \in U} (\psi_*(\mathcal{O}_D).$$

Since $\operatorname{Spec^r} D$ is Hausdorff, there exist disjoint open neighborhoods $V_1, \dots, V_r$ of $y_1, \dots, y_r$. Let U be an open neighborhood of x such that $U \cap \psi(Y - V_1 \cup \cdots \cup V_r) = \emptyset$. Then

$$\psi^{-1}(U) = (V_1 \cap \psi^{-1}(U)) \sqcup \cdots \sqcup (V_r \cap \psi^{-1}(U)).$$

The above is still true if we replace U with any open set contained in U. So we have defined a basis of neighborhoods of x such that for each open neighborhood U in the basis, $\psi^{-1}(U)$ is the disjoint union of neighborhoods of the y_i. Note that for $W = \sqcup W_i$, where W_i is a neighborhood of y_i, and f_i is a characteristic function for W_i,

$$\begin{aligned}\mathcal{O}_D(W) &= (D\{f_1 + \cdots + f_k\}^{-1})^{fa} = (\oplus_i D\{f_i\}^{-1})^{fa} \\ &= \oplus_i \mathcal{O}_D(W_i).\end{aligned}$$

Now we restrict to this disjoint basis in the above direct limit and use the above equation to conclude that:

$$D_x = \varinjlim_{x \in U} \psi_* \mathcal{O}_D(U) = \varinjlim_{x \in U} \mathcal{O}_D(\psi^{-1}(U)),$$

$$D_x = \varinjlim_{x \in U} \mathcal{O}_D(V_1) \oplus \cdots \oplus \mathcal{O}_D(V_r) = D_{y_1} \oplus \cdots \oplus D_{y_r}.$$

We have shown that D_{y_i} is finite over C_x, so by the last line, D_x is finite over C_x. Recall that $D/m_x D \cong D_{y_1}/m_x D_{y_1} \oplus \cdots \oplus D_{y_r}/m_x D_{y_r}$, and as ψ has bounded degree, there is an integer d such that $\dim D/m_x D \leq d$ for all $x \in \mathrm{Spec}^{\mathrm{r}}\, A$. As D is fair, $D \cong C^\infty(\mathbb{R}^m)/J$ for some m, J. Let $t_1, \ldots, t_m$ be the generators of $C^\infty(\mathbb{R}^m)$. The set $\{1, t_i, t_i^2, \ldots, t_i^d\}$ is linearly dependent in $D/m_x D$, so every monomial in the t_i in $D/m_x D$ can be written as a linear combination of

$$\{t_1^{a_1} \cdots t_m^{a_m} | 0 \leq a_i < d\}.$$

So the vector subspace of polynomials in $D/m_x D$ is finite dimensional, and hence it is closed in the Fréchet topology on $D/m_x D$. But the space of polynomials is dense in $C^\infty(\mathbb{R}^m)$, and hence in $D/m_x D$, so in fact monomials of this form generate $D/m_x D$. Let M' be the C_x submodule of D_x generated by these monomials. Recall that $D_x = D_{y_1} \oplus \cdots \oplus D_{y_r}$. So $M' + m_x D_x = D_x$. As D_x is a finite dimensional C_x-module, we can apply the Nakayama lemma and conclude that $M' = D_x$. So these monomials generate D_x as a C_x-module..

Define a morphism of sheaves of $\mathcal{O}_C$-modules $\mathcal{O}_C^r \to \psi_* \mathcal{O}_D$ defined by these monomials. Then at the level of stalks, $(\mathcal{O}_C^r)_x \to D_x$ is surjective since the image contains all of these monomials. Since surjectivity can be checked at the level of stalks, this is an epimorphism of ringed spaces, and by taking global sections, $C^r \to D$ is surjective. So D is a finite C-module, and by definition, ϕ is a finite morphism. $\square$

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