To my parents,

Vivien Wei Ngan Chu
and
Yuen Hing Lam.
The sea of knowledge knows no bounds,
No shore’s in sight even if the sailor turns around.

Anonymous.

To pursue the unlimited with the limited,
Is it not perilous?

Master Chong, Carving up an Ox.

Remember not the gains and losses,
Nor the love and hatred.
Remember only the good old days,
Once we’re acquaintances.

Jim Wong, The Giant.

Looking back at where the winds and rains began,
– I’ll go home –
Now there’s no more wind, rain, nor sunshine therein.

So Sik, Calming the Winds and Waves.
Abstract

This thesis studies derived equivalences between total spaces of vector bundles and dg-quivers.

A dg-quiver is a graded quiver whose path algebra is a dg-algebra. A quiver with superpotential is a dg-quiver whose differential is determined by a “function” $\Phi$. It is known that the bounded derived category of representations of quivers with superpotential with finite dimensional cohomology is a Calabi–Yau triangulated category. Hence quivers with superpotential can be viewed as noncommutative Calabi–Yau manifolds.

One might then ask if there are derived equivalences between Calabi–Yau manifolds and quivers with superpotential. In this thesis, we answer this question and, generalizing Bridgeland [15], give a recipe on how to construct such derived equivalences.

Let $\pi : V \to X$ be an anti-semiample vector bundle over a smooth projective variety $X$, i.e., $S^k V^\vee$ is globally generated for $k \gg 0$. Given a full exceptional sequence $E$ on $D^b(\text{Coh}(X))$, under some cohomological vanishing conditions, we construct a dg-quiver $Q_E$ in terms of the dual exceptional sequence of $E$ such that $D^b(\text{Coh}(V)) \cong D^b_{fg}(\text{Rep}(Q_E))$. Moreover, this equivalence restricts to an equivalence between $D^b_{cs}(\text{Coh}(V))$, the full subcategory containing complexes of compact support, and $D^b_{fd}(\text{Rep}(Q_E))$, the full subcategory containing complexes with finite dimensional cohomology. If $V$ is non-compact Calabi–Yau, we show that $Q_E$ is equipped with a superpotential $\Phi$, i.e., the differential on $Q_E$ is determined by the “function” $\Phi$. In this case, the triangulated categories $D^b_{cs}(\text{Coh}(V))$ and $D^b_{fd}(\text{Rep}(Q_E))$ are both Calabi–Yau.

We can also construct derived equivalences equivariantly. Suppose a finite group $G$ acts on $X$ and this action lifts to $V$, endowing $\pi : V \to X$ the structure of an equivariant vector bundle. Suppose further that each object in the exceptional sequence $E$ is equipped with a $G$-linearization. Then we can construct a quotient dg-quiver $Q_E/G$ from $Q_E$, generalizing the construction of the McKay quiver, such that $D^b(\text{Coh}^G(V)) \cong D^b(\text{Rep}_{fg}(Q_E/G))$. If $V$ is non-compact Calabi–Yau equivariantly, then $Q_E/G$ is also equipped with a superpotential.

We also give a product construction for derived equivalences. Suppose we have vector bundles $\pi_V : V \to X$ and $\pi_W : W \to Y$, with full exceptional sequences $E$ on $D^b(\text{Coh}(V))$ (resp. $F$ on $D^b(\text{Coh}(W))$), then we can construct a product dg-quiver $Q_E \times Q_F$ such that $D^b(\text{Coh}(V \times W)) \cong D^b(\text{Rep}_{fg}(Q_E \times Q_F))$. If both $V$ and $W$ are Calabi–Yau, then $Q_E \times Q_F$ is also equipped with a superpotential.

Using these constructions, we can produce a lot of beautiful pictures of quivers with superpotential derived equivalent to the total spaces of vector bundles which are Calabi–Yau. Examples include $T^2_\mathbb{P}^2$, $K_{\mathbb{P}^2}$, and $O_{\mathbb{P}^2}(-1) \oplus O_{\mathbb{P}^2}(-2)$ etc.

Finally, we try to connect quivers with superpotential to the recent work by Pantev, Toën, Vaquié and Vezzosi [58] and Ben-Bassat, Brav, Bussi and Joyce [4] on shifted symplectic structures. We outline a strategy of proof for the existence of shifted symplectic structures in a standard ‘Darboux form’ on the derived moduli stack of representations of quivers with superpotential.
Acknowledgements

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The writing up period of this thesis saw Hong Kong experienced its most significant political protest ever – the Umbrella Revolution. I wish to thank all my fellow Hong Kongers who are fighting for freedom and democracy at this very moment. Without them, I would have lost my beloved home. I am ashamed
of myself not being physically there and can only act as a “keyboard warrior”. I hope everyone stays safe and takes good care of themselves.

Lastly, but most importantly, I would like to give my heartfelt thanks to my parents for their love and unconditional support throughout my life. They have provided me an excellent and carefree environment to grow up, as well as all the freedom I need to pursue whatever I want in life. I must also thank them for their tolerance and understanding, for in the last twenty-something years, which I am sure, I have not been very productive, and have not given much back to the family. I promise to be a more responsible son in the years to come.

May I end this note by quoting a dialogue from the *Ashes of Time*, one of my favourite movies, directed by Wong Kar-Wai:

“What’s beyond this desert?”

“Just another desert.”
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Introduction

In Calabi–Yau geometry, one important object is the category of coherent sheaves. Coherent sheaves on $m$-Calabi–Yau manifolds enjoy a special form of Serre duality: for any coherent sheaves $E$ and $F$, \[ \text{Ext}^i(E, F) \cong \text{Ext}^{m-i}(F, E) \vee. \] This property is written solely in terms of the Ext groups and hence can be axiomatised to a definition of Calabi–Yau abelian categories. As algebraic geometry progresses, we soon know that it is more flexible to work in the derived category of coherent sheaves (which is a triangulated category) rather than the abelian category. Kontsevich [46] later modified the definition of Calabi–Yau abelian categories to give a notion of Calabi–Yau triangulated categories.

The earliest examples of Calabi–Yau categories other than coherent sheaves on Calabi–Yau manifolds were probably given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22] as representations of some quivers with relations coming from the derivatives of a linear sum of closed paths (the superpotential). One problem of this definition of quivers with superpotential, however, is that they do not always produce Calabi–Yau categories. This problem was later solved by Ginzburg [27], Keller [41] and van den Bergh [71] (see also van den Bergh [70]) by adding derived structures, i.e., considering dg-quivers (graded quivers whose path algebra is endowed with the structure of dg-algebras) rather than quivers with relations. This new definition of quivers with superpotential as dg-quivers always produces Calabi–Yau categories and yields the old definition of quivers with superpotential as quivers with relations when taking the zeroth cohomology of the dg-quiver.

As is well known, using the notion of exceptional sequences, one can construct derived equivalences between varieties and quivers with relations. One then ask if there are derived equivalences between Calabi–Yau varieties and quivers with superpotential. This thesis is a study on how to construct such derived equivalences. There are three major difficulties. To start off, compact Calabi–Yau manifolds do not have any exceptional sequences due to Serre duality and therefore, one cannot directly employ exceptional sequences to construct such derived equivalences. Bridgeland [15] found a slick way of getting around this problem by considering noncompact examples, namely, the total space of vector bundles, and pulling back exceptional sequences on the base manifolds to produce derived equivalences between the total spaces of vector bundles with quivers with relations. This is essentially Corollary 5.2.9.

The second difficulty is to resolve quivers with relations by dg-quivers. This is done by using $A_{\infty}$-Koszul duality. Any dg-quiver can be characterized as the Koszul dual of an augmented $A_\text{fin}$-category ($A_\infty$-category with $m_n = 0$ for $n \gg 0$). This construction sends an $A_\text{fin}$-category $A$ to the quiver constructed by taking the vertices to be $\text{Obj} (A)$ and degree $i$ edges between vertices $u$ and $v$ to be a basis of $\tilde{A}^{i-1}(u, v) \vee$, where $\tilde{A}$ denotes the kernel of the augmentation map. The path algebra of the quiver is then given by the Koszul dual dg-algebra $E(A)$, with differential given by $d = \bigoplus m_n \vee$. In Theorem 5.4.5, we produce an $A_\text{fin}$-category in terms of dual exceptional sequence and show that its Koszul dual is the desired dg-quiver. Readers familiar with the theory of $A_\infty$-algebras would perhaps find it awkward to work with $A_\text{fin}$-categories as the property of being $A_\text{fin}$ is not invariant under homotopy perturbation. Indeed, much of the trouble here is to show that we always end up with an $A_\text{fin}$-category rather than just an $A_\infty$-category. This is done by considering an additional grading on the $A_\infty$-category known as the Adams grading.

The last difficulty is to show that the dg-quiver built this way is a quiver with superpotential. We use the characterization that quivers with superpotential are precisely the Koszul dual of positively graded
augmented $A_{in}$-categories with cyclic structure. We then show the existence of a cyclic structure on our $A_{in}$-category by using a result of van den Bergh [70] characterizing a class of algebras known as exact Calabi–Yau algebras. This is Proposition 5.5.2.

This yields the following theorem, which is the basis of this thesis and is essentially a generalization of Bridgeland [15, Proposition 4.1].

**Theorem A** (Theorem 5.4.5 and Proposition 5.5.2). Let $X$ be a smooth projective variety and $\pi : V \to X$ be an anti-semiample vector bundle, i.e., $S^k V^\vee$ is globally generated for $k \gg 0$. Let $\mathcal{E} = (E_0, \ldots, E_n)$ be an exceptional sequence on $D^b(\text{Coh}(X))$. Suppose the vanishing condition
\[ \text{Hom}^\ell(E_i, E_j \otimes S^k V^\vee) = 0 \]
is satisfied for all $i, j, k$ and all $\ell \geq 1$. Denote by $\mathcal{F} = (F_n, \ldots, F_0)$ the dual exceptional sequence to $\mathcal{E}$. Then there is an $A_{in}$-category $A_{\mathcal{E}}$ where $\text{Obj}(A_{\mathcal{E}}) = \mathcal{F}$ and
\[ A_{\mathcal{E}}(F_i, F_j) = \bigoplus_{k \geq 0} \text{Hom}^{\ell-k}(F_i, F_j \otimes \wedge^k V) \]
such that $E(A_{\mathcal{E}})$ is cohomologically Noetherian and $D^b(\text{Coh}(V)) \cong D^b_{fg}(E(A_{\mathcal{E}})^{\text{op}})$. Moreover, this equivalence restricts to a derived equivalence $D^b_{\text{fg}}(\text{Coh}(V)) \cong D^b_{\text{fg}}(E(A_{\mathcal{E}})^{\text{op}})$. Furthermore, if $V$ is noncompact Calabi–Yau, $A_{\mathcal{E}}$ has a cyclic structure.

Here, by cohomologically Noetherian, we mean that the algebra $H^\bullet(E(A)^{\text{op}})$ is Noetherian. The derived category $D^b_{\text{fg}}(E(A_{\mathcal{E}})^{\text{op}})$ denotes the full triangulated subcategory of $D^b(E(A_{\mathcal{E}})^{\text{op}})$ consisting of complexes whose cohomologies are finitely generated over $H^\bullet(E(A_{\mathcal{E}})^{\text{op}})$; the derived category $D^b_{\text{fg}}(E(A_{\mathcal{E}})^{\text{op}})$ consists of complexes whose cohomologies are finite dimensional; and $D^b_{\text{fg}}(\text{Coh}(V))$ denotes the full triangulated subcategory of $D^b(\text{Coh}(V))$ consisting of complexes whose cohomologies are compactly supported.

In other words, we can produce a dg-quiver derived equivalent to the total space of the vector bundle, and in the case when the total space of the vector bundle is Calabi–Yau, the dg-quiver is equipped with a superpotential. The underlying graded quiver can be described explicitly in terms of the dual exceptional sequence and the vector bundle, although the differential (or the superpotential in the Calabi–Yau case) is not readily known. However, in the Calabi–Yau examples of dimension no greater than 4, knowing the classical quiver with relations derived equivalent to the vector bundle is enough to determine the superpotential, since there are enough constraints. One can generalize Theorem A and remove the vanishing condition. But then we no longer have a concrete description for the underlying graded quiver.

We can also construct derived equivalences equivariantly. Let $G$ be a finite group and $A$ an $A_{in}$-category, with $G$ acting on $A$ by strict $A_{in}$-isomorphisms. Then one can construct a quotient $A_{in}$-category $A/G$. We have the following

**Theorem B** (Theorem 5.6.5). In the situation in Theorem A, suppose there is a finite group $G$ acting on $X$, and this action lifts to $\mathcal{E}$. Assume further that each object $E_i \in \mathcal{E}$ can be equipped with a $G$-linearization. Then there is a $G$-action on $A_{\mathcal{E}}$ by strict $A_{in}$-isomorphisms and an equivalence $D^b(\text{Coh}(G(V))) \cong D^b_{\text{fg}}(E(A_{\mathcal{E}}/G)^{\text{op}})$. Furthermore, if $K_V$ is trivial as an equivariant vector bundle, or equivalently if $\det V \cong K_X$ equivariantly, then $A_{\mathcal{E}}/G$ also has a cyclic structure.

In other words, if we know the dg-quiver derived equivalent to $V$, we can construct the dg-quiver derived equivalent equivariantly to $V$. The action of the differential (or the superpotential in the Calabi–Yau case) on the equivariant dg-quiver is also determined. In fact, this quotient construction on quiver is a generalization of the McKay quiver.

We also have the following recipe for products. Given two $A_{\infty}$-algebras $A$ and $A'$, there is an $A_{\infty}$-tensor product $A \otimes A'$ defined by Amorim [2] which preserves cyclic structures in the sense that if both $A$ and $A'$ have cyclic structures, then so does $A \otimes A'$.
**Theorem C** (Theorem 5.7.3). Let \( \pi_V : V \to X \) and \( \pi_W : W \to Y \) be anti-semiample vector bundles. Let \( E \) be an exceptional sequence on \( X \) and \( F \) be an exceptional sequence on \( Y \). Suppose they satisfy the vanishing conditions

\[
\text{Hom}(E_i, E_j \otimes S^kV^\vee) = 0 \quad \text{and} \quad \text{Hom}(F_i, F_j \otimes S^kW^\vee) = 0
\]

for all \( i, j, k \) and \( \ell \geq 1 \). Then the \( A_\infty \)-structure on \( A_E \otimes A_F \) is \( A_{\text{fin}} \) and there is an equivalence \( D^b(\text{Coh}(V \times W)) \cong D^b(fg)(E(A_E \otimes A_F)^\text{op}). \) If both \( V \) and \( W \) are Calabi–Yau, then \( A_E \otimes A_F \) has a cyclic structure.

In other words, if we know the dg-quivers derived equivalent to vector bundles, we also know the dg-quiver derived equivalent to their product. However, the differential on this product quiver is not uniquely determined as tensor product of \( A_\infty \)-categories are only defined up to \( A_\infty \)-quasi-isomorphisms. In the case when one of the \( A_{\text{fin}} \)-category has \( m_n = 0 \) for \( m \geq 3 \), then there is a natural choice of \( A_{\text{fin}} \)-structure on the tensor product, and hence a natural choice of differential on the product quiver. In the Calabi–Yau case, this means that when one of the superpotentials is cubic, then there is a natural choice of superpotential on the product quiver.

Using these three theorems, we can produce a lot of beautiful pictures of quivers with superpotential derived equivalent to the total space of vector bundles which are Calabi–Yau. Examples include \( T^\vee P^2 \), \( K_{P^n} \), and \( O_{P^2}(-1) \oplus O_{P^2}(-2) \) etc.

Quivers with superpotential also connect to the recent work by Pantev, Toën, Vaquié and Vezzosi [58] and Ben-Bassat, Brav, Bussi and Joyce [4] on shifted symplectic structures in that the moduli space of representations of quiver with superpotential seems to be equipped with a shifted symplectic structure which is in a standard Darboux form. The precise statement is stated in Conjecture 7.4.1, and an attempt to prove the conjecture is sketched in Section 7.4. The main difficulty here is finding a way to describe symplectic forms on the moduli space, which is a quotient stack of a derived scheme by a linear algebraic group, by invariant symplectic forms on the atlas. We propose a way to do this via Lie algebra cohomology in Section 7.3.

**A Guide to the Chapters**

**Chapter 1** is a review on noncommutative geometry.

In Section 1.1, we introduce quivers and their representations, and discuss how they can be thought of as objects in noncommutative geometry.

Section 1.2 reviews some calculus on noncommutative space.

**Chapter 2** is a review on the theory of triangulated categories.

Section 2.1-2.3 define triangulated category, derived categories and \( t \)-structures.

Section 2.4 defines Serre functors and Calabi–Yau triangulated categories.

Section 2.5-2.7 give a brief review of different notions of compact generators, admissible subcategories and mutation functors.

Section 2.8-2.9 define exceptional sequences and tilting objects, which are the main ingredients to construct derived equivalences.

**Chapter 3** is a survey on \( A_\infty \)-algebras and operations on them.

In section 3.1, we introduce \( A_\infty \)-algebras and other related notions such as minimal models, \( A_\infty \)-modules, and their derived categories.

Section 3.2 defines the notion of minimal \( A_\infty \)-algebras, and discusses how to construct minimal models by using homotopy perturbation.
Section 3.3 defines the notion of cyclic structure on $A_{\infty}$-algebras and describes how it gives rise to Calabi–Yau categories.

Section 3.4 defines the Koszul functor, which is essentially a way of producing dg-quivers from $A_{\infty}$-algebras. There are two versions of this functor: the completed one and the incomplete one. The completed one is defined on $A_{\infty}$-algebras and yields the completed path algebra of a dg-quiver. The incomplete one is only defined on $A_{\text{fin}}$-algebras, and yields the incomplete path algebra of a dg-quiver. The difference between the two versions is similar to the difference between power series and polynomials. Admittedly, working with $A_{\text{fin}}$-algebras and hence the incomplete Koszul functor is awkward in the world of $A_{\infty}$-algebras, as being $A_{\text{fin}}$ is not a homotopy invariant property. For example, the minimal model of an $A_{\text{fin}}$-algebra is not necessarily $A_{\text{fin}}$. However, as we will see in Chapter 5, the incomplete Koszul functor is central to our problem of constructing derived equivalences between dg-quivers and total spaces of vector bundles.

Section 3.5 defines the quotient of an $A_{\infty}$-algebra by a finite group, and the smash product of an $A_{\infty}$-algebra by a finite group. Although the definition of quotient construction is straightforward, it appears to be new. This quotient construction is central to constructing derived equivalences equivariantly as described in Section 5.6. We then prove a relation between the quotient construction and the smash product, and shows how these two constructions commute with the Koszul dual functor, i.e., the Koszul dual of the quotient (resp. smash product) of an $A_{\text{fin}}$-algebra is the quotient (resp. smash product) of the Koszul dual of an $A_{\text{fin}}$-algebra. This section is inspired by the work of Bocklandt, Schedler, and Wemyss [7].

Section 3.6 surveys different constructions of $A_{\infty}$-tensor product. Since $A_{\infty}$-tensor products are only unique up to $A_{\infty}$-quasi-isomorphisms, there is in general no natural formulae for computing the tensor product, although there is one in the case when one of the $A_{\infty}$-algebras is $A_2$, i.e., a dg-algebra. Particularly important to us is the tensor product constructed by Amorim and Tu [2], since their construction preserves cyclic structures. We then prove that, under some local finiteness conditions, the Koszul functor commutes with the tensor product, i.e., Koszul dual of tensor product of $A_{\infty}$-algebras is quasi-isomorphic to tensor product of Koszul duals of $A_{\infty}$-algebras as dg-algebras.

Chapter 4 is devoted to the study of quivers with superpotential.

In section 4.1, we define quivers with superpotentials. Our definition of quivers with superpotential is taken from van den Bergh [70], where the completed path algebra of a quiver with superpotential is known as a deformed DG-preprojective algebra there.

Section 4.2 gives a correspondence between quivers with superpotential and the Koszul dual of $A_{\text{fin}}$-categories with cyclic structures. Using this correspondence, we define the notion of product of quivers with superpotential and the notion of quotient of quivers with superpotential by finite groups.

In section 4.3, we follow van den Bergh [71] and prove that the path algebras of quivers with superpotential are Calabi–Yau algebras, and hence the categories of representations of quivers with superpotential are also Calabi–Yau.

Finally, Section 4.4 describes quivers with superpotential of dimensions 1 to 4. In particular, we describe in dimension 3 how our definition of quivers with superpotential as dg-quivers is connected to the old definition of quivers with superpotential as quivers with relations given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22].

Chapter 5 is the heart of the thesis where we prove our main results.

Section 5.1 gives a review on equivariant sheaves.

In Section 5.2, we generalize a result by Bridgeland [15, Proposition 4.1] and show that if $\pi : V \to X$ is an anti-semiample vector bundle on a smooth projective manifold with an exceptional poset $\mathcal{E}$, then under some cohomological vanishing conditions, the total space $V$ is derived equivalent to an algebra $\Lambda_{\mathcal{E}}$ which is the path algebra of a quiver with relations. If we remove the cohomological vanishing condition, we end up with an $A_{\text{fin}}$-algebra rather than a quiver with relations.
Section 5.3 tries to resolve $\Lambda_E$, the path algebra of a quiver with relations (or more generally the $A_{\text{fin}}$-algebra) by a dg-quiver $Q_E$.

Section 5.4 gives a concrete description of the underlying graded quiver of the dg-quiver $Q_E$ in terms of the dual exceptional poset of $E$.

Section 5.5 proves the existence of a superpotential on $Q_E$ when $V$ is noncompact Calabi–Yau.

Section 5.6 considers the $G$-equivariant situation and constructs from $Q_E$ a quotient quiver $Q_E/G$, generalizing the construction of the McKay quiver, which is derived equivalent to $D^b(\text{Coh}^G(V))$. In the case when $V$ is equivariantly Calabi–Yau, $Q_E/G$ is also equipped with a superpotential.

Section 5.7 proves the product construction. We start with two dg-quivers $Q_E$ and $Q_F$ derived equivalent to vector bundles $V$ and $W$ respectively, and construct a product quiver $Q_E \times Q_F$ which is derived equivalent to $V \times W$. When both $V$ and $W$ are Calabi–Yau, we show that the product quiver $Q_E \times Q_F$ is also equipped with a superpotential.

Chapter 6 is a list of examples illustrating theorems in Chapter 5.

Section 6.1 contains examples illustrating Theorem 5.2.9 which produces the quivers with relations derived equivalent to total space of vector bundles.

Section 6.2 and 6.3 contain examples illustrating Theorem 5.4.5. Section 6.2 introduces a class of algebras called Koszul algebras whose dg-resolution is particularly easy to describe. We also give some examples of vector bundles whose classical tilting algebras are Koszul.

Section 6.3 contains some worked-out examples of derived equivalences between total spaces of vector bundles and dg-quivers, and if the total spaces of vector bundles are Calabi–Yau, quivers with superpotential. These examples are calculated by first determining the classical tilting algebras, then try to work out the dg-resolutions to determine the dg-quiver. In Calabi–Yau examples of dimension no greater than 4, there are enough constraints and hence the classical tilting algebras determine their dg-resolutions.

Section 6.4 contains a list of examples by applying the product construction in Theorem 5.7.3. Since the general formulae for the cyclic $A_\infty$-tensor product defined by Amorim and Tu [2] is not known, we only work with the case when one of the $A_\infty$-algebras is a classical algebra.

Section 6.5 contains a list of examples illustrating the quotient construction in Theorem 5.6.5.

Chapter 7 contains some unfinished work which aims to make a connection between quivers with superpotential and the recent work on shifted symplectic structures by Pantev, Toën, Vaquié and Vezzosi [58], and Ben-Bassat, Brav, Bussi and Joyce [4].

Section 7.1 reviews the theory on derived algebraic geometry developed by Toën and Vezzosi [64, 65, 66] and Pantev, Toën, Vaquié and Vezzosi [58].

Section 7.2 develops the Lie algebra cohomology for dg-modules by modifying the usual Lie algebra cohomology theory.

Section 7.3 defines the $G$-invariant de Rham complex of on a derived scheme Spec $R$ by using the Lie algebra cohomology developed in Section 7.2. We conjecture that the $G$-invariant de Rham complex should describe forms and closed forms on the quotient stack [Spec $R/G$] and outline a strategy of proof.

Section 7.4 describes the moduli space of representations of quiver with superpotential, and outline a strategy of proof on showing the existence of a shifted symplectic structure which is in a standard Darboux form by using the $G$-invariant de Rham complex introduced in Section 7.3.

Chapter 8 discusses possible future research directions following the thesis.

Appendix A gives some cohomological formulae for computing examples in Chapter 6.
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Chapter 1

Noncommutative Geometry

This chapter is a review on noncommutative geometry. In Section 1.1, we introduce quivers and their representations, and discuss how they can be viewed as objects in noncommutative geometry. Section 1.2 reviews some calculus on noncommutative space.

1.1 Quivers and Representations

We review the definition of quivers and their representations. The lecture notes by Crawley-Boevey [19] are a good reference.

Definition 1.1.1 (Quiver). A finite $\mathbb{Z}$-graded quiver $Q$ consists of the following data:

- A finite set of vertices $V_Q$;
- For any vertices $v, w \in V_Q$, a finite set of edges $E_Q(v, w) = \bigcup_{i \in \mathbb{Z}} E_i Q(v, w)$.
  
If $e \in E_i Q(v, w)$, $v$ and $w$ are called the tail and head of $e$, denoted by $t(e)$ and $h(e)$ respectively, and $i$ is called the degree of $e$, denoted by $\text{deg}(e)$. Pictorially, we view $e$ as an arrow going from $v$ to $w$.

Definition 1.1.2 (Path). A path $p$ of length $n$ in $Q$ is a sequence of edges $e_n e_{n-1} \cdots e_2 e_1$ with $t(e_{i+1}) = h(e_i)$ for $1 \leq i \leq n - 1$. The tail of $p$ is $t(p) = v_0$ and the head is $h(p) = v_n$. Pictorially, we view $p$ as a sequence of arrows going from $v_0$ to $v_n$.

The degree of path is the sum of degrees of its component edges. Each vertex $v$ will also be viewed as a path of both length and degree 0 going from $v$ to $v$.

Definition 1.1.3 (Path Category). Let $\mathbb{K}$ be a field. The path category of $Q$ over $\mathbb{K}$, denoted by $\mathbb{K}Q$, is the $\mathbb{K}$-linear category defined by

- $\text{Obj}(\mathbb{Q}) = V$, and
- $\mathbb{K}Q(v, w) = \mathbb{K}\{\text{paths going from } v \text{ to } w\}$,
- the composition map $\circ : \mathbb{K}Q(u, v) \times \mathbb{K}Q(v, w) \to \mathbb{K}Q(u, w)$ is given by concatenation of paths, with
- the identity given by the empty path at each vertex.
Definition 1.1.4 (Path algebra). The path algebra of $Q$ over $\mathbb{K}$, denoted by $\mathbb{K}Q$, is the unital associative $\mathbb{K}$-algebra spanned over $\mathbb{K}$ by all paths of length $k \geq 0$, with multiplication of paths $p$ and $q$ given by the concatenation $qp$ if $h(p) = t(q)$ and zero otherwise. The identity is the sum of empty paths over the set of vertices. The vector subspace spanned by all paths of length $k \geq n$ is a two-sided ideal and is denoted by $\mathbb{K}Q_{(n)}$.

The path category and the path algebra are essentially the same thing, as from the path category we can get the path algebra by taking the direct sum of all morphism spaces and define multiplication to be composition if two morphisms are composable and zero otherwise. If we view the path algebra as an algebra over the discrete $\mathbb{K}$-algebra generated by the vertices, one can recover the path category by defining the morphism space from $v$ to $w$ to be the vector space $w\mathbb{K}Qv$ and composition of morphism by multiplication.

Definition 1.1.5 (Quiver with relations). A quiver with relation $(Q,I)$ is a quiver $Q$ with a two-sided ideal $I$ in $\mathbb{K}Q$ with $I \subseteq \mathbb{K}Q(Q)$. The path algebra of $(Q,I)$ is the unital associative algebra $\mathbb{K}Q/I$.

Definition 1.1.6 (Differential-graded Quiver). A dg-quiver is a graded quiver together with a $S$-linear differential $d : \mathbb{K}Q \to \mathbb{K}Q$ of degree 1.

Path algebras as tensor algebras. Let $S$ be the discrete $\mathbb{K}$-algebra over $V_Q$, i.e., the path algebra of the quiver with vertex set $V_Q$ and no edges, and $\mathbb{K}E_Q$ be the $\mathbb{K}$-vector space spanned by $E_Q$. Then $\mathbb{K}E_Q$ is naturally a $S$-bimodule with scalar multiplication given by path multiplication and $\mathbb{K}Q$ is isomorphic as an unital associative $\mathbb{K}$-algebra to $TS(\mathbb{K}E_Q)$.

Path categories as tensor categories. Analogously, path categories can be written in the form of tensor categories [41, §3.5].

Definition 1.1.7 (Representation of Quiver). Let $Q$ be a quiver. A finite dimensional representation $(W,\rho)$ of $Q$ consists of finite dimensional $\mathbb{K}$-vector spaces $W_v$ for each vertex $v \in V_Q$ and linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ for each edge $e \in E_Q$. A finite dimensional representation $(W,\rho)$ of a quiver with relations $(Q,I)$ is a finite dimensional representation of $Q$ such that for all $r = \sum a_{e_n \cdots e_1} e_n \cdots e_1 \in I$ linear combinations of paths having common head and tail vertices, the corresponding linear maps are trivial:

$$\sum a_{e_n \cdots e_1} \rho_{e_n} \circ \cdots \circ \rho_{e_1} = 0.$$

A morphism of representations $\phi : (W,\rho) \to (U,\sigma)$ consists of linear maps $\phi_v : W_v \to U_v$ for each $v \in V_Q$ such that $\phi_{t(e)} \circ \rho_e = \sigma_e \circ \phi_{h(e)}$ for all $e \in E_Q$.

Proposition 1.1.8. Let $(Q,I)$ be a quiver with relations. The category of representations of $(Q,I)$ is equivalent to the category of finite dimensional left $\mathbb{K}Q/I$-module.

Proof. Refer to [19]. □

Definition 1.1.9 (Representation of dg-quiver). A representation of a dg-quiver consists of chain complexes $W_v$ for each vertex $v$ and graded linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ for each edge $e$ such that for any $m \in W_{t(e)}$, the following identity holds:

$$d_{W_{h(e)}}(\rho_e(m)) = (dgQ\rho_e)(m) + (-1)^{|e|}\rho_e(d_{W_{h(e)}}m).$$

Proposition 1.1.10. The category of dg-representations of a dg-quiver $(Q,d)$ is equivalent to the category of dg-modules over $(\mathbb{K}Q,d)$.

Path algebra as noncommutative analogue of affine variety. One may view the path algebra of a quiver as a noncommutative analogue of a polynomial algebra. For example, let $Q$ be a vertex with 2 loops $x$ and $y$, then $\mathbb{K}Q = \mathbb{K}(x,y)$. Path algebra of a quiver with relations is then analogous to finitely generated commutative algebras, for instance, $\mathbb{K}(x,y)/I$ for some ideal $I$. Representations of quivers are then the noncommutative analogue of coherent sheaves. Dg-quivers can be think of as the noncommutative analogue of derived schemes.
1.2 Noncommutative Calculus

In this section, we review the tools of noncommutative calculus. Proofs in this section are omitted. For proofs and further details, please refer to the paper by Crawley-Boevey, Etingof and Ginzburg [20], the lecture notes by Ginzburg [26] and the paper by van den Bergh [69].

Notation and Convention. We will work in the relative setting. Fix a ground field $\mathbb{K}$. Let $S$ be a $\mathbb{K}$-algebra, $A$ be a $\mathbb{Z}$-graded unital associative $S$-algebra and $M, N$ be $\mathbb{Z}$-graded $A$-bimodules. First, we fix some notation and convention. Anything unadorned will always mean relative to the ground field $\mathbb{K}$, e.g., unadorned tensor product $\otimes$ means $\otimes_{\mathbb{K}}$. The shifted module $\Sigma M$ is defined by $(\Sigma M)^i = M^{i+1}$. Throughout this thesis, the Koszul sign rule will be enforced: when moving an element $a$ past another element $b$, the sign $(-1)^{|a||b|}$ will appear. For example, the tensor product $\varphi \otimes \phi$ of two morphisms of graded $A$-bimodule is defined as:

$$(\varphi \otimes \phi)(a \otimes b) = (-1)^{|\phi||a|} \varphi(a) \otimes \phi(b).$$

We will employ the Sweedler’s notation [63] and write any element $b \in A \otimes A$ as $b = b' \otimes b''$ instead of the more accurate $\sum b_i' \otimes b_i''$. For instance, if $\Theta$ is a $\mathbb{K}$-linear map with target in $A \otimes A$, say $\Theta : A \rightarrow A \otimes A$, we will write $\Theta(a) = \Theta'(a) \otimes \Theta''(a)$. The outer and inner $A$-bimodule structure on the free $A$-bimodule $A \otimes A$ will be denoted respectively by

$$a(b' \otimes b'')c = (ab') \otimes (b''c)$$

$$a \ast (b' \otimes b'') \ast c = (-1)^{|a||b'|+|c|} (ab'') \otimes (b' \ast c)$$

If we denote the interchange operator $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ by $\sigma$, the two $A$-bimodule structures are then related by

$$\sigma(a(b' \otimes b''))c = a \ast \sigma(b' \otimes b'') \ast c.$$

The multiplication map $a \otimes b \rightarrow ab$ will always be denoted by $m$.

Definition 1.2.1 (Derivations). Let $K$ be a field, $S$ be a $K$-algebra, $A$ be a $\mathbb{Z}$-graded $S$-algebra and $M$ be an $\mathbb{Z}$-graded $A$-bimodule. A $S$-linear derivation of degree $n$ from $A$ to $M$ is a $S$-bimodule morphism $f : A \rightarrow M$ of degree $n$ which satisfy Leibnitz’s rule, i.e., for all $a, b \in A$,

$$f(ab) = f(a)b + (-1)^{|a||b|} a f(b).$$

The set of all $S$-linear derivations from $A$ to $M$ is denoted by $\text{Der}_S(A, M)$. In the special case $M = A \otimes A$ with its outer bimodule structure, such a derivation is said to be a $S$-linear double derivation of $A$. The set of all $S$-linear double derivations of $A$ is denoted by $\text{Der}_S(A)$.

Remark 1.2.2. In general, $\text{Der}_S(A, M)$ is only an abelian group. However, for double derivation, $\text{Der}(A)$ is still an $A$-bimodule due to the inner bimodule structure of $A \otimes A$. In other words, for any double derivation $f$ and $a, b \in A$, we define the double derivation $a f b$ by $(a f b)(-) = a \ast f(-) \ast b$.

Example 1.2.3. To see how double derivations arise naturally, consider $A = \mathbb{K}Q$ for some quiver $Q$. For any edge $e$ in $Q$, we can define a double derivation $\partial_e$ acting on any edge $f$ by

$$\partial_e(f) = \begin{cases} h(e) \otimes t(e) & \text{if } f = e, \\ 0 & \text{otherwise}. \end{cases}$$

From the double derivation $\partial_e$, one can define a derivation $\partial^*_e : \mathbb{K}Q \rightarrow \mathbb{K}Q$ by $\partial^*_e = m \circ \sigma \circ \partial_e$. This derivation vanishes on commutators and hence descends to a derivation $\partial^*_e : \mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q] \rightarrow \mathbb{K}Q$.

Definition 1.2.4 (Noncommutative cotangent bundle). The $A$-algebra $\Theta^*_S(A) = T_A(\text{Der}_S(A))$ is called the $S$-relative noncommutative cotangent bundle of $A$.

Definition 1.2.5 (Differential 1-form). The $A$-bimodule of noncommutative 1-forms relative to $S$, denoted by $\Omega^1_S(A)$, is the $A$-bimodule generated by symbols of the form $da$ for any $a \in A$, subject to the relation

$$d_{dR}(ab) = (d_{dR}a)b + a(d_{dR}b).$$
Proposition 1.2.6. The functor \( M \mapsto \text{Der}_S(A, M) \) is representable by the \( A \)-bimodule \( \Omega^1_S(A) \). In other words, there are canonical isomorphisms
\[
\text{Der}_S(A, M) \cong \text{Hom}_{A-\text{Bimod}}(\Omega^1_S(A), M).
\]

Proposition 1.2.7. The following sequence is exact:
\[
0 \to \Omega^1_S(A) \xrightarrow{i} A \otimes_A A \xrightarrow{m} A \to 0,
\]
where \( \varphi(da) = a \otimes 1 - 1 \otimes a \) and \( m(a \otimes b) = ab \).

Definition 1.2.8 (Noncommutative tangent bundle). The \( A \)-algebra \( \Omega^n_S(A) = T_A \Omega^1_S(A) \) is called the \( S \)-relative noncommutative tangent bundle of \( A \). The algebra \( \Omega^n_S(A) \) is \( \mathbb{Z} \times \mathbb{Z} \)-graded, with the first \( \mathbb{Z} \)-grading \( | - | \) coming from the \( \mathbb{Z} \)-grading on \( A \), and the second one \( \| - \| \) coming from the “form” degree, i.e., the number of \( dA \)'s appearing in the element. The Koszul sign rule will be enforced according to the following rule: when moving an element \( \alpha \in \Omega^n_S(A) \) past another element \( \beta \in \Omega^n_S(A) \), the sign \((-1)^{|\alpha||\beta|+|\alpha||\beta|}\) will appear. For example, the commutator is defined as
\[
[\alpha, \beta] = \alpha \beta - (-1)^{|\alpha||\beta|+|\alpha||\beta|} \beta \alpha.
\]

Proposition 1.2.9. The de Rham differential map \( d_{dR} : A \to \Omega^1_{dR}A \) extends to an \( S \)-linear derivation \( d_{dR} : \Omega^n_S(A) \to \Omega^n_S(A) \) of degree \((0, 1)\) which satisfies \( d^2_{dR} = 0 \) and
\[
d_{dR}(a \circ d_{dR} a_1 \cdots d_{dR} a_n) = d_{dR} a_0 d_{dR} a_1 \cdots d_{dR} a_n.
\]

Definition 1.2.10 (Noncommutative de Rham complex). The \( S \)-relative noncommutative de Rham complex of \( A \) is the graded vector space defined by
\[
\text{DR}^*_S(A) = \Omega^*_S(A)/[\Omega^*_S(A), \Omega^*_S(A)].
\]

Definition 1.2.11 (Contraction). Let \( \Theta \in \text{Der}_S(A) \). The contraction map \( i_\Theta : \Omega^n_S(A) \to A \otimes A \) defined to be the \( A \)-bimodule morphism given by
\[
i_\Theta(d_{dR}a) = \Theta(a).
\]
This map extends to a derivation of degree \((-1)\) \( i_\Theta : \Omega^n_S(A) \to \Omega^n_S(A) \circ \Omega^n_S(A) \). The reduced contraction map \( \iota_\Theta : \Omega^n_S(A) \to \Omega^n_S(A) \) is the degree \((-1)\) derivation defined by \( \iota_\Theta = m \circ \sigma \circ i_\Theta \).

Definition 1.2.12 (Lie derivative). The Lie derivative \( L_\Theta : \Omega^n_S(A) \to \Omega^n_S(A) \circ \Omega^n_S(A) \) is by the Cartan formula
\[
L_\Theta = d_{dR} i_\Theta + i_\Theta d_{dR}.
\]
The reduced Lie derivative \( L_\Theta : \Omega^n_S(A) \to \Omega^n_S(A) \) is defined by \( L_\Theta = m \circ \sigma \circ L_\Theta \).

Proposition 1.2.13. Given any double derivation \( \Theta \in \text{Der}(A) \), the reduced contraction \( \iota_\Theta : \Omega^n_S(A) \to \Omega^n_S(A) \) descends to \( \iota_\Theta : \text{DR}^*_S(A) \to \Omega^n_S(A) \).

Definition 1.2.14 (Symplectic 2-form). A closed noncommutative 2-form \( \omega \in \text{DR}^2_S(A) \) is said to be symplectic if the map \( \iota_\omega : \text{Der}_S(A) \to \Omega^1_S(A) \) defined by \( \Theta \mapsto \iota_\Theta \omega \) is an isomorphism.

Definition 1.2.15 (Double Poisson bracket). A double Poisson bracket of degree \( n \) is a linear map \( \{ - , - \} : A \otimes A \to A \otimes A \) of degree \( n \) which satisfies
\[
\{a, b\} = \{a + b, c\} - \{a, c\} + \{b, c\},
\]
and
\[
0 = \{a, b, c\} + \{a, b, c\} + \{b, a, c\} + \{a, b, c\} - \{a, b, c\} + \{b, a, c\} + \{a, b, c\}.
\]

where \( \tau : A \otimes A \otimes A \to A \otimes A \otimes A \) is the map sending \( a \otimes b \otimes c \to (-1)^{|a||b|+|c|} c \otimes a \otimes b \). If \( A \) is a dg-algebra, then a double Poisson bracket of degree \( n \) is a dg-double Poisson bracket of degree \( n \) if it further satisfies
\[
d\{a, b\} = \{da, b\} + (-1)^{|a|+n} \{a, db\}.
\]
Remark 1.2.16. Note that a double bracket also satisfies
\[
\{ ab, c \} = a \ast \{ b, c \} + (-1)^{|b|(|c|+n)} \{ a, c \} \ast b.
\]

Proposition 1.2.17. Every symplectic 2-form $\omega$ give rise to a double Poisson bracket.

Proof. For any $a \in A$, let $H_a$ be the corresponding Hamiltonian double derivation, i.e., $H_a$ is the unique double derivation satisfying
\[
\iota_{H_a} \omega = d dR_a.
\]

Define $\{ a, b \} = H_a(\omega) = i_{H_a(d dR b)} = i_{H_a i_{H_b} \omega}$.

Proposition 1.2.18 ([69] Proposition 1.4). A double Poisson bracket of degree $n$ defines a Kontsevich bracket of degree $n$
\[
\{-, -\} = m \circ \{ -, -\} : A \otimes A \rightarrow A
\]
which satisfy the following properties:

1. $\{-, -\}$ is a derivation in the second argument, i.e.,
\[
\{ a, b c \} = \{ a, b \} c + (-1)^{|a|+|b|} \{ b, a, c \}.
\]

2. $\{-, -\}$ vanishes on commutators in the first argument and hence descends to map
\[
\{-, -\} : A/[A, A] \otimes A \rightarrow A.
\]

3. $\{-, -\}$ satisfy the Jacobi identity
\[
\{ a, \{ b, c \} \} = \{ \{ a, b \}, c \} + (-1)^{|a|+|b|} \{ b, \{ a, c \} \}.
\]

4. $\{-, -\}$ descends to a Lie bracket
\[
\]

The double Poisson bracket and the Kontsevich bracket are related by the following

Proposition 1.2.19 ([69] Proposition 2.4.2). We have the following identity:
\[
\{ a, \{ b, c \} \} = \{ \{ a, b \}, c \} + (-1)^{|b|+n} \{ b, \{ a, c \} \}.
\]
Chapter 2
Triangulated Categories

This chapter is a review of the theory on triangulated categories. Sections 2.1-2.3 define triangulated category, derived categories and $t$-structures. Section 2.4 defines Serre functors and Calabi–Yau triangulated categories. Sections 2.5-2.7 give a brief review of different notions of compact generators, admissible subcategories, and mutation functors. Sections 2.8-2.9 define exceptional sequences and tilting objects, which are the main ingredients to construct derived equivalences.

2.1 Triangulated Categories

We review the definition of triangulated categories. Further details can be found in Gelfand and Manin [25].

Definition 2.1.1 ($K$-categories). Let $K$ be a field. A $K$-category is a category $A$ where for any $X, Y \in \text{Obj} A$, $A(X, Y)$ is endowed with the structure of a $K$-module such that the composition maps

$$A(X, Y) \times A(Y, Z) \to A(X, Z)$$

are $K$-bilinear. A $K$-category $A$ is said to be $K$-linear if $A$ has a zero object and the product of any two objects in $A$ exists.

Definition 2.1.2 (Graded $K$-categories). A graded $K$-category $(A, \Sigma)$ is a $k$-linear category $A$ together with an automorphism $\Sigma$ of $A$. We define the graded Hom-sets by

$$A^i(X, Y) = A(A, \Sigma^i B).$$

Definition 2.1.3 (Graded functors). A graded functor $(F, \eta) : (A, \Sigma_A) \to (B, \Sigma_B)$ between graded categories is a $k$-linear functor $F : A \to B$ together with a natural isomorphism $\eta : F\Sigma_A \to \Sigma_B F$.

Definition 2.1.4 (Triangles). A triangle in a graded $k$-category $(A, \Sigma)$ is a sequence $A \to B \to C \to \Sigma A$. A morphism between two triangles is a commutative diagram

$$
\begin{array}{ccc}
A & \overset{u}{\longrightarrow} & B \\
\downarrow & & \downarrow \\
A' & \overset{u'}{\longrightarrow} & B'
\end{array}
\quad
\begin{array}{ccc}
C & \overset{v}{\longrightarrow} & \Sigma A \\
\downarrow & & \downarrow \\
C' & \overset{v'}{\longrightarrow} & \Sigma A'
\end{array}
$$

Definition 2.1.5 (Triangulated categories). A triangulated $k$-category is a graded $k$-linear category $(A, \Sigma)$ equipped with a set of distinguished triangles which is stable under isomorphisms and satisfying the following axioms:

T0. For any $A \in \text{Obj} A$, the triangle

$$A \xrightarrow{id} A \xrightarrow{0} \Sigma A$$

is distinguished.
T1. Any morphism $\phi : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{\phi} B \xrightarrow{} C \xrightarrow{} \Sigma A$$

T2. A triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

is distinguished if and only if the triangle

$$B \xrightarrow{v} C \xrightarrow{w} \Sigma A \xrightarrow{-\Sigma u} \Sigma B$$

is distinguished.

T3. If there is a commutative diagram of distinguished triangles with vertical morphisms $a : A \to A'$ and $b : B \to B'$

$$\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{b} & B'
\end{array}$$

there exists a morphism $c : C \to C'$ making the diagram commute.

T4. The triangles satisfy the octahedral axiom: Given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} \Sigma X,$$

$$Y \xrightarrow{v} Z \xrightarrow{\ell} X' \xrightarrow{i} \Sigma Y,$$

$$X \xrightarrow{\Sigma u} Z \xrightarrow{m} Y' \xrightarrow{n} \Sigma X,$$

there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} \Sigma Z'$$

such that

$$\ell = g \circ m, \quad k = n \circ f, \quad h = \Sigma j \circ i, \quad i \circ g = \Sigma u \circ n, \quad f \circ j = m \circ v.$$ 

The name “octahedral axiom” comes from the fact that the above distinguished triangles can be packed into an octahedron:

![Octahedron Diagram]

Remark 2.1.6. Note that we do not assume a priori that two morphisms in a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ compose to zero. However, it is a consequence of axioms T1 and T3 that they do: there exists a morphism $h$ making the diagram commute.
i.e., the composition $g \circ f$ factors through the zero object, and hence must be a zero morphism.

**Remark 2.1.7.** Axiom $T2$ postulates every morphism $A \xrightarrow{f} B$ fits into a distinguished triangle $A \xrightarrow{f} B \xrightarrow{} C \xrightarrow{} \Sigma A$. It is a consequence of the axioms [25, Corollary IV.1.4] that the object $C$ is unique up to non-unique isomorphism. This object $C$ is called a cone of the morphism $f$. We can then rephrase Axiom $T4$ in the following way: Let $f, g$ be morphisms in a triangulated category $\mathcal{A}$ and $C(f), C(g)$ and $C(g \circ f)$ be a cone of the morphism $f, g$ and $g \circ f$ respectively. Then there exists a distinguished triangle

$$C(f) \xrightarrow{} C(g \circ f) \xrightarrow{} C(g) \xrightarrow{} \Sigma C(f).$$

**Definition 2.1.8** (Triangle Functor). Let $(\mathcal{A}, \Sigma \mathcal{A})$ and $(\mathcal{B}, \Sigma \mathcal{B})$ be triangulated $k$-categories. A triangle functor is a graded $k$-linear functor $(F, \eta) : (\mathcal{A}, \Sigma \mathcal{A}) \rightarrow (\mathcal{B}, \Sigma \mathcal{B})$ preserving distinguished triangles, i.e., for each distinguished triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A,$$

the triangle

$$FA \xrightarrow{Fv} FB \xrightarrow{Fu} FC \xrightarrow{\eta(C,Fw)} \Sigma FC$$

is also distinguished.

**Remark 2.1.9.** In general, $(\Sigma, \text{id}_{\Sigma^2})$ is not a triangle functor. However, $(\Sigma, -\text{id}_{\Sigma^2})$ is a triangle functor. If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ is a distinguished triangle, by (T2), so is $\Sigma A \xrightarrow{-\Sigma u} \Sigma B \xrightarrow{-\Sigma v} \Sigma C \xrightarrow{-\Sigma w} \Sigma^2 A$. The following isomorphism of triangle

$$\begin{array}{c}
\Sigma A \xrightarrow{-\Sigma u} \Sigma B \xrightarrow{-\Sigma v} \Sigma C \xrightarrow{-\Sigma w} \Sigma^2 A \\
\text{id}_{\Sigma A} \downarrow \quad \quad \quad \quad \quad \downarrow \text{id}_{\Sigma^2 A} \\
\Sigma A \xrightarrow{-\Sigma u} \Sigma B \xrightarrow{-\Sigma v} \Sigma C \xrightarrow{-\Sigma w} \Sigma^2 A
\end{array}$$

shows $\Sigma A \xrightarrow{\Sigma u} \Sigma B \xrightarrow{\Sigma v} \Sigma C \xrightarrow{\Sigma w} \Sigma^2 A$ is a distinguished triangle. Hence $(\Sigma, -\text{id}_{\Sigma^2})$ is a triangle functor.

**Definition 2.1.10** (Morphism of triangle functors). Let $(\mathcal{A}, \Sigma \mathcal{A}), (\mathcal{B}, \Sigma \mathcal{B})$ be triangulated category and $(F, \phi), (G, \psi) : (\mathcal{A}, \Sigma \mathcal{A}) \rightarrow (\mathcal{B}, \Sigma \mathcal{B})$ be triangle functors. A morphism of triangle functors $\alpha : (F, \phi) \rightarrow (G, \psi)$ is a natural transformation $\alpha : F \rightarrow G$ such that for all $X \in \text{Obj} \mathcal{A}$, the following square commutes

$$\begin{array}{c}
F \Sigma X \xrightarrow{\phi} \Sigma FX \\
\text{id}_{\Sigma X} \downarrow \quad \quad \quad \quad \quad \downarrow \text{id}_{\Sigma X} \\
G \Sigma X \xrightarrow{\psi} \Sigma GX.
\end{array}$$

**Definition 2.1.11** (Triangulated subcategory). A triangulated subcategory of a triangulated category $\mathcal{A}$ is a subcategory $\mathcal{B}$ of $\mathcal{A}$ such that the inclusion functor $i : \mathcal{B} \rightarrow \mathcal{A}$ is a triangle functor. A triangulated subcategory $\mathcal{B}$ is said to be thick if it is stable under taking direct summands, i.e., $A \oplus B \in \text{Obj} \mathcal{B}$ implies $A, B \in \text{Obj} \mathcal{B}$.

**Definition 2.1.12** (Orthogonal Complement). Let $\mathcal{B}$ be a triangulated subcategory of a triangulated category $\mathcal{A}$. The right orthogonal complement of $\mathcal{B}$ is the full subcategory $\mathcal{B}^\perp$ of $\mathcal{A}$ containing all objects $A \in \text{Obj} (\mathcal{A})$ such that $\text{Hom}(B, A) = 0$ for all $B \in \text{Obj} (\mathcal{B})$. Similarly, the left orthogonal complement of $\mathcal{B}$ is the full subcategory $^\perp \mathcal{B}$ of $\mathcal{A}$ containing all objects $A \in \text{Obj} (\mathcal{A})$ such that $\text{Hom}(A, B) = 0$ for all $B \in \text{Obj} (\mathcal{B})$.

### 2.2 Derived Categories

The main examples of triangulated categories come from deriving abelian categories as we will describe in this section. We will only sketch the constructions. Readers are referred to Gelfand and Manin [25] for details and proofs.
Let $\mathcal{A}$ be an abelian category. Denote by $K(\mathcal{A})$ the homotopy category of $\mathcal{A}$, whose objects are chain complexes in $\mathcal{A}$ and morphisms are chain maps modulo chain homotopies. The collection of quasi-isomorphisms in $K(\mathcal{A})$, i.e., chain maps which induce isomorphisms between homology, forms a multiplication system which satisfies the following three axioms:

1. All identity morphisms are quasi-isomorphisms and compositions of quasi-isomorphisms are quasi-isomorphisms;

2. If $t : Z \to Y$ is a quasi-isomorphism, then for any morphism $g : X \to Y$ in $K(\mathcal{A})$, there exists $f : W \to Z$ and a quasi-isomorphism $s : W \to X$ making the diagram commutative:

$$
\begin{array}{ccc}
W & \xrightarrow{f} & Z \\
\downarrow{s} & & \downarrow{t} \\
X & \xrightarrow{g} & Y.
\end{array}
$$

Similarly, if $s : W \to X$ is a quasi-isomorphism and $f : W \to Z$ any morphism in $K(\mathcal{A})$, there exists $g : X \to Y$ and a quasi-isomorphism $t : Z \to Y$ making the diagram commutative:

$$
\begin{array}{ccc}
W & \xrightarrow{f} & Z \\
\downarrow{s} & & \downarrow{t} \\
X & \xrightarrow{g} & Y.
\end{array}
$$

3. If $f, g : X \to Y$ are morphisms in $K(\mathcal{A})$, then the following two conditions are equivalent:

(a) $sf = sg$ for some quasi-isomorphism $s$,

(b) $ft = gt$ for some quasi-isomorphism $t$.

The derived category $D(\mathcal{A})$ is constructed from $K(\mathcal{A})$ by formally inverting all quasi-isomorphisms: Objects in $D(\mathcal{A})$ are the same as $K(\mathcal{A})$, i.e., chain complexes of objects in $\mathcal{A}$. Morphisms between an object $X$ and $Y$ are given by equivalence classes of diagrams in the form $X \leftarrow Z \to Y$ where $s : Z \to X$ is a quasi-isomorphism and $f : Z \to Y$ is a chain map. Two diagrams $X \xleftarrow{s_1} Z \xrightarrow{f_1} Y$ and $X \xleftarrow{s_2} W \xrightarrow{g} Y$ are equivalent if there is a diagram $W \xleftarrow{r} U \xrightarrow{h} Z$ which fits into a commutative diagram:

$$
\begin{array}{ccc}
& & X \\
& r & \searrow{g} \\
W & \xrightarrow{f} & Z \\
\downarrow{s} & & \downarrow{t} \\
& s & \swarrow{h} \\
U & \xleftarrow{f} & Y.
\end{array}
$$

Composition of maps are given by

$$(X \xleftarrow{s} U \xrightarrow{f} Y) \circ (Y \xleftarrow{t} V \xrightarrow{g} Z) = (X \xleftarrow{s} W \xrightarrow{gf} Z)$$

where

$$
\begin{array}{ccc}
W & \xrightarrow{\lambda} & V \\
\downarrow{t} & & \downarrow{r} \\
U & \xrightarrow{f} & Y
\end{array}
$$

is a commutative diagram we get by Axiom 2, with $r$ a quasi-isomorphism. One can check this definition is well-defined. The set $\text{Hom}_{D(\mathcal{A})}(X, Y)$ forms a vector space over $\mathbb{K}$:

1. $k \cdot (X \xleftarrow{s} Z \xrightarrow{f} Y) = (X \xleftarrow{k s} Z \xrightarrow{k f} Y)$ for any $k \in \mathbb{K}$;
2. \((X \xleftarrow{s} Z \xrightarrow{f} Y) + (X \xleftarrow{t} W \xrightarrow{g} Y) = (X \xleftarrow{r} Z \xrightarrow{fu+gv} Y)\) where

\[
\begin{array}{ccc}
U & \xrightarrow{v} & W \\
\downarrow{u} & & \downarrow{t} \\
Z & \xrightarrow{r} & X
\end{array}
\]

is a commutative diagram we get by Axiom 2, with \(u\) and \(r = su = tv\) quasi-isomorphisms.

This shows \(D(A)\) is a \(K\)-linear category. A triangle \(X \rightarrow Y \rightarrow Z \rightarrow \Sigma X\) in \(D(A)\) is said to be distinguished if it is isomorphic to a triangle of the form \(A \xrightarrow{f} B \rightarrow \text{cone}(f) \rightarrow \Sigma A\). Equipped with this class of distinguished triangles, \(D(A)\) has the structure of a triangulated category. The bounded derived category \(D^b(A)\) is defined as the smallest triangulated subcategory of \(D(A)\) containing all bounded complexes.

Example 2.2.1. Let \(X\) be a variety. Then \(\text{Coh}(X)\) is an abelian category and \(D^b(\text{Coh}(X))\) is the bounded derived category of coherent sheaves on \(X\). We will denote by \(D^b_{cs}(\text{Coh}(X))\) the full triangulated subcategory of \(D^b(\text{Coh}(X))\) consisting of complexes whose cohomologies are compactly supported.

Example 2.2.2. Let \(A\) be a dg-algebra. We denote by \(D^b(A)\) the bounded derived category of dg-modules over \(A\). We also denote by \(D^b_{fg}(A)\) the full subcategory of \(D^b(A)\) consisting of complexes whose cohomologies are finitely generated modules over \(H^\bullet(A)\). When \(A\) is an ordinary Noetherian algebra, the category of finitely generated \(A\)-modules \(A\)-mod is abelian, and we have an equivalence \(D^b(A\text{-mod}) \cong D^b_{fg}(A)\).

2.3 \(t\)-structures

In this section, we discuss \(t\)-structures. We follow Manin [25, §IV.4]. It is known that two different abelian categories might yield the same triangulated category. The formalism of \(t\)-structure was invented to see different abelian subcategories inside a triangulated category.

Definition 2.3.1. Let \(\mathcal{T}\) be a triangulated category. A \(t\)-structure on \(\mathcal{T}\) is a pair of strictly full subcategories \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) satisfying

1. \(\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}\) and \(\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}\),
2. \(\text{Hom}(X, Y) = 0\) for \(X \in \text{Obj} \mathcal{T}^{\leq 0}\) and \(Y \in \text{Obj} \mathcal{T}^{\geq 1}\),
3. For any \(X \in \text{Obj} \mathcal{T}\) there exists a distinguished triangle \(A \rightarrow X \rightarrow B \rightarrow A[1]\) with \(A \in \text{Obj} \mathcal{T}^{\leq 0}\) and \(B \in \text{Obj} \mathcal{T}^{\geq 1}\), where \(\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]\) and \(\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]\).

The full subcategory \(\mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}\) is called the heart of the \(t\)-structure.

Theorem 2.3.2. The heart of any \(t\)-structure on a triangulated category is an abelian category.

Proof. See [25, §IV.4, Theorem 4].

Remark 2.3.3. In general, given a triangulated category \(\mathcal{T}\) with a \(t\)-structure whose heart is \(\mathcal{A}\), the derived category \(D(\mathcal{A})\) might not be equivalent to \(\mathcal{T}\). Moreover, in general, there is no obvious relation of \(\mathcal{T}\) with the category of complexes over \(\mathcal{A}\). This is caused by the non-functorality of the cone [25, §IV.4 Remark 13].
Proposition 2.4.2. Let autoequivalence (Definition 2.4.1) be assumed to be $\mathbb{K}$-linear and Hom-finite, i.e., all the morphism spaces are finite dimensional over $\mathbb{K}$. References for this section are the paper by Keller [45, §2] and the book by Huybrechts [34, §1].

Definition 2.4.1 (Serre functor). A Serre functor on a Hom-finite triangulated category $\mathcal{A}$ is a triangle autoequivalence $(S, \sigma) : \mathcal{A} \to \mathcal{A}$ together with a family of isomorphisms

$$\eta_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{A}(Y,SX)^\vee$$

natural in both $X$ and $Y$, such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}(X,Y) & \xrightarrow{\eta_{X,Y}} & \mathcal{A}(Y,SX)^\vee \\
\downarrow & & \downarrow \sigma_X^* \\
\mathcal{A}(X,Y) & \xrightarrow{\Sigma} & \mathcal{A}(\Sigma X, \Sigma Y) \xrightarrow{\eta_{\Sigma X,\Sigma Y}} \mathcal{A}(\Sigma Y, \Sigma SX)^\vee.
\end{array}$$

In other words, for any $X, Y \in \text{Obj} \mathcal{A}$, $f \in \mathcal{A}(X,Y)$ and $g \in \mathcal{A}(Y,SX)$, we have

$$\langle \eta_{X,Y}(f), \Sigma^{-1}(\sigma_X \circ g) \rangle = -\langle \eta_{\Sigma X,\Sigma Y}(f), g \rangle.$$

Proposition 2.4.2. Let $(\mathcal{A}, \Sigma_A)$ be a Hom-finite triangulated category and $(S, \sigma)$ be an autoequivalence. Then $S$ is a Serre functor if and only if there is a family of linear maps $\text{tr}_X : \mathcal{A}(X, SX) \to k$ such that the family of induced pairings $\mathcal{A}(X,Y) \times \mathcal{A}(Y,SX) \to k$ given by $(f,g) \to \text{tr}_X(g \circ f)$ are nondegenerate and they satisfy

$$\text{tr}_X(g \circ f) = \text{tr}_Y(Sf \circ g)$$

and for all $h : \Sigma X \to S\Sigma X$

$$\text{tr}_X(\Sigma^{-1}(\sigma_X \circ h)) = -\text{tr}_{\Sigma X}(h).$$

Proof. Suppose $(S, \sigma)$ is a Serre functor. Define $\text{tr}_X = \eta_{X,X}(\text{id}_X)$. By naturality of $\eta$, we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{A}(X,X) & \xrightarrow{f_*} & \mathcal{A}(X,Y) \\
\downarrow \eta_{X,X} & & \downarrow \eta_{X,Y} \\
\mathcal{A}(X,SX)^\vee & \xrightarrow{f_*} & \mathcal{A}(Y,SX)^\vee \xrightarrow{(sf)^*} \mathcal{A}(Y,SY)^\vee.
\end{array}$$

Then

$$\text{tr}_X(g \circ f) = \langle g \circ f, \eta_{X,X}(\text{id}_X) \rangle = \langle g, \eta_{X,Y}(f) \rangle = \langle Sf \circ g, \eta_{Y,Y}(\text{id}_Y) \rangle = \text{tr}_Y(Sf \circ g).$$

Also, from the above equation, we see that for a fixed $f$, $\text{tr}_X(g \circ f) = 0$ for all $g$ implies $\eta_{X,Y}(f) = 0$ which in turn implies $f = 0$ since $\eta_{X,Y}$ is an isomorphism. Similarly, if for a fixed $g$, $\text{tr}_X(g \circ f) = 0$ for all $f$, then $g = 0$ since $\eta_{X,Y}$ is an isomorphism and hence $\eta_{X,Y}(f)$ is arbitrary.

For the second equality, by definition, for all $h : \Sigma X \to S\Sigma X$, we have

$$\langle \eta_{X,X}(\text{id}_X), \Sigma^{-1}(\sigma_X \circ h) \rangle = -\langle \eta_{\Sigma X,\Sigma X}(\text{id}_X), h \rangle$$

which implies

$$\text{tr}_X(\Sigma^{-1}(\sigma_X \circ h)) = -\text{tr}_{\Sigma X}(h).$$

Conversely, we define $\eta_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{A}(Y,SX)^\vee$ by $\eta_{X,Y}(f) = \text{tr}_X(- \circ f)$. Since the induced pairing is assumed to be non-degenerate and both vector spaces are finite dimensional, $\eta_{X,Y}$ is an isomorphism. Naturality of $\eta_{X,Y}$ in both $X$ and $Y$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{A}(Z,Y) & \xrightarrow{h_*} & \mathcal{A}(X,Y) \\
\downarrow \eta_{Z,Y} & & \downarrow \eta_{X,Y} \\
\mathcal{A}(Y,SZ)^\vee & \xrightarrow{(h^*)^*} & \mathcal{A}(Y,SX)^\vee \xrightarrow{k_*} \mathcal{A}(W, SX)^\vee.
\end{array}$$
for arbitrary $W, Z \in \text{Obj} \mathcal{A}$ and $h \in \mathcal{A}(X, Z)$ and $k \in \mathcal{A}(Y, W)$. This is checked readily by unwinding the definition: for all $u \in \mathcal{A}(Z, Y)$

\[
((Sh)^* \circ \eta_{Z,Y})(u) = tr_Z(Sh \circ - \circ u) = tr_X(- \circ u \circ h) = \eta_{X,Y}(u \circ h) = (\eta_{X,Y} \circ h^*)(u)
\]

and for all $v \in \mathcal{A}(X, Y)$,

\[
(\eta_{X,W} \circ k_*)(v) = \eta_{X,W}(k \circ v) = (\eta_X \circ h)(v).
\]

Also, by (2.4.2),

\[
\langle \eta_{X,Y}(f), \Sigma^{-1}(\sigma_X \circ g) \rangle = tr_X(\Sigma^{-1}(\sigma_X \circ g) \circ f) = tr_X(\Sigma^{-1}(\sigma_X \circ g \circ \Sigma f)) = -\langle \eta_{X,Y}(\Sigma f), g \rangle.
\]

Remark 2.4.3. The negative sign in Equation (2.4.2) is explained by van den Bergh in [68, Remark A.4.2].

Definition 2.4.4 (Calabi–Yau triangulated category). A triangulated category is said to be $d$-Calabi–Yau if $(\Sigma, -\text{id}_\mathcal{A})^d$ is a Serre functor.

Proposition 2.4.5. A triangulated $\mathbb{K}$-category $(\mathcal{A}, \Sigma)$ is $d$-Calabi–Yau if and only if for each $X \in \text{Obj} \mathcal{A}$ there is a linear map $tr_X : \mathcal{A}(X, \Sigma^d X) \to \mathbb{K}$ such that for all $X, Y \in \text{Obj} \mathcal{A}$, and integers $p, q$ with $p + q = d$, the induced pairings $\langle \cdot, \cdot \rangle : \mathcal{A}(X, \Sigma^p Y) \times \mathcal{A}(Y, \Sigma^q X) \to \mathbb{K}$ given by $(f, g) \to tr_X((\Sigma^p g) \circ f)$ are nondegenerate and they satisfy

\[
tr_X((\Sigma^p g) \circ f) = (-1)^{pq} tr_Y((\Sigma^q f) \circ g).
\]

Proof. Suppose $(\mathcal{A}, \Sigma)$ is $d$-Calabi–Yau. Without loss of generality, we may assume the Serre functor is given by $(\Sigma, -\text{id})^d = (\Sigma^d, (-1)^d \text{id})$. Thus $(-1)^d tr_X(\Sigma^{-1} h) = -tr_{\Sigma X}(h)$ for any $h : \Sigma X \to \Sigma^d X$, and

\[
tr_X((\Sigma^p g) \circ f) = (-1)^{pq} tr_Y((\Sigma^q f) \circ g)
\]

by equation (2.4.2) in Proposition 2.4.2

\[
= (-1)^{pq} tr_{\Sigma X}((\Sigma^q f) \circ g)
\]

by equation (2.4.1) in Proposition 2.4.2

We show the converse by showing equation (2.4.1) and (2.4.2) in Proposition 2.4.2. Equation (2.4.1) is a special case for $p = 0$ and $q = d$. For equation (2.4.2), given any $h : \Sigma X \to \Sigma^d+1 X$, if we put $p = -1$ and $q = d + 1$, and view the identity map on $X$ as $\text{id}_X : X \to \Sigma^{-1}(X)$, then equation (2.4.2) is verified:

\[
tr_X((\Sigma^p g) \circ f) = (-1)^{pq} tr_X(\Sigma^{-1} h \circ \text{id}_X)
\]

\[
= (-1)^{d+1} tr_{\Sigma X}((\Sigma^d+1 \text{id}_X) \circ h)
\]

\[
= -tr_{\Sigma X}(h).
\]

Example 2.4.6 (Serre duality). Let $X$ be a smooth quasi-projective variety. Denote by $D^b_{cs}(\text{Coh}(X))$ as the smallest full triangulated subcategory of $D^b(\text{Coh}(X))$ which contains all complexes with compact support. Then $(-) \otimes K_X[\dim X] : D^b_{cs}(\text{Coh}(X)) \to D^b_{cs}(\text{Coh}(X))$ is a Serre functor. In particular, if $K_X$ is trivial, $D^b_{cs}(\text{Coh}(X))$ is a Calabi–Yau triangulated category.

Example 2.4.7 ([45], Lemma 4.1). Let $A$ be a dg-algebra which is homologically smooth, i.e., $A \in \text{Per}(A^{op} \otimes A)$. Denote by $D^b_{fd}(A)$ as the full triangulated subcategory of $D^b(A)$ which contains all complexes with finite dimensional cohomologies. Define the dualizing complex $\Omega = \text{RHom}_{A^{op} \otimes A}(A, A^{op} \otimes A)$. Then $(-) \otimes \Omega : D^b_{fd}(A) \to D^b_{fd}(A)$ is a Serre functor. In particular, if we have a isomorphism $\Omega \cong \Sigma^{-d} A$ as objects in $D(A^{op} \otimes A)$, then $D^b_{fd}(A)$ is $d$-Calabi–Yau.
2.5 Compact Generators

This section surveys different notions of generators in a triangulated category and results that we will be using in the remaining thesis. It is essentially a summary of Bondal and van den Bergh [9].

**Definition 2.5.1 (Compact Objects).** Let $\mathcal{D}$ be a triangulated category which admits arbitrary direct sums. In general, for any object $X \in \text{Obj}(\mathcal{D})$, the functor $\text{Hom}(X, -)$ only commutes with finite direct sums. An object $X$ is said to be compact if $\text{Hom}(X, -)$ commutes with arbitrary direct sums. The full subcategory containing all compact objects is denoted by $\mathcal{D}^c$.

**Example 2.5.2 ([9], Thm 3.1.1).** Let $X$ be a projective variety and $G$ a finite group acting on $X$. The compact objects in $\text{D}(\text{QCoh}^G(X))$ are precisely the $G$-equivariant perfect complexes, i.e., complexes that are locally quasi-isomorphic to a bounded complex of equivariant vector bundles. If $X$ is smooth, all complexes in $\text{D}^b(\text{Coh}^G(X))$ are perfect, hence $\text{D}(\text{QCoh}^G(X))^c = \text{D}^b(\text{Coh}^G(X))$.

**Example 2.5.3 ([42], Prop 8.3).** Let $A$ be a dg-algebra and $\text{D}(A)$ the derived category of dg-modules over $A$. Then $\text{D}(A)^e = \text{Per}(A)$, where $\text{Per}(A)$ is the smallest thick subcategory in $\text{D}(A)$ containing the free dg-module $A$.

**Definition 2.5.4 (Generators).** A set of objects $\mathcal{E}$ in $\mathcal{D}$ classically generates $\mathcal{D}$ if $\mathcal{D}$ is the smallest thick subcategory in $\mathcal{D}$ containing $\mathcal{E}$. We say $\mathcal{E}$ generates $\mathcal{D}$ if $\mathcal{E}^\perp = 0$. We say $\mathcal{D}$ is compactly generated if $\mathcal{D}$ is generated by $\mathcal{D}^c$.

**Theorem 2.5.5 ([9], Thm 2.1.2).** Let $\mathcal{D}$ be a compactly generated triangulated category. Then a set of compact objects in $\mathcal{D}$ classically generates $\mathcal{D}^c$ if and only if it generates $\mathcal{D}$.

**Example 2.5.6 ([9], Thm 3.1.1).** Let $X$ be a variety and $G$ be a finite group acting on $X$. Then $\text{D}(\text{QCoh}^G(X))$ is compactly generated.

**Corollary 2.5.7.** Let $X$ be a smooth variety and $G$ be a finite group acting on $X$ by automorphisms. Then a set of objects $\mathcal{E}$ in $\text{D}^b(\text{Coh}^G(X))$ classically generates $\text{D}^b(\text{Coh}^G(X))$ if and only if $\mathcal{E}$ generates $\text{D}(\text{QCoh}^G(X))$.

**Proof.** This is a consequence of Example 2.5.2, Theorem 2.5.5 and Example 2.5.6.

2.6 Admissible Subcategories

**Definition 2.6.1 (Admissible Subcategories).** A full triangulated subcategory $\mathcal{B} \subseteq \mathcal{D}$ is said to be left admissible (resp. right admissible) if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{D}$ has a left (resp. right) adjoint. A full triangulated subcategory is said to be admissible if it is both left and right admissible.

**Proposition 2.6.2 ([8], Lemma 3.1).** Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. The following are equivalent:

1. $\mathcal{B}$ and $\mathcal{B}^\perp$ classically generate $\mathcal{D}$;

2. For any object $X \in \mathcal{D}$, there exists $B \in \mathcal{B}$ and $C \in \mathcal{B}^\perp$ and a distinguished triangle
   $$B \rightarrow X \rightarrow C \rightarrow B[1];$$

3. $\mathcal{B}$ is right admissible, and the right adjoint $q : \mathcal{D} \rightarrow \mathcal{B}$ sends $X$ to $B$;

4. $\mathcal{B}^\perp$ is left admissible, and the left adjoint $p : \mathcal{D} \rightarrow \mathcal{B}^\perp$ sends $X$ to $C$.

There is of course the similar

**Proposition 2.6.3.** The following are equivalent:

1. $\mathcal{B}^\perp$ and $\mathcal{B}$ classically generate $\mathcal{D}$;
2. For any object $X \in \mathcal{D}$, there exists $B \in \mathcal{B}$ and $C \in \perp \mathcal{B}$ and a distinguished triangle

$$C \to X \to B \to C[1];$$

3. $\mathcal{B}$ is left admissible, and the left adjoint $p : \mathcal{D} \to \mathcal{B}$ sends $X$ to $B$;

4. $\perp \mathcal{B}$ is right admissible, and the right adjoint $q : \mathcal{D} \to \perp \mathcal{B}$ sends $X$ to $C$.

**Corollary 2.6.4.** Let $\mathcal{B}$ be a right (resp. left) admissible subcategory of $\mathcal{D}$. Then $\mathcal{B} = \mathcal{D}$ if and only if $\perp \mathcal{B} = 0$ (resp. $\perp \mathcal{B} = 0$).

### 2.7 Mutation Functors

This section defines mutation functors. We follow the convention of Bridgeland and Stern [16], which differs from the more standard convention of Bondal and Kapranov [8, 10] by a shift functor, but it simplifies some of our formulæ.

Let $\mathcal{B}$ be an admissible full triangulated subcategory of $\mathcal{D}$. By Propositions 2.6.2 and 2.6.3, there are left adjoint $p : \mathcal{D} \to \perp \mathcal{B}$ to the inclusion $i : \perp \mathcal{B} \to \mathcal{D}$, and right adjoint $q : \mathcal{D} \to \perp \mathcal{B}$ to the inclusion $j : \perp \mathcal{B} \to \mathcal{D}$.

**Definition 2.7.1.** The left mutation functor $L_B : \perp \mathcal{B} \to \mathcal{B}$ is defined to be $L_B = p \circ j$. Similarly, the right mutation functor $R_B : \mathcal{B} \to \perp \mathcal{B}$ is defined to be $R_B = q \circ i$.

If $E$ is an object in $\mathcal{D}$, we define $L_E = L_{\langle E \rangle}$ and $R_E = R_{\langle E \rangle}$, where $\langle E \rangle$ is the smallest full triangulated subcategory in $\mathcal{D}$ containing $E$.

**Proposition 2.7.2.** Let $X \in \perp \mathcal{B}$ and $Y \in \mathcal{B}$. Then $Y = L_B(X)$ if and only if there is an object $B \in \mathcal{B}$ and a triangle

$$B \to X \to Y \to B[1].$$

Similarly, $X = R_B(Y)$ if and only if there is an object $B \in \mathcal{B}$ and a triangle

$$X \to Y \to B \to X[1].$$

The two mutation functors $L_B$ and $R_B$ are inverse to each other.

**Proof.** The first two claims are immediate from Proposition 2.6.2 and 2.6.3. The last claim follows from the first two claims.

### 2.8 Exceptional Sequences

This section introduces the notion of exceptional poset in a $\mathbb{K}$-linear triangulated category $\mathcal{D}$. For any objects $A, B \in \mathcal{D}$, we denote $\text{Hom}^k(A, B) = \text{Hom}(A, B[k])$ for $k \in \mathbb{Z}$ and

$$\text{Hom}^\bullet(A, B) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}^k(A, B)[-k]$$

the chain complex of vector spaces with trivial differential.

**Definition 2.8.1 (Exceptional Poset).** An object $E \in \mathcal{D}$ is said to be exceptional if

$$\text{Hom}^\bullet(E, E) = \mathbb{K}.$$

Let $(I, \preceq)$ be a finite poset. A finite set of exceptional objects $\mathcal{E} = \{E_i\}_{i \in I}$ in $\mathcal{D}$ indexed by $(I, \preceq)$ is an exceptional poset if

$$\text{Hom}^\bullet(E_i, E_j) = 0$$

unless $i \preceq j$. 
If \((I, \preceq)\) is a totally ordered set, i.e., in the form \(\{1, \ldots, n\}, \preceq\), we say \(\mathcal{E}\) is an exceptional sequence. An exceptional poset is said to be strong if
\[
\text{Hom}^k(E_i, E_j) = 0 \text{ for all } k \geq 1 \text{ and all } i, j \in I.
\]

It is said to be full if \(\mathcal{E}\) classically generates \(\mathcal{D}\), i.e., \(\mathcal{D}\) is the smallest full triangulated category containing \(\mathcal{E}\). We say an exceptional poset \(\mathcal{E}\) has length \(n\) if it has \(n\) objects. For any \(j \in I\) the subset \(\{E_i : i \prec j\} \subseteq \mathcal{E}\) is also an exceptional poset and will be denoted by \(\mathcal{E}_{<j}\). The exceptional poset \(\mathcal{E}_{\not\prec j}\) and \(\mathcal{E}_{\not\succ j}\) are similarly defined.

Remark 2.8.2. Note that every poset \((I, \preceq)\) can be refined into a totally ordered set, i.e., there exists a (non-unique) monotone bijection \((I, \preceq) \to (\{1, 2, \ldots, |I|\}, \preceq)\). Note also that if \(i\) and \(j\) are incomparable elements in \(I\), then one can always find total order refinements \(\phi\) and \(\psi\) such that \(\phi(i) < \phi(j)\) whilst \(\psi(i) > \psi(j)\). Hence, by considering exceptional posets with all total order refinements, there is no loss of generality by considering only exceptional sequences. However, since sometimes we have only a natural partial order instead of a total order on \(I\), we will stick to the notion of exceptional poset.

The following proposition tells us that the length of a full exceptional poset is an invariant of the derived category.

Proposition 2.8.3 ([16], Lemma C.2). Let \(\mathcal{E}\) be a full exceptional poset on \(\mathcal{D}\) with length \(n\). Then \([\mathcal{E}] = \{[E_i] \}_{i \in I}\) form a \(\mathbb{Z}\)-linear basis of \(K(\mathcal{D})\). In particular, \(\text{length} \ \mathcal{E} = \text{rank} \ K(\mathcal{D})\).

Proposition 2.8.4. The full triangulated subcategory \((\mathcal{E})\) classically generated by an exceptional poset \(\mathcal{E}\) is admissible. Moreover, if \(\mathcal{E}\) is an exceptional object, then \(L_E X\) is the cone of the evaluation map
\[
\text{Hom}^*(E, X) \otimes E \to X \to L_EX.
\]

If \(\mathcal{E}\) is an exceptional poset and \(\phi : (I, \preceq) \to (\{1, \ldots, n\}, \preceq)\) is a monotone bijection, then
\[
L_E = L_{E_{\phi^{-1}(1)}} \cdots L_{E_{\phi^{-1}(n)}}.
\]

Proof. Without loss of generality, we may assume \(\mathcal{E}\) is an exceptional sequence. We induct on the length of exceptional poset \(\mathcal{E}\). Suppose \(n = 1\) and \(\mathcal{E} = \{E\}\). For any \(X \in \mathcal{D}\), we have a natural evaluation map \(\text{Hom}^*(E, X) \otimes E \to X\). Extending it to a triangle
\[
\text{Hom}^*(E, X) \otimes E \to X \to Y
\]
and applying \(\text{Hom}(E, -)\), we get \(Y \in (\mathcal{E})^\perp\). By Proposition 2.6.2, \(\langle \mathcal{E} \rangle\) is right admissible and by Proposition 2.7.2, \(Y = L_EX\). Left admissibility of \(\langle \mathcal{E} \rangle\) is similarly proven. Now suppose \(\mathcal{E} = \{E_1, \ldots, E_n\}\). By induction assumption, \(\langle \mathcal{E}_{\geq 2} \rangle\) is admissible, and \(L_{E_{\geq 2}} = L_{E_{\geq 2}} \cdots L_{E_n}\). By Proposition 2.6.2, for any \(X\), there are triangles
\[
A \to X \xrightarrow{\alpha} L_{E_{\geq 2}} X
\]
and applying \(\text{Hom}(E, -)\), we get \(Y \in (\mathcal{E})^\perp\). By Proposition 2.6.2, \(\langle \mathcal{E} \rangle\) is right admissible and by Proposition 2.7.2, \(Y = L_EX\). Left admissibility of \(\langle \mathcal{E} \rangle\) is similarly proven. Now suppose \(\mathcal{E} = \{E_1, \ldots, E_n\}\). By induction assumption, \(\langle \mathcal{E}_{> 2} \rangle\) is admissible, and \(L_{E_{> 2}} = L_{E_{> 2}} \cdots L_{E_n}\). By Proposition 2.6.2, for any \(X\), there are triangles
\[
A \to X \xrightarrow{\alpha} L_{E_{> 2}} X
\]
where \(A \in \langle \mathcal{E}_{> 2} \rangle\). Applying \(\text{Hom}^*(E_i, -)\) to the second sequence, we get \(L_{E_i} L_{E_{> 2}} X \in \langle E_i \rangle^\perp\) for all \(i\) and hence in \(\langle \mathcal{E} \rangle^\perp\). Extend the map \(X \xrightarrow{\beta} L_{E_i} L_{E_{> 2}} X\) to a triangle
\[
B \to X \xrightarrow{\beta \circ \alpha} L_{E_i} L_{E_{> 2}} X.
\]

By the octahedral axiom, \(B \in \langle \mathcal{E} \rangle\) and by Proposition 2.6.2 \(\langle \mathcal{E} \rangle\) is right admissible. By Proposition 2.7.2, \(L_EX = L_{E_i} L_{E_{> 2}} X = L_{E_i} \cdots L_{E_n} X\). □

Proposition 2.8.5 (Dual Exceptional Poset). Let \(\mathcal{E} = \{E_i\}_{i \in I}\) be a full exceptional poset. Then there is a unique set of objects \(\mathcal{F} = \{F_i\}_{i \in I^\circ}\) such that
\[
\text{Hom}^*(E_i, F_j) = \begin{cases} 
K & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, the object \(F_i\) is given by the formula \(F_i = L_{E_i} E_i\), and the set \(\mathcal{F}\) is a full exceptional poset which is called the full exceptional poset dual to \(\mathcal{E}\).
Proof. The proof is a simple modification of [16, Lemma 2.5]. First, we show existence: By the definition of exceptional poset, \( E_j \in \langle E_{k}\rangle^{\perp} \), hence \( F_j = L_{E_{k}}(E_j) \) is well defined and we have a chain of inclusion
\[
F_j \in \langle E_{k}\rangle^{\perp} \subseteq \langle E_{E_j}\rangle \subseteq \langle E_{j}\rangle^{\perp}.
\]
In particular,
\[
\text{Hom}^{*}(E_i, F_j) = 0
\]
whenever \( i \prec j \) or \( i \not\prec j \), i.e., whenever \( i \not= j \). By Proposition 2.7.2, there exists \( Y \in \langle E_{j}\rangle \) and a triangle
\[
Y \to E_j \to F_j \to Y[1].
\]
Applying the exact functor \( \text{Hom}^{*}(E_j, -) \), since \( \text{Hom}^{*}(E_j, Y) = 0 \), we conclude
\[
\text{Hom}^{*}(E_j, F_j) = \text{Hom}^{*}(E_j, E_j) = \mathbb{K}.
\]
To show \( \mathcal{F} = \{ F_i \}_{i \in I} \) is an exceptional poset, we have to show \( j \not= i \) implies \( \text{Hom}^{*}(F_i, F_j) = 0 \). Since \( F_i \in \langle E_{E_i}\rangle \), we are done if \( \text{Hom}^{*}(E_k, F_j) = 0 \) for all \( k \not< i \), which is true since \( j \not< i \) and \( k \not< i \) implies \( j \not= k \). For uniqueness, choose a nonzero map \( E_j \to F_j \) and extend it to a triangle
\[
Y \to E_j \to F_j \to Y[1].
\]
Applying the functor \( \text{Hom}^{*}(E_i, -) \) shows \( \text{Hom}^{*}(E_i, Y) = 0 \) for all \( i \not= j \) and \( i = j \). Hence \( Y \in \langle E_{j}\rangle \).
Proposition 2.7.2 now shows \( F_j = L_{E_{E_j}}(E_j) \).

\[\tag*{\text{Remark 2.8.6.}}\]
Note that the partial order on \( \mathcal{F} \) is reversed. Note also that in general, the dual exceptional poset of a strong exceptional poset is NOT strong.

\[\tag*{\text{Example 2.8.7.}}\]
Take \( \mathbb{P}^{n} \) and the full strong exceptional sequence \( \mathcal{E} = (\mathcal{O}, \ldots, \mathcal{O}(n)) \). By computing cohomologies using the Bott’s formula [11], one can check the dual full exceptional poset is given by
\[
\mathcal{F} = (\Omega^{n}(n)[n], \Omega^{n-1}(n-1)[n-1], \ldots, \Omega^{1}(1)[1], \mathcal{O}),
\]
where \( \Omega^{i} \) denotes the \( i \)-th wedge power of the cotangent sheaf.

\[\tag*{\text{Example 2.8.8.}}\]
Take \( \mathbb{P}^{1} \times \mathbb{P}^{1} \) and the full strong exceptional poset
\[
\mathcal{E} = (\mathcal{O} < \mathcal{O}(1, 0), \mathcal{O}(0, 1) < \mathcal{O}(1, 1)).
\]
By computing cohomologies, one can check the dual full exceptional poset is given by
\[
\mathcal{F} = (\mathcal{O}(-1,-1)[2] < \mathcal{O}(0,-1)[1], \mathcal{O}(-1,0)[1] < \mathcal{O}).
\]

\[\tag*{\text{Remark 2.8.9.}}\]
Exceptional posets do not exist on \( \mathbb{D}^{b}(\text{Coh}(X)) \) for any Calabi–Yau \( X \). This is because by Serre duality,
\[
\text{Hom}^{\dim X}(E, E) = \text{Hom}^{0}(E, E)
\]
which contains at least a copy of \( \mathbb{K} \), hence cannot vanish.

\section{2.9 Tilting Objects}

In the rest of this thesis, we will be working with algebraic triangulated categories in the sense of Keller [43, §3.6]. The precise definition of an algebraic triangulated category will not bother us much, but let me point out that, in Keller’s words, “all’ triangulated categories occurring in algebra and geometry are algebraic.” The main examples we have in our mind will be the derived category of equivariant sheaves on smooth varieties and the derived category of dg-modules over dg-algebras.

\[\tag*{\text{Definition 2.9.1 (Tilting Object).}}\]
An object \( T \) in an algebraic triangulated category \( \mathcal{D} \) is said to be \textit{tilting} if it is
• compact, i.e., the functor \( \text{Hom}(T, -) \) commutes with arbitrary coproduct, and

• generating, i.e., the only object \( X \) with \( \text{Hom}^\bullet(T, X) = 0 \) is the zero object.

A tilting object \( T \) is said to be \textit{classical} if \( \text{Hom}^k(T, T) = 0 \) for all \( k \neq 0 \).

Here is the main theorem of the section.

**Theorem 2.9.2** ([42], Theorem 8.7). Let \( T \) be a tilting object in an algebraic triangulated category \( D \) which admits all set-indexed coproducts. Then there is a dg-algebra \( \text{RHom}(T, T) \) with cohomology \( \text{Hom}^\bullet(T, T) \) and a triangle equivalence

\[
\Phi : D \to D(\text{RHom}(T, T)^\text{op})
\]

which takes \( T \) to the free module of rank one, and whose composition with cohomology is given by

\[
H^\bullet \circ \Phi : D \to \text{Grmod}(\text{Hom}^\bullet(T, T)^\text{op}), \quad X \mapsto \text{Hom}^\bullet(T, X).
\]

Furthermore, this equivalence restricts to an equivalence between the perfect derived categories

\[
\Phi : \text{Per}(T) \to \text{Per}(\text{RHom}(T, T)^\text{op}).
\]

One way to construct tilting objects is from full exceptional posets.

**Proposition 2.9.3.** Let \( \mathcal{E} = \{E_i\}_{i \in I} \) be a full exceptional poset of compact objects in \( D \). Then \( \bigoplus_{i \in I} E_i \) is a tilting object in \( D \). If \( \mathcal{E} \) is further assumed to be strong, then \( E \) is classical tilting.

**Proof.** Since \( \mathcal{E} \) is an exceptional poset, \( \langle \mathcal{E} \rangle \) is admissible by Proposition 2.8.4, \( \langle \mathcal{E} \rangle = D \) if and only if \( \langle \mathcal{E} \rangle^\perp = 0 \). Hence \( E \) is tilting since \( \mathcal{E} \) is full. If \( \mathcal{E} \) is strong, then \( \text{Hom}^k(E_i, E_j) = 0 \) for all \( k \geq 1 \) and thus \( E \) is classical tilting. \( \blacksquare \)
Chapter 3

$A_\infty$-Algebras

This chapter is a survey on $A_\infty$-algebras and operations on them.

In section 3.1, we introduce $A_\infty$-algebras and other related notions such as minimal models, $A_\infty$-modules and their derived categories.

Section 3.2 defines the notion of minimal $A_\infty$-algebras, and discusses how to construct minimal models by using homotopy perturbation.

Section 3.3 defines the notion of cyclic structure on $A_\infty$-algebras and describes how it gives rise to Calabi–Yau categories.

Section 3.4 defines the Koszul functor, which is essentially a way of producing dg-quivers from $A_\infty$-algebras. There are two versions of this functor: the completed one and the incomplete one. The completed one is defined on $A_\infty$-algebras and yields the completed path algebra of a dg-quiver. The incomplete one is only defined on $A_{\text{fin}}$-algebras, and yields the incompletely path algebra of a dg-quiver. The difference between the two versions is similar to the difference between power series and polynomials. Admittedly, working with $A_{\text{fin}}$-algebras and hence the incomplete Koszul functor is awkward in the world of $A_\infty$-algebras as being $A_{\text{fin}}$ is not a homotopy invariant property. For example, the minimal model of an $A_{\text{fin}}$-algebra is not necessarily $A_{\text{fin}}$. However, as we will see in Chapter 5, the incomplete Koszul functor is central to our problem of constructing derived equivalences between dg-quivers and total spaces of vector bundles.

Section 3.5 defines the quotient of an $A_\infty$-algebra by a finite group, and the smash product of an $A_\infty$-algebra by a finite group. Although the definition of quotient construction is straightforward, it seems to be new. This quotient construction is central to constructing derived equivalences equivariantly as described in Section 5.6. We then prove a relation between the quotient construction and the smash product, and shows how these two constructions commute with the Koszul dual functor, i.e., the Koszul dual of the quotient (resp. smash product) of an $A_{\text{fin}}$-algebra is the quotient (resp. smash product) of its Koszul dual. This section is inspired by the work of Bocklandt, Schedler, and Wemyss [7].

Section 3.6 surveys different constructions of $A_\infty$-tensor product. Since $A_\infty$-tensor products are only unique up to $A_\infty$-quasi-isomorphisms, there is in general no natural formulae for computing the tensor product, although there is one in the case when one of the $A_\infty$-algebras is $A_2$, i.e., a dg-algebra. Particularly important to us is the tensor product constructed by Amorim and Tu [2], since their construction preserves cyclic structures. We then prove that, under some local finiteness conditions, the Koszul functor commutes with the tensor product, i.e., Koszul dual of tensor product of $A_\infty$-algebras is quasi-isomorphic to tensor product of Koszul duals of $A_\infty$-algebras as dg-algebras.
3.1 \(A_\infty\)-Algebras and \(A_\infty\)-Modules

This section introduces \(A_\infty\)-algebras. We follow the sign conventions of Lefèvre-Hasegawa [48]. We refer to Keller [44] for a short introduction and Lefèvre-Hasegawa [48] for a comprehensive reference.

**Definition 3.1.1 \((A_\infty\)-algebras\).** Let \(K\) be a field and \(S\) be a semisimple algebra over \(K\). An \(A_\infty\)-algebra over \(S\) is a \(\mathbb{Z}\)-graded \(S\)-bimodule,

\[
A = \bigoplus_{i \in \mathbb{Z}} A^i
\]

together with, for each \(n \in \mathbb{N}\), an \(S\)-bimodule homomorphisms of degree \(2 - n\)

\[
m_n : A^{\otimes n} \rightarrow A
\]

satisfying the the \(A_\infty\)-relations

\[
\sum_{a+b+c=n \atop b \geq 1} (-1)^{abc} m_{a+1+c} \circ (\text{id}^{\otimes a} \otimes m_b \otimes \text{id}^{\otimes c}) = 0.
\]

The first few \(A_\infty\)-relations read as follows:

- When \(n = 1\), we have \(m_1 \circ m_1 = 0\), i.e., \((A, m_1)\) is a chain complex.
- When \(n = 2\), we have \(m_1 \circ m_2 = m_2 \circ (m_1 \otimes \text{id} + \text{id} \otimes m_1)\), i.e., \(m_1\) is a derivation with respect to \(m_2\).
- When \(n = 3\), we have

\[
m_2 \circ (\text{id} \otimes m_2 - m_2 \otimes \text{id}) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \text{id} \otimes \text{id} + \text{id} \otimes m_1 \otimes \text{id} + \text{id} \otimes \text{id} \otimes m_1),
\]

i.e., \(m_2\) is associative up to homotopy given by \(m_3\).

A morphism \(f : A \rightarrow B\) of \(A_\infty\)-algebras over \(S\) is a family of \(S\)-bimodule morphisms of degree \(1 - n\)

\[
f_n : A^{\otimes n} \rightarrow B,
\]

satisfying the \(A_\infty\)-relations

\[
\sum_{a+b+c=n \atop b \geq 1} (-1)^{abc} f_{a+1+c} \circ (\text{id}^{\otimes a} \otimes m_b \otimes \text{id}^{\otimes c}) = \sum_{r=1}^{n} \sum_{i_1 + \cdots + i_r = n} (-1)^s m_r \circ (f_{i_1} \otimes \cdots \otimes f_{i_r}),
\]

where \(s = \sum_{u=2}^{n} (1 - i_u) \sum_{v=1}^{u} i_v \). Composition of two morphisms \(f : B \rightarrow C\) and \(g : A \rightarrow B\) is defined by

\[
(f \circ g)_n = \sum_{r=1}^{n} \sum_{i_1 + \cdots + i_r = n} (-1)^s f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r}).
\]

A morphism of \(A_\infty\)-algebra is called strict if \(f_n = 0\) for \(n \geq 2\). In this case, the \(A_\infty\)-relations simplifies to \(f_1 m_n = m_n \circ f_1^{\otimes n}\). The identity morphism of an \(A_\infty\)-algebra is the strict \(A_\infty\)-morphism with \(f_1 = \text{id}\). An \(A_\infty\)-quasi-isomorphism \(f\) is an \(A_\infty\)-morphism whose \(f_1\) induce isomorphism on the homology on the chain complex \((A, m_1)\).

An \(A_\infty\)-algebra is strictly unital if there is an element \(1_A \in A\) of degree 0 such that \(m_2(1_A, a) = a = m_2(a, 1_A)\) for all \(a \in A\) and \(m_n(a_1, \cdots, a_n) = 0\) whenever \(n \neq 2\) and one of the \(a_i = 1_A\). A morphism of strictly unital \(A_\infty\)-algebra \(f : A \rightarrow B\) is strictly unital if \(f_1(1_A) = 1_B\) and \(f_n(a_1, \cdots, a_n) = 0\) whenever \(n \neq 1\). Note that for any strictly unital \(A_\infty\)-algebra, there is a canonical strict morphism \(\eta : S \rightarrow A\) mapping \(1_S\) to \(1_A\). An \(A_\infty\)-algebra is augmented if it is strictly unital and there is a strictly unital morphism \(\epsilon : A \rightarrow S\) such that \(\epsilon \circ \eta = \text{id}_S\). A morphism of augmented \(A_\infty\)-algebras is a strictly unital
morphism $f : A \to B$ such that $\epsilon_B \circ f = \epsilon_A$. When an $A_\infty$-algebra is augmented, there is a decomposition $A = S \oplus A$, where $A = \ker \epsilon$ is called the augmentation ideal.

An $A_\infty$-algebra is said to be $A_{n\infty}$ if $m_i = 0$ for $i \gg 0$. An $A_\infty$-algebra is said to be $A_n$ if $m_i = 0$ for all $i \geq n$. In particular, an $A_1$-algebra is a chain complex, and an $A_2$-algebra is a dg-algebra. Morphisms between chain complexes (resp. dg-algebras) are the same as strict $A_\infty$-morphisms between $A_1$-algebras (resp. $A_2$-algebras). A morphism is said to be $A_n$ if $f_r = 0$ for $r > n$ and is said to be $A_{n\infty}$ if $f_r = 0$ for $r \gg 0$.

**Definition 3.1.2** ($A_\infty$-homotopy). Let $A$ and $B$ be two $A_\infty$-algebras and $f, g : A \to B$ be $A_\infty$-morphisms. An $A_\infty$-homotopy between $f$ and $g$ is a family of morphisms of degree $-n$

$$h_n : A^\otimes n \to B$$

satisfying the equation

$$f_n - g_n = \sum_{i_1 + \cdots + i_r + k + j_1 + \cdots + j_s = n} (-1)^{s}m_{r+1+t} \circ (f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_s})$$

$$+ \sum_{j+k+\ell = n} (-1)^{r+s}h_{j+1+\ell} \circ (\text{id}^\otimes \otimes m_k \otimes \text{id}^\otimes \otimes \ell),$$

where

$$s = t + \sum_{1 \leq u \leq t} (1 - j_u)(n - \sum_{u \geq a} j_u) + k \sum_{1 \leq u \leq r} i_u + \sum_{2 \leq u \leq r} (1 - i_u) \sum_{u < a} j_u.$$

Similarly, an $A_\infty$-homotopy is said to be $A_n$ if $h_r = 0$ for $r > n$ and $A_{n\infty}$ if $h_r = 0$ for $r \gg 0$.

One of the salient features of $A_\infty$-algebras is that all $A_\infty$-quasi-isomorphisms are invertible up to homotopy:

**Theorem 3.1.3** ([48] Corollary 1.3.1.3). An $A_\infty$-quasi-isomorphism is an $A_\infty$-homotopy equivalence.

**Definition 3.1.4** ($A_\infty$-modules). An $A_\infty$-right module over $A$ is a $Z$-graded $S$-bimodule $M$, together with a family of $S$-bimodule morphisms of degree $2 - n$

$$m^M_n : M \otimes_S A^\otimes n \to M$$

satisfying the $A_\infty$-relations

$$\sum_{a+b+c=n} \sum_{a,b \geq 1} (-1)^{ab+c}m^M_{a+1+c} \circ (\text{id}^\otimes \otimes m_b \otimes \text{id}^\otimes c) + \sum_{b+cn} \sum_{b \geq 1} (-1)^{c}m^M_{1+c} \circ (m^M_b \otimes \text{id}^\otimes c) = 0.$$

**Definition 3.1.5** ($A_\infty$-morphism). Let $A$ be an $A_\infty$-algebra and $M, N$ be $A_\infty$-modules. An $A_\infty$-morphism $f : M \to N$ is a family of $S$-bimodule morphisms of degree $1 - n$

$$f_n : M \otimes_S A^\otimes n \to N$$

which satisfy the equations

$$\sum_{a+b+c=n} \sum_{b \geq 1} (-1)^{ab+c}f_{a+1+c} \circ (\text{id}^\otimes \otimes m_b \otimes \text{id}^\otimes c) = \sum_{r+s=n} m_{s+1} \circ (f_r \otimes \text{id}^\otimes s).$$

An $A_\infty$-morphism is said to be an $A_\infty$-quasi-isomorphism if $f_1$ induces an isomorphism on cohomology. An $A_\infty$-morphism is said to be strict if $f_i = 0$ for all $i \geq 2$.

**Definition 3.1.6** ($A_\infty$-homotopy). Let $A$ be an $A_\infty$-algebra and $M, N$ be $A_\infty$-modules. Let $f, g : M \to N$ be $A_\infty$-morphisms. An $A_\infty$-homotopy between $f$ and $g$ is a family of morphisms of degree $-n$

$$h_n : M \otimes A^\otimes n \to N$$

satisfying the equations

$$f_n - g_n = \sum_{r+s=n} (-1)^{r}m_{r+s}(h_r \otimes \text{id}^\otimes s) + \sum_{a+b+c=n} \sum_{b \geq 1} (-1)^{ab+c}h_{a+1+c}(\text{id}^\otimes a \otimes m_b \otimes \text{id}^\otimes c).$$
Theorem 3.1.7 ([48] Proposition 2.4.1.1). An $A_\infty$-quasi-isomorphism between $A_\infty$-modules is an $A_\infty$-homotopy equivalence.

Derived Category of $A_\infty$-modules. Recall that for an honest algebra $A$, the derived category $\text{D}(A)$ is obtained by localizing the homotopy category of $A$-modules $\text{K}(A)$ at the class of quasi-isomorphism. One would like to define the derived category of an $A_\infty$-algebra in a similar way. Since by Theorem 3.1.7, $A_\infty$-quasi-isomorphisms are already homotopy equivalence, we have

Definition 3.1.8 (Derived category of $A_\infty$-modules). The derived category $\text{D}_\infty(A)$ of an $A_\infty$-algebra $A$ is defined to be the homotopy category of $A_\infty$-modules over $A$, i.e., objects in $\text{D}_\infty(A)$ are $A_\infty$-modules over $A$ and morphisms between two $A_\infty$-modules $M$ and $N$ are $A_\infty$-morphisms modulo $A_\infty$-homotopies.

Definition 3.1.9 (Perfect derived category). Let $A$ be an $A_\infty$-algebra. Then $A$ can be regarded as an $A_\infty$-module over itself. The smallest triangulated category generated by the $A$-module $A$ is called the perfect derived category of $A$ and is denoted by $\text{Per}_{\infty}(A)$.

$A_\infty$-categories

Definition 3.1.10. An $A_\infty$-category $\mathcal{A}$ consists of the following data:

1. a set of objects $\text{Obj}(\mathcal{A})$,
2. for any $X, Y \in \text{Obj}(\mathcal{A})$, a $\mathbb{Z}$-graded vector space $\mathcal{A}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i(X, Y)$,
3. for each $n = 1, 2, 3, \ldots$, and any $X_0, \ldots, X_n \in \text{Obj}(\mathcal{A})$, a linear map homogeneous of degree $2 - n$ $m_n : \mathcal{A}(X_{n-1}, X_0) \otimes \mathcal{A}(X_{n-2}, X_{n-1}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_n)$, satisfying the equations
\[
\sum_{a+b+c=n} (-1)^{ab+rc} m_{a+1+c} \circ (\text{id}^a \otimes m_b \otimes \text{id}^c) = 0.
\]

If $m_1 = 0$, then $\mathcal{A}$ is said to be minimal. An $A_\infty$-category is said to be strictly unital if for each object $X \in \text{Obj} \mathcal{A}$, there is $e_X \in \mathcal{A}(X, X)$ such that $m_2(e_X \otimes b) = b$, $m_2(a \otimes e_Y) = a$ for any $a \in \mathcal{A}(Y, X)$ and $b \in \mathcal{A}(X, Y)$, and $m_n(a_1 \otimes \cdots \otimes a_1) = 0$ whenever some $a_j = e_X$ for some $X$. Let $\mathcal{S}$ be the discrete $\mathbb{K}$-category on $\text{Obj} \mathcal{A}$, i.e., $\text{Obj} \mathcal{S} = \text{Obj} \mathcal{A}$ and $\mathcal{S}(X, X) = \mathbb{K}$ and $\mathcal{S}(X, Y) = 0$ whenever $X \neq Y$. Then each strictly unital $A_\infty$-category is endowed with an $A_\infty$-functor $\epsilon : \mathcal{S} \to \mathcal{A}$ which sends each $1_X \in \mathcal{S}(X, X)$ to the identity element $e_X \in \mathcal{A}(X, X)$. If there is an $A_\infty$-functor $\eta : \mathcal{A} \to \mathcal{S}$ such that $\eta \circ \epsilon = \text{id}_\mathcal{S}$, then $\mathcal{A}$ is said to be augmented. An $A_\infty$-category is said to be finite if $\text{Obj} \mathcal{A}$ is finite and for any objects $X, Y$, $\mathcal{A}(X, Y)$ is finite dimensional.

Conventions. When the base algebra of an $A_\infty$-algebra is in the form $\mathcal{S} = \mathbb{K}^r$, the data of an $A_\infty$-algebra is the same as an $A_\infty$-category $\mathcal{A}$ with $r$ objects: Let $e_i$ be the vector in $\mathcal{S} = \mathbb{K}^r$ with 1 in the $i$-th place and zero elsewhere. Given an $A_\infty$-algebra over $\mathcal{S}$, we can define an $A_\infty$-category $\mathcal{A}$ by taking $\text{Obj}(\mathcal{A}) = \{e_1, \ldots, e_r\}$ and $\mathcal{A}(e_i, e_j) = e_i e_j$. Conversely, given an $A_\infty$-category with $r$ objects, choose a bijection between $\text{Obj}(\mathcal{A})$ and $\{e_1, \ldots, e_r\}$ and take $\mathcal{A} = \bigoplus_{i,j} \mathcal{A}(e_i, e_j)$. Each $e_i$ acts on left on $\mathcal{A}$ by projecting to the vector subspace $\bigoplus_{j} \mathcal{A}(e_i, e_j)$. This defines a left $\mathcal{S}$-module structure on $\mathcal{A}$. Right action is defined similarly. Since in this thesis we will only deal with $A_\infty$-category with finite objects, we will not distinguish $A_\infty$-algebras and $A_\infty$-categories and will use the two terminologies interchangeably.
3.2 Minimal Model

An $A_\infty$-algebra is said to be minimal if $m_1 = 0$. In this case the multiplication map $m_2$ is associative. By Kadeishvili’s theorem, every $A_\infty$-algebra is quasi-isomorphic to a minimal one:

**Theorem 3.2.1** (Kadeishvili [36], see also [48] Corollary 1.4.1.4). Let $A$ be an $A_\infty$-algebra. Then the cohomology $H^*(A)$ has a unique (up to isomorphism of $A_\infty$-algebras) $A_\infty$-structure such that

- $m_1 = 0$ and $m_2$ is induced by $m_2^A$,
- there is an $A_\infty$-quasi-isomorphism $i : H^*(A) \to A$ lifting the identity map on $H^*(A)$.

The $A_\infty$-structure on the minimal model can be described more explicitly. The following construction is given by Markl [54, §7] and is known as homotopy perturbation. Denote by $B$ and $Z$ the $S$-bimodule of coboundaries and cocycles in $A$. Since $S$ is semisimple, we can choose a splitting ($S$-subbimodule) $H$ and $L$ such that

$$Z = B \oplus H \text{ and } A = Z \oplus L = B \oplus H \oplus L,$$

where $H(A) \cong H$. Denote by $p : A \to H$ the projection map and by $i : H \to A$ the inclusion map. Define a linear map $h : A \to A$ by $h = 0$ on $L \oplus H$ and $h = (m_1^A|_L)^{-1}$ on $B$. It follows that $hm_1^A$ (resp. $m_1^A h$) is the projection to $L$ (resp. $B$), and forms a homotopy from $id_A$ to $i \circ p$ i.e., $id_A - i \circ p = m_1^A h + hm_1^A$.

Define a sequence of linear maps $\lambda_n : A \otimes^n A \to A$ of degree $(2 - n)$ for all $n \geq 2$ inductively as follow: Take $\lambda_2 = m_2^A$, and for $n \geq 3$, take

$$\lambda_n = \sum_I (-1)^{\theta(r_1, \ldots, r_k)} m_k^A \circ ((h \circ \lambda_{r_1}) \otimes \cdots \otimes (h \circ \lambda_{r_k})), \tag{3.2.1}$$

where the sum is over the set

$$I = \{(k, r_1, \ldots, r_k) : 2 \leq k \leq n, r_1, \ldots, r_k \geq 1, r_1 + \cdots + r_k = n\},$$

the sign is given by

$$\theta(r_1, \ldots, r_k) = \sum_{1 \leq \alpha < \beta \leq k} r_\alpha (r_\beta + 1)$$

and $h\lambda_1$ is defined formally to be $id_A$. Now, define $m_n^{H^*(A)} : H^*(A) \otimes^n \to H^*(A)$ of degree $2 - n$ by

$$m_n^{H^*(A)} = p \circ \lambda_n \circ i \otimes^n.$$

These maps satisfy the $A_\infty$-relations and $H^*(A)$ equipped with linear maps $m_i^{H^*(A)}$ for all $i \geq 2$ is the desired minimal model of $A$. The $A_\infty$-quasi-isomorphism $i : H^*(A) \to A$ is given by

$$i_n = h \circ \lambda_n \circ i \otimes^n.$$

**Remark 3.2.2.** As pointed out by Markl [54, §4], the recursive formula in Equation 3.2.1 can be reformulated as a sum of trees: Let $P_n$ be the set of all rooted planar directed trees with $n$ leaves and each internal vertex has at least two incoming edges. For each $T \in P_n$, one can assign a linear map $F_T : A \otimes^n \to A$ by interpreting each internal vertex with $k$ incoming edges by $m_k$, and each internal edge by $h$. Next, we would like to define a number $\theta(T)$. Let $v$ be an internal vertex of $T$ with $k$ incoming edges. Denote by $r_i$ the number of paths going from any leaves of $T$ to the root of $T$ which passes through the $i$-th edge of $v$. Define $\theta_T(v) = \theta(r_1, \cdots, r_k)$ and let $\theta(T) = \sum_{\text{internal vertices}} \theta_T(v)$. Then Equation 3.2.1 can be rewritten as

$$\lambda_n = \sum_{T \in P_n} (-1)^{\theta(T)} F_T.$$
3.3 Cyclic Structure

Let $\mathcal{A}$ be a minimal $A_\infty$-category whose morphism spaces are finite dimensional. A cyclic structure of degree $d$ on $\mathcal{A}$ consists of, for each $X,Y \in \text{Obj} \, \mathcal{A}$, a supersymmetric nondegenerate bilinear form homogeneous of degree $(-d)$

$$(-,-) : \mathcal{A}(X,Y) \times \mathcal{A}(Y,X) \to \mathbb{K}$$

satisfying the Koszul sign rule

$$\langle m_n(a_1 \otimes \cdots \otimes a_n), a_{n+1} \rangle = (-1)^{a_1(|a_2| + \cdots + |a_{n+1}|)} \langle m_n(a_2 \otimes \cdots \otimes a_{n+1}), a_1 \rangle.$$

In general, cyclic structure is not preserved by $A_\infty$-quasi-isomorphisms, i.e., if $\mathcal{A}$ and $\mathcal{A}'$ are $A_\infty$-quasi-isomorphic, and $\mathcal{A}$ has a cyclic structure, $\mathcal{A}'$ might not have a cyclic structure. To preserve the cyclic structure, we need a class of $A_\infty$-morphisms which respect the cyclic structure. This gives rise to the notion of cyclic $A_\infty$-morphism, which was first defined by Kajiura [37, Definition 2.13].

**Definition 3.3.1** (Cyclic $A_\infty$-morphism). Let $\mathcal{A}$ and $\mathcal{A}'$ be cyclic algebras. An $A_\infty$-morphism $f : \mathcal{A} \to \mathcal{A}'$ is said to be cyclic if for all $a_1, \ldots, a_n \in \mathcal{A}$,

$$\langle a_1, a_2 \rangle_\mathcal{A} = \langle f_1(a_1), f_1(a_2) \rangle_{\mathcal{A}'}$$

and for $n \geq 3$

$$\sum_{i+j=n} (-1)^{i+\sum \ell \cdot (i-\ell+1)|a_\ell| + \sum_{\ell+1} \ell \cdot (k-\ell)|x_k|} \langle f_\ell(a_1, \ldots, a_i), f_j(a_{i+1}, \ldots, a_n) \rangle = 0$$

Following Keller [45, §5], we describe a way of producing Calabi–Yau triangulated categories from $A_\infty$-categories with cyclic structure. We denote by $D(\mathcal{A})$ the derived category of $A_\infty$-module over $\mathcal{A}$. The perfect derived category $\text{per}(\mathcal{A})$ is the thick triangulated subcategory of $D(\mathcal{A})$ generated by the representable $A_\infty$-modules $\mathcal{A}(-, X)$ for all $X \in \text{Obj} \, \mathcal{A}$. In other words, $\text{per}(\mathcal{A})$ is the smallest full triangulated subcategory which is stable under taking direct summands which contains all representable $A_\infty$-modules $\mathcal{A}(-, X)$. In case $\mathcal{A}$ is an ordinary $\mathbb{K}$-algebra, then $\text{Per}(\mathcal{A})$ is the full subcategory of $D(\mathcal{A})$ formed by perfect complexes, i.e., those quasi-isomorphic to a bounded complex of finitely generated projective modules.

**Proposition 3.3.2** ([45], §5). Let $\mathcal{A}$ be a Hom-finite minimal $A_\infty$-category with a cyclic structure of degree $d$. Then $\text{Per}(\mathcal{A})$ is Hom-finite $d$-Calabi–Yau triangulated category.

3.4 Koszul Functor

In this section, we will introduce the Koszul functor which produces quasi-free dg-algebras from $A_\infty$-algebras. There are potentially two ways to do this. The first one is to take the bar construction followed by taking dual. The second one is to take dual followed by the cobar construction. If we start with an $A_{\infty}$-algebra, then under some locally finiteness conditions, the two constructions end up giving the same dg-algebra.

For our purpose, the second approach seems to be conceptually simpler and this is the road we will take. In this case when our $A_{\infty}$-algebra is finite dimensional, this is all good and product a dg-algebra which is the path algebra of a dg-quiver. However, it runs into problem as soon as we consider algebras which are not finite dimensional. This is because in general $(V \otimes V)^*$ and $V^* \otimes V^*$ are not isomorphic, and hence taking the dual of an $A_\infty$-algebra does not necessarily produce an $A_\infty$-coalgebra.

Following Lu, Palmieri, Wu and Zhang [51], it is useful to impose some local finiteness condition by equipping $A_\infty$-algebras with an additional grading, called the Adams grading, by an abelian group $G$. We will write the degree of a bihomogeneous element $a$ in the form $\text{deg} \, a = (\text{deg}_1 a, \text{deg}_2 a) \in \mathbb{Z} \times G$. The $(i,j)$-th component of $A$ will be denoted by $A^i_j$. Henceforth in this section, all $A_\infty$-algebra will be assumed to be locally finite in the sense that each $(i,j)$-th component $A^i_j$ is finite dimensional. The
multiplication maps $m_n$’s will be assumed to preserve the Adams grading, i.e., of degree $(2 - n, 0)$. $A_\infty$-morphisms, $A_\infty$-homotopies will also be assumed to preserve the Adams grading. The suspension functor $\Sigma$ will only shift the first grading and ignore the Adams grading, i.e., $(\Sigma A)_j = A_{j+1}$.

For any Adams graded $A_\infty$-algebra, its minimal model $H^\bullet(A)$ can be chosen to be Adams graded by choosing the splitting $A = B \oplus H \oplus L$ described in Section 3.2 to be Adams graded.

For any locally finite $\mathbb{Z} \times G$-graded vector space $V = \bigoplus V_j^i$, one can take the graded dual $V^\# = \bigoplus \text{Hom}(V_j^{-i}, \mathbb{K}) = \bigoplus (V_j^{-i})^*$. The graded dual is better behaved than the usual dual since they enjoy the isomorphisms $(V^\#)^\# \cong V$ and $(V^{\otimes n})^\# \cong (V^\#)^{\otimes n}$.

In what follows, we will assume all $A_\infty$-algebras are augmented over the semiample algebra $\mathbb{K}^n$. The kernel of the augmentation map will be denoted by $A$. 

**Adams graded $A_{\text{fin}}$-algebras.** We will say an Adams graded $A_\infty$-algebra $A_{\text{fin}}$ if for any $j \in G$ in the Adams grading, there is an $r_j \in \mathbb{N}$ such that the $j$-th component of the map $m_n$, i.e.,

$$m_n : \bigoplus_{i_1, \ldots, i_n, j_1 + \ldots + j_n = j} A_{i_1}^{i_1} \otimes \cdots \otimes A_{i_n}^{i_n} \to \bigoplus_i A_i^j$$

is zero for $n > r_j$. Similarly, for two $A_\infty$-algebras Adams graded by the same abelian group $G$, an $A_\infty$-morphism $f : A \to B$ is said to be $A_{\text{fin}}$ if for any $j \in G$, there is an $r_j \in \mathbb{N}$ such that the $j$-th component of the map $f_n$ is zero for $n > r_j$. The notion of $A_{\text{fin}}$-homotopy is similarly defined.

**Taking graded dual followed by cobar construction.** Let $A$ be an Adams graded $A_\infty$-algebra. Taking graded dual of the multiplication maps $m_n$’s, we get linear maps of degree $(2 - n, 0)$

$$m_n^\# : A^\# \longrightarrow (A^{\otimes n})^\# \cong (A^\#)^{\otimes n}.$$

Shifting degree, we define linear maps $b_n$ of degree $(1, 0)$ via the following commutative diagram

$$
\begin{array}{c}
(\Sigma \tilde{A})^\# \xrightarrow{b_n^\#} ((\Sigma \tilde{A})^\#)^{\otimes n} \\
\downarrow \Sigma^\# \\
\tilde{A}^\# \xrightarrow{m_n^\#} (\tilde{A}^\#)^{\otimes n}
\end{array}
$$

Putting them together we get a linear map of degree $(1, 0)$

$$d = \prod b_n^\# : (\Sigma \tilde{A})^\# \longrightarrow \prod ((\Sigma \tilde{A})^\#)^{\otimes n} = \hat{T}_S(\Sigma A)^\#.$$

which extends to a (continuous) derivation

$$d : \hat{T}_S(\Sigma A)^\# \longrightarrow \hat{T}_S(\Sigma A)^\#.$$

By the $A_\infty$-relations of the $m_n$’s, the map $d$ is a differential, i.e., $d^2 = 0$. We thus get a dg-algebra $\Omega(A^\#) = (\hat{T}_S(\Sigma A)^\#, d)$. This construction can also be applied to $A_\infty$-morphisms to get dg-algebra morphisms. Thus, we have set up a functor, which we call the completed Koszul functor,

$$\hat{E} : (\text{locally finite } A_\infty\text{-algebras}) \longrightarrow (\text{quasi-free dg-algebras})^{op}.$$ 

This functor sends $A_\infty$-morphisms to dg-algebra morphisms and $A_\infty$-homotopies to dg-homotopies.
A variant of the Koszul functor. There is a variant of the Koszul functor for $A_{\infty}$-algebras: instead of taking direct sum of the $b_n$’s to get a linear map of degree (1, 0)

\[ d = \bigoplus b_n^\#: (\Sigma A)^\# \longrightarrow \bigoplus((\Sigma A)^{\otimes e_{n+1}})^\# = T_S(\Sigma A)^\#, \]

which extends to a differential

\[ d : T_S(\Sigma A)^\# \longrightarrow T_S(\Sigma A)^#. \]

The component of $d$ which maps $(\Sigma A)^\# \otimes \Sigma A^{\otimes n} \longrightarrow (\Sigma A)^\#$ is given by

\[ \sum_{r+1+t=p \atop r+s+t=q} id^{\otimes r} \otimes b_n^\# \otimes id^{\otimes t}. \]

This set up a functor

\[ E : (\text{locally finite } A_{\infty}-\text{algebras}) \longrightarrow (\text{quasi-free dg-algebras})^{op} \]

which sends $A_{\infty}$-morphisms to dg-algebra morphisms and $A_{\infty}$-homotopies to dg-homotopies. For an $A_{\infty}$-algebra, the difference between $E(A)$ and $E(A)$ is analogous to the difference between formal power series and polynomials. The dg-algebra $E(A)$ will be called the Koszul dual of $A$.

Koszul functor as a construction of dg-quiver. Recall that an $A_{\infty}$-category is said to be finite if $A$ has a finite set of objects and all morphism spaces are finite dimensional. Note that a finite $A_{\infty}$-category is characterised by the property $m_n = 0$ for $n \gg 0$. When the Koszul functor is applied to a finite augmented $A_{\infty}$-category $A$, it can be viewed as a construction which produces a dg-quiver $Q_A$: Obj $(A)$ correspond to vertex set of $Q_A$; degree $i$ edges between two vertices $u$ and $v$ correspond to a basis of the vector space $A^{1-i}(u,v)^\vee$. Then $KQ_A = E(A)$ and $KQ_A = E(A)$ and the dg-structure on $E(A)$ turns $Q_A$ into a dg-quiver. We may sometimes abuse notation and denote $Q_A$ by $E(A)$, i.e., we are identifying a quiver with its path algebra.

Conversely, every dg-quiver can be constructed this way. Given a dg-quiver $Q$, we construct an augmented $A_{\infty}$-category by taking

\[ \text{Obj}(A) = \{\text{vertices in } Q\}, \]

and for any vertices $u, v$, we take the augmentation ideal $\mathcal{A}$ to be

\[ A^i(u, v) = \mathcal{K}\{\text{degree } (1 - i) \text{ edges in } Q\}^\vee, \]

or in other words,

\[ A^i(u, v) = \begin{cases} \mathcal{K}v \oplus \mathcal{K}\{\text{degree } (1 - i) \text{ edges in } Q\}^\vee & \text{if } u = v \text{ and } i = 0, \\ \mathcal{K}\{\text{degree } (1 - i) \text{ edges in } Q\}^\vee & \text{if } u \neq v \text{ and } i = 0, \\ \mathcal{K}\{\text{degree } (1 - i) \text{ edges in } Q\}^\vee & \text{if } i \neq 0. \end{cases} \]

We then define the shifted higher multiplication maps $b_n$ by

\[ b_n(e_1^\vee, \ldots, e_n^\vee) = \sum_{e \text{ edges in } Q} (\text{Coefficient of } e_1 \cdots e_n \text{ in } de)e^\vee, \]

\[ b_n(e_1^\vee, \ldots, v, \ldots, e_n^\vee) = 0 \quad \text{for } n \neq 2 \]

\[ b_2(v, e^\vee) = (-1)^{|v||e^\vee|}b_2(e^\vee, v) = e^\vee \]

where $e_i$ are edges in $Q$ and $v$ are vertices in $Q$. The higher multiplication maps $m_n$ are then given by

\[ m_n(e_1^\vee, \ldots, e_n^\vee) = (-1)^{|e_1^\vee|+\cdots+2|e_{n-1}^\vee|+|e_n^\vee|} \sum_{e \text{ edges in } Q} (\text{Coefficient of } e_1 \cdots e_n \text{ in } de)e^\vee, \]

\[ m_n(e_1^\vee, \ldots, v, \ldots, e_n^\vee) = 0 \quad \text{for } n \neq 2 \]

\[ m_2(v, e^\vee) = m_2(e^\vee, v) = e^\vee. \]

The $A_{\infty}$-relations follows from $d^2 = 0$, and by construction we have $E(A) = KQ$. 

Chapter 3. $A_{\infty}$-Algebras
Bar construction followed by taking graded dual. As mentioned in the beginning of this section, there is another approach to produce quasi-free dg-algebras from $A_\infty$-algebras. Shifting the degree of the multiplication map $m_n$, we get a linear map $b_n$ of degree $(1,0)$ by the following commutative diagram

$$
\xymatrix{
\tilde{A} \otimes_{\mathbb{Z}} \otimes n \ar[r]^{m_n} & \tilde{A} \\
\Sigma \otimes n \ar[r] & \Sigma \\
(\Sigma \tilde{A}) \otimes_{\mathbb{Z}} \otimes n \ar[r]^{b_n} & \Sigma \tilde{A}
}
$$

In other words, $b_n : (\Sigma \tilde{A}) \otimes_{\mathbb{Z}} \otimes n \longrightarrow \Sigma \tilde{A}$ is the map which sends

$$
a_1 \otimes \cdots \otimes a_n \longmapsto (-1)^{(n-1)a_1+\cdots+2a_{n-2}+a_{n-1}}m_n(a_1, \cdots, a_n),
$$

where the sign comes from the Koszul sign rule. Putting them together, we get

$$
b = \bigoplus b_n : T_S(\Sigma \tilde{A}) = \bigoplus (\Sigma \tilde{A}) \otimes_{\mathbb{Z}} \otimes n \longrightarrow \Sigma \tilde{A}
$$

which extends to a coderivation map

$$
b : T_S(\Sigma \tilde{A}) \longrightarrow T_S(\Sigma \tilde{A}).
$$

whose component mapping $(\Sigma \tilde{A}) \otimes_{\mathbb{Z}} q \longrightarrow (\Sigma \tilde{A}) \otimes_{\mathbb{Z}} p$ is given by

$$
\sum_{r+1+t=p} \text{id}^{\otimes r} \otimes b_n \otimes \text{id}^{\otimes t}
$$

By the $A_\infty$-relations of $m_n$’s, this map is a codifferential, i.e., $b^2 = 0$. We will denote this dg-coalgebra by $BA$. Taking graded dual, we get a differential

$$
b^\# : (BA)^\# \longrightarrow (BA)^\#
$$

and a dg-algebra $(BA)^\#$.

Equivalence of the two constructions. Suppose $A$ is $A_\infty$ and $E(A)$ is locally finite. Then since $E(A)^\#_j$ is finite dimensional, the right hand side of

$$
[E(A)]^\#_j = \bigoplus_{n \geq 1} \bigoplus_{i_1+\cdots+i_n = j} ([\Sigma \tilde{A}]^\#_{i_1}) \otimes \cdots \otimes ([\Sigma \tilde{A}]^\#_{i_n})
$$

is a finite sum. Hence

$$
E(A)^\# = \bigoplus_{i,j} \text{Hom}(E(A)^\#_j, K)
$$

$$
= \bigoplus_{i,j} \bigoplus_{n \geq 1} \bigoplus_{i_1+\cdots+i_n = j} \text{Hom}([\Sigma \tilde{A}]^\#_{i_1}) \otimes \cdots \otimes ([\Sigma \tilde{A}]^\#_{i_n}), K)
$$

$$
= \bigoplus_{i,j} \bigoplus_{n \geq 1} \bigoplus_{i_1+\cdots+i_n = j} ([\Sigma \tilde{A}]^\#_{i_1}) \otimes \cdots \otimes ([\Sigma \tilde{A}]^\#_{i_n})
$$

$$
= T_S(\Sigma \tilde{A}) = BA.
$$

Remark 3.4.1. Note that in general $E(A)^\# \not\cong BA$ since for infinite sum, we have $\text{Hom}(\bigoplus V_i, W) = \prod \text{Hom}(V_i, W)$. Hence the assumption $E(A)$ is locally finite is crucial.
Comparing the differential, we arrive at

**Proposition 3.4.2.** Let $A$ be a locally finite $A_{\text{fin}}$-algebra. Suppose $E(A)$ is also locally finite. Then $E(A) = (BA)^\#$.

**Proof.** We have already shown $E(A) = (BA)^\#$ as a vector space. To see the two maps $b^#$ and $d$ are the same, it suffices to compare their components mapping $((\Sigma A)^\#)^{op} \rightarrow ((\Sigma A)^\#)^{op}$ and we are done. □

**Koszul duality.** The following theorem justifies calling $E(A)$ the Koszul dual of $A$.

**Theorem 3.4.3 ([51], Theorem 2.4).** 1. Let $A$ be a locally finite $A_{\text{fin}}$-algebra. Suppose $E(A)$ is locally finite. Then $E(E(A))$ is $A_{\infty}$-quasi-isomorphic to $A$.

2. Let $A$ be a locally finite dg-algebra. Suppose $E(A)$ is also locally finite. Then there is a quasi-isomorphism of dg-algebras $E(E(A)) \rightarrow A$.

**Proof.** Denote by $\Omega$ the cobar construction on dg-algebras. Then by definition, $E(A) = \Omega(A^\#)$ for any locally finite dg-algebras $A$. By Proposition 3.4.2, $E(E(A)) = E((BA)^\#) = \Omega BA$. By [48, Lemma 2.3.4.3], $\Omega BA$ and $A$ are $A_{\infty}$-quasi-isomorphic and we are done.

Now if $A$ is a dg-algebra, we have a quasi-isomorphism of dg-algebras $\Omega BA \rightarrow A$ by [48, Lemma 1.3.2.3]. By Proposition 3.4.2, we are done. □

### 3.5 Quotient and Smash Product

In this section, we will fix a finite group $G$ and an algebraically closed field $K$ with char$(K) \nmid \text{ord}(G)$, and try to construct a quotient $A_{\infty}$-category. The reason for the assumption on the characteristic of $K$ is the following theorem which can be found in almost any standard textbooks on representation theory of finite groups.

**Theorem 3.5.1 (Maschke).** Every finite dimensional representation over $K$ of $G$ is completely reducible, i.e., every $G$-invariant subspace has a $G$-invariant complement.

**Proof.** Let $V$ be a finite dimensional representation and $W$ an invariant subspace. Choose a projection $\pi : V \rightarrow W$. Define a $G$-equivariant map $\pi_G : V \rightarrow W$ by

$$\pi_G(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v).$$

Take $W^\perp = \ker \pi_G$ and we are done. □

The set of all isomorphism classes of irreducible representations of $G$ will be denoted by $\text{Irr}(G)$, and irreducible representations by Greek alphabets $\rho, \sigma, \tau$ etc.

**Quotient construction.** The quotient construction described below is essentially an incarnation of the McKay quiver. Suppose $G$ acts on an $A_{\infty}$-category $A$ by fixing all objects of $A$ and acting on the morphism spaces by strict $A_{\infty}$-isomorphisms. One can construct a quotient $A_{\infty}$-category $A/G$ as follows:

$$\text{Obj}(A/G) = \text{Irr}(G) \times \text{Obj}(A)$$

$$(A/G)^+(\rho \times u, \sigma \times v) = \text{Hom}_G(\rho, A^+(u, v) \otimes \sigma)$$

To define the multiplication maps $m_{n/G}^A$, observe that $A^+(\rho \times u, \sigma \times v) = [\text{Hom}(\rho, \sigma) \otimes A^+(u, v)]^G$. We have natural maps of degree $2 - n$

$$\circ \otimes m_n^A : [\text{Hom}(\rho_{n-1}, \rho_n) \otimes A(v_{n-1}, v_n)] \otimes \cdots \otimes [\text{Hom}(\rho_0, \rho_1) \otimes A(v_0, v_1)] \rightarrow \text{Hom}(\rho_0, \rho_n) \otimes A(v_0, v_n)$$

$$(a_n \otimes a_n) \otimes \cdots \otimes (a_1 \otimes a_1) \mapsto (a_n \circ \cdots \circ a_1) \otimes m_n^A(a_n, \ldots, a_1),$$
which satisfy the $A_\infty$-relations. Since $G$ acts on $A$ through strict $A_\infty$-isomorphisms,

\[
g \cdot (\circ \otimes m_n^A)((\alpha_n \otimes a_n) \otimes \cdots \otimes (\alpha_1 \otimes a_1)) = g \cdot [(\alpha_n \circ \cdots \circ \alpha_1) \otimes m_n^A(a_n, \cdots, a_1)] \\
= [g \cdot (\alpha_n \circ \cdots \circ \alpha_1)] \otimes g \cdot m_n^A(a_n, \cdots, a_1) \\
= [(g \cdot \alpha_n) \circ \cdots \circ (g \cdot \alpha_1)] \otimes m_n^A(g \cdot a_n, \cdots, g \cdot a_1) \\
= (\circ \otimes m_n^A)(g \cdot (\alpha_n \otimes a_n), \cdots, g \cdot (\alpha_1 \otimes a_1)).
\]

This shows $\circ \otimes m_n^A$ descends to the $G$-invariant part to linear maps of degree $2 - n$

\[
m_n^{A/G} : (A/G)(\rho_{n-1} \times v_{n-1}, \rho_n \times v_n) \otimes (A/G)(\rho_0 \times v_0, \rho_1 \times v_1) \to (A/G)(\rho_0 \times v_0, \rho_n \times v_0)
\]
satisfying the $A_\infty$-relations.

Thus we obtained a new $A_\infty$-category $A/G$. Note that the quotient construction preserves most properties of the original $A_\infty$-category: if $A$ is finite dimensional/$A_n/A_{\text{in}}$/unital/augmented/connected, then so is $A/G$. If $A$ has a $G$-invariant cyclic structure of degree $m$, then so does $A/G$: the bilinear forms on $A$

\[
\langle -, - \rangle_A : A(u, v) \otimes A(v, u) \to K
\]

induce bilinear forms

\[
\langle -, - \rangle : [\text{Hom}(\rho, \sigma) \otimes A(u, v)] \otimes [\text{Hom}(\sigma, \rho) \otimes A(v, u)] \to K
\]

\[
(\alpha \otimes a) \otimes (\beta \otimes b) \mapsto \text{tr}(\alpha \beta)(a, b)
\]

which restricts to the $G$-invariant part

\[
\langle -, - \rangle_{A/G} : (A/G)(\rho \times u, \sigma \times v) \otimes (A/G)(\sigma \times v, \rho \times u) \to K.
\]

These bilinear forms are also non-degenerate and cyclically invariant since the trace maps $\text{tr}$ are.

**A variant of the quotient construction.** There is a variant of the quotient construction. Given an $A_\infty$-algebra $A$ over a semisimple algebra $S = K^r$, on which $G$ acts by strict $A_\infty$-isomorphism, one can define an $A_\infty$-algebra over $S \otimes KG$ as follows. The underlying $\mathbb{Z}$-graded $S \otimes KG$-bimodule is given by

\[
\text{Hom}_G(KG, A \otimes KG),
\]

with $S \otimes KG$ acting on the left and right by

\[
((u \otimes g) \varphi(v \otimes h))(-) = u \varphi(-g)(v \otimes h).
\]

The multiplication maps $m_n$ are defined similarly as in $A/G$. Since $\text{End}_G(KG, A \otimes KG) = \bigoplus_{\rho \in \text{Irr}(G)} \rho^{\text{dim } \rho}$, we see that

\[
\text{Hom}_G(KG, A \otimes KG) = \bigoplus_{\rho, \sigma \in \text{Irr}(G)} \text{Hom}_G(\rho, A \otimes \sigma)^{\text{dim } \rho, \text{dim } \sigma},
\]

i.e., this construction is a variant of $A/G$ which takes into account the multiplicity of each irreducible representation in the regular representation of $G$. Note that when $G$ is abelian, the two constructions coincide since every irreducible representation of $G$ are 1-dimensional. In general, the two constructions are related by a Morita functor. Recall that by Maschke’s theorem, $KG$ is a semisimple algebra. In fact, $KG \cong \bigoplus_{\rho \in \text{Irr}(G)} \text{End}_{\rho}$ canonically as an algebra. Denote by $e_\rho$ the matrix in $\text{End}_{\rho}$ with 1 in the

(1,1)-entry and 0 in all other entries. Then $\rho = KG e_\rho$ and all the $e_\rho$’s are orthogonal idempotents, i.e.,

\[
(e_\rho)^2 = e_\rho \text{ and } e_\rho e_\sigma = 0 \text{ if } \rho \neq \sigma.
\]

The element $e = \sum_{\rho \in \text{Irr}(G)} e_\rho \in KG$ is an idempotent element which is full in the sense that $KG e KG = KG$. Moreover, the algebra $S_G := e KG e$ is commutative and is spanned by all the $e_\rho$’s. Since $\check{c} = (1_g \otimes e)$ is also a full idempotent in the semisimple algebra $S \otimes KG$ with

\[
\check{c}(S \otimes KG) \check{c} = S \otimes S_G,
\]

we have the following
Theorem 3.5.2 (Morita Equivalence, [7] Lemma 2.2). The functor
\[ F : \text{Bimod } (S \otimes \mathbb{K}G) \to \text{Bimod } (S \otimes S_G) \]
\[ M \mapsto \tilde{c}M\tilde{c} \]
is an equivalence which commutes with tensor product in the sense that
\[ F(M \otimes N) \cong F(M) \otimes F(N) \]
are naturally isomorphic through the isomorphism
\[ \tilde{c}(m \otimes n)\tilde{c} \mapsto \tilde{c}m\tilde{c} \otimes \tilde{c}n\tilde{c}. \]

Since the Morita functor \( F : \text{Bimod } S \otimes \mathbb{K}G \to \text{Bimod } S \otimes S_G \) commutes with tensor product, the \((S \otimes S_G)-bimodule \( F(\text{Hom}_G(KG, A \otimes KG)) \) has an \( A_\infty \)-structure given by \( F(m_n) \).

Proposition 3.5.3. The \( A_\infty \)-algebras \( F(\text{Hom}_G(KG, A \otimes KG)) \) and \( A/G \) are strictly \( A_\infty \)-isomorphic.

Proof. This follows by observing that
\[ F(\text{Hom}_G(KG, A \otimes KG)) = \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}_G(KG_{\rho, \sigma}, A \otimes KG_{\rho, \sigma}) = \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}(\rho, A \otimes \sigma) = A/G \]
and that the \( m_n \)'s of the two \( A_\infty \)-algebras are defined in the same way. \( \blacksquare \)

Quotient of minimal model. We show that if \( G \) acts on \( A \) by strict \( A_\infty \)-isomorphisms, then it also acts on its minimal model \( H^\bullet(A) \) by strict \( A_\infty \)-isomorphisms and moreover, they give \( A_\infty \)-isomorphic quotient. Recall that in the construction of minimal model (Proposition 3.2.1), one has to choose a splitting
\[ Z = B \oplus H \text{ and } A = Z \oplus L = B \oplus H \oplus L. \] (3.5.1)

Since the group action commutes with \( m_1^A \), the space of cocycles and coboundaries \( Z \) and \( B \) are \( G \)-subrepresentations. If we choose the splitting equivariantly, which is possible by the Maschke Theorem 3.5.1, we get the diagram
\[ \begin{array}{ccc}
A & \xrightarrow{p} & H \\
\downarrow{h} & & \downarrow{h} \\
\end{array} \]
where all maps are equivariant. Since the \( A_\infty \)-structure on \( H^\bullet(A) \) is given by \( m_n^{H^\bullet(A)} = p \circ \lambda_n \circ i^\otimes n \), where \( \lambda_n \) is defined inductively by \( \lambda_2 = m_2^A \) and equation 3.2.1
\[ \lambda_n = \sum_i (-1)^{p(r_1, \ldots, r_k)} m_1^A \circ ((h \circ \lambda_{r_1}) \otimes \cdots \otimes (h \circ \lambda_{r_k})), \]
we see that \( G \) also acts on \( H^\bullet(A) \) by strict \( A_\infty \)-isomorphisms. Hence we can form the quotient \( H^\bullet(A)/G \).

We show that one can choose an \( A_\infty \)-structure on \( H^\bullet(A)/G \) which is strictly \( A_\infty \)-isomorphic to \( H^\bullet(A)/G \).

Recall that \( m_1^{A/G} \) is the restriction of
\[ \text{id} \otimes m_1^A : \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}(\rho, \sigma) \otimes A \to \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}(\rho, \sigma) \otimes A \]
to its \( G \)-invariant part. Hence if we choose the same equivariant splitting (equation 3.5.1), the space of cocycles and coboundaries of \((A/G, m_1^{A/G})\) are
\[ Z/G = \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}_G(\rho, \sigma \otimes Z) \text{ and } B/G = \bigoplus_{\rho, \sigma \in \text{Irr } (G)} \text{Hom}_G(\rho, \sigma \otimes B) \]
respectively. Similarly, if we denote
\[ H/G = \bigoplus_{\rho,\sigma \in \text{irr}(G)} \text{Hom}_G(\rho, \sigma \otimes H) \quad \text{and} \quad L/G = \bigoplus_{\rho,\sigma \in \text{irr}(G)} \text{Hom}_G(\rho, \sigma \otimes L), \]
we have splittings
\[ Z/G = B/G \oplus H/G \quad \text{and} \quad A/G = Z/G \oplus L/G = B/G \oplus H/G \oplus L/G \]
and a diagram
\[
\begin{array}{ccc}
A/G & \xrightarrow{(id \otimes h)^G} & H/G \\
\downarrow & & \downarrow \\
A/G & \xrightarrow{(id \otimes p)^G} & H/G \\
\end{array}
\]
satisfying
\[ \text{id}_{A/G} - (id \otimes i)^G \circ (id \otimes p)^G = m^A_G \circ (id \otimes h)^G + (id \otimes h)^G \circ m^A_G. \]
We then have as in Equation 3.5.1 a family of linear maps \( \tilde{\lambda}_n : (A/G)^{\otimes n} \to A/G \). One can then show by induction that \( \tilde{\lambda}_n \) is the restriction of the linear maps
\[ \circ \otimes \lambda_n : \left( \bigoplus_{\rho,\sigma \in \text{irr}(G)} \text{Hom}(\rho, \sigma) \otimes A \right)^{\otimes n} \to \bigoplus_{\rho,\sigma \in \text{irr}(G)} \text{Hom}(\rho, \sigma) \otimes A \]
to its \( G \)-invariant part. Hence \( m^H_n(\cdot)^G \) is the restriction of \( \circ \otimes m^H_n(\cdot) \), i.e., \( m^H_n(\cdot)^G = m^H_n(\cdot) \). We have thus proved the following

**Proposition 3.5.4.** Let \( A \) be an \( A_{\infty} \)-category on which a finite group \( G \) acts by \( A_{\infty} \)-isomorphisms. Then one can choose a minimal model \( H^*(A) \) of \( A \) on which \( G \) acts by \( A_{\infty} \)-isomorphisms, and a minimal model \( H^*(A/G) \) of \( A/G \), such that there is an \( A_{\infty} \)-isomorphism \( H^*(A/G) \cong H^*(A/G) \). 

**Smash product.** Suppose \( G \) acts on an \( A_{\infty} \)-algebra \( A \) over \( S = K^r \) by strict \( A_{\infty} \)-isomorphisms. One can construct the smash product \( A_{\infty} \)-algebra \( A \# G \) as follows. As a vector space, \( A \# G = A \otimes K^G \). There is a \( (S \otimes K^G) \)-bimodule structure on \( A \otimes K^G \) by
\[
(u \otimes g)(a \otimes x)(v \otimes h) = u(ga)v \otimes g(xh),
\]
for any \( u, v \in S_A, g, h \in G \) and \( x \in K^G \). The multiplication maps \( m^A_{\# G} \) are defined for, by, \( g_i \in G, \)
\[
m^A_{\# G}(a_1 \otimes g_1, \ldots, a_n \otimes g_n) = m^A(a_1, g_1a_2, g_1g_2a_3, \ldots, g_1 \cdots g_n-a_n) \otimes g_1 \cdots g_n.
\]
Using the isomorphism
\[
(A \otimes K^G)^{\otimes n} \cong A^{\otimes n} \otimes K^G
\]
\[
(a_1 \otimes g_1) \cdots (a_n \otimes g_n) \mapsto (a_1 \otimes g_1a_2 \otimes g_2 \cdots g_n-a_n) \otimes g_1 \cdots g_n,
\]
we have the commutative diagram
\[
\begin{array}{ccc}
(A \otimes K^G)^{\otimes n} & \xrightarrow{m^A_{\# G}} & A^{\otimes n} \otimes K^G \\
\downarrow & & \downarrow \\
A \otimes K^G & \xrightarrow{m^A_{\# G} \circ \text{id}_G} & A \otimes K^G \\
\end{array}
\]
Again, the smash product preserves most properties of the original \( A_{\infty} \)-algebra: if \( A \) is finite dimensional/\( A_{\infty} \)/unital/augmented/connected/cyclic of degree \( m \), then so is \( A \# G \).

**Proposition 3.5.5.** The two dg-algebras \( E(A\# G) \) and \( E(A)\# G \) are isomorphic. The two algebras \( H^*(E(A\# G)) \) and \( H^*(E(A))\# G \) are isomorphic.

**Proof.** The first claim comes from the commutative diagram (3.5.2). Taking cohomology of the first claim, we have \( H^*(E(A\# G)) = H^*(E(A))\# G \). Since the functor \((-) \otimes K^G\) is exact, and hence preserves cohomologies, we have \( H^*(E(A)\# G) = H^*(E(A))\# G \) and the second claim follows.
Relation between Quotient and Smash Product. The following proposition relates the smash product and the quotient construction with multiplicities.

**Proposition 3.5.6.** The \( A_{\infty} \)-algebras \( \text{Hom}_G(KG, A \otimes KG) \) and \( (A^{op} \# G)^{op} \) are strictly \( A_{\infty} \)-isomorphic.

**Proof.** Observe that \( \text{Hom}_G(KG, A \otimes KG) \cong A \otimes KG \) as an \( S_A \otimes KG \)-bimodule via the map \( \varphi \mapsto \varphi(1) \), with inverse given by sending \( a \otimes g \in A \otimes KG \) to the map \( (h \mapsto ha \otimes hg) \) for all \( h \in G \). We wish to write down the induced \( A_{\infty} \)-structure on \( A \otimes KG \) under this isomorphism. To do this, we first write down explicitly the map \( h \mapsto ha \otimes hg \in \text{Hom}_G(KG, A \otimes KG) \) as an element in \([A \otimes \text{Hom}(KG, KG)]^G\). Define the \( \mathbb{K} \)-linear map \( \varphi_k^g : KG \to KG \) by

\[
\varphi_k^g(h) = \begin{cases} 
kg & \text{if } h = k \\
0 & \text{otherwise.}
\end{cases}
\]

Then the map \( h \mapsto ha \otimes hg \) correspond to the element \( \sum_{k \in G} ka \otimes \varphi_k^g \). We will need the following simple lemma for computation.

**Lemma 3.5.7.**

1. The group \( G \) acts on \( \varphi_k^g \) by \( \ell \cdot \varphi_k^g = \varphi_{\ell k}^g \).

2. Compositions are given by

\[
\varphi_k^g \circ \varphi_n^h = \begin{cases} 
\varphi_k^{ng} & \text{if } k = \ell h \\
0 & \text{otherwise.}
\end{cases}
\]

Now we can compute the induced \( m_n^{A \otimes KG} \) on \( A \otimes KG \). Since

\[
\sum_{k_1, \ldots, k_n \in G} m_n(k_n a_n \otimes \varphi_{k_n}^g, \ldots, k_1 a_1 \otimes \varphi_{k_1}^g) = \sum_{k_1, \ldots, k_n \in G} m_n(k_n a_n, \ldots, k_1 a_1) \otimes \varphi_{k_n}^g \circ \cdots \circ \varphi_{k_1}^g
\]

\[
= \sum_{k_1 \in G} m_n^A(k_1 g_1 \cdots g_{n-1} a_n, \ldots, k_1 a_1) \otimes \varphi_{g_n}^{g \cdots g_n}
\]

\[
= \sum_{k_1 \in G} k_1 m_n^A(g_1 \cdots g_{n-1} a_n, \ldots, a_1) \otimes \varphi_{g_n}^{g \cdots g_n},
\]

we conclude that

\[
m_n^{A \otimes KG}(a_n \otimes g_n, \ldots, a_1 \otimes g_1) = m_n^A(g_1 \cdots g_{n-1} a_n, \ldots, a_1) \otimes (g_1 \cdots g_n).
\]

Now observe that \( (A^{op} \# G)^{op} \) has the same objects and morphism spaces as \( A \otimes KG \), with \( A_{\infty} \)-structure given by

\[
m_n^{(A^{op} \# G)^{op}}(a_n \otimes g_n, \ldots, a_1 \otimes g_1) = m_n^{A^{op} \# G}(a_1 \otimes g_1, \ldots, a_n \otimes g_n)
\]

\[
= m_n^{A^{op}}(a_1, g_1 a_2, \ldots, g_1 \cdots g_{n-1} a_n) \otimes (g_1 \cdots g_n)
\]

\[
= m_n^A(g_1 \cdots g_{n-1} a_n, \ldots, a_1) \otimes (g_1 \cdots g_n)
\]

which is the same as that of \( A \otimes \mathbb{K}G \), as desired. ■

**Corollary 3.5.8.** The \( A_{\infty} \)-algebras \( A/G \) and \( F((A^{op} \# G)^{op}) \) are strictly \( A_{\infty} \)-isomorphic.

**Proof.** This is immediate since \( A/G \cong F(\text{Hom}_G(KG, A \otimes KG)) \) by Proposition 3.5.3. ■

**Proposition 3.5.9.** The two dg-algebras \( E(A/G) \) and \( E(A)/G \) are isomorphic. The two \( A_{\infty} \)-algebras \( H^*(E(A/G)) \) and \( H^*(E(A))/G \) are \( A_{\infty} \)-quasi-isomorphic.

**Proof.** By Corollary 3.5.8, \( E(A/G) \cong E(F((A^{op} \# G)^{op})) \). Since the Morita equivalence functor \( F \) in Theorem 3.5.2 commutes with tensor product, it also commutes with the Koszul functor \( E \), and we have \( E(F((A^{op} \# G)^{op})) \cong F(E(A^{op} \# G)^{op}) \). By Proposition 3.5.5, \( E(A^{op} \# G)^{op} \cong (E(A)^{op} \# G)^{op} \). By Corollary 3.5.8 again, \( F((E(A)^{op} \# G)^{op}) \cong E(A)/G \) and we arrive at the first claim. Taking cohomology, we conclude \( H^*(E(A/G)) \) and \( H^*(E(A))/G \) are \( A_{\infty} \)-quasi-isomorphic. By Proposition 3.5.4, \( H^*(E(A/G)) \) and \( H^*(E(A))/G \) are \( A_{\infty} \)-quasi-isomorphic and we are done. ■
3.6 Tensor Product

In this section, we give a review on how to construct tensor product of $A_{\infty}$-algebras. Except Proposition 3.6.1, this is essentially a summary of Amorim and Tu [2].

Tensor product of a dg-algebra with an $A_{\infty}$-algebra

Before looking at the general case, let us look at the special case where one of them is only a dg-algebra rather than a full-fledged $A_{\infty}$-algebra. When both of them are dg-algebras, this is well-known and we take

$$m_1^{A \otimes A'} = m_1^A \otimes id_{A'} + id_A \otimes m_1^{A'} \quad \text{and} \quad m_2^{A \otimes A'} = m_2^A \otimes m_2^{A'}$$  (3.6.1)

If $A$ is a dg-algebra and $A'$ is an $A_{\infty}$-algebra, there is a natural generalization:

$$m_1^{A \otimes A'} = m_1^A \otimes id_A + id_A \otimes m_1^{A'} \quad \text{and for } n \geq 2 \quad m_n^{A \otimes A'} = m_n^A \circ (m_2^A \circ (m_2^A \circ \cdots)) \otimes m_n^{A'}.$$  (3.6.2)

For simplicity, we will denote the $n$-th iteration of $m_n^A$ by $(m_n^A)^{\circ n}$. Direct computation using chain rule on $A$ and the $A_{\infty}$-relations on $A'$ shows the above definition of $m_n^{A \otimes A'}$ satisfies the $A_{\infty}$-relations:

$$\sum_{a+b+c=n} (-1)^{a+b+c} m_{a+b+c}^{A \otimes A'} \circ \left( \right) = m_1^{A \otimes A'} \circ m_n^{A \otimes A'} + \sum_{a+b+c=n} (-1)^{a+b+c} m_{a+b+c}^{A \otimes A'} \circ \left( \right) + \sum_{a+b+c=n} (-1)^{a+b+c} m_{a+b+c}^{A \otimes A'} \circ \left( \right) + \sum_{a+b+c=n} (-1)^{a+b+c} m_{a+b+c}^{A \otimes A'} \circ \left( \right) = 0.$$

Tensor product of $A_{\infty}$-algebras

For general $A_{\infty}$-algebras, there is no natural way to define tensor product. There are various ways to construct an $A_{\infty}$-structure on $A \otimes A'$ which are $A_{\infty}$-quasi-isomorphic to each other, but in general not strictly $A_{\infty}$-quasi-isomorphic. In the following, we describe some of the constructions. Saneblidze and Umble [61] were the first to construct an $A_{\infty}$-structure on $A \otimes A'$ whose $m_n^{A \otimes A'}$ are given by a closed formula in terms of $m_j^A$ and $m_j^{A'}$ where $j \leq n$. Markl and Shnider [55] later reformulated their construction as a diagonal map on the $A_{\infty}$-operad $A_{\infty}$: Given a chain complex $(A, d)$, there is an associated operad $End_A$. Any $A_{\infty}$-structure $m_n^A$ on $A$ with $m_1^A = d$ can then be described as an operad homomorphism $\rho : A_{\infty} \to End_A$. Now, given any two chain complexes $A$ and $A'$, there is a natural map of operads $End_A \otimes End_{A'} \to End_{A \otimes A'}$. The problem of constructing an $A_{\infty}$-structure on $A \otimes A'$ then becomes the problem of constructing a “canonical” diagonal $\Delta : A_{\infty} \to A_{\infty} \otimes A_{\infty}$. For if such a diagonal exists, one can simply take the composition

$$\rho_{A \otimes A'} : A_{\infty} \xrightarrow{\Delta} A_{\infty} \otimes A_{\infty} \xrightarrow{\rho_A \otimes \rho_{A'}} End_A \otimes End_{A'} \longrightarrow End_{A \otimes A'}$$

to give an $A_{\infty}$-structure on $A \otimes A'$. Amorim [1] also reformulated the Saneblidze-Umble construction in terms of dg-algebras: For every $A_{\infty}$-algebra $A$, one can construct a dg-algebra $Hom(A, A)$ whose space of cycles are the $A_{\infty}$-endomorphisms of $A$ and whose homology are $A_{\infty}$-endomorphisms of $A$ up to $A_{\infty}$-homotopy. This dg-algebra $Hom(A, A)$ is $A_{\infty}$-quasi-isomorphic to $A$. One then forms the tensor product dg-algebra $Hom(A, A) \otimes Hom(A', A')$ and uses homology perturbation to transfer the dg-structure on $Hom(A, A) \otimes Hom(A', A')$ to an $A_{\infty}$-structure on $A \otimes A'$. All $A_{\infty}$-structure on tensor products defined above are $A_{\infty}$-quasi-isomorphic.
Tensor product for Adams-graded $A_\infty$-algebras. In the case of Adams-graded $A_\infty$-algebras, the tensor product $A_\infty$-structure also preserves the tensor product Adams-grading on $A \otimes A'$ as $m_n^{A \otimes A'}$ are given by a closed formula in terms of $m_j^A$ and $m_j^{A'}$ where $j \leq n$.

Cyclic structure. Given two cyclic $A_\infty$-algebras $A$ and $A'$, there is a natural inner product on $A \otimes A'$ defined by

$$\langle a_1 \otimes a_1, a_2 \otimes a_2 \rangle_{A \otimes A'} = (-1)^{|a_1||a_2|} \langle a_1, a_2 \rangle_A \langle a_1, a_2 \rangle_{A'}.$$  \hspace{1cm} (3.6.3)

One might ask whether the tensor product constructions described above are cyclic with respect to this natural inner product. In the special case where one of the $A_\infty$-algebra is a dg-algebra, it can be checked directly that the natural tensor product structure given by Equation 3.6.2 is cyclic. However, for full-fledged $A_\infty$-algebras, all the tensor product constructions described above do not preserve cyclicity in general. As Tradler pointed out in [67], $m_3^{A \otimes A'}$ in the Saneblidze-Umbre construction is already not cyclic. The first construction of tensor product of cyclic $A_\infty$-algebras which respect the natural inner product as in Equation 3.6.3 seems to be given by Amorim and Tu in [2]. There, they constructed a cyclic diagonal on the $A_\infty$-operad and showed that any tensor product $A_\infty$-structure defined by a cyclic diagonal are cyclically $A_\infty$-quasi-isomorphic to each other. As the $A_\infty$-structure is constructed by a diagonal, the $m_{n}^{A \otimes A'}$ can in principle be written as a sum of tensor products and compositions of $m_j^A$ and $m_j^{A'}$ for $j \leq n$. However, Amorim and Tu only gave explicit formulae for $m_n^{A \otimes A'}$ up to $n = 4$ and stated that the general formulae appeared to be a very complicated combinatorial problem.

Convention. As all $A_\infty$-structures on tensor products we described above are $A_\infty$-quasi-isomorphic, in principle it makes no difference to which one uses. Hence we will not distinguish them and only write $A \otimes A'$ to denote $A_\infty$-tensor product. However, the two constructions given by Amorim and Tu [2] and Amorim [1] would be the most important in this thesis as the first one preserves cyclic structures and the second one reduces tensor product of $A_\infty$-algebras to tensor product on dg-algebras.

Koszul functor and tensor product. The following proposition essentially says that under some locally finite conditions, the Koszul functor commutes with the $A_\infty$-tensor product. Recall the bar construction functor $B$ sending $A_\infty$-algebras to dg-coalgebras defined in Section 3.4.

Proposition 3.6.1. 1. Let $A$ and $A'$ be two $A_\infty$-algebras. Then there is a quasi-isomorphism of dg-algebras

$$B(A \otimes A')^# \rightarrow B(A)^# \otimes B(A')^#$$

and an $A_\infty$-quasi-isomorphism of $A_\infty$-algebras

$$H^*(B(A \otimes A')^#) \cong H^*((BA)^#) \otimes H^*((BA')^#).$$

2. Let $A$ and $A'$ be locally finite Adams graded $A_{fin}$-algebras. Suppose one of the $A_\infty$-tensor product structure on $A \otimes A'$ is $A_{fin}$ and that $E(A)$, $E(A')$ and $E(A \otimes A')$ are all locally finite. Then there is a quasi-isomorphism of dg-algebras

$$E(A \otimes A') \cong E(A) \otimes E(A')$$

and an $A_\infty$-quasi-isomorphism of $A_\infty$-algebras

$$H^*(E(A \otimes A')) \cong H^*(E(A)) \otimes H^*(E(A')).$$

Proof. Using Amorim’s version of $A_\infty$-tensor product which reduces $A_\infty$-tensor product to dg tensor product in [1], it suffices to prove the proposition when both $A$ and $A'$ are dg-algebras. Both

$$B(A) \otimes A \otimes B(A') \otimes A' \rightarrow S_A \otimes S_{A'}$$

and

$$B(A \otimes A') \otimes A \otimes A' \rightarrow S_A \otimes S_{A'}$$

are semi-free resolution of $S_A \otimes S_{A'}$. Hence there is a chain homotopy equivalence between

$$B(A \otimes A') \otimes A \otimes A' \xrightarrow{f} B(A) \otimes A \otimes B(A') \otimes A'.$$
Now,
\[ B(A \otimes A') = B(A \otimes A') \otimes (A \otimes A')_{\otimes A'}(S_A \otimes S'_A) \]
and similarly
\[ B(A) \otimes B(A') = (B(A) \otimes A \otimes B(A') \otimes A')_{\otimes A'}(S_A \otimes S'_A). \]
Hence we have a chain homotopy equivalence
\[ B(A \otimes A') \xrightarrow{f \otimes \text{id}_{A \otimes A'}} B(A) \otimes B(A'). \]
By MacLane [53, Chapter X, Theorem 12.2], the map \( g \otimes \text{id}_{A \otimes A'} \) is given by inclusion maps followed by the shuffle product on \( B(A \otimes A') \)
\[ B(A) \otimes B(A') \rightarrow B(A \otimes A') \otimes B(A \otimes A') \xrightarrow{sh} B(A \otimes A') \]
\[ a \otimes a' \mapsto (a \otimes 1) \otimes (1 \otimes a') \mapsto (a \otimes 1) \ast (1 \otimes a') \]
which is a coalgebra map. Taking dual, we obtain a dg-algebra morphism
\[ \phi : B(A \otimes A')^\# \rightarrow B(A)^\# \otimes B(A')^\# \]
which is a quasi-isomorphism. By homology perturbation, we have the following chain of \( A_{\infty} \)-quasi-isomorphisms
\[ H^\bullet(B(A \otimes A')^\#) \cong B(A \otimes A')^\# \cong BA^\# \otimes BA'^\# \cong H^\bullet(BA^\#) \otimes H^\bullet(BA'^\#). \]
The second statement follows from the observation that under the assumptions, we have \( E(A) = (BA)^\# \), \( E(A') = (BA)^\# \) and \( E(A \otimes A') = (B(A \otimes A'))^\# \) as dg-algebras by Proposition 3.4.2. \( \blacksquare \)
Chapter 4

Quivers with Superpotential

This chapter is devoted to the study of quivers with superpotential.

In section 4.1, we define quivers with superpotentials. Our definition of quivers with superpotential is taken from van den Bergh [70], where the completed path algebra of a quiver with superpotential is known as a deformed DG-preprojective algebra there.

Section 4.2 gives a correspondence between quivers with superpotential and the Koszul dual of $A_n$-categories with cyclic structures. Using this correspondence, we define the notion of product of quivers with superpotential and the notion of quotient of quivers with superpotential by finite groups.

In section 4.3, we follow van den Bergh [71] and prove that the path algebras of quivers with superpotential are Calabi–Yau algebras, and hence the categories of representations of quivers with superpotential are also Calabi–Yau.

Finally, Section 4.4 describes quivers with superpotential of dimensions 1 to 4. In particular, we describe in dimension 3 how our definition of quivers with superpotential as dg-quivers is connected to the old definition of quivers with superpotential as quivers with relations given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22].

4.1 Quivers with Superpotential

In this section, we introduce the notion of quiver with superpotential. The presentation here essentially follows van den Bergh [70, 71].

Notations and Conventions. We will fix a field $\mathbb{K}$ of characteristic zero. By a cycle in a quiver $Q$, we will mean a closed path forgetting the starting and ending points, or more precisely, a closed path as an element in the vector space $\mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$. For simplicity, we will write $\mathbb{K}Q_{\text{cyc}} = \mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$. For any edge $e$ in a graded quiver $Q$, we will denote by $\partial_e : \mathbb{K}Q \to \mathbb{K}Q \otimes \mathbb{K}Q$ the double derivation of degree $|e|$ acting on any edge $f$ in $Q$ by

$$\partial_e f = \begin{cases} h(e) \otimes t(e) & \text{if } e = f \\ 0 & \text{otherwise.} \end{cases}$$

We will also denote by $\partial^e : \mathbb{K}Q \to \mathbb{K}Q$ the derivation $m \circ \sigma \circ \partial_e$, where $m$ is the multiplication map in $\mathbb{K}Q$ and $\sigma : \mathbb{K}Q \otimes \mathbb{K}Q$ is the interchange operator sending $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$. This derivation vanishes on commutators and hence descends to a derivation $\partial^e : \mathbb{K}Q_{\text{cyc}} \to \mathbb{K}Q$. For other notions in noncommutative calculus such as noncommutative symplectic form and double Poisson bracket, please refer to Chapter 1.
Construction of Quivers with Superpotential. Let \( \tilde{Q} \) be a graded quiver with degree of all edges lying in the interval \([2 - m, 0]\). Suppose \( \tilde{Q} \) is equipped with a pairing \( (-, -) \) of degree \( 2 - m \) on the set of edges in \( \tilde{Q} \) such that

- \((a, b) = -(1)^{|a||b|}(b, a)\)
- \((a, b) = 0\) unless \(t(a) = h(b)\) and \(t(b) = h(a)\)
- the matrix \(\{(a, b)\}\) is invertible.

Let \(\langle -, -\rangle\) denote the dual quadratic form, i.e., the pairing on the dual space of the set of edges in \(\tilde{Q}\) such that the matrix \(\{\langle a^\vee, b^\vee\rangle\}\) is given by the inverse of \(\{(a, b)\}\). Then there is a noncommutative symplectic 2-form of degree \(2 - m\)

\[
\omega = \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle d_{dR}xd_{dR}y.
\]

Proposition 4.1.1. The 2-form \(\omega\) is a noncommutative symplectic form.

Proof. We show the map \(\iota_\omega : \text{Der}_S(K\tilde{Q}_A) \to \Omega^1_S(K\tilde{Q}_A)\) defined by \(\Theta \mapsto \iota_\Theta \omega\) is an isomorphism by exhibiting an inverse. For any edge \(a \in \tilde{Q}_A\),

\[
\iota_\omega(\partial_a) = \iota_{\partial_a} \omega = \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}_A} \langle x^\vee, y^\vee\rangle \left( (\partial_a^x x)d_{dR}y + (-1)^{1 + |a||x|}d_{dR}x(\partial_a^y y) \right)
= \sum_{y \text{ edges in } \tilde{Q}_A} \langle a^\vee, y^\vee\rangle d_{dR}y.
\]

By direct calculation, one can see that the inverse of \(i_\omega\) is given by

\[
d_{dR}a \mapsto \sum_{b \text{ edges in } \tilde{Q}_A} (a, b)\partial_b.
\]

The symplectic form \(\omega\) induces a Poisson double bracket \(\{\{\cdot, \cdot\}\} : K\tilde{Q} \otimes K\tilde{Q} \to K\tilde{Q} \otimes K\tilde{Q}\).

Proposition 4.1.2. For any edge \(a \in \tilde{Q}\), we have the formula

\[
\{\{a, -\}\} = \sum_{b \text{ edges in } \tilde{Q}} (a, b)\partial_b.
\]

Proof. By definition of the double bracket, this follows from

\[
\sum_{b \text{ edges in } \tilde{Q}} (a, b)\iota_{\partial_b} \omega
= \sum_{b \text{ edges in } \tilde{Q}} \sum_{x, y \text{ edges in } \tilde{Q}_A} \frac{1}{2} (a, b)\langle x^\vee, y^\vee\rangle \left( (\partial_a^x x)d_{dR}y + (-1)^{1 + |b||x|}d_{dR}x(\partial_a^y y) \right)
= \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}_A} (a, x)\langle x^\vee, y^\vee\rangle d_{dR}y + (a, y)\langle y^\vee, x^\vee\rangle d_{dR}x
= \sum_{x, y \text{ edges in } \tilde{Q}} (a, x)\langle x^\vee, y^\vee\rangle d_{dR}y
= d_{dR}a.
\]
The double Poisson bracket induces the Kontsevich bracket (c.f. Section 1) \((-\cdot,-\cdot) : \mathbb{K}Q \otimes \mathbb{K}Q_{cyc} \to \mathbb{K}Q\), which descends to a Lie bracket \((-\cdot,-\cdot) : \mathbb{K}Q_{cyc} \otimes \mathbb{K}Q_{cyc} \to \mathbb{K}Q_{cyc}\). Let \(\Phi \in \mathbb{K}Q_{cyc}\) be a sum of cycles of degree \(3 - m\) which satisfies the master equation \(\{\Phi, \Phi\} = 0\) in \(\mathbb{K}Q_{cyc}\). Then one can define \(d : \mathbb{K}Q \to \mathbb{K}Q\) by \(d = \{\Phi, -\}\).

**Proposition 4.1.3.** The linear map \(d\) is a differential of degree 1. For any edge \(a\) in \(Q\), we have
\[
da = (-1)^{1+a+m} \sum_{b \text{ edges in } Q} (a, b) \partial^a_b \Phi.
\]

**Proof.** Since \(\Phi\) is of degree 3 and \((-\cdot,-\cdot)\) is of degree \(m - 2\), \(d\) is a map of degree 1. Now, by Jacobi identity,
\[
d^2 (a) = \{\Phi, \{\Phi, a\}\} = \frac{1}{2} \{\{\Phi, \Phi\}, a\} = 0.
\]

Hence \(d\) is a differential. For any edge \(a\), we have
\[
\{\Phi, a\} = m \circ \{\Phi, a\} = (-1)^{(1+|a|+2-m)(|\Phi|+2-m)} m \circ a \circ a, \Phi = (-1)^{1+a+m} \sum_{b \text{ edges in } Q} (a, b) \partial^a_b \Phi
\]
where the last equality is by Proposition 4.1.2. 

This differential \(d\) on \(\mathbb{K}Q\) is compatible with the double Poisson bracket.

**Proposition 4.1.4.** The double Poisson bracket \(\{-\cdot, -\cdot\}\) is a dg-double Poisson bracket of degree \(m - 2\), i.e.,
\[
d\{-a, b\} = \{da, b\} + (-1)^{|a|+m-2} \{a, db\}.
\]

**Proof.** The proposition follows readily from Proposition 1.2.19 and the definition \(d = \{\Phi, -\}\). 

Let \(Q\) be the quiver constructed from \(\tilde{Q}\) by adding to each vertex \(v\) in \(Q\) a loop \(v^*\) of degree \(1 - m\). Define a differential on \(\mathbb{K}Q\) by
\[
d(v^*) = v \ell v \text{ for any vertex } v, \text{ where } \ell = \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle xy,
\]
\[
d(a) = \{a, \Phi\} \text{ for any edge } a \text{ with degree } a \leq 2 - m.
\]

The following lemma shows that we have indeed defined a differential on \(\mathbb{K}Q\).

**Lemma 4.1.5.** \(d^2(v^*) = 0\).

**Proof.** It suffices to show \(d \ell = 0\), for then \(d^2(v^*) = d(v \ell v) = v(d\ell)v = 0\). Using Proposition 4.1.3,
\[
d \ell = \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle ((dx)y + (-1)^{|x|} xdy)
\]
\[
= \sum_{b,x,y \text{ edges in } \tilde{Q}} (-1)^{1+|x||y|+1+m+|x|} \langle y^\vee, x^\vee \rangle (x, b) (\partial^x_b \Phi) y + (-1)^{|x|+1+m+|y|} \langle x^\vee, y^\vee \rangle (y, b) x (\partial^y_b \Phi)
\]
\[
= \sum_{y \text{ edges in } \tilde{Q}} (-1)^{(3-m-|y|)/|y|} (\partial^x_y \Phi) y - \sum_{x \text{ edges in } \tilde{Q}} x (\partial^x_y \Phi)
\]
\[
= \text{(sum of all cyclic permutations of } \Phi\text{)} - \text{(sum of all cyclic permutations of } \Phi\text{)}
\]
\[
= 0.
\]

The dg-quiver \(Q\) built in this way is called a quiver with superpotential \(\Phi\) of dimension \(m\).
Definition 4.1.6. A quiver with superpotential of dimension $m$ is a dg-quiver $Q$ together with an element $\Phi \in KQ_{\text{cyc}}$ of degree $3 - m$ in the following form:

1. The degrees of all the edges of $Q$ lie in the interval $[1 - m, 0]$, i.e.,
   
   $E^i_Q = \emptyset$ unless $1 - m \leq i \leq 0$.

2. For each vertex $v$ there is exactly 1 loop $v^*$ of degree $1 - m$ at the vertex $v$, and there are no other edges of degree $1 - m$, i.e.,
   
   $E^{1-m}_Q = \bigcup_{v \in V_Q} E^{1-m}_Q(v, v)$ with $E^{1-m}_Q(v, v) = \{v^*\}$.

3. For any $i \in [2 - m, 0]$, there is a pairing
   
   $(-, -) : E^i_Q \times E^{2-m-i}_Q \to K$

   between edges of degree $i$ and edges of degree $2 - m - i$ such that
   
   \begin{itemize}
   \item $(a, b) = -(-1)^{|a||b|}(b, a)$
   \item $(a, b) = 0$ unless $t(a) = h(b)$ and $t(b) = h(a)$
   \item the matrix $\{(a, b)\}$ is invertible.
   \end{itemize}

4. For any edge $a$ with $\deg a \in [2 - m, 0]$, the differential is given by
   
   $da = \{\Phi, a\}$.

5. For any degree $1 - m$ loop $v^*$,
   
   $dv^* = v \left( \sum_{x, y \text{ edges in } Q \text{ with } \deg x, \deg y \in [2-m,0]} \langle x^\vee, y^\vee \rangle xy \right) v$,

   where $\langle - , - \rangle$ is the dual pairing of $(-, -)$.

Remark 4.1.7. Note that the above definition automatically implies that the superpotential $\Phi$ satisfies the master equation $\{\Phi, \Phi\} = 0$ in $KQ_{\text{cyc}}$ as $d^2(a) = \{\Phi, \{\Phi, a\}\} = \frac{1}{2} \{\Phi, \Phi\}, a = 0$ implies $d^2\{\Phi, \Phi\} = 0$ for all edges $a$ in $Q$. Hence the data of the subquiver $\tilde{Q}$ together with the antisymmetric pairing $(-, -)$ and the superpotential $\Phi$ satisfying the master equation $\{\Phi, \Phi\} = 0$ in $KQ_{\text{cyc}}$ uniquely determine the quiver with superpotential $Q$ by the above construction.

4.2 Characterisation of Quivers with Superpotential

In this section, we would like to characterize quivers with superpotential in terms of $A_{\text{fin}}$-algebras. Recall from Section 3 that for any dg-quiver $Q$, its path algebra $\mathbb{K}Q$ is isomorphic to $E(A)$ for a unique (up to strict $A_{\infty}$-isomorphism) $A_{\text{fin}}$-algebra $A$. We would like to show that for a quiver with superpotential $Q$, the corresponding $A_{\text{fin}}$-algebra is equipped with a cyclic structure. Conversely, given any $A_{\text{fin}}$-algebra with cyclic structure, its Koszul dual $E(A)$ is the path algebra of a quiver with superpotential.

All $A_{\text{fin}}$-algebras in this section will be assumed to be augmented, finite (it has finite number of objects and finite dimensional morphism space) and positively graded ($A^i = 0$ for $i \leq 0$, where $A$ is the kernel of the augmentation map).

Let $Q$ be a quiver with superpotential. Recall the $A_{\text{fin}}$-category $\mathcal{A}$ such that $\mathbb{K}Q = E(A)$ is constructed by

$\text{Obj}(\mathcal{A}) = \{\text{vertices in } Q\}$.
Then for a \( \langle \Phi \rangle = \sum \text{all cyclic permutations of } \Phi \). Similarly, a positively graded structure of degree \( m \) is denoted by \( \bar{A}^i \). A \( - \)-category \( A \) is said to be finite if \( \text{Obj}(A) \) is finite and \( A(u,v) \) is finite dimensional for all \( u,v \in \text{Obj}(A) \). Also, an augmented \( A_{\infty} \)-category \( \bar{A} \) is said to be positively graded if \( \bar{A}^i = 0 \) for all \( i \leq 0 \). Summarizing, we have

\[
\langle a^\vee, b^\vee \rangle_{\bar{A}} = (-1)^{|a^\vee|} \langle a^\vee, b^\vee \rangle.
\]

Then for \( a_0, \ldots, a_n \) edges in \( Q \),

\[
\langle m_{n}(a_n^\vee, \ldots, a^\vee_0), a_0^\vee \rangle_{\bar{A}} = (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_i^\vee|+|a_j^\vee|+1+|a|+m}(a, b)(\text{Coefficient of } a_n \cdots a_1 \text{ in } \partial^i \Phi)(a^\vee, a^\vee_0)
\]

where \( \Phi = \sum \text{all cyclic permutations of } \Phi \). Similarly,

\[
\langle m_{n-1}(a_n^\vee, \ldots, a^\vee_0), a^\vee_0 \rangle_{\bar{A}} = (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_i^\vee|+|a|+m(1)(|a|+1)}(\text{Coefficient of } a_n \cdots a_1 \text{ in } \partial^i \Phi)(a^\vee, a^\vee_0)
\]

Since \( \Phi \) has degree \( 3 - m \), we have \( \langle m_{n}(a_n^\vee, \ldots, a^\vee_0), a_0^\vee \rangle_{\bar{A}} = \langle m_{n-1}(a_n^\vee, \ldots, a^\vee_0), a^\vee_0 \rangle_{\bar{A}} = 0 \) unless \( |a_n| + \cdots + |a_0| = 3 - m \). If \( |a_n| + \cdots + |a_0| = 3 - m \), using \( |a^\vee| = 1 - |a| \), we can verify that

\[
\langle m_{n}(a_n^\vee, \ldots, a^\vee_0), a_0^\vee \rangle_{\bar{A}} = (-1)^{n+|a^\vee_0|(|a_n^\vee|+\cdots+|a^\vee_0|)}(m_{n-1}(a_n^\vee, \ldots, a^\vee_0), a^\vee_0)_{\bar{A}}.
\]

Hence \( \langle \cdot, \cdot \rangle_{\bar{A}} \) is a cyclic structure. Recall that an \( A_{\infty} \)-category is said to be finite if \( \text{Obj}(A) \) is finite and \( A(u,v) \) is finite dimensional for all \( u,v \in \text{Obj}(A) \). Also, an augmented \( A_{\infty} \)-category \( \bar{A} \) is said to be positively graded if \( \bar{A}^i = 0 \) for all \( i \leq 0 \). Summarizing, we have

**Proposition 4.2.1.** Let \( Q \) be a dg-quiver. Then there exists a finite and positively graded \( A_{\infty} \)-category \( A \) such that \( \mathbb{K}Q = E(A) \). If \( Q \) is a quiver with superpotential of dimension \( m \), then there is a cyclic structure of degree \( m \) on \( A \).

Next, we would like to prove the converse of Proposition 4.2.1. Let us start with a finite and positively graded \( A_{\infty} \)-category \( A \) with a cyclic structure of degree \( m \). Recall from Section 3 that one can construct from \( A \) a quiver \( Q \) as follows: take the vertex set of \( Q \) to be \( \text{Obj}(A) \) and for any vertices \( u,v \), the set of degree \( i \) edges from \( u \) to \( v \) is given by a fixed basis of \( A^{i-1}(u,v)^\vee \). We will denote elements in the chosen basis of \( A \) by \( a^\vee \), and its dual basis (as elements in \( Q \)) by \( a \). The degrees of \( a \) and \( a^\vee \) are related by \( |a| = 1 - |a^\vee| \). By the cyclic structure of degree \( m \) on \( A \), we have \( A^i \cong A^{m-i} \). Since \( A \) is positively graded, we conclude that the degree of all edges in \( Q \) lies in the interval \([1 - m, 0]\), for each vertex \( v \) there is exactly one loop \( v^\vee \) of degree \( 1 - m \), and that these are precisely all the degree \( 1 - m \) edges in
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Q. The cyclic structure \((-,-)\) on \(A\) determines a super-antisymmetric pairing \((-,-)\) of degree \(2 - m\) on the set of edges of \(Q\) by defining its dual pairing \((-, -)\) to be

\[
(a^\nu, b^\nu) = (-1)^{|a^\nu|} (a^\nu, b^\nu)_A.
\]

Let \(\hat{Q}\) be the subquiver of \(Q\) which contains all vertices of \(Q\) and all edges in \(Q\) with degree in the interval \([2 - m, 0]\). Proposition 4.1.1 then shows

\[
\omega = \frac{1}{2} \sum_{x,y \text{ edges in } \hat{Q}} \langle x^\nu, y^\nu \rangle d_{dR} x d_{dR} y
\]

defines a noncommutative symplectic form on \(\hat{Q}\), and the corresponding double Poisson bracket \(\{ - , - \}\) is given by Proposition 4.1.2. It remains to show that the differential on \(Q\) is in the form of a quiver with superpotential. Denote by \(b_n\) the shift of the \(A_\infty\)-structure \(m_n\) on \(A\). By the cyclic invariance of \((-, -)_A\), we conclude that the quadratic form \((-, -)_A\) is invariant under cyclic symmetry:

\[
\langle b_n(a_1^\nu, \ldots, a_n^\nu), a_{n+1}^\nu \rangle = \langle b_n(a_2^\nu, \ldots, a_{n+1}^\nu), a_1^\nu \rangle.
\]

Define

\[
\Phi = \sum_{n \geq 1} \sum_{a_1, \ldots, a_{n+1} \text{ edges in } \hat{Q}} (b_n(a_1^\nu, \ldots, a_n^\nu), a_{n+1}^\nu) / n + 1 a_1 \cdots a_{n+1},
\]

which is a finite sum of closed paths of since \(A\) is \(A_\infty\), hence \(b_n = 0 \) for \(n \gg 0\). Note that \(\Phi\) is of degree \(3 - m\).

**Proposition 4.2.2.** We have the following descriptions on the differential \(d\) on \(\mathbb{K}\hat{Q}\):

1. For any edge \(a\) in \(\hat{Q}\), we have

\[
da = (-1)^{1+|a|+m} \sum_{b \text{ edges in } \hat{Q}} (a, b) \partial_b^\nu \Phi.
\]

2. The differential can be written in terms of the Kontsevich bracket by

\[
d = \{ \Phi, - \}.
\]

3. The differential \(d\) is a Hamiltonian vector field with Hamiltonian \(\Phi\), i.e.,

\[
d_{dR} \Phi = i_d \omega.
\]

**Proof.** Denote by \((-, -)\) the inverse quadratic form of \((-, -)_A\). Then

\[
a = \sum_{b \text{ edges in } \hat{Q}} (b, a)(-, b^\nu).
\]

By definition,

\[
da = \sum_n b_n^\nu (a)
\]

\[
= \sum_{n \in \mathbb{N}} \sum_{b, a_1, \ldots, a_n \text{ edges in } \hat{Q}} (b, a)(b_n(a_1^\nu, \ldots, a_n^\nu), b^\nu) a_1 \cdots a_n
\]

\[
= \sum_{b, a_1, \ldots, a_n \text{ edges in } \hat{Q}} (-1)^{|b| (3-m-|b|)} (b, a)(b_n(a_1^\nu, \ldots, a_n^\nu), b^\nu) / n + 1 \partial_b^\nu (a_1 \cdots a_n b)
\]

\[
= \sum_{b \text{ edges in } \hat{Q}} (-1)^{|\nu+b|+1+|a||b|} (a, b) \partial_b^\nu \Phi
\]

\[
= \sum_{b \text{ edges in } \hat{Q}} (-1)^{1+|a|+m} (a, b) \partial_b^\nu \Phi,
\]
where we have used that \((a, b) \neq 0\) if and only if \(|a| = |b| + m \mod 2\). The third equality perhaps requires more explanation. If the loop has cyclic symmetry of degree \(n + 1\), i.e., \(a_1 \cdots a_n b = e^{n+1}\) for some edge \(e\), then \(\partial_x(e^{n+1}) = (n + 1)e^n\). Up to cyclic permutation, there is only one such loop in the sum in the third line hence the \((n + 1)\) factor in the denominator is cancelled. If the loop has no cyclic symmetry, then up to cyclic permutation, there are \(n + 1\) such loops in the sum and hence the \(n + 1\) factor in the denominator is also cancelled. In the general case where a loop has a cyclic symmetry of degree \(k\), then we can write \(a_1 \cdots a_n b = p^k\), where \(p\) has no cyclic symmetry. Up to cyclic permutation there are \((n + 1)/k\) such loop in the sum, and \(\partial_x(p^k) = k(\partial_x p) p^{k-1}(\partial_x p)\). Hence the \(n + 1\) factor in the denominator is also cancelled. This proves (a).

Now, for any edge \(a\), we have

\[
\{\Phi, a\} = m \circ \{\Phi, a\} = -(-1)^{(|a|+2-m)(|\Phi|+2-m)} m \circ \sigma \{\Phi, a\} = (-1)^{1+a+m} \sum_{b \text{ edges in } \tilde{Q}} (a, b) \partial^a \Phi = da,
\]

where the second last equality is by part (a) of the statement and Proposition 4.1.2. Since both \(\{\Phi, -\}\) and \(d\) are derivations, we conclude that \(d = \{\Phi, -\}\). This proves part (b). For part (c), we have

\[
i_d \omega = \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle((dx)(dR_dR) + (-1)^{|a|}(dR_dR)(dy))
= \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} (-1)^{|a|+|y|+|x|+|x^\vee|+|y^\vee|}(dR_dR)(dx) + (-1)^{|a|+|y^\vee|}(x^\vee, y^\vee)(dR_dR)(dy)
= \sum_{x, y \text{ edges in } \tilde{Q}} (-1)^{|a|+|y|+|x|+|x^\vee|+|y^\vee|}(x^\vee, y^\vee)(dR_dR)(dy)
= \sum_{b, x, y \text{ edges in } \tilde{Q}} (-1)^{|a|+|y|+|x|+|x^\vee|+|y^\vee|+|y^\vee|}(x^\vee, y^\vee)(y, b)(dR_dR)\partial^a \Phi
= \sum_{y \text{ edges in } \tilde{Q}} (dR_dR)\partial^a \Phi
= dR_dR \Phi.
\]

It remains to describe the action of the differential on \(KQ\) on the degree \(1-m\) loops attached to each vertex. For this purpose, we first prove the following

**Lemma 4.2.3.** Let \(A\) be an \(A_{\infty}\)-category with a cyclic structure of degree \(m\). Then for all \(n \geq 3\), \(\deg m_n(a_1, \cdots, a_n) = m\) implies \(m_n(a_1, \cdots, a_n) = 0\).

**Proof.** For any \(v \in A^0\) and \(n \geq 3\),

\[
\langle m_n(a_1, \cdots, a_n), v \rangle_A = \pm \langle m_n(v, a_1, \cdots, a_{n-1}), a_n \rangle_A = 0
\]

since \(m_n(v, a_1, \cdots, a_{n-1}) = 0\). Since \(\langle -, - \rangle_A\) is nondegenerate, we conclude \(m_n(a_1, \cdots, a_n) = 0\). \(\square\)

**Proposition 4.2.4.** Let \(v^*\) be the degree \(1-d\) loop at the vertex \(v\) in \(Q\). Then

\[
dv^* = vtv \quad \text{where} \quad \ell = \sum_{x, y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle xy.
\]

**Proof.** Write \(t = \sum_{v \text{ vertices in } Q} v^*\). It suffices to show \(dt = \ell\), for then \(dv^* = d(vtv) = v(dt)v = vtv\). We
have
\[ dt = \sum_{v \text{ vertices in } \tilde{Q}} \sum_{n \in \mathbb{N}} b_n^\#(v^\ast) \]
\[ = \sum_{v \text{ vertices in } \tilde{Q}} \sum_{a_1, \ldots, a_n \text{ edges in } \tilde{Q}} \sum_{n \in \mathbb{N}} \langle v^\ast, b_n(a_1^\ast, \ldots, a_n^\ast) \rangle a_1 \cdots a_n \]
\[ = \sum_{v \text{ vertices in } \tilde{Q}} \sum_{a_1, a_2 \text{ edges in } \tilde{Q}} \langle a_1^\ast, b_2(a_1^\ast, a_2^\ast) \rangle a_1 a_2 \]
\[ = \sum_{a_1, a_2 \text{ edges in } \tilde{Q}} (a_1^\ast, a_2^\ast) a_1 a_2 \]
\[ = \ell \]
where in the third equality, we have used Lemma 4.2.3.

\[ \text{Remark 4.2.5.} \quad \text{Note here that } \ell \text{ is an element homogeneous of degree } 2 - m. \]

Summarizing, we have proven the converse of Proposition 4.2.1 and arrived at

\[ \text{Proposition 4.2.6.} \quad \text{A dg-quiver } Q \text{ is a quiver with superpotential if and only if } \mathbb{K} Q = E(A) \text{ for a finite positively graded cyclic } A_{\text{fin}}\text{-category } A. \]

With the characterisation of quivers with superpotential by \( A_{\text{fin}} \)-categories with cyclic structure, we can define products and quotients of quivers with superpotential.

\[ \text{Products of Quivers.} \quad \text{Products of quiver are not always defined. But as we will see in Chapter 5, they are always defined for quivers arising from exceptional sequences. Recall from Chapter 3 that for every dg-quiver } Q, \text{ one can construct an } A_{\text{fin}}\text{-algebra augmented over the discrete } \mathbb{K}\text{-algebra spanned by the vertices of } Q, \text{ such that } \mathbb{K} Q = E(A). \text{ For any two quivers } Q \text{ and their associated } A_{\text{fin}}\text{-algebras } A \text{ and } A', \text{ one can form the } A_\infty\text{-tensor product } A \otimes A'. \text{ In general, this tensor product is not } A_{\text{fin}}. \text{ If it is, one can define the product dg-quiver of } Q \text{ and } Q' \text{ to be the dg-quiver arising from } A \otimes A'. \text{ In other words, even if the product is well-defined, the product dg-quiver is only defined up to quasi-isomorphism. The underlying product quiver is given by} \]
\[ \{\text{Degree } i \text{ edge in } Q \times Q' \} = \{ a \otimes v : a \text{ degree } i \text{ edges in } Q \text{ and } v \text{ vertex in } Q' \} \]
\[ \cup \{ a \otimes b : a \text{ edge in } Q \text{ and } b \text{ edge in } Q' \text{ with } \deg a + \deg b - 1 = i \} \]
\[ \cup \{ u \otimes b : u \text{ vertex in } Q' \text{ and } b \text{ degree } i \text{ edge in } Q' \} \]

The differential, however, is not uniquely determined. If both quivers \( Q \) and \( Q' \) admit superpotentials, then Amorim’s construction of tensor product of \( A_\infty \)-algebras shows that one can choose a differential on \( Q \times Q' \) which comes from a superpotential. In general, to write down the product superpotential, one has to know an explicit formula for the cyclic tensor product \( A_\infty \)-structure. Unfortunately, the author does not know of such a general formula. When one of the superpotentials is cubic, i.e., one of the \( A_\infty \)-algebras involved is \( A_2 \), i.e., \( m_n = 0 \) for \( n \geq 3 \), one can use the formula given by Equation (3.6.2). Examples of product quivers are given in Chapter 6.

\[ \text{Quotients of Quivers.} \quad \text{Let } G \text{ be a finite group and } Q \text{ be a dg-quiver. Suppose } G \text{ acts on } \mathbb{K} Q \text{ by dg-isomorphisms. Write } \mathbb{K} Q = E(A) \text{ for an } A_{\text{fin}}\text{-algebra. Then } G \text{ acts on } A \text{ by strict } A_{\text{fin}}\text{-isomorphisms and hence one can form the quotient } A/G. \text{ The quotient quiver } Q/G \text{ is by definition the quiver associated to } A/G. \text{ Now suppose } Q \text{ is a quiver with superpotential and the antisymmetric pairing on } Q \text{ is } G\text{-invariant, then the cyclic structure on } A \text{ is also } G\text{-invariant. Thus } A/G \text{ also inherits a cyclic structure, making } Q/G \text{ a quiver with superpotential. Examples of quotients of quiver with superpotential can be found in Chapter 6.} \]
4.3 Quivers with Superpotential are $n$-Calabi–Yau

In this section, we would like to show that the path algebra of a quiver with superpotential is Calabi–Yau, and that as a consequence, its derived category is Calabi–Yau.

Calabi–Yau Algebras. Following Kontsevich, Ginzburg defined in [27, §3] the notion of Calabi–Yau algebras which arise naturally in the geometry of Calabi–Yau manifolds to transplant most of traditional Calabi–Yau geometry to the noncommutative setting.

Recall that in Example 2.4.7, for a homologically smooth dg-algebra $A$, we have the notion of dualizing complex: Denote by $D^b(A)$ the bounded derived category of dg-modules over $A$ and $D^b_{fd}(A)$ the full subcategory of $D^b(A)$ consisting of those dg $A$-modules whose homology is of finite total dimension. Define the dualizing complex $\Omega = R\text{Hom}_{A^{op}\otimes A}(A, A^{op} \otimes A)$. Then $(-) \otimes \Omega : D^b_{fd}(A) \rightarrow D^b_{fd}(A)$ is a Serre functor. In particular, if we have an isomorphism $\Omega \cong \Sigma^{-d}A$ as objects in $D(A^{op} \otimes A)$, then $D^b_{fd}(A)$ is $d$-Calabi–Yau. This motivates the following definition:

**Definition 4.3.1** (Calabi–Yau algebras). A dg-algebra $A$ is said to be $m$-Calabi–Yau if

- $A$ is homologically smooth, i.e., $A \in \text{Per}(A \otimes A)$, and
- there is a quasi-isomorphism of $A$-bimodules

$$\eta : R\text{Hom}_{A^{op}\text{-Bimod}}(A, A \otimes A) \rightarrow \Sigma^{-m}A.$$

In the original definition, Ginzburg requires in addition that $\eta$ is self dual, but it was later shown by van den Bergh [70, Prop. C.1] that this is automatic. We have the following

**Proposition 4.3.2.** If $A$ is an $m$-Calabi–Yau dg-algebra, then $D^b_{fd}(A)$ is a $m$-Calabi–Yau triangulated category.

**Proof.** See [41, Lemma 3.4] and [45, Lemma 4.1].

Let $Q$ be a quiver with superpotential of dimension $m$. Our proof that $\mathbb{K}Q$ is $m$-Calabi–Yau is a direct adaptation of van den Bergh’s proof [71] in the 3-Calabi–Yau case. First, we show that $\mathbb{K}Q$ is homologically smooth.

**Proposition 4.3.3.** Let $Q$ be a quiver and $S$ be the discrete $\mathbb{K}$-algebra on the vertices of $Q$. The exact sequence

$$0 \rightarrow \Omega^1_S(\mathbb{K}Q) \xrightarrow{\varphi} \mathbb{K}Q \otimes_S \mathbb{K}Q \rightarrow 0,$$

where $\varphi(dp) = p \otimes_S 1 - 1 \otimes_S p$ and $m(p \otimes_S q) = pq$, is a resolution of $\mathbb{K}Q$ by modules in $\text{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$. In other words, $\mathbb{K}Q$ is homologically smooth.

**Proof.** The sequence in the proposition is exact by Proposition 1.2.7. Let $E$ be the $\mathbb{K}$-vector space spanned by all edges in $Q$ and $E_{jk}$ the $\mathbb{K}$-vector space spanned by all edges from $k$ to $j$. Then

$$\mathbb{K}Q \otimes E \otimes \mathbb{K}Q \cong \bigoplus_{i,j,k \in V_Q} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q$$

$$\cong \bigoplus_{i,l \in V_Q} \mathbb{K}Qi \otimes E_{il} \otimes \ell\mathbb{K}Q \bigoplus_{i \neq j \text{ or } k \neq l} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q$$

$$\cong \mathbb{K}Q \otimes_S E \otimes_S \mathbb{K}Q \bigoplus_{i \neq j \text{ or } k \neq l} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q$$

$$\cong \Omega^1_S(\mathbb{K}Q) \bigoplus_{i \neq j \text{ or } k \neq l} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q$$

Hence $\Omega^1_S$ is a summand of the free $\mathbb{K}Q$-bimodule $\mathbb{K}Q \otimes E \otimes \mathbb{K}Q$, i.e., $\Omega^1_S(\mathbb{K}Q) \in \text{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$. Similarly,

$$\mathbb{K}Q \otimes \mathbb{K}Q \cong \bigoplus_{i,j \in V} \mathbb{K}Qi \otimes j\mathbb{K}Q \cong \mathbb{K}Qi \otimes i\mathbb{K}Q \bigoplus \mathbb{K}Qi \otimes j\mathbb{K}Q \cong \mathbb{K}Q \otimes_S \mathbb{K}Q \bigoplus \mathbb{K}Qi \otimes j\mathbb{K}Q$$
shows $KQ \otimes_S KQ$ is in $\text{Per}(KQ \otimes KQ)$. Since $\text{Per}(KQ \otimes KQ)$ is closed under extension, $KQ$ is also in $\text{Per}(KQ \otimes KQ)$, i.e., $KQ$ is homologically smooth. ■

If we define
\[
d_{\text{KQ} \otimes_S \text{KQ}} : KQ \otimes_S KQ \to KQ \otimes_S KQ, \quad a \otimes_S b \mapsto d(a) \otimes_S b + (-1)^{|a|}a \otimes_S d(b),
\]

and assign a grading on $\Omega^1_S(KQ)$ by $|d_a|_{\Omega^1_S(KQ)} = |a|_{KQ}$, then $d_{\text{KQ} \otimes_S \text{KQ}}$ and $d_{\Omega^1_S(KQ)}$ are differentials of degree $(-1)$ on both spaces, and the resolution becomes a dg-bimodule resolution, i.e., both $\varphi$ and $m$ are morphisms of dg-bimodules.

**Cofibrant replacement of $KQ$.** Denote the subquiver of $Q$ consisting of all edges of degree in the interval $[2-n, 0]$ by $\overline{Q}$. Let $v^*$ be the degree $1-n$ loop at vertex $v$ and $t$ be the sum of all degree $1-n$ loops. The resolution of $KQ$ in Proposition 4.3.3 shows $[25, \text{Prop. III.3.5}] KQ$ is quasi-isomorphic to

\[
P = \text{cone}(\varphi)
\]

\[
= KQ \otimes_S KQ \otimes \Sigma \Omega^1_S(KQ) \tag{4.3.1}
\]

\[
= KQ \otimes_S KQ \otimes (\Sigma(KQ \otimes_K Q, \Omega_S(KQ) \otimes_K KQ)) \otimes (\Sigma(KQ \otimes_S \Omega^1_S(S[t]) \otimes_S KQ),
\]

which is in $\text{Per}(KQ \otimes KQ)$. Hence $P$ is a cofibrant replacement of $KQ$ as a $KQ$-bimodule. The differential of the cone $d_P$ is given by

\[
d_P = \begin{pmatrix}
\Sigma \varphi & 0 \\
\Sigma d_{\Omega^1_S(KQ)}
\end{pmatrix}
\]

Observe that $|a \otimes b|_P = |a|_{KQ} + |b|_{KQ}$ and $|d_{\text{KQ} \otimes_S \text{KQ}}|_P = |a|_{KQ} - 1$ for any $a \in KQ$. To avoid too many subscripts, in what follows, the subscript in $d_{\text{KQ} \otimes_S \text{KQ}}$ and $d_{\Omega^1_S(KQ)}$ will be suppressed, but, to avoid confusion, the subscript $P$ in $d_P$ is always written.

Since $P$ is a cofibrant replacement of $KQ$, we have

\[
R\text{Hom}_{KQ-\text{Bimod}}(KQ, KQ \otimes KQ) = \text{Hom}_{KQ-\text{Bimod}}(P, KQ \otimes KQ).
\]

Hence to construct a quasi-isomorphism

\[
\eta : R\text{Hom}_{KQ-\text{Bimod}}(KQ, KQ \otimes KQ) \to \Sigma^{-m}KQ,
\]

it suffices to construct an isomorphism of dg-modules

\[
\text{Hom}_{KQ-\text{Bimod}}(P, KQ \otimes KQ) \cong \Sigma^{-m}P.
\]

This is done by constructing a nondegenerate pairing of bimodules in the following sense:

**Pairing of Bimodules.** We follow the conventions of van den Bergh [71]. A pairing of degree $n$ between $A$-bimodules $M$ and $N$ is a bilinear homogeneous map of degree $n$

\[
\langle -,- \rangle : M \times N \to A \otimes A
\]

such that $\langle p, - \rangle$ is linear for the outer bimodule structure on $A \otimes A$ and $\langle - , q \rangle$ is linear for the inner bimodule structure on $A \otimes A$, i.e.,

\[
\langle apb, q \rangle = (-1)^{|b||p|+n}a \ast \langle p, q \rangle \ast b
\]

\[
\langle p, aq \rangle = (-1)^{|a||p|+n}a \langle p, q \rangle b.
\]

If $M$ and $N$ modules in $\text{Per}(A \otimes A)$, we say the pairing is nondegenerate if the map

\[
M \to \Sigma^n \text{Hom}_{A-\text{Bimod}}(N, A \otimes A), \quad p \mapsto \langle p, - \rangle
\]
is an isomorphism. If $A$ is a dg-algebra and $M, N$ are dg-bimodules, then a dg-pairing is a pairing which satisfy Leibniz’s rule
\[ \xi(p, q) = \xi_p q + (-1)^{|p|+|q|} q p. \]
A dg-pairing which is nondegenerate induces an isomorphism of dg-modules
\[ M \cong \Sigma^n \text{Hom}_{A-Bimod}(N, A \otimes A). \]
In the special case $M = N$, a pairing is said to be (super)-symmetric if
\[ \langle [p, q] \rangle = (-1)^{|p|(|p|+|q|)\sigma}|q, p). \]

**Example 4.3.4.** Let $Q$ be a quiver and $S$ be the discrete $K$-algebra on the vertices of $Q$. Then $KQ \otimes_S KQ \in \text{Per}(KQ \otimes KQ)$ by Proposition 4.3.3. Define a pairing
\[ \Sigma^n(KQ \otimes_S KQ) \times (KQ \otimes_S KQ) \to KQ \otimes KQ \]
by
\[ \langle [1 \otimes_S 1, 1 \otimes_S 1] \rangle = \sum_{v \in V_Q} v \otimes v. \]
Then it is a nondegenerate pairing of degree $n$. To see this, observe that any $KQ$-bimodule morphism $\varphi : \Sigma^n(KQ \otimes_S KQ) \to KQ \otimes KQ$ is determined on its value on $1 \otimes_S 1$. Let $\varphi(1 \otimes_S 1) = a' \otimes a''$. Then
\[ \varphi(1 \otimes_S 1) = \sum_{v \in V_Q} \varphi(v \otimes_S v) = \sum_{v \in V_Q} v \varphi(1 \otimes_S 1)v = \sum_{v \in V_Q} va' \otimes a''v = a'' \sum_{v \in V_Q} va' \otimes \varphi. \]
In other words, $\varphi = \langle [1 \otimes_S 1, 1 \otimes_S 1] \rangle_{a'}$.

Now, we return to our proof of that path algebra of quiver with superpotential are Calabi–Yau. Recall from Equation (4.3.1) that $P = \text{cone} \varphi$ is a cofibrant replacement of $KQ$. We define a symmetric pairing $\langle [\cdot, \cdot] : \mathbb{P} \times \mathbb{P} \to KQ \otimes KQ$ of degree $-m$ by defining on generators
\[
\langle [d_{i, R}^e, 1 \otimes_S 1] \rangle = (-1)^m v \otimes v
\]
\[
\langle [1 \otimes_S 1, d_{i, R}^e] \rangle = (-1)^m v \otimes v
\]
\[
\langle [d_{i, R}^m a, d_{i, R}^m b] \rangle = (-1)^{|d_{i, R}^m a| r} \langle [a, b] \rangle
\]
for all $a, b$ edges in $\mathbb{Q}$ and assigning the value zero for all other combinations. Note that the first two equations of Equation (4.3.2) is equivalent to
\[
\langle [d_{i, R}^m, 1 \otimes_S 1] \rangle = (-1)^m \sum_{v \in V_Q} v \otimes v
\]
\[
\langle [1 \otimes_S 1, d_{i, R}^m] \rangle = (-1)^m \sum_{v \in V_Q} v \otimes v
\]
by linearity of the pairing.

**Proposition 4.3.5.** The pairing $\langle [\cdot, \cdot] \rangle$ is a symmetric nondegenerate dg-pairing of degree $-m$.

**Proof.** First, we check symmetry:
\[
\langle [d_{i, R}^m, 1 \otimes_S 1] \rangle = (-1)^m \sum_{v \in V_Q} v \otimes v = (-1)^{(|1 \otimes_S 1| r - m)(|d_{i, R}^m| r - m)\sigma(1 \otimes_S 1, dt)}
\]
since $|d_{i, R}^m| p = |t| x_Q - 1 = -m$.
\[
\langle [d_{i, R}^m a, d_{i, R}^m b] \rangle = (-1)^{|d_{i, R}^m a| r} \langle [a, b] \rangle
\]
\[
= (-1)^{|d_{i, R}^m a| r + (|a| x_Q - m)^2(|b| x_Q - m + 2)\sigma(b, a)}
\]
\[
= (-1)^{|d_{i, R}^m a| r - m(|d_{i, R}^m b| r - m)\sigma(\langle [d_{i, R}^m b, d_{i, R}^m a] \rangle).
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since \( |d_{dR} a|_P + (|a|_{K_Q} - m + 2)(|b|_{K_Q} - m + 2) + 1 + |d_{dR} b|_P = (|d_{dR} a|_P - m)(|d_{dR} b|_P - m) \mod 2 \). Next we show the pairing is nondegenerate. Since for any edge \( a \),

\[
\langle \langle a, b \rangle \rangle = (-1)^{|a|-1} \cdot \langle \langle a, b \rangle \rangle \partial_b,
\]

we see that the pairing is nondegenerate on \( \Omega^1_S(K\tilde{Q}) \). Since \( K\tilde{Q} \otimes S \Omega^1_S(S[t]) \otimes S K\tilde{Q} \cong K\tilde{Q} \otimes S K\tilde{Q} \), together with Example 4.3.4, this shows the pairing is nondegenerate on \( P \).

It remains to check the compatibility of \( \langle \langle -,- \rangle \rangle \) with \( d_P \), i.e., we have the Leibnitz’s rule

\[
d\langle \langle p,q \rangle \rangle = \langle \langle dp,pq \rangle \rangle + (-1)^{|p|_P-|m|} \langle \langle dp,q \rangle \rangle.
\]

This is a direct calculation divided into six cases.

1. We have

\[
d\langle \langle d_{dR} t, d_{dR} t \rangle \rangle = 0,
\]

\[
\langle \langle dpd_{dR} t, d_{dR} t \rangle \rangle = \langle \langle t \otimes S 1 - 1 \otimes S t - d_{dR} \ell, d_{dR} t \rangle \rangle
\]

\[
= t \ast \langle \langle 1 \otimes S 1, d_{dR} t \rangle \rangle - \langle \langle 1 \otimes S 1, d_{dR} t \rangle \rangle \ast t
\]

\[
= (-1)^m \sum_{v \text{ vertices in } Q} (v \otimes t_v - t_v \otimes v)
\]

and

\[
\langle \langle d_{dR} t, dpd_{dR} t \rangle \rangle = \langle \langle d_{dR} t, t \otimes S 1 - 1 \otimes S t - d_{dR} \ell \rangle \rangle
\]

\[
= t \langle \langle d_{dR} t, 1 \otimes S 1 \rangle \rangle - \langle \langle d_{dR} t, 1 \otimes S 1 \rangle \rangle t
\]

\[
= (-1)^m \sum_{v \text{ vertices in } Q} (t_v \otimes v - v \otimes t_v)
\]

Hence

\[
d\langle \langle d_{dR} t, d_{dR} t \rangle \rangle = \langle \langle dpd_{dR} t, d_{dR} t \rangle \rangle + (-1)^{|d_{dR} t|_{P}-|m|} \langle \langle d_{dR} t, dpd_{dR} t \rangle \rangle.
\]

2. We have for any \( a \in K\tilde{Q} \),

\[
d\langle \langle d_{dR} a, d_{dR} t \rangle \rangle = 0,
\]

\[
\langle \langle dpd_{dR} a, d_{dR} t \rangle \rangle = \langle \langle a \otimes S 1 - 1 \otimes S a - d_{dR} (da), d_{dR} t \rangle \rangle
\]

\[
= a \ast \langle \langle 1 \otimes S 1, d_{dR} t \rangle \rangle - \langle \langle 1 \otimes S 1, d_{dR} t \rangle \rangle \ast a
\]

\[
= (-1)^m \sum_{v \text{ vertices in } Q} (v \otimes av - va \otimes v)
\]

\[
= (-1)^m (t(a) \otimes a - a \otimes h(a))
\]

and

\[
\langle \langle d_{dR} a, dpd_{dR} t \rangle \rangle = \langle \langle d_{dR} a, t \otimes S 1 - 1 \otimes S t - d_{dR} \ell \rangle \rangle
\]

\[
= - \langle \langle d_{dR} a, d_{dR} \ell \rangle \rangle
\]

\[
= (-1)^{|d_{dR} a|_{P}-1} \langle \langle a, \ell \rangle \rangle.
\]

Now

\[
\langle \langle a, \ell \rangle \rangle = \sum_{b \text{ edges in } \tilde{Q}} (a, b) \partial_b \ell
\]

\[
= \sum_{b, x, y \text{ edges in } \tilde{Q}} (a, b) \langle \langle x^y, y^x \rangle \rangle \partial_b (xy)
\]

\[
= \sum_{x, y \text{ edges in } \tilde{Q}} (a, x) \langle \langle x^y, y^x \rangle \rangle h(x) \otimes t(x)y + (-1)^{|x|_U} (a, y) \langle \langle x^y, y^x \rangle \rangle h(y) \otimes t(y)
\]

\[
= t(a) \otimes a - a \otimes h(a).
\]

Hence

\[
d\langle \langle d_{dR} a, d_{dR} t \rangle \rangle = \langle \langle dpd_{dR} a, d_{dR} t \rangle \rangle + (-1)^{|d_{dR} a|_{P}-m} \langle \langle d_{dR} t, dpd_{dR} a \rangle \rangle.
\]
3. We have
\[ \xi(\varphi dt, 1 \otimes 1) = 0, \]
\[ \langle \varphi dt, 1 \otimes 1 \rangle = \langle t \otimes 1 - 1 \otimes t - dt, 1 \otimes 1 \rangle = 0 \]
and
\[ \langle dt, \xi(1 \otimes 1) \rangle = 0. \]
Hence Leibnitz’s rule is verified.

4. For \( a \in \mathbb{K}Q \),
\[ d\langle\langle d\varphi a, 1 \otimes 1 \rangle\rangle = 0 = \langle\langle d\varphi a, d(1 \otimes 1) \rangle\rangle \]
and
\[ \langle\langle dP(d\varphi a), 1 \otimes 1 \rangle\rangle = \langle\langle a \otimes 1 - 1 \otimes a - d(d\varphi a), 1 \otimes 1 \rangle\rangle = 0. \]
Hence Leibnitz’s rule is verified.

5. The case
\[ d\langle\langle 1 \otimes 1, 1 \otimes 1 \rangle\rangle = \langle\langle dP(1 \otimes 1), 1 \otimes 1 \rangle\rangle + (-1)^{1+1} \langle\langle 1 \otimes 1, dP(1 \otimes 1) \rangle\rangle \]
is trivial since all terms are zero.

6. We have for \( a, b \in \mathbb{K}Q \), since \( \{-,-\} \) is a double dg-bracket,
\[ d\langle\langle d\varphi a, d\varphi b \rangle\rangle = (-1)^{|d\varphi a||d\varphi b|} d\langle\langle a, b \rangle\rangle \]
\[ = (-1)^{|d\varphi a||d\varphi b|} \langle\langle da, b \rangle\rangle + (-1)^{|d\varphi a||d\varphi b|+|a||KQ} \langle\langle a, db \rangle\rangle \]
\[ = (-1)^{|d\varphi a||d\varphi b|} \langle\langle da, b \rangle\rangle + (-1)^{1+m} \langle\langle a, db \rangle\rangle \]
\[ \langle\langle dP(d\varphi a), d\varphi b \rangle\rangle = \langle\langle a \otimes 1 - 1 \otimes a - d(d\varphi a), d\varphi b \rangle\rangle \]
\[ = (-1)^{|d\varphi a||d\varphi b|} \langle\langle a, b \rangle\rangle \]
\[ = (-1)^{|d\varphi a||d\varphi b|} \langle\langle a, b \rangle\rangle \]
and
\[ \langle\langle d\varphi a, dP(d\varphi b) \rangle\rangle = \langle\langle d\varphi a, b \otimes 1 - 1 \otimes b - d(d\varphi b) \rangle\rangle \]
\[ = (-1)^{|d\varphi a||d\varphi b|} \langle\langle a, db \rangle\rangle. \]
Hence
\[ d\langle\langle d\varphi a, d\varphi b \rangle\rangle = \langle\langle dP(d\varphi a), d\varphi b \rangle\rangle + (-1)^{|d\varphi a||d\varphi b|+1} \langle\langle d\varphi a, dP(d\varphi b) \rangle\rangle. \]

\textbf{Theorem 4.3.6.} \( \mathbb{K}Q \) is a \( m \)-Calabi–Yau dg-algebra.

\textit{Proof.}
\[ \text{RHom}(\mathbb{K}Q, \mathbb{K}Q \otimes \mathbb{K}Q) \cong \text{Hom}(P, \mathbb{K}Q \otimes \mathbb{K}Q) \]
\[ \cong \Sigma^{-m} P \]
\[ \cong \Sigma^{-m} \mathbb{K}Q, \]
where the first and third isomorphisms hold since \( P \) is a cofibrant replacement for \( \mathbb{K}Q \) and the second isomorphism is induced by the nondegenerate dg-pairing \( \langle\langle -,-\rangle\rangle \).

\textbf{Theorem 4.3.7.} The bounded derived category \( D^b_{\text{dg}}(\mathbb{K}Q) \) with its standard t-structure is a \( m \)-Calabi–Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over \( H^0(\mathbb{K}Q) \).
Proof. By Proposition 4.3.2, $D^b_{fd}(KQ)$ is an $m$-Calabi–Yau triangulated category. The heart of the standard $t$-structure on $D^b_{fd}(KQ)$ consists of finite dimensional $dg$ $KQ$-modules concentrated at degree 0. We show that the action of $KQ$ on such $KQ$-modules factors through $H^0(KQ)$. For any $a \in (KQ)^1$, $d(am) = (da)m + (-1)^{|a|}adm$ implies $(da)m = 0$ since the other two terms are of degree $(-1)$ and vanish automatically. This shows $d(KQ)^1$ acts trivially, and hence the action factors through $H^0(KQ)$. This gives a finite dimensional $H^0(KQ)$-module and yields the required equivalence. 

Calabi–Yau algebras not coming from quivers with superpotential. We have shown that every quiver with superpotential produces a Calabi–Yau algebra. One might be interested in the converse, i.e., if all Calabi–Yau algebras arise in this way. Davison [21] has shown that fundamental group algebras of compact hyperbolic manifolds of dimension greater than one are Calabi–Yau algebras which do not arise in this way. Thus it is not true that every Calabi–Yau algebra comes from a quiver with superpotential. In fact, van den Bergh [70] has defined a class of Calabi–Yau algebras, which he called exact Calabi–Yau algebras, using cyclic and Hochschild homology, and characterized them as Calabi–Yau algebras which are quasi-isomorphic to quivers with superpotential (which he called deformed $dg$-preprojective algebras).

4.4 Quivers with Superpotential of Low Dimensions

In this section, we describe quivers with superpotential of dimensions 1 to 4.

Quivers with Superpotential of Dimension 1. Quivers with superpotential of dimension 1 are given by a finite number of vertices, and for each vertex $v$, a loop $v^*$ of degree 0, with trivial differential.

Quivers with Superpotential of Dimension 2. Quivers with superpotential $Q$ of dimension 2 are in the following form: for each vertex $v$, there is a loop $v^*$ of degree $-1$, and these are all the degree $-1$ edges in $Q$. There is an antisymmetric nondegenerate pairing on degree 0 edges $(\cdot, \cdot)$. The symplectic form on $\tilde{KQ}$ reads

$$\omega = \frac{1}{2} \sum_{e,f \text{ edges of degree 0}} (e^\vee, f^\vee) d_{dR}e d_{dR}f.$$ 

The differential reads

$$d(e) = 0 \text{ for any degree 0 edge } e;$$

$$d(v^*) = v^\ell v \text{ for any degree } -1 \text{ loop } v^*,$$

where $\ell = \sum (e^\vee, f^\vee) ef$.

Quivers with Superpotential of Dimension 3. We will put quivers with superpotential of dimension 3 into a standard form and show that they are precisely the Ginzburg algebras introduced by Ginzburg in [27]. There is an anti-symmetric nondegenerate pairing between degree 0 edges and degree $-1$ edges. By changing basis if necessary, we may assume that for each degree 0 edge $e$, there is a degree $-1$ edge $e^*$ in the opposite direction, such that the pairing is given by $(e, f^*) = \delta_{e,f}$. The symplectic form on $\tilde{KQ}$ then reads

$$\omega = \sum_{e \text{ edges of degree 0 in } Q} d_{dR}e d_{dR}e^*,$$

and the superpotential $\Phi$ is of degree 0, i.e., only depends on the degree 0 edges $e$'s. The differential $d : KQ \to KQ$ reads

$$d(e) = 0 \text{ for any degree 0 edge } e;$$

$$d(e^*) = -\partial_e \Phi;$$

$$d(v^*) = v^\ell v \text{ for any degree } -2 \text{ loop } v^*,$$

where $\ell = \sum [e, e^*]$. \hfill (4.4.1)

In particular, the dg-quiver $Q$ is determined by its degree 0 sub-quiver $Q^*$ and the superpotential $\Phi$ which is an element in $KQ_{cyc}^*$ as follow: Given $Q^*$ and $\Phi$, one can construct a 3-Calabi–Yau quiver with
superpotential \( Q \) by first adding to \( Q^* \) a reversed arrow \( e^* \) of degree \(-1\) for each arrow \( e \) and define a degree 1 map \( d : \mathbb{K}Q \to \mathbb{K}Q \) as in equation 4.4.1 using \( \Phi \). The map \( d \) such defined is a differential as \( \{ \Phi, \Phi \} \) has degree 1, hence the master equation \( \{ \Phi, \Phi \} = 0 \) is empty. This reverse construction is exactly the one given by Ginzburg in [27]. Thus our definition of quiver with superpotential is a generalization of the Ginzburg construction. Note that \( H^0(\mathbb{K}Q, \Phi) = \mathbb{K}Q^*/(\partial_r \Phi : e \text{ edges in } Q^*) \) is the so called Jacobi algebra of \( (Q^*, \Phi) \). Theorem 4.3.7 then readily yields

**Theorem 4.4.1.** The bounded derived category \( D^b_{\mathbb{K}Q}(\mathbb{K}Q) \) with its standard \( t \)-structure is a 3-Calabi–Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over the Jacobi algebra \( \mathbb{K}Q^*/(\partial_r \Phi : e \text{ edges in } Q^*) \).

**Quivers with Superpotential of Dimension 4.** We give a standard form for 4-Calabi–Yau quiver with superpotential and a Ginzburg algebra-like construction for dimension 4.

There is an anti-symmetric nondegenerate pairing between degree 0 edges and degree \(-2\) edges. By a change of basis if necessary, we may assume that for each degree 0 edge \( e \), there is a reversed degree \(-2\) edge \( e^* \), such that the pairing is given by \((e, f^*) = \delta_{ef} \). There is a symmetric nondegenerate quadratic form \( q \) on the degree \(-1\) edges. The symplectic form on \( \mathbb{K}Q \) then reads

\[
\omega = \sum_{e \text{ edges in degree } 0} d_{dR}ed_{dR}e^* + \frac{1}{2} \sum_{r, s \text{ edges of degree } -1} q(r, s)d_{dR}rd_{dR}s.
\]

and the superpotential \( \Phi \) is of degree \(-1\), i.e., only depends on the degree 0 edges \( e \)'s and the degree \(-1\) edges \( r \)'s. If we denote by \( A_r = dr \) for degree \(-1\) edges \( r \), then the superpotential \( \Phi \) can be written in the form

\[
\Phi = \sum_{s \text{ edges of degree } -1} s(\partial_s^2 \Phi) = \sum_{u, r \text{ edges of degree } -1} (u, r)(r^\vee, s^\vee)s\partial_s^2 \Phi = \sum_{r, s \text{ edges of degree } -1} q(r, s)sA_r.
\]

The differential \( d : \mathbb{K}Q \to \mathbb{K}Q \) reads

\[
d(e) = 0 \quad \text{for any degree 0 edge } e; \\
d(r) = A_r; \\
d(e^*) = \partial_r \Phi; \\
d(v^*) = v^\ell v \quad \text{for any degree } -3 \text{ loop } v^*, \quad (4.4.2)
\]

The condition \( d^2 e^* = 0 \) gives rise to an equation:

\[
\partial_e^2 \left( \sum_{r, s \text{ edges of degree } -1} q(r, s)A_r A_s \right) = \sum_{r, s \text{ edges of degree } -1} q(r, s)(\partial_e^p A_r) A_s (\partial_e q A_s) + q(r, s)(\partial_e^p A_s) A_r (\partial_e q A_r) \\
= d \left( \sum_{r, s \text{ edges of degree } -1} q(r, s)(\partial_e^p A_r) s(\partial_e q A_r) + q(r, s)(\partial_e^p A_s) r(\partial_e q A_s) \right) \\
= 2d \partial_e^2 \left( \sum_{r, s \text{ edges of degree } -1} q(r, s)A_r A_s \right) = 2d^2 e^* = 0
\]

Since this is true for all degree 0 edges \( e \) in \( Q \), we conclude

\[
\sum_{r, s \text{ edges of degree } -1} q(r, s)A_r A_s = 0 \text{ in } \mathbb{K}Q_{\text{cyc}}, \text{ i.e., modulo cyclic permutation.}
\]
Note that the above equation is the equation \( \{ \Phi, \Phi \} = 0 \) written out in coordinates. Now, if we take \( Q^* \) to be the subquiver of \( Q \) consisting of all vertices and all degree 0 edges in \( Q \), \( R \) to be the set of degree \(-1\) edges, we get the following data:

1. A finite quiver \( Q^* \),
2. A finite set \( R \) (of indices),
3. Maps \( h : R \to V_{Q^*} \) and \( t : R \to V_{Q^*} \) (heads and tails),
4. A map \( A : R \to \mathbb{K}Q \) such that \( A_r = A(r) \in h(r)KQ^*t(r) \),
5. A symmetric function \( q : R \times R \to \mathbb{K} \) with the following properties:
   
   (a) \( q(r, s) = 0 \) unless \( h(r) = t(s) \) and \( h(s) = t(r) \),
   (b) \( q \) is nondegenerate in the sense that the matrix \( \{ q(r, s) \}_{r, s \in R} \) is invertible,
   (c) \( \sum_{r, s \in R} q(r, s)A_rA_s = 0 \) mod \( [\mathbb{K}Q^*, \mathbb{K}Q^*] \).

Conversely, if one starts with the above data, one can reverse the construction and produce a quiver with superpotential \( Q \) of dimension 4 as follows: Take \( Q \) to be the quiver constructed by adding to \( Q^* \) a degree \(-1\) edge \( r \) from \( t(r) \) to \( h(r) \) for each element \( r \in R \), a degree \(-2\) edge \( e^* \) in the reverse direction for each edge \( e \) in \( Q^* \), and a degree \(-3\) loop \( v^* \) for each vertex \( v \) in \( Q^* \). Let \( \Phi \in \mathbb{K}Q_{cyc} \) denotes the element \( \sum_{r, s \in R} q(r, s)A_rA_s \). Define a degree 1 map \( d : \mathbb{K}Q \to \mathbb{K}Q \) by Equation (4.4.2). This map \( d \) is a differential as we have assume our data satisfy the master equation

\[
\{ \Phi, \Phi \} = \sum_{r, s \in R} q(r, s)A_rA_s = 0 \mod [\mathbb{K}Q^*, \mathbb{K}Q^*].
\]

Then \( Q \) is a quiver with superpotential of dimension 4 with superpotential \( \Phi \) and a super-antisymmetric pairing \((-, -)\) on the subquiver \( Q \) containing all vertices and all edges with degree 0, \(-1\), \(-2\) given by

\[
(e, e^*) = 1 \\
(r, s) = q(r, s) \\
(e^*, e) = -1
\]

and zero otherwise. Since \( H^0(\mathbb{K}Q) = \mathbb{K}Q^*/\langle A_r : r \in R \rangle \), Theorem 4.3.7 readily yields

**Theorem 4.4.2.** The bounded derived category \( D^{\text{bd}}_\text{fd}(\mathbb{K}Q) \) with its standard t-structure is a 4-Calabi-Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over the algebra \( \mathbb{K}Q^*/\langle A_r : r \in R \rangle \).
Chapter 5
Derived Equivalences between Vector Bundles and DG-Quivers

This chapter is the heart of the thesis where we prove our main results.

Section 5.1 gives a review on equivariant sheaves.

In Section 5.2, we generalize a result by Bridgeland [15, Proposition 4.1] and show that if \( \pi : V \to X \) is an anti-semiample vector bundle on a smooth projective manifold with an exceptional poset \( E \), then under some cohomological vanishing conditions, the total space \( V \) is derived equivalent to an algebra \( \Lambda_E \) which is the path algebra of a quiver with relations. If we remove the cohomological vanishing condition, we end up with an \( A_{\text{fin}} \)-algebra rather than a quiver with relations.

Section 5.3 tries to resolve \( \Lambda_E \), the path algebra of a quiver with relations (or more generally the \( A_{\text{fin}} \)-algebra) by a dg-quiver \( Q_E \).

Section 5.4 gives a concrete description of the underlying graded quiver of the dg-quiver \( Q_E \) in terms of the dual exceptional poset of \( E \).

Section 5.5 proves the existence of a superpotential on \( Q_E \) when \( V \) is noncompact Calabi–Yau.

Section 5.6 considers the \( G \)-equivariant situation and constructs from \( Q_E \) a quotient quiver \( Q_E/G \), generalizing the construction of the McKay quiver, which is derived equivalent to \( D^b(\text{Coh}^G(V)) \). In the case when \( V \) is equivariantly Calabi–Yau, \( Q_E/G \) is also equipped with a superpotential.

Section 5.7 proves the product construction. We start with two dg-quivers \( Q_E \) and \( Q_F \) derived equivalent to vector bundles \( V \) and \( W \) respectively, and construct a product quiver \( Q_E \times Q_F \) which is derived equivalent to \( V \times W \). When both \( V \) and \( W \) are Calabi–Yau, we show that the product quiver \( Q_E \times Q_F \) is also equipped with a superpotential.

5.1 Equivariant Sheaves

In this section, we define \( G \)-equivariant sheaves. We will fix a finite group \( G \) acting on a smooth variety \( X \) of finite type over \( K \) by automorphisms, with \( |G| \nmid \text{char}(K) \). We will denote the multiplication map on \( G \) by \( \mu : G \times G \to G \) and the group action map by \( \sigma : G \times X \to X \). We will also need the projection map \( \pi_X : G \times X \to X \) and the map \( \pi_{23} : G \times G \times X \to G \times X \) projecting onto the last two factors.

Definition 5.1.1 (G-linearization). A \( G \)-linearization of a quasi-coherent sheaf \( E \) on \( X \) is an isomorphism \( \lambda^E : \pi_X^*E \to \sigma^*E \) satisfying the cocycle condition

\[
(\mu \times \text{id}_X)^*\lambda = \pi_{23}^\ast \lambda \circ (\text{id}_G \times \sigma)^*\lambda.
\]

This amounts to the following data: For each \( g \in G \), there is an isomorphism \( \lambda_g : E \to g^*E \), satisfying the cocycle condition \( \lambda_{hg}^E = g^*(\lambda_h^E) \circ \lambda_g^E \).
**Definition 5.1.2** (Equivariant sheaves). A $G$-equivariant quasi-coherent sheaf (resp. coherent sheaf) on $X$ is a quasi-coherent sheaf (resp. coherent sheaf) $E$ on $X$ together with a $G$-linearization $\lambda^E$. If $(E, \lambda^E)$, $(F, \lambda^F)$ are $G$-equivariant sheaves, then $G$ acts on $\text{Hom}_X(E, F)$ by $g \cdot \phi = (\lambda^E_g)^{-1} \circ (g^{-1})^* \phi \circ \lambda^F_g$.

The Hom space in the categories $\text{QCoh}^G(X)$ and $\text{Coh}^G(X)$ are given by the $G$-equivariant maps, i.e., $\text{Hom}_G(E, F) = [\text{Hom}_X(E, F)]^G$. In other words, a morphism $\phi : E \to F$ is equivariant if for all $g \in G$ we have a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & F \\
\downarrow{\lambda^E_g} & & \downarrow{\lambda^F_g} \\
g^*E & \xrightarrow{g^*\phi} & g^*F.
\end{array}
$$

By Grothendieck [30, Proposition 5.1.2], $\text{QCoh}^G(X)$ has enough injectives. Hence we can resolve any equivariant sheaf by equivariant injective resolution. Moreover, since $X$ is projective and $G$ is finite, one can find a $G$-equivariant ample line bundle on $X$, and hence every $G$-equivariant sheaf has a $G$-equivariant locally free resolution. There is a forgetful functor $U : \text{QCoh}^G(X) \to \text{Coh}(X)$ which sends an equivariant sheaf $(E, \lambda^E)$ to its underlying sheaf $E$ and a equivariant morphism $f$ to itself, now regarded as a morphism between sheaves. Clearly, the forgetful functor also restricts to a functor $U : \text{Coh}^G(X) \to \text{Coh}(X)$.

Next, we would like to review how to derive the $\text{Hom}_G$ functor.

**Proposition 5.1.3.** The functor $[-]^G : G\text{-Mod} \to \mathbb{K}\text{-Mod}$ taking $G$-invariant part is exact.

**Proof.** Let $0 \to U \to V \to W$ be an exact sequence of $G$-representations. By Maschke’s theorem, we have a decomposition

$$
0 \to U^G \oplus U' \to V^G \oplus V' \to W^G \oplus W' \to 0
$$

where each summand is a subrepresentation. Thus taking $G$-invariant part yields an exact sequence. ■

**Corollary 5.1.4.** For any $(E, \lambda^E), (F, \lambda^F) \in \text{D}(\text{QCoh}^G(X))$, we have

$$
\text{RHom}_G((E, \lambda^E), (F, \lambda^F)) = \text{RHom}(E, F)^G.
$$

**Proof.** Write $\text{Hom}_G = [-]^G \circ \text{Hom} \circ U$. The corollary follows from exactness of $[-]^G$. ■

Should there be no confusion, from now on we will, by abusing notation, denote an equivariant sheaf $(E, \lambda^E)$ by only its underlying sheaf $E$.

### 5.2 Tilting Objects on Equivariant Vector Bundles

In this section, we will be working over the following setting: Let $G$ be a finite group and $\mathbb{K}$ be a field with $\text{ord}(G) \not| \text{char}(\mathbb{K})$. Let $X$ be a smooth projective variety over $\mathbb{K}$ with $G$ acting by automorphisms. Let $\pi : V \to X$ be a $G$-equivariant vector bundle. Unless otherwise stated, throughout this section, $\mathcal{E} = \{E_i\}_{i \in I}$ will denote a full exceptional poset on $\text{D}^b(\text{Coh}^G(X))$. We will also write $E = \bigoplus_{i \in I} E_i$.

**Lemma 5.2.1.**

1. The map $\pi^*$ is exact.

2. The map $\pi_*$ is exact.

3. The functor $\pi_*\pi^*$ is naturally isomorphic to $- \otimes S^*V^\vee$.

4. The functor $\pi_*\pi^*$ preserves injectives.

**Proof.** Statement 1 follows from flatness of $\pi$. For statement 2, by Hartshorne [31, §Ex. II.5.17] and that $\pi$ is affine, the map $\pi_*$ defines an equivalence between the category of quasi-coherent $\mathcal{O}_V$-modules and the category of quasi-coherent $\pi_*\mathcal{O}_V$-modules, which is a subcategory of quasi-coherent $\mathcal{O}_X$-modules. Thus $\pi_*$ is exact. Statement 3 follows from projection formula: For any $\mathcal{O}_X$-modules $M$, we have functorial isomorphisms

$$
\pi_*\pi^*M = \pi_*(\pi^*M \otimes \mathcal{O}_V) = M \otimes \pi_*\mathcal{O}_V = M \otimes S^*V^\vee.
$$
Statement 4 follows from statement 3 since for any injective $I$ and any $\mathcal{O}_X$-modules $M$,
\[
\text{Ext}^i(M, \pi_*\pi^*I) = \text{Ext}^i(M, I \otimes S^*V^*) = \text{Ext}^i(M \otimes S^*V, I) = 0
\]
whenever $i \neq 0$.

Bridgeland pointed out in [15, Prop. 4.1] (see also [16, Thm. 3.6]) that one can construct tilting objects on vector bundles by pulling back tilting objects on the base variety.

**Lemma 5.2.2.** If $T$ is a tilting object in $D(\text{QCoh}^G(X))$, then $\pi^*T$ is a tilting object in $D(\text{QCoh}^G(V))$.

**Proof.** By the adjunction $\pi^* \dashv \pi_*$, we have $\text{Hom}^\bullet_X(\pi^*T, F) = \text{Hom}^\bullet_X(T, \pi_*F)$ for all objects $F$. In particular, for an arbitrary coproduct $\bigoplus_{i \in I} F_i$,
\[
\text{Hom}^\bullet_Y(\pi^*T, \bigoplus_{i \in I} F_i) = \text{Hom}^\bullet_X(T, \pi_* \bigoplus_{i \in I} F_i) = \bigoplus_{i \in I} \text{Hom}^\bullet_X(T, \pi_* F_i) = \bigoplus_{i \in I} \text{Hom}^\bullet_Y(\pi^*T, F_i)
\]
and hence $\pi^*T$ is compact. To show $\pi^*T$ is generating, suppose $F$ has the property $\text{Hom}^\bullet_Y(\pi^*T, F) = 0$. Then $\text{Hom}^\bullet_X(T, \pi_* F) = 0$ and hence $\pi_* F = 0$. Since $\pi_*$ is exact, it preserves cohomologies and this implies $F$ has no cohomologies, i.e., $F = 0$ as desired.

**Lemma 5.2.3.** Suppose a finite set of objects $\mathcal{E} = \{E_i\}_{i \in I}$ in $D^b(\text{Coh}^G(X))$ classically generates $D^b(\text{Coh}^G(X))$, then the finite set of objects $\pi^*\mathcal{E} = \{\pi^*E_i\}_{i \in I}$ classically generates $D^b(\text{Coh}^G(V))$.

**Proof.** By Example 2.5.2, $\mathcal{E}$ are compact objects in $D(\text{QCoh}^G(X))$. By Corollary 2.5.7, $\mathcal{E}$ classically generates $D^b(\text{Coh}^G(X))$ if and only if $\mathcal{E}$ generates $D(\text{QCoh}^G(X))$. Thus $E = \bigoplus_{i \in I} E_i$ is a tilting object in $D(\text{QCoh}^G(X))$. By Lemma 5.2.2, $\pi^*E$ is a tilting object in $D(\text{QCoh}^G(V))$. By Corollary 2.5.7 again, the set $\pi^*\mathcal{E}$ classically generates $D^b(\text{Coh}^G(V))$.

The following theorem is a generalization of Bridgeland and Stern [16, Thm. 3.6] and is essentially an application of Theorem 2.9.2.

**Theorem 5.2.4.** Let $X$ be a smooth projective variety together with an action by a finite group $G$. Let $\pi : V \rightarrow X$ be a $G$-equivariant vector bundle and $\mathcal{E} = \{E_i\}_{i \in I}$ be a full exceptional poset on $D^b(\text{Coh}^G(X))$. Write $E = \bigoplus_{i \in I} E_i$. Then there is a dg-algebra $R$ whose underlying chain complex represents $R\text{Hom}_G(E, \pi_*\pi^*E) = R\text{Hom}_G(E, E \otimes S^*V^*)$ and an equivalence
\[
\Psi : D^b(\text{Coh}^G(V)) \rightarrow \text{Per}(R^{op})
\]
which after composing with the forgetful functor $U : \text{Per}(R^{op}) \rightarrow D(\mathbb{K})$ yields $R\text{Hom}_G(E, \pi_*(-))$, i.e., $U \circ \Psi = R\text{Hom}_G(E, \pi_*(-))$.

**Proof.** For each $i \in I$, choose a $G$-equivariant injective resolution $I_{E_i}$ for $E_i$. Write $I_E = \bigoplus_{i \in I} I_{E_i}$. Define a dg-algebra $R = \text{Hom}_G(I_E, I_E \otimes S^*V^*)$ as follows: As a chain complex, the $i$-th graded piece is given by
\[
R^i = \text{Hom}^i_G(I_E, I_E \otimes S^*V^*) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_G(I^p_E, I^{i+p}_E \otimes S^*V^*),
\]
with differential
\[
d_{R^i}(f) = (d_{I_E} \otimes \text{id}_{S^*V^*}) \circ f - (-1)^i f \circ d_{I_E}.
\]
The multiplication map on $R$ is given by composing morphisms followed by symmetric product on the $S^*V^*$ factor, i.e.,
\[
R^i \otimes R^j \rightarrow R^{i+j}
\]
f \otimes g \mapsto (\text{id}_{I_E} \otimes m)(g \otimes \text{id}_{S^*V^*}) \circ f,
\]
where the map $m : S^*V^* \otimes S^*V^* \rightarrow S^*V^*$ is the symmetric product.
The underlying chain complex of \( R \) represents \( \text{RHom}(E, E \otimes S^*V^\vee) \): Since \( - \otimes S^*V^\vee \) is exact, \( \text{Hom}(E, - \otimes S^*V^\vee) \) is a left exact functor. Hence the right derived functor is given by
\[
\text{RHom}_G(E, E \otimes S^*V^\vee) = \text{RHom}_G(I_E, E \otimes S^*V^\vee) = \text{Hom}_G(I_E, I_E \otimes S^*V^\vee)
\]
since \( I_E \) is an injective resolution of \( E \).

Next, we define a functor
\[
\Psi : D^b(\text{Coh}^G(V)) \to D(R^{\text{op}}-\text{Mod}).
\]

Recall that by the natural isomorphism \( \pi_\ast \pi^\ast \cong - \otimes S^*V^\vee \) in Lemma 5.2.1, we have an isomorphism \( R = \text{Hom}(I_E, \pi_\ast \pi^\ast I_E) \) as a dg-algebra. For any \( M \in D^b(\text{Coh}^G(V)) \), choose an injective resolution \( I_M \).

The chain complex \( \text{Hom}_G(I_E, \pi_\ast I_M) \) has a right dg-module structure over \( R \) given by
\[
\text{Hom}_G(I_E, \pi_\ast I_M) \otimes \text{Hom}_G(I_E, \pi_\ast \pi^\ast I_E) \to \text{Hom}_G(I_E, \pi_\ast I_M)
\]
\[
(g, f) \mapsto \left( I_E \xrightarrow{f} \pi_\ast \pi^\ast I_E \right) \xrightarrow{\pi_\ast \pi^\ast g} \pi_\ast \pi^\ast \pi_\ast I_M \cong \pi_\ast \mathcal{O}_V \otimes \pi_\ast I_M \cong \pi_\ast \mathcal{O}_V \otimes \pi_\ast I_M,
\]
where \( \mu_M : \mathcal{O}_V \otimes I_M \to I_M \) is the \( \mathcal{O}_V \)-module structure map on \( I_M \).

We then define
\[
\Psi(M) = \text{Hom}_G(I_E, \pi_\ast I_M)
\]
\[
\Psi(M \xrightarrow{f} N) \mapsto (\text{Hom}_G(I_E, \pi_\ast I_M) \xrightarrow{\pi_\ast f} \text{Hom}_G(I_E, \pi_\ast I_N)).
\]

This functor is exact since it is the composition of the following four exact functors:
\[
I : D^b(\text{Coh}^G(V)) \to K(\text{Coh}^G(V)), \quad M \mapsto I_M
\]
\[
\pi_\ast : D^b(\text{Coh}^G(V)) \to D^b(\text{Coh}^G(X))
\]
\[
\text{Hom}_G(I_E, -) : K(\text{Coh}^G(X)) \to K(R^{\text{op}}-\text{Mod})
\]

and the natural projection functor
\[
Q : K(R^{\text{op}}-\text{Mod}) \to D(R^{\text{op}}-\text{Mod}).
\]

By the same reason as \( R \) represents \( \text{RHom}(E, E \otimes S^*V^\vee) \), we see that \( U \circ \Psi = \text{RHom}(E, \pi_\ast(-)) \). Next, we show \( \Psi \) is fully faithful. Recall that by Lemma 5.2.1, \( \pi_\ast \pi^\ast \) preserves injectives. For any \( E_i, E_j \in \mathcal{E} \),
\[
\text{Hom}_{D^b(\text{Coh}^G(V))}(\pi_\ast E_i, \pi_\ast E_j) = \text{Hom}_{D^b(\text{Coh}^G(X))}(E_i, \pi_\ast \pi^\ast E_j) = \text{Hom}_{K(\text{Coh}^G(X))}(I_E, \pi_\ast \pi^\ast I_{E_j}) = H^0(\text{Hom}_G(I_E, \pi_\ast \pi^\ast I_{E_j})) = \text{Hom}_R(\text{Hom}_G(I_E, \pi_\ast \pi^\ast I_{E_j}), \text{Hom}_G(I_E, \pi_\ast \pi^\ast I_{E_j})/\{\text{homotopy equivalence}\}) = \text{Hom}_{K(R^{\text{op}}-\text{Mod})}(\text{Hom}_G(I_E, \pi_\ast \pi^\ast I_{E_j}), \text{Hom}_G(I_E, \pi_\ast \pi^\ast I_{E_j})) = \text{Hom}_{D(R^{\text{op}}-\text{Mod})}(\Psi^\ast E_i, \Psi^\ast E_j)
\]

By Lemma 5.2.3, the set \( \{\pi_\ast E_i\}_{i \in I} \) classically generates \( D^b(\text{Coh}^G(V)) \). Hence \( \Psi \) is fully faithful. The essential image is the triangulated category classically generated by \( \Psi^\ast E_i \). Since each \( \Psi^\ast E_i \) is a direct summand of \( \Psi^\ast E = R \), the essential image of \( \Psi \) is the triangulated category classically generated by \( R \), i.e., \( \text{Per}(R^{\text{op}}) \). Hence we have our desired equivalence \( \Psi : D^b(\text{Coh}^G(V)) \to \text{Per}(R^{\text{op}}) \).

By homology perturbation, we have

**Corollary 5.2.5.** Let \( \pi : V \to X \) be a \( G \)-equivariant vector bundle and \( \mathcal{E} \) a full exceptional collection on \( D^b(\text{Coh}^G(V)) \), then there is a minimal \( A^\infty \)-algebra
\[
A_\mathcal{E} = \text{Hom}^\ast_{\mathcal{E}}(E, E \otimes S^*V^\vee)
\]
and an equivalence
\[
\Phi : D^b(\text{Coh}^G(V)) \to \text{Per}_\infty(A_\mathcal{E}^{\text{op}}).
\]
Proof. Since the underlying chain complex of $R$ represents $\text{RHom}(E, E \otimes S^*V^\vee)$, its cohomology is given by

$$\Lambda_\mathcal{E} = \text{Hom}^*_\mathbb{Z}(E, E \otimes S^*V^\vee).$$

By Theorem 3.2.1, there is a minimal $A_\infty$-structure on $\Lambda_\mathcal{E}$ making it quasi-isomorphic to $R$ as an $A_\infty$-algebra. Lemma 5.3.4 below shows that this $A_\infty$-structure is indeed $A_R$. Since $D(R^{op}) \cong D_\infty(R^{op})$, and this equivalence restricts to an equivalence between perfect derived categories, we have an equivalence $D^b(\text{Coh}^G(V)) \cong \text{Per}(R^{op}) \cong \text{Per}_\infty(\Lambda^{op})$ as desired.

\section*{Definition 5.2.6.} Let $\pi : V \to X$ be a $G$-equivariant vector bundle. A full exceptional poset $\mathcal{E} = \{E_i\}_{i \in I}$ on $D^b(\text{Coh}^G(X))$ is said to be if $\text{Hom}_{\mathcal{E}}^k(E_i, E_j \otimes S^*V^\vee) = 0$ for all $k \neq 0$ and all $i, j \in I$.

\section*{Remark 5.2.7.} Note that a $V$-geometric exceptional poset is necessarily strong.

\section*{Remark 5.2.8.} When $G$ is trivial and $V = K_X$ is the canonical bundle, a $K_X$-geometric full exceptional collection is the same as a geometric helix defined in Bridgeland and Stern [16]. This is where our terminology comes from.

\section*{Corollary 5.2.9.} Let $\pi : V \to X$ be a $G$-equivariant vector bundle which is anti-semiample i.e., $S^kV^\vee$ is globally generated for $k \gg 0$. Let $\mathcal{E}$ be a full $V$-geometric exceptional poset on $D^b(\text{Coh}^G(V))$. Then $\Lambda_\mathcal{E}$ is an ordinary algebra which is Noetherian and there is an equivalence

$$\Phi : D^b(\text{Coh}^G(V)) \to D^b_{\mathcal{E}}(\Lambda^{op}),$$

where $D^b_{\mathcal{E}}(\Lambda^{op})$ is the full triangulated subcategory of $D^b(\Lambda^{op})$ consisting of complexes whose cohomologies are finitely generated $\Lambda^{op}$-modules.

\section*{Proof.} Since $\mathcal{E}$ is $V$-geometric, $\Lambda_\mathcal{E}$ is an ordinary algebra. It remains to show $\Lambda_\mathcal{E}$ is Noetherian and of finite global dimension, for then $\text{Per}(\Lambda^{op}) = D^b(\Lambda^{op}\text{-mod}) = D^b_{\mathcal{E}}(\Lambda^{op})$. Since $S^kV^\vee$ is globally generated for $k \gg 0$, we have an exact sequence

$$\mathcal{O} \otimes H^0(X, S^kV^\vee) \to S^kV^\vee \to 0$$

Taking dual, we see that $S^kV \hookrightarrow \mathcal{O} \otimes H^0(X, S^kV^\vee)^\vee$ embeds $S^kV$ into a trivial bundle as a subbundle. Now choose, $k, \ell \gg 0$ and coprime. Then we have an embedding

$$V \hookrightarrow S^kV \oplus S^\ell V \hookrightarrow \mathcal{O} \otimes (H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee)$$

of $V$ into a trivial bundle. Composing this embedding with the map

$$\mathcal{O} \otimes (H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee) \to H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee$$

which projects the trivial bundle to its fiber yields a projective morphism

$$V \to H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee.$$

Thus $V$ is projective over an affine scheme of finite type. The algebra $\Lambda_\mathcal{E} \cong \text{End}_V(\pi^*E)$ is then a finitely generated module over a finitely generated algebra, thus itself Noetherian. By Hille and van den Bergh [32, Thm 7.6], $\Lambda_\mathcal{E}$ has finite global dimension as desired.

\section*{Corollary 5.2.10.} The above equivalence restricts to an equivalence

$$\Phi : D^b_{\mathcal{E}}(\text{Coh}^G(V)) \to D^b_{\mathcal{E}}(\Lambda^{op}).$$

\section*{Proof.} If $M \in D^b_{\mathcal{E}}(\text{Coh}^G(V))$, then $\text{Hom}^*(\pi^*E, M)$ is finite dimensional. As

$$H^*(\Phi(M)) = \text{Hom}^*(\Lambda_\mathcal{E}, \Phi(M)) = \text{Hom}^*(\pi^*E, M),$$
we conclude $\Phi(M) \in \text{D}^b_f(A_{op}^p)$. Now suppose $\Phi(M) \in \text{D}^b_f(A_{op}^p)$. Then by assumption, $\text{Hom}^\bullet(\pi^* E, M)$ is finite dimensional. Since $M$ lies in the thick category generated by $\pi^* E$, we conclude that $\text{Hom}^\bullet(M, M)$ is also finite dimensional. Via the multiplication map

$$S^\bullet H^0(X, S^kV^\vee) \otimes S^\bullet H^0(X, S^lV^\vee) \rightarrow \bigoplus_{r \geq 0} H^0(X, S^rV^\vee) = H^0(V, O_V),$$

$\text{Hom}^\bullet(M, M)$ becomes a finite dimensional module over $S^\bullet H^0(X, S^kV^\vee) \otimes S^\bullet H^0(X, S^lV^\vee)$, and thus it is supported in finitely many points $\{p_1, \ldots, p_r\}$ on $H^0(X, S^kV^\vee) \oplus H^0(X, S^lV^\vee)$. Recall that we have a projective morphism $\phi : V \rightarrow H^0(X, S^kV^\vee) \oplus H^0(X, S^lV^\vee)$. We would like to show $\text{supp} M \subseteq \phi^{-1}(\{p_1, \ldots, p_r\})$. If this is true, $\text{supp} M$ would be compact as $\phi$ is a proper morphism. Let $q \notin \phi^{-1}(\{p_1, \ldots, p_r\})$ be a $K$-point on the vector bundle $V$. Then $\phi(q) \notin \text{supp} \text{Hom}^\bullet(M, M)$. Since $\text{supp} \text{Hom}^\bullet(M, M)$ is defined by the annihilator ideal of $\text{Hom}^\bullet(M, M)$, this means there exists $f \in S^\bullet H^0(X, S^kV^\vee) \otimes S^\bullet H^0(X, S^lV^\vee)$ with $f \cdot \text{id}_M = 0$ but $f(\phi(q)) \neq 0$. In particular, we see that the identity map on $M$ localizes to the zero map at the point $q$. Hence $M$ must be supported in $\phi^{-1}(\{p_1, \ldots, p_r\})$ as desired. 

**Tilting algebra as quiver with relations.** As pointed out by Bridgeland and Stern [16], in the case when $\Lambda_\mathcal{E}$ is an ordinary algebra, one can construct a quiver with relations $(Q, I)$ whose path algebra is isomorphic to $\Lambda_\mathcal{E}$. The vertex set of $Q$ corresponds to the full exceptional poset $\mathcal{E}$, while the edges between two vertices $v$ and $w$ corresponding to exceptional object $E_v, E_w \in \mathcal{E}$ is given by a basis of the cokernel of the map

$$\bigoplus_{E_v, E_w, E \in \mathcal{E}} \text{Hom}(E_v, E_i \otimes S^rV^\vee) \otimes \text{Hom}(E_i, E_w \otimes S^sV^\vee) \rightarrow \text{Hom}(E_v, E_w \otimes S^rV^\vee).$$

There is then a natural surjective map $\varphi : KQ \rightarrow \Lambda_\mathcal{E}$ and we can take our desired quiver with relations to be $(Q, \ker \varphi)$.

The following examples are applications of Corollary 5.2.9.

**Example 5.2.11.** Let $\mathcal{E} = \text{Spec} K$ and $V = K^n$. Then $\mathcal{E} = \mathcal{O}_X$ is a $V$-geometric exceptional collection on $\text{D}^b_!(\text{Coh}(\text{Spec} K))$. The classical tilting algebra is given by

$$\Lambda_\mathcal{E} = \text{Hom}^\bullet(\mathcal{O}, \mathcal{O} \otimes S^rV^\vee) = S^rV^\vee = K[x_1, \ldots, x_n]$$

and the classical tilting quiver is given by one vertex with $n$ loops $x_i$, with relations given by $x_i x_j = x_j x_i$.

**Example 5.2.12.** If we take $V$ to be the zero vector bundle, we recover the tilting quiver for $X$. In the case $X = \mathbb{P}^n$, with the full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$, we have the Beilinson quiver

\[
\begin{array}{ccccccc}
& & a_{10} & & & & \\
& & \vdots & & a_{20} & & \\
& a_{11} & & \cdots & & \cdots & & a_{n0} \\
\vdots & & & & & & \vdots & & \vdots \\
& \cdots & & \cdots & & \cdots & & \cdots \\
& \cdots & & \cdots & & \cdots & & \cdots \\
& & & & & & v_n \\
\end{array}
\]

with relations $a_{i+1,j} = a_{i+1,k} a_{i,j}$. The vertex $v_i$ corresponds to the bundle $\mathcal{O}(i)$, and the arrows $a_{ij}$ correspond to $\text{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \cong H^0(\mathbb{P}^n, \mathcal{O}(1))$, i.e., the homogeneous coordinates on $\mathbb{P}^n$. This also explain the relations $a_{i+1,j+1} = a_{i+1,j} a_{i,j+1}$: they correspond to the relation $Z_j Z_k = Z_k Z_j$ in the homogeneous coordinate ring.

More examples will be given in Section 6.1.

### 5.3 Quasi-free Resolution of Tilting Algebra

In this section, we will resolve the tilting $A_{\infty}$-algebra $\Lambda_\mathcal{E}$ by an augmented quasi-free dg-algebra.

To simplify notations, we will drop the subscript $\mathcal{E}$ and write $\Lambda = \Lambda_\mathcal{E}$. The idea is to use Koszul duality, which is not new and has been explored by Keller [40], Lu, Palmieri, Wu and Zhang [50], Segal
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[62], and van der Bergh [70]. By Koszul duality, if $E(\Lambda)$ is locally finite, then $E(E(\Lambda))$ is $A_{\infty}$-quasi-isomorphic to $\Lambda$. However, since $E(\Lambda)$ is a very large dg-algebra, $E(E(\Lambda))$ cannot be a minimal dg model of $\Lambda$. To make it smaller, we take its cohomology $H^*(E(\Lambda))$ and transfer the dg-structure on $E(\Lambda)$ to an $A_{\infty}$-structure on $H^*(E(\Lambda))$. It turns out that $H^*(E(\Lambda)) = \text{Ext}_A^*(S, S)$ is finite dimensional and $E(\text{Ext}_A^*(S, S))$ would be the desired quasi-free dg-algebra resolving $\Lambda$. Making all of these rigorous involves an Adams grading on $\Lambda$. In the following, we will denote by $n$ the length of $E$ and $E$ isomorphic to $\Lambda$. However, since $A_{\infty}$-algebras Adams graded by $0_0 \to \Lambda$.

Lemma 5.3.1. Let $E = \{E_1, \ldots, E_n\}$ be a full exceptional sequence on $D^b(\text{Coh}^G(X))$. The tilting $A_{\infty}$-algebra

$$\Lambda = \bigoplus_{k=0}^\infty \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{a=-n}^{n} \bigoplus_{j-i=a} \text{Hom}_G^\ell(E_i, E_j \otimes S^k V^\vee)$$

is Adams graded by $(a, k) \in \mathbb{Z} \times \mathbb{Z}$, and is a locally finite augmented algebra over $A_{0,0} = S$. The Adams grading of the augmentation ideal $\bar{\Lambda}$ is supported in

$$J = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : -(n-1) \leq a \leq n-1 \text{ and } b \geq 1\} \cup \{(a, 0) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq a \leq n-1\}.$$

Proof. The tilting algebra $\Lambda$ is locally finite since $A_{0,0}^{a,k} = \bigoplus_{j-i=a} \text{Hom}_G^\ell(E_i, E_j \otimes S^k V^\vee)$ is finite dimensional. The rest follows from the fact that $(E_0, \ldots, E_n)$ is an exceptional sequence.

A partition of $j \in J$ is a way of writing $j$ as a finite sum of elements in $J$. Denote by $p(j)$ the number of such partitions.

Lemma 5.3.2. The number $p(j)$ is finite.

Proof. A general partition of $j = (a, b) \in J$ is in the form

$$(a, b) = (a_1, b_1) + \cdots + (a_k, b_k) + (c_1, 0) + \cdots + (c_\ell, 0),$$

where $-n \leq a_i \leq n$, $b_i \geq 1$ and $c_i \geq 1$. Rewriting, we have

$$b = b_1 + \cdots + b_k$$

$$a = a_1 - \cdots - a_k = c_1 + \cdots + c_\ell$$

which are both partitions of natural numbers. Since $0 \leq a - a_1 - \cdots - a_k \leq a + (n-1)k \leq a + (n-1)b$, we conclude

$$p(j) = p(a, b) \leq \text{(number of partitions of } b) \cdot \text{(number of partitions of } a + (n-1)b).$$

Lemma 5.3.3. Let $A$ and $A'$ be augmented Adams graded $A_{\infty}$-algebras Adams graded by $\mathbb{Z} \times \mathbb{Z}$. Suppose their augmentation ideals are both supported in $J$. Then

1. $A$ and $A'$ are both $A_{\infty}$.
2. Any Adams-graded $A_{\infty}$-morphism $f : A \to A'$ is $A_{\infty}$.
3. Any Adams-graded $A_{\infty}$-homotopy $h : A \to A'$ between $A_{\infty}$-morphisms $f, g : A \to A'$ is $A_{\infty}$.
4. The Adams-graded minimal model $H^*(A)$ is $A_{\infty}$ and is $A_{\infty}$-homotopic to $A$. 

Proof. The rest follows from the fact that $(E_0, \ldots, E_n)$ is an exceptional sequence.

The number $p(j)$ is finite.
Proof. For statement 1, since \( m_r \) preserves the Adams degree, for any fixed \( j \in J \),

\[
m_r : A^j_{\ell_1} \otimes_S \cdots \otimes_S A^j_{\ell_r} \to A^j
\]
can be nonzero only if \( j = j_1 + \cdots + j_r \), with \( j, j_1, \ldots, j_n \in J \). By Lemma 5.3.2, there can be only a finite number of partitions of \( j \). Hence the \( j \)-th component of \( m_r \) vanishes for \( r > 0 \), i.e., \( A \) is \( A_{\text{fin}} \). The other second and third statement are proven similarly.

For the last statement, observe that the Adams grading on \( A \) induce an Adams grading on \( H^\bullet(A) \), and recall that by choosing a splitting respecting the Adams grading in Section 3.2, one can get an \( A_{\infty} \)-structures on \( H^\bullet(A) \) preserving the induced Adams grading , and which is \( A_{\infty} \)-quasi-isomorphic to \( A \). By Theorem 3.1.3, \( H^\bullet(A) \) is \( A_{\infty} \)-homotopic to \( A \). By statement 3, this \( A_{\infty} \)-homotopy is \( A_{\text{fin}} \) and we are done.

**Lemma 5.3.4.** The \( A_{\infty} \)-structure on \( \Lambda \) is in an \( A_{\text{fin}} \)-structure, i.e., \( m_r = 0 \) for \( r > 0 \).

**Proof.** This is a special case of Lemma 5.3.3. 

The Adams grading on \( \Lambda \) induces an Adams grading on its Koszul dual \( E(\Lambda) \), and the combinatorial Lemma 5.3.2 is crucial in showing the following lemma describing the Adams grading on \( E(\Lambda) \).

**Lemma 5.3.5.** 1. The dg-algebra \( E(\Lambda) \) is a \( \mathbb{Z} \times \mathbb{Z} \)-graded locally finite algebra augmented over \( S \). The \( \mathbb{Z} \times \mathbb{Z} \)-grading of its augmentation ideal is supported in \( J \).

2. The same also holds for the dg-algebra \( E(E(\Lambda)) \).

**Proof.** The \( j \)-th component of the augmentation ideal of \( E(\Lambda) \) is

\[
\bigoplus_m ((\Sigma \Lambda)^\#)^{\otimes S} \frac{\ell_1 + \cdots + \ell_m = \ell}{j_1 + \cdots + j_m = j}
\]

\[
= \bigoplus_m \bigoplus_{\ell_1 + \cdots + \ell_m = \ell \atop j_1 + \cdots + j_m = j} ((\Sigma \Lambda)^{\#})_{j_1} \otimes_S \cdots \otimes_S ((\Sigma \Lambda)^{\#})_{j_m}.
\]

If \( j \in J \), the direct sum on the right hand side is a finite sum since the number of partitions of \( j \in J \) is finite. Since \( \Lambda \) is locally finite, each \( \Lambda_{j_i} \) is finite dimensional and hence the \( j \)-th component of the augmentation ideal of \( E(\Lambda) \) is also finite dimensional.

If \( j \notin J \), we would like to show the \( j \)-th component vanishes, which is equivalent to the vanishing of each vector space \( \Lambda_{j_1} \otimes_S \cdots \otimes_S \Lambda_{j_m} \) whenever \( j_1 + \cdots + j_m \notin J \). We show the case when \( m = 2 \), i.e., \( \Lambda_{j_1} \otimes S \Lambda_{j_2} = 0 \) whenever \( j_1 + j_2 \notin J \). The proof for general \( m \) is similar. Write \( j = (j', j'') \) and \( k = (k', k'') \). Note that \( j + k \notin J \) is equivalent to \( j' + k' \notin [-(n-1), n-1] \), where \( n \) is the length of the exceptional sequence \( E \). Now

\[
\tilde{\Lambda}_{j_1} \otimes_S \Lambda_k = \bigoplus_{0 \leq i, i+j' \leq n} \Hom(E_{i_1}, E_{i_1+j'} \otimes S^j V^V) \otimes_S \Hom(E_{i_2}, E_{i_2+k'} \otimes S^{k'} V^V)
\]

\[
= \bigoplus_{0 \leq i, i+j' \leq n} \Hom(E_{i_1}, E_{i_1+j'} \otimes S^j V^V) \otimes_S \Hom(E_{i_2+j'}, E_{i_1+j'+k'} \otimes S^{k'} V^V).
\]

If \( j' + k' \geq n \), then \( i + j' + k' \geq n \) for all \( i = 1, \ldots, n \). If \( j' + k' \leq -n \), then \( i + j' + k' < 0 \) for all \( i = 1, \ldots, n \). Hence \( \tilde{\Lambda}_{j_1} \otimes_S \Lambda_k = 0 \). In particular, the support of the augmentation ideal lies in \( J \) and we have proven the first statement.

Now, in proving the first statement, we have only used the fact that \( \Lambda \) is locally finite with Adams grading of its augmented ideal supported in \( J \). Since this is also true for \( E(\Lambda) \), the same also holds for \( E(E(\Lambda)) \). 

\[\blacksquare\]
Lemma 5.3.6 ([52] Lemma 11.1). $H^k(E(\Lambda)) = \text{Ext}^k_{\Lambda}(S, S)$.

Proof. By Lemma 5.3.5 and Koszul duality (Theorem 3.4.3), $E(\Lambda) = \Omega BA$ is a $\Lambda$-algebra which is $A_{\infty}$-quasi-isomorphic to $\Lambda$. By Lemma 5.3.3, they are $A_{\text{fin}}$-homotopic. Since the Koszul functor $E$ sends $A_{\text{fin}}$-homotopies to dg-homotopies, $E(\Omega BA)$ is dg-homotopic to $E(\Lambda)$ and hence $H^k(E(\Omega BA)) \cong H^k(E(\Lambda))$. By [48, Lemma 2.3.4.4 and 2.4.2.3], we have an equivalence $D(\Omega BA) \cong D_{\infty}(\Lambda)$ which sends $S \mapsto S[-1]$. Hence $\text{Ext}^k_{\Omega BA}(S, S) = \text{Ext}^k_{\Lambda}(S, S)$. Thus, passing to $\Omega BA$ if necessary, we may assume $\Lambda$ is a dg-algebra. Since $E(\Lambda)$ is locally finite, $E(\Lambda) = (BA)^\#$. Now, by [24, Prop. 19.2], $B(\Lambda) \otimes S \Lambda$ is a semi-free resolution of the right $\Lambda$-module $S$. Then

$$R\text{Hom}(S, S) = \text{Hom}_{\Lambda}(BA \otimes_{S} \Lambda, S) = \text{Hom}_{S}(BA, S) = (BA)^\# = E(\Lambda).$$

Taking cohomologies on both sides, we are done. \hfill \blacksquare

Lemma 5.3.7. Any $A_{\infty}$-structure on the Yoneda algebra $\text{Ext}^*_\Lambda(S, S)$ preserving the Adams grading is $A_{\text{fin}}$. In particular, the $A_{\text{fin}}$-structure obtained on $\text{Ext}^*_\Lambda(S, S)$ by taking minimal model of $E(\Lambda)$ is turns $\text{Ext}^*_\Lambda(S, S)$ into an augmented $A_{\text{fin}}$-algebra over $S$ which is $A_{\text{fin}}$-homotopic equivalent to $E(\Lambda)$.

Proof. This is a consequence of Lemma 5.3.3 and 5.3.6. \hfill \blacksquare

Henceforth the Yoneda algebra will always be equipped with this natural $A_{\text{fin}}$-structure coming from taking minimal model of $E(\Lambda)$.

Proposition 5.3.8. 1. There is a $A_{\text{fin}}$-quasi-isomorphism $E(\text{Ext}^*_\Lambda(S, S)) \cong \Lambda$.

2. If $\Lambda$ is a classical algebra, then there is a quasi-isomorphism

$$E(\text{Ext}^*_\Lambda(S, S)) \rightarrow H^0(E(\text{Ext}^*_\Lambda(S, S))) \cong \Lambda,$$

i.e., $E(\text{Ext}^*_\Lambda(S, S))$ is a quasi-free dg-algebra resolving $\Lambda$.

Proof. By Lemma 5.3.7, $\text{Ext}^*_\Lambda(S, S)$ is $A_{\text{fin}}$-homotopic equivalent to $E(\Lambda)$. Since the Koszul functor $E$ sends $A_{\text{fin}}$-homotopies to dg-homotopies, $E(\text{Ext}^*_\Lambda(S, S))$ is dg-homotopic to $E(E(\Lambda))$. Now $E(E(\Lambda))$ is $A_{\infty}$-quasi-isomorphic to $\Lambda$ by Koszul duality (Theorem 3.4.3), and hence $A_{\text{fin}}$-quasi-isomorphic to $\Lambda$ by Lemma 5.3.3 and 5.3.5. This proves the first statement.

If $\Lambda$ is a classical algebra, by the same argument, $E(\text{Ext}^*_\Lambda(S, S))$ is dg-homotopic to $E(E(\Lambda))$. By Koszul duality of dg-algebras (Theorem 3.4.3), there is a quasi-isomorphism of dg-algebras $E(E(\Lambda)) \rightarrow \Lambda$. Hence there is a quasi-isomorphism as desired. \hfill \blacksquare

5.4 Computing $\text{Ext}^*_\Lambda(S, S)$

In this section, we compute $\text{Ext}^*_\Lambda(S, S)$ in terms of the dual exceptional sequence to $E$ and the vector bundle $V$.

When $\Lambda_{\mathcal{E}}$ is an algebra rather than an $A_{\infty}$-algebra, which happens for example when $E$ is $V$-geometric, we can compute $\text{Ext}^*_\Lambda(S, S)$ in terms of the dual exceptional sequence $F$ of $E$. Among all modules over $\Lambda_{\mathcal{E}}$, the following projective modules and simple modules are of utmost importance for us as they correspond to objects in $E$ and $F$. Let $P_i = \text{Hom}_{G}(E, E_i \otimes S^i V^\vee)$. Then $P_i$ is a right module over $\Lambda_{\mathcal{E}}$ which is a direct summand of $\Lambda_{\mathcal{E}}$. Let $S_i = \text{Hom}_{G}(E_i, E_i \otimes S^i V^\vee) = \text{Hom}_{G}(E_i, E_i) = \mathbb{K}$. Then $S_i$ is a simple right module over $\Lambda_{\mathcal{E}}$. When we write $\Lambda_{\mathcal{E}}$ in the form of a quiver with relations $(Q, I)$, then $P_i$ corresponds to the vector space of all paths starting at the vertex $i$, and $S_i$ corresponds to the 1-dimensional vector space sitting at the vertex $i$. We will denote $S = \bigoplus_{i \in I} S_i$. Then $\Lambda_{\mathcal{E}}$ is an $A_{\text{fin}}$-algebra augmented over $S$.

The following proposition describes the preimage of the modules $P_i$ and $S_i$ under the isomorphism $\Phi : D^b(\text{Coh}^p(V)) \rightarrow \text{Per}(A_{\mathcal{E}}^\mathbb{Z})$. 


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Proposition 5.4.1 ([16], Lemma 3.7). Let $\pi : V \to X$ be a $G$-equivariant vector bundle. Let $E = \{E_i\}_{i \in I}$ be a full $V$-geometric exceptional poset on $D^b(\text{Coh } G(V))$ and $\mathcal{F} = \{F_i\}_{i \in I^{\text{op}}}$ be the dual exceptional poset of $E$. The equivalence $\Phi : D^b(\text{Coh } G(V)) \to \text{Per}(\Lambda_E^{\text{op}})$ sends

$$\Phi(\pi^*E_i) = P_i \quad \text{and} \quad \Phi(s,F_i) = S_i,$$

where $s : X \to V$ is the zero section.

Proof. The equivalence $\Psi : D^b(\text{Coh } G(V)) \to \text{Per}(R^{\text{op}})$ in Theorem 5.2.4 sends

$$\Psi(E_i) = \text{Hom}(I_E, I_i \otimes S^*V^\vee).$$

Now

$$H^\bullet(\text{Hom}(I_E, I_i \otimes S^*V^\vee)) = \text{Hom}^\bullet(E, E_i \otimes S^*V^\vee).$$

Hence $\Phi(\pi^*E_i) = P_i$.

Next, we choose an injective resolution $I_{F_j}$ for $F_j$. Then $s_*I_{F_j}$ is an injective resolution for $s_*F_j$ and hence $\Psi(s,F_j) = \text{Hom}(I_{E_i}, \pi_*s_*I_{F_j}) = \text{Hom}(I_{E_i}, I_{F_j})$. Taking cohomology, we conclude $\Phi(s,F_j) = \text{Hom}^\bullet(E, F_j) = S_j$ as vector spaces. To show they are isomorphic as $\Lambda_E$-modules, it suffices to show that $\text{Hom}(E, E \otimes S^kV^\vee)$ acts trivially on $\Phi(s,F_j)$ whenever $k \geq 1$. The action of $R$ on $\Psi(s,F_j)$ factors through the map $S^kV^\vee \otimes I_{F_j} \to I_{F_j}$ which in turns comes from the pushforward under $\pi$ of the $\mathcal{O}_V$-module structure on $s_*F_j$. This shows that the action of $S^kV^\vee$ on $I_{F_j}$, and hence the action of $\text{Hom}(I_E, I \otimes S^kV^\vee)$ on $\text{Hom}(I_{E_i}, I_{F_j})$ is trivial for $k \geq 1$. Taking cohomology, we see that $\text{Hom}(E, E \otimes S^kV^\vee)$ also acts trivially on $\Phi(s,F_j)$ whenever $k \geq 1$, as desired. \hfill \blacksquare

Remark 5.4.2. This proposition is the only place where we need $\Lambda_E$ to be an algebra rather than an $A_\infty$-algebra in order to calculate $\text{Ext}_E^\bullet(S,S)$ in terms of the dual exceptional collection of $E$. In the general $A_\infty$-algebra case, we do not know if the higher multiplication maps acts trivially, and hence we do not know if $\Phi(s,F_j) = S_j$.

Proposition 5.4.3. Let $s : X \to V$ be the zero section. Then for all $E \in D(\text{QCoh}(X))$,

$$Ls^*s_*E = \bigoplus_{k=0}^{\text{rank} V} \bigwedge^k V^\vee[k] \otimes E = S^\bullet(V[-1])^\vee \otimes E.$$

Proof. Consider $X$ as a subvariety of $V$ via the zero section. Then $X$ is given by the zero locus of the tautological section $\sigma$ of the tautological vector bundle $\pi^*V$ on $V$. The sheaf $s_*\mathcal{O}_X$ can be resolved by the Koszul complex

$$0 \to \bigwedge^V (\pi^*V)^\vee \to \cdots \to (\pi^*V)^\vee \to \mathcal{O}_V \to s_*\mathcal{O}_X \to 0,$$

where the maps are given by contraction with the section $\sigma$. In other words, we have a quasi-isomorphism $\bigwedge^\bullet(\pi^*V)^\vee \cong s_*\mathcal{O}_X$. Since $\bigwedge^\bullet(\pi^*V)^\vee$ is locally free, its derived pullback under $s$ is the same as the underived pullback, i.e., restriction to $X$. Since the maps in the complex $\bigwedge^\bullet(\pi^*V)^\vee$ is given by contraction with $\sigma$, they restrict to zero on $X$. This yields

$$s^*s_*\mathcal{O}_X \cong s^*(\bigwedge^\bullet(\pi^*V)^\vee) = \bigwedge^\bullet(\pi^*V)^\vee|_X = \bigoplus_{k=0}^{\text{rank} V} \bigwedge^k V^\vee[k].$$

The general case follows from

$$s^*s_*E = \pi_*(s_*s^*s_*E) = \pi_*(s_*\mathcal{O}_X \otimes s_*E) = \pi_*(s^*s_*\mathcal{O}_X \otimes E) = \bigoplus_{k=0}^{\text{rank} V} \bigwedge^k V^\vee[k] \otimes E = S^\bullet(V[-1])^\vee \otimes E.$$

\hfill \blacksquare
Corollary 5.4.4.

\[
\text{Hom}_E^\bullet(s,E,s,F) = \bigoplus_{\ell \in \mathbb{Z}} \sum_{k=0}^{\text{rank } V} \text{Hom}_E^{\ell-k}(\Lambda^k V^\vee \otimes E, F) = \bigoplus_{k=0}^{\text{rank } V} \text{Hom}_E^{\ell-k}(E, F \otimes S^\bullet(V[-1]))
\]

Proof. The first equality is by the adjunction \( s^* \vdash s_* \), the second by Proposition 5.4.3, the third by locally-freeness of \( V \).

We therefore arrived at the following

Theorem 5.4.5. Let \( X \) be a smooth \( G \)-variety and \( \pi : V \to X \) be a \( G \)-equivariant anti-semiample vector bundle. Suppose \( \mathcal{E} = \{ E_i \}_{i \in I} \) is a full \( V \)-geometric exceptional poset on \( \mathbb{D}^b(\text{Coh}_G(X)) \) with dual exceptional poset \( \mathcal{F} = \{ F_i \}_{i \in I^{\text{op}}} \). Then there is an \( A_{\text{fin}} \)-structure on

\[
A_E \cong \bigoplus_{i,j \in I} \sum_{\ell \in \mathbb{Z}} \text{Hom}_E^{\ell-k}(F_i, F_j \otimes \Lambda^k V),
\]

making it a finite dimensional \( A_{\text{fin}} \)-algebra augmented over \( S \), such that \( E(A_E) \) is a dg-algebra resolving \( \Lambda_E \) and hence \( \mathbb{D}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}(E(A_E)^{\text{op}}) \), where \( \mathbb{D}_{fg}(E(A_E)^{\text{op}}) \) denotes the full triangulated subcategory of \( \mathbb{D}^b(E(A_E)^{\text{op}}) \) consisting of complexes whose cohomologies are finitely generated over \( H^*(E(A_E)^{\text{op}}) \). Moreover, this equivalence restricts to an equivalence \( \mathbb{D}_{cs}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(E(A_E)^{\text{op}}) \).

Proof. Theorem 5.2.9 gives an \( A_{\text{fin}} \)-algebra \( A_E \) and a derived equivalence \( \mathbb{D}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(A_{E}^{\text{op}}) \). Proposition 5.3.7 and 5.3.8 shows there is an \( A_{\text{fin}} \)-structure on \( H^*(E(A_E)) = \text{Ext}^*_A(S,S) \) such that \( E(\text{Ext}_{A_E}^*(S,S)) \) is \( A_{\text{fin}} \)-isomorphic to \( \Lambda_E \). By Proposition 5.4.1 and 5.4.4,

\[
\text{Ext}_{A_E}^*(S,S) \cong \bigoplus_{i,j} \text{Hom}_{A_E}^*(s,F_i, s,F_j) \cong A_E
\]
as graded \( S \)-bimodules. Transferring the \( A_{\text{fin}} \)-structure on \( \text{Ext}_{A_E}^*(S,S) \) to \( A_E \), we have derived equivalences

\[
\mathbb{D}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(A_E^{\text{op}}) \cong \mathbb{D}_{fg}^b(E(\text{Ext}_{A_E}^*(S,S))^{\text{op}}) \cong \mathbb{D}_{fg}^b(E(A_E)^{\text{op}}).
\]
The fact that this equivalence restricts to an equivalence \( \mathbb{D}_{cs}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(E(A_E)^{\text{op}}) \) follows from Corollary 5.2.10.

Remark 5.4.6. One can also transfer the Adams grading on \( \text{Ext}_{A_E}^*(S,S) \) to \( A_E \). A natural guess for this Adams grading on \( A_E \) would be given by \( (j-i,k) \). Since we will not need this explicit description, we will not prove the statement.

Recall \( E(A_E) \) is the path algebra of a dg-quiver \( Q_E \). Hence we have shown that \( V \) is derived equivalent to a dg-quiver.

Corollary 5.4.7. Let \( X \) be a smooth \( G \)-variety and \( \pi : V \to X \) be a \( G \)-equivariant vector bundle. Suppose \( \mathcal{E} = \{ E_i \}_{i \in I} \) is a full \( V \)-geometric exceptional poset on \( \mathbb{D}^b(\text{Coh}_G(X)) \), then there exists a dg-quiver \( Q_E \) such that \( \mathbb{D}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(Q_E) \) and \( \mathbb{D}_{cs}^b(\text{Coh}_G(V)) \cong \mathbb{D}_{fg}^b(Q_E) \).

Example 5.4.8. Let \( X = \mathbb{P}^2 \) and \( V = 0 \). Let \( \mathcal{E} = (O, O(1), O(2)) \) be the Beilinson sequence. The classical tilting quiver is given by

\[
\begin{array}{c}
\bullet & \overrightarrow{a_0} & \bullet \\
v_0 & \overrightarrow{\sigma_1} & \overrightarrow{v_1} & \overrightarrow{b_1} & \overrightarrow{\sigma_2} & \overrightarrow{v_2} & \overrightarrow{b_2}
\end{array}
\]

with relations \( b_0 a_j = b_j a_0 \). Using the cohomology formula in the Appendix (Section A.2), the \( A_{\infty} \)-category \( A_E \) is given by

\[
A^\ell(v_0, v_0) = \text{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) = H^\ell(\mathbb{P}^2, O) = \begin{cases} 
\mathbb{C} & \text{if } \ell = 0 \\
0 & \text{otherwise}.
\end{cases}
\]
\( \mathcal{A}'(v_0, v_1) = \text{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) = H^{\ell-1}(\mathbb{P}^2, \Omega(2)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \)

\( \mathcal{A}'(v_0, v_2) = \text{Hom}^\ell(\Omega^2(2)[2], \mathcal{O}) = H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \)

\( \mathcal{A}'(v_1, v_0) = \text{Hom}^\ell(\Omega(1)[1], \Omega^2(2)[2]) = H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2)) = 0 \)

\( \mathcal{A}'(v_1, v_1) = \text{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) = H^\ell(\mathbb{P}^2, \mathcal{T} \otimes \Omega) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases} \)

\( \mathcal{A}'(v_1, v_2) = \text{Hom}^\ell(\Omega(1)[1], \mathcal{O}) = H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \)

\( \mathcal{A}'(v_2, v_0) = \text{Hom}^\ell(\mathcal{O}, \Omega^2(2)[2]) = H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1)) = 0. \)

\( \mathcal{A}'(v_2, v_1) = \text{Hom}^\ell(\mathcal{O}, \Omega(1)[1]) = H^{\ell+1}(\mathbb{P}^2, \Omega(1)) = 0. \)

\( \mathcal{A}'(v_2, v_2) = \text{Hom}^\ell(\mathcal{O}, \mathcal{O}) = H^\ell(\mathbb{P}^2, \mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases} \)

Hence the dg-quiver is given by

\[
\begin{array}{c}
\bullet & \overset{p}{\longrightarrow} & \bullet & \overset{a}{\longrightarrow} & \bullet & \overset{b}{\longrightarrow} & \bullet \\
\end{array}
\]

where black edges are of degree 0 and red edges are of degree \(-1\). Next, we would like to determine the differential. Since \( H^0(\mathcal{KQ}) \) is the classical tilting quiver in Equation (5.4.1), by a change of basis if necessary, we may assume the differential sends

\[
dp_i = b_{i+1}a_{i+2} - b_{i+2}a_{i+1}, \quad da_i = 0, \quad db_i = 0.
\]

More examples will be given in Section 6.3.

### 5.5 Superpotential

In this section, we construct a superpotential on the dg-tilting quiver \( Q_\mathcal{E} \) in the case when the total space of \( V \) is Calabi–Yau.

**Serre Functor on Equivariant Derived Category** We describe how to obtain equivariant Serre duality on a equivariant vector bundle \( V \), which the author learnt from Bridgeland-King-Reid [14]. Since \( X \) is projective and \( G \) is finite, one can find a \( G \)-invariant ample line bundle \( \mathcal{O}_X(1) \) on \( X \). Consider the embedding \( i : V \hookrightarrow \mathbb{P}(V \oplus \mathcal{O}_X(1)) \). Then \( i_* \) embeds \( D^b_{\text{coh}}(\text{Coh}^G(V)) \) into \( D^b_{\text{coh}}(\text{Coh}^G(\mathbb{P}(V \oplus \mathcal{O}_X(1)))) \) as a full subcategory. The Serre functor \((-) \otimes K_{\mathbb{P}(V \oplus \mathcal{O}_X(1))}[\dim V] \) on \( D^b_{\text{coh}}(\text{Coh}^G(\mathbb{P}(V \oplus \mathcal{O}_X(1)))) \) then restricts to a Serre functor \((-) \otimes K_V[\dim V] \) on \( D^b_{\text{coh}}(\text{Coh}^G(V)) \). In particular, if \( K_V \) is trivial as a \( G \)-equivariant vector bundle, \( D^b_{\text{coh}}(\text{Coh}^G(V)) \) becomes a Calabi–Yau category.

**Proposition 5.5.1.** Let \( \pi : V \to X \) be a \( G \)-equivariant vector bundle. Then \( K_V \) is trivial as an equivariant vector bundle if and only if \( \det V = K_X \) as an equivariant vector bundle. In particular, when \( G \) is trivial, \( V \) is (non-compact) Calabi–Yau if and only if \( \det V = K_X \).

**Proof.** We have a short exact sequence

\[
0 \to \pi^*V \to TV \to \pi^*TX \to 0.
\]

We have therefore

\[
\det TV = \det \pi^*V \otimes \det \pi^*TX = \pi^*(\det V \otimes K_X').
\]
If det $V = K_X$, then det $TV = \pi^*O_X$ which is trivial. If $V$ is Calabi-Yau, we have

$$O_X = O_V|_X = \pi^*(\det V \otimes K_X^\vee)|_X = \det V \otimes K_X^\vee.$$ 

Hence det $V = K_X$. \hfill \blacksquare

**Pairing on $A_\mathcal{E}$.** If $V$ is noncompact Calabi–Yau, there is a pairing on $A_\mathcal{E}$ as follows: consider the composition map

$$\text{Hom}^{\ell-k}(E, F \otimes \wedge^k V) \otimes \text{Hom}^{\dim X+\ell}(F, E \otimes \wedge^{\rank V-k} V) \rightarrow \text{Hom}^{\dim X}(E, E \otimes \det V)$$

$$f \otimes g \mapsto (\text{id} \otimes \wedge) \circ (g \otimes \text{id}_{\wedge^k V}) \circ f$$

Since $V$ is Calabi–Yau, det $V = K_X$. Composing with the trace map of Serre duality

$$\text{Hom}^{\dim X}(E, E \otimes K_X) \rightarrow \mathbb{K},$$

we obtain a pairing on $A_\mathcal{E}$ of degree $\dim V$

$$A_\mathcal{E}^\vee \otimes A_\mathcal{E}^{\dim V-\ell} \rightarrow \mathbb{K}. $$

One might ask whether the $A_{in}$-structure on $A_\mathcal{E}$ comes from a cyclic structure. The following theorem answers the question:

**Proposition 5.5.2.** Let $\pi : V \rightarrow X$ be an anti-semiample vector bundle which is $m$-Calabi–Yau. Let $\mathcal{E}$ be a $V$-geometric exceptional poset on $D^b(\text{Coh} (X))$. Then $A_\mathcal{E}$ has a cyclic structure of degree $m$ and hence $E(A)$ has a superpotential, i.e., $V$ is derived equivariant to a quiver with superpotential.

**Proof.** The tilting object $\bigoplus \pi^*E_i$ is classical since

$$\text{Hom}^{\ell}(\pi^*E_i, \pi^*E_j) = \bigoplus_{i,j \in I} \text{Hom}^{\ell}(\pi^*E_i, \pi^*E_j) = \text{Hom}^{\ell}(E_i, E_j \otimes S^\vee V^\vee) = 0$$

By [45, Prop. 3.3.1], $A_\mathcal{E}$ is an Calabi–Yau algebra. Now, since $A_\mathcal{E}$ is graded, by [70, Corollary 9.3, Theorem 12.1], $A_\mathcal{E}$ is equipped with a cyclic structure. \hfill \blacksquare

Examples of quivers with superpotential constructed this way will be given in Section 6.4.

### 5.6 Quotient Construction

In the last section, we have shown that if $V \rightarrow X$ is an anti-semiample vector bundle over a smooth projective variety, and $\mathcal{E}$ is a $V$-geometric exceptional poset on $D^b(\text{Coh} (X))$, then there exists a dg-quiver $Q_\mathcal{E}$ such that $D^b(\text{Coh} (X)) \cong D^b_q(Q_\mathcal{E}).$ In this section we study the following problem: Suppose there is a finite group $G$ acting on $X$ by automorphisms, and the group action lifts to an action on the vector bundle $V$ and the exceptional poset $\mathcal{E}$. We ask if we can construct a new quiver $Q_\mathcal{E}/G$ such that $D^b(\text{Coh}^G(V)) \cong D^b_q(Q_\mathcal{E}/G)$. Theorem 5.6.5 answers this affirmatively: recall the $Q_\mathcal{E}$ is the Koszul dual of an $A_{in}$-algebra $A_\mathcal{E}$. The correct construction for $Q_\mathcal{E}/G$ is to apply the quotient construction of $A_{in}$-algebras to $A_\mathcal{E}$ and take its Koszul dual, i.e., $Q_\mathcal{E}/G = E(A_\mathcal{E}/G)$. This quotient construction generalizes the McKay quiver.

**Proposition 5.6.1.** Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a finite poset of objects on $D^b(\text{Coh}^G(X))$ whose underlying objects form a full exceptional poset on $D^b(\text{Coh} (X))$. Then $\mathcal{E} \otimes \text{Irr} (G) = \{E_i \otimes \rho \}_{i \in I, \rho \in \text{Irr} (G)}$, with the partial order $E_i \otimes \rho \prec E_j \otimes \sigma$ if and only if $i \prec j$, is a full exceptional poset on $D^b(\text{Coh}^G(X))$. Moreover, if $\mathcal{E}$ is strong, then so is $\mathcal{E} \otimes \text{Irr} (G)$.
Proof. From
\[ \text{Hom}_d^k(E_i \otimes \rho, E_j \otimes \sigma) = [\text{Hom}_X^k(E_i \otimes \rho, E_j \otimes \sigma)]^G = [\text{Hom}_X^k(E_i, E_j) \otimes \text{Hom}(\rho, \sigma)]^G, \]
and Schur’s lemma, we see that each object \( E_i \otimes \rho \) is exceptional, and that \( \text{Hom}^*(E_i \otimes \rho, E_j \otimes \sigma) \) vanishes unless \( i \leq j \), hence \( \mathcal{E} \otimes \text{Irr}(G) \) is an exceptional poset. Also, if \( \mathcal{E} \) is strong, then so is \( \mathcal{E} \otimes \text{Irr}(G) \). It remains to show \( \mathcal{E} \otimes \text{Irr}(G) \) classically generates \( D^b(\text{Coh}^G(X)) \) if \( \mathcal{E} \) classically generates \( D^b(\text{Coh}^G(X)) \). Let \( F \in D(\mathcal{Q}\text{Coh}(X)) \) be such that \( \text{Hom}^*_G(E_i \otimes \rho, F) = 0 \) for all \( i \in I \) and \( \rho \in \text{Irr}(G) \). Then \( \text{Hom}^*_G(\rho, \text{Hom}^*(E_i, F)) = [\rho^* \otimes \text{Hom}^*(E_i, F)]^G = \text{Hom}^*_G(E_i \otimes \rho, F) = 0 \). Now, decomposing \( \text{Hom}^*(E_i, F) \) into a direct sum of irreducible representations of \( G \), we see that \( \text{Hom}^*(E_i, F) = 0 \) for all \( i \in I \). Since \( \mathcal{E} \) is full, this shows \( F = 0 \). Hence \( \mathcal{E} \otimes \text{Irr}(G) \) generates \( D^b(\text{Coh}^G(X)) \).

**Proposition 5.6.2.** Let \( \mathcal{F} \) be the dual sequence to \( \mathcal{E} \) in \( D^b(\text{Coh}(X)) \). Then \( \mathcal{F} \) has a natural lift to \( D^b(\text{Coh}^G(X)) \) and \( \mathcal{F} \otimes \text{Irr}(G) \) is the dual sequence to \( \mathcal{E} \otimes \text{Irr}(G) \) in \( D^b(\text{Coh}^G(X)) \).

**Proof.** Without loss of generality, we may assume \( \mathcal{E} \) is an exceptional sequence. Recall that by Proposition 2.8.5, \( F_i = L_{E_i \sigma} E_i = L_{E_i} \cdots L_{E_i \sigma} E_i \). Hence to show \( F_i \) has a natural lift to \( D^b(\text{Coh}^G(X)) \), it suffices to show that if any \( E, X \in D^b(\text{Coh}(X)) \) lift to \( D^b(\text{Coh}^G(X)) \), then so does \( L_E X \). This follows from \( G \)-equivariance of the evaluation map \( \text{Hom}^*(E, X) \otimes E \rightarrow X \). Since \( F_i = L_{E_i \sigma} E_i \), we have a triangle
\[ A \rightarrow E_i \rightarrow F_i \rightarrow A[1] \]
where both maps are equivariant and \( A \in \{ \mathcal{E}, \mathcal{I} \} \). Applying \( \text{Hom}^*(E_i, -) \), we see that \( \text{Hom}^*(F_i, F_i) = \text{Hom}^*(E_i, E_i) = \mathbb{K} \) is the trivial representation. Hence
\[ \text{Hom}^*_G(E_i \otimes \rho, F_j \otimes \sigma) = [\text{Hom}^*_G(E_i, F_j) \otimes \text{Hom}(\rho, \sigma)]^G = \begin{cases} \text{Hom}_G(\rho, \sigma) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \mathbb{K} & \text{if } i = j \text{ and } \rho = \sigma \\ 0 & \text{otherwise.} \end{cases} \]

By uniqueness of dual exceptional sequence in Proposition 2.8.5, \( \mathcal{F} \otimes \text{Irr}(G) \) is the dual exceptional sequence to \( \mathcal{E} \otimes \text{Irr}(G) \).

**Corollary 5.6.3.** Let \( \mathcal{E} \) be a finite poset of objects in \( D^b(\text{Coh}^G(V)) \). Suppose the underlying objects of \( \mathcal{E} \) in \( D^b(\text{Coh}(V)) \) form a \( V \)-geometric full exceptional poset on \( D^b(\text{Coh}(V)) \), then \( \mathcal{E} \otimes \text{Irr}(G) \) is a full \( V \)-geometric exceptional poset on \( D^b(\text{Coh}^G(V)) \).

**Proof.** It suffices to show \( \text{Hom}^*_G(E_i \otimes \rho, E_j \otimes \sigma \otimes S^k V^*) \) vanishes for all \( \ell \geq 1, k \geq 0 \) and all \( \rho, \sigma \in \text{Irr}(G) \). This follows immediately since
\[ \text{Hom}^*_G(E_i \otimes \rho, E_j \otimes \sigma \otimes S^k V^*) = [\text{Hom}(\rho, \sigma) \otimes \text{Hom}^*_X(E_i, E_j \otimes S^k V^*)]^G. \]

**Proposition 5.6.4.** Let \( \pi: V \rightarrow X \) be a \( G \)-equivariant vector bundle and \( \mathcal{E} \) be a finite poset of objects in \( D^b(\text{Coh}^G(X)) \) such that the underlying objects in \( D^b(\text{Coh}(X)) \) form a full exceptional poset. Then there exists an \( A_{\mathcal{E}} \)-structure on \( \mathcal{A}_{\mathcal{E}} \) such that

1. there is an equivalence \( \Phi: D^b(\text{Coh}(V)) \cong \text{Per}_\infty(\Lambda_{\mathcal{E}}^{op}) \).
2. If \( M, N \in D^b(\text{Coh}(V)) \) lift to \( D^b(\text{Coh}^G(V)) \), then \( G \) acts on \( \Phi(M) \) and \( \Phi(N) \) and the isomorphism \( \Phi: \text{Hom}(M, N) \cong \text{Hom}(\Phi(M), \Phi(N)) \) is equivariant.
3. The finite group \( G \) acts on \( \mathcal{A}_{\mathcal{E}} \) by strict \( A_{\mathcal{E}} \)-isomorphisms, and there is a derived equivalence \( D^b(\text{Coh}^G(V)) \cong \text{Per}_\infty((\Lambda_{\mathcal{E}}/G)^{op}) \).
4. If $\mathcal{E}$ is $V$-geometric and $V^\vee$ is semiaimple, there is an equivalence $D^b(\text{Coh}^G(V)) \cong D^b_{fg}(\text{L}_E/G)^{op}$. Moreover, this equivalence restricts to an equivalence $D^b_{fg}(\text{Coh}^G(V)) \cong D^b_{fg}(\text{L}_E/G)^{op}$.

**Proof.** Statement 1 is the content of Proposition 3.5.4. For statement 2, recall that in Theorem 5.2.4, we constructed a dg-algebra $R = \text{Hom}(I_E, I_E \otimes S^\bullet V^\vee)$. $I_E$ is a $G$-equivariant injective resolution for $E$, and an equivalence $\Psi : D^b(\text{Coh}(V)) \cong \text{Per}(R^p)$ by setting

$$
\Psi(M) = \text{Hom}(I_E, \pi_* I_M)
$$

$$
\Psi(M \xrightarrow{\phi} N) = (\text{Hom}(I_E, \pi_* I_M) \xrightarrow{\pi_* \phi} \text{Hom}(I_E, \pi_* I_N))
$$

where $I_M$ and $I_N$ are $G$-equivariant injective resolution for $M$ and $N$ respectively. Since $I_E$ and $I_M$ are both equipped with a $G$-linearization, $\Psi(M) = \text{Hom}(I_E, \pi_* I_M)$ is equipped with a $G$-action. We will choose all maps equivariantly: for example, we will choose the lifting $I_M$ to be $\lambda^M_\phi$ etc. For any $\phi \in \text{Hom}(M, N)$,

$$
\Psi(g \cdot (M \xrightarrow{\phi} N)) = \Psi(M \xrightarrow{\lambda^M_\phi (g^{-1})} M \xrightarrow{(g^{-1})^* \phi} (g^{-1})^* N \xrightarrow{(g^{-1})^* \phi} N)
$$

sends the map $h \in \text{Hom}(I_E, \pi_* I_M)$ to the morphism in $\text{Hom}(I_E, \pi_* I_N)$ given by

$$
I_E \xrightarrow{h} \pi_* I_M \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* N} \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_N.
$$

On the other hand, $g \cdot \Psi(M \xrightarrow{\phi} N)$ sends $h \in \text{Hom}(I_E, \pi_* I_M)$ to the morphism in $\text{Hom}(I_E, \pi_* I_N)$ given by

$$
g \cdot (I_E \xrightarrow{I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* M} \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* N} = g \cdot (I_E \xrightarrow{g^* h} g^* I_E \xrightarrow{g^* I_{(g^{-1})^* \phi}} g^* I_{(g^{-1})^* M} \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* N})
$$

\begin{align*}
&= (I_E \xrightarrow{I_{(g^{-1})^* \phi}} \pi_* I_M \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* N}) = (I_E \xrightarrow{\pi_* I_{(g^{-1})^* \phi}} \pi_* I_{(g^{-1})^* N})
\end{align*}

since everything is equivariant. Hence the isomorphism $\Psi : \text{Hom}(M, N) \cong \text{Hom}(\Psi(M), \Psi(N))$ is equivariant. Taking cohomology, we arrive at statement 2. For statement 3, choose $G$-invariant injective resolution $I_E$, for each $E_i$ then for any $\rho \in \text{Irr}(G)$, $I_E \otimes \rho$ is an injective resolution for $E_i \otimes \rho$. Write $I_E = \bigoplus_{i \in I} I_{E_i}$ and let $R = \text{RHom}(I_E, I_E \otimes S^\bullet V^\vee)$. Then

$$
R/G = \bigoplus_{\rho, \sigma \in \text{Irr}(G)} \text{RHom}_G(I_E \otimes \rho, I_E \otimes S^\bullet V^\vee \otimes \sigma)
$$

and by Theorem 5.2.4 and Corollary 5.6.3, there is an equivalence $D^b(\text{Coh}^G(V)) \cong \text{Per}(R/G)^{op}$.

By Proposition 3.5.4, we can choose minimal models of $R/G$ and $R$ such that there are $A_\infty$-quasi-isomorphisms

$$
R/G \cong H^\bullet(R/G) \cong H^\bullet(R)/G = \Lambda_E/G
$$

and we have the desired equivalence. For statement 4, by Proposition 5.2.9, $\Lambda_E$ is an algebra. It suffices to show $\Lambda_E/G$ has finite global dimension, for then $\text{Per}(\Lambda_E(G)^{op}) = D^b((\Lambda_E/G)^{op})$. By Proposition 5.2.9, $\Lambda_E$ has finite global dimension. By [3, III Theorem 4.4], the global dimension of $\Lambda_E^{op} \# G$ is the same as that of $\Lambda_E^{op}$, which is the same as that of $\Lambda_E$, hence also finite. Now by Theorem 3.5.2 and Proposition 3.5.5, $(\Lambda_E^{op} \# G)^{op}$ and $\Lambda_E/G$ are Morita equivalent, and hence have the same global dimension. Thus $\Lambda_E/G$ has finite global dimension.
The product vector bundle $\pi$ is equivalent to $V$.

In this section, we study the following problem: Suppose we have vector bundles $5.7$ Product Construction

**Proposition 5.7.1.** Let $\pi : V \to X$ be a strong $G$-equivariant anti-semiample vector bundle. Suppose $E$ is a finite set of objects in $D^b(Coh^G(X))$ such that the underlying objects form a full exceptional poset on $D^b(Coh(X))$. Then $E$ acts on $A_E$ by strict $A_{\infty}$-isomorphisms and $D^b(Coh^G(V)) = D^b_j(E(A_E/G)^{op})$.

**Proof.** By Proposition 5.6.4, $D^b(Coh^G(V)) \cong D^b_j((A_E/G)^{op})$. Now by Corollary 5.5.9 and Proposition 3.5.4, $H^*(E(A_E/G)) = H^*(E(A_E)/G) = A_E/G$. Hence $E(A_E/G)$ is quasi-isomorphic to $A_E/G$ and we have $D^b(Coh^G(V)) = D^b_j(E(A_E/G)^{op})$.

Examples illustrating this section will be given in Section 6.5.

### 5.7 Product Construction

In this section, we study the following problem: Suppose we have vector bundles $\pi : V \to X$ and $\pi : W \to Y$, together with exceptional posets $E$ on $X$ and $F$ on $Y$ which are respectively $V$-geometric and $W$-geometric. Then we can construct dg-quivers $Q_E = E(A_E)$ and $Q_F = E(A_F)$ which are derived equivalent to $V$ and $W$ respectively. We ask if there is a product construction of dg-quivers such that the product vector bundle $\pi : V \times W : V \times W \to X \times Y$ is derived equivalent to $Q_E \times Q_F$. Proposition 6.5.3 answers this question. The correct product construction should be applied at the $A_{\infty}$-algebra level: We should set $Q_E \times Q_F = E(A_E \otimes A_F)$.

Note in the following that the projection map from $X \times Y$ to both $X$ and $Y$ are flat, hence pullback by both maps are exact.

**Proposition 5.7.1.** Let $X$ and $Y$ be smooth projective varieties and $E = \{E_i\}_{i \in I}$ and $F = \{F_j\}_{j \in J}$ be full exceptional posets on $X$ and $Y$ respectively. Then $E \boxtimes F = \{E_i \otimes F_j\}_{i \times j}$ is a full exceptional poset on $X \times Y$, where the partial order on $I \times J$ is given by $(i, j) \preceq (i', j')$ if and only if $i \preceq i'$ and $j \preceq j'$. If both $E$ and $F$ are strong, then so is $E \boxtimes F$.

**Proof.** Using the Künneth formula

\[
\text{RHom}(E_i \boxtimes F_j, E_i' \boxtimes F_j') = R\Gamma(X \times Y, \text{RHom}(E_i \boxtimes F_j, E_i' \boxtimes F_j'))
\]

\[
= R\Gamma(X \times Y, (E_i' \boxtimes E_j') \boxtimes (F_j' \boxtimes F_j'))
\]

\[
= R\Gamma(X, E_i' \boxtimes E_j') \otimes R\Gamma(Y, F_j' \boxtimes F_j')
\]

\[
= R\Gamma(X, \text{RHom}(E_i', E_j)) \otimes R\Gamma(Y, \text{RHom}(F_j', F_j'))
\]

\[
= \text{RHom}(E_i, E_j) \otimes \text{RHom}(F_j, F_j'),
\]

we have

\[
\text{Hom}^0_{X \times Y}(E_i \boxtimes F_j, E_i' \boxtimes F_j') = \bigoplus_{p+q=\ell} \text{Hom}^0_{X}(E_i, E_j) \otimes \text{Hom}^0_{Y}(F_j, F_j').
\]

Hence $E \boxtimes F$ is an exceptional poset. Also, if both $E$ and $F$ are strong, then so is $E \boxtimes F$. Next, we show $E \boxtimes F$ generates $D(Coh(X))$. The following proof is given by Bondal and van den Bergh [9, Lemma
3.4.1]. Write $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{j \in J} F_j$. Let $A \in D(QCoh(X))$ be such that $\text{Hom}^\bullet(E \boxtimes^L F, A) = 0$. Let $\pi_X$ and $\pi_Y$ be projections of $X \times Y$ on the first and second factor. We have a commutative diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_X} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi_Y} & Y
\end{array}
$$

Since

$$0 = \text{RHom}(E \boxtimes^L F, A) = \text{RHom}(\pi_Y^* E, \text{RHom}(\pi_Y^* F, A)) = \text{RHom}(E, \text{R} \pi_X \ast \text{RHom}(\pi_Y^* F, A))$$

and $E$ generates $D(QCoh(X))$, we have $\text{R} \pi_X \ast \text{RHom}(\pi_Y^* F, A) = 0$. This implies that

$$0 = \text{R} \Gamma(X \times Y, \text{R} \pi_X \ast \text{RHom}(\pi_Y^* F, A)) = \text{RHom}(\pi_Y^* F, A) = \text{RHom}(F, \text{R} \pi_Y \ast A)$$

and hence $\text{R} \pi_Y \ast A = 0$ since $F$ generates $D(QCoh(Y))$. Now, from

$$\text{R} \Gamma(X \times Y, A) = \text{R} \Gamma(Y, \text{R} \pi_Y \ast A) = 0,$$

we conclude $A = 0$ and hence $E \boxtimes^L F$ generates $D(QCoh(X \times Y))$. $\blacksquare$

**Proposition 5.7.2.** Let $E$ and $F$ be full exceptional posets and $\mathcal{M}$ and $\mathcal{N}$ be their dual exceptional posets. Then $\mathcal{M} \boxtimes^L \mathcal{N}$ is the dual exceptional poset of $E \boxtimes^L F$.

**Proof.** This follows from the Kunneth theorem

$$\text{RHom}(E_i \boxtimes^L F_j, M_k \boxtimes^L N_\ell) = \text{RHom}(E_i, M_k) \otimes \text{RHom}(F_j, N_\ell).$$

$\blacksquare$

**Theorem 5.7.3.** Let $\pi_Y : V \to X$ and $\pi_W : W \to Y$ be vector bundles. Let $E$ be a $V$-geometric full exceptional poset on $D^b(Coh(X))$ and $F$ be a $W$-geometric full exceptional poset on $Y$. Then $E \boxtimes^L F$ is $V \times W$-geometric, the tensor product $A_\infty$-structure on $A_E \otimes A_F$ is $A_{\text{fin}}$, and there is an equivalence

$$D^b(Coh(V \times W)) \cong D^b_\text{fin}(E(A_E \otimes A_F)^{op}).$$

**Proof.** Since

$$\text{Hom}^\bullet(E_i \boxtimes^L F_j, E'_i \boxtimes^L F'_j \otimes S^*(V \times W)^\vee) = \text{Hom}^\bullet(E_i \boxtimes^L F_j, E'_i \boxtimes^L F'_j \otimes S^*V^\vee \boxtimes S^*W^\vee)$$

$$= \text{Hom}^\bullet(E_i, E'_i \otimes S^*V^\vee) \otimes \text{Hom}^\bullet(F_j, F'_j \otimes S^*W^\vee)$$

we conclude that $E \boxtimes^L F$ is $V \times W$-geometric and $A_{E \boxtimes^L F} = A_E \otimes A_F$ as graded algebras.

By Lemma 5.3.5, $A_E$, $A_F$ and $A_{E \boxtimes^L F} = A_E \otimes A_F$ are all locally finite. Hence by Proposition 3.6.1, we have a chain of $A_\infty$-quasi-isomorphisms

$$A_{E \boxtimes^L F} \cong H^\bullet(E(A_E \boxtimes^L F)) \cong H^\bullet(E(A_E) \otimes A_F) \cong H^\bullet(E(A_E)) \otimes H^\bullet(E(A_F)) = A_E \otimes A_F.$$

To see that the tensor product $A_\infty$-structure on $A_E \otimes A_F$ is $A_{\text{fin}}$, observe that since $A_E \otimes A_F$ is isomorphic to $A_{E \boxtimes^L F}$ as vector spaces, $A_E \otimes A_F$ inherits the same Adams grading on $A_{E \boxtimes^L F}$ (which in turns is inherited from the algebra $\text{Ext}_{A_{E \boxtimes^L F}}^\bullet(S, S)$). Since the tensor product $A_\infty$-structure is given by a finite sum of tensor products and compositions of the two $A_\infty$-structures, the tensor product $A_\infty$-structures also preserves the Adams grading and by Lemma 5.3.7, the tensor product $A_\infty$-structure is also $A_{\text{fin}}$. $\blacksquare$

Examples illustrating this theorem will be given in Section 6.4.
Chapter 6

Examples

This chapter is a list of examples illustrating theorems in Chapter 5.

Section 6.1 contains examples illustrating Theorem 5.2.9 which produces quivers with relations derived equivalent to total spaces of vector bundles.

Section 6.2 introduces a class of algebras called Koszul algebras whose dg-resolution is particularly easy to describe. We also give some examples of vector bundles whose classical tilting algebras are Koszul.

Section 6.3 contains some worked out examples illustrating Theorem 5.4.5, constructing derived equivalences between total spaces of vector bundles and dg-quivers, and if the total space of vector bundles are Calabi–Yau, quivers with superpotential. These examples are calculated by first determining the classical tilting algebras, then try to work out the dg-resolutions to determine the dg-quiver. In Calabi–Yau examples of dimension no greater than 4, there are enough constraints and hence the classical tilting algebras determine their dg-resolutions.

Section 6.4 contains a list of examples by applying the product construction in Theorem 5.7.3. Since the general formulae for the cyclic \( A_\infty \)-tensor product defined by Amorim and Tu [2] are not known, we only work with the case when one of the \( A_\infty \)-algebras is an honest algebra.

Section 6.5 contains a list of examples illustrating the quotient construction in Theorem 5.6.5. Unless otherwise specified, all cohomological formulae can be found in the Appendix.

6.1 Classical Tilting Quivers

Example 6.1.1. Take \( X = \mathbb{P}^n \), the full exceptional collection \( \mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n)) \) and \( V = T_{\mathbb{P}^n}^\vee \). Write \( W = \mathbb{C}^{n+1} \). The Euler exact sequence
\[
0 \to \mathcal{O} \to W(1) \to T_{\mathbb{P}^n} \to 0
\]
induces a short exact sequence via Koszul resolution [35, 12.12] for all \( k \geq 1 \)
\[
0 \to S^{k-1}W(k-1) \to S^kW(k) \to S^kT_{\mathbb{P}^n} \to 0.
\]
(6.1.1)

Tensoring the above sequence with \( \mathcal{O}(j-i) \) and using the vanishing \( H^i(\mathbb{P}^n, \mathcal{O}(p)) = 0 \) whenever \( i \geq 1 \) and \( p \geq -n \), we conclude that for all \( k \geq 0, \ell \geq 1 \) and \( 0 \leq i, j \leq n \)
\[
\text{Ext}_{T_{\mathbb{P}^n}}^\ell(\mathcal{O}(i), \mathcal{O}(j) \otimes S^kT_{\mathbb{P}^n}) = H^\ell(\mathbb{P}^n, S^kT_{\mathbb{P}^n}(j-i)) = 0.
\]
Hence \( \mathcal{E} \) is \( T_{\mathbb{P}^n}^\vee \)-geometric. Since \( T_{\mathbb{P}^n} \) is ample, by Corollary 5.2.9, we have a derived equivalence \( D^b(\text{Coh} (T_{\mathbb{P}^n}^\vee)) \cong D^b(\Lambda^{T_{\mathbb{P}^n}}_\mathcal{E}) \), where
\[
\Lambda_{\mathcal{E}} = \bigoplus_{i,j=0}^n \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^jT_{\mathbb{P}^n}).
\]
We claim that $\Lambda_{E}$ is isomorphic to the path algebra of the quiver

\[
\begin{array}{ccc}
\bullet & \xleftarrow{a_{10}} & \cdots \\
\vdots & \vdots & \vdots \\
\bullet & \xleftarrow{a_{2n}} & \cdots \\
\bullet & \xleftarrow{b_{1n}} & \cdots \\
\end{array}
\]

with relations

\[
\begin{align*}
\sum_{j=0}^{n} a_{ij} b_{ij} &= 0 \\
a_{i+1,j} a_{ij} &= a_{i+1,j+1} a_{i,j} \\
b_{i+1,j} a_{i+1,j} &= b_{i,j} a_{i+1,j+1}
\end{align*}
\]

First, we show that $\Lambda_{E}$ is generated as a $\bigoplus_{i=1}^{n} \text{Hom}(\mathcal{O}(i),\mathcal{O}(i))$ algebra by the vector space

\[
U = \bigoplus_{i=0}^{n} \text{Hom}(\mathcal{O}(i-1),\mathcal{O}(i)) \oplus \text{Hom}(\mathcal{O}(i),\mathcal{O}(i-1) \otimes T_{P_n}).
\]

We will need the following lemma.

**Lemma 6.1.2.** The multiplication map

\[
H^{0}(\mathbb{P}^{n}, S^{k} T_{P_n}(\ell)) \otimes H^{0}(\mathbb{P}^{n}, S^{k'} T_{P_n}(\ell')) \to H^{0}(\mathbb{P}^{n}, S^{k+k'} T_{P_n}(\ell + \ell'))
\]

is surjective if $k + \ell \geq 0$ and $k' + \ell' \geq 0$.

**Proof.** Twisting the short exact sequence (6.1.1) by $\mathcal{O}(\ell)$, we get a short exact sequence

\[
0 \to S^{k-1} W(k-1+\ell) \to S^{k} W(k+\ell) \to S^{k} T_{P_n}(\ell) \to 0.
\]

By naturality, we have the following commutative diagram

\[
\begin{array}{ccc}
H^{0}(\mathbb{P}^{n}, S^{k} W(k+\ell)) \otimes H^{0}(\mathbb{P}^{n}, S^{k'} W(k'+\ell')) & \to & H^{0}(\mathbb{P}^{n}, S^{k+k'} W(k+k'+\ell+\ell')) \\
\downarrow & & \downarrow \\
H^{0}(\mathbb{P}^{n}, S^{k} T_{P_n}(\ell)) \otimes H^{0}(\mathbb{P}^{n}, S^{k'} T_{P_n}(\ell')) & \to & H^{0}(\mathbb{P}^{n}, S^{k+k'} T_{P_n}(\ell + \ell')).
\end{array}
\]

The top horizontal map is surjective since both $S^{k} W \otimes S^{k'} W \to S^{k+k'} W$ and $H^{0}(\mathbb{P}^{n}, \mathcal{O}(k+\ell)) \otimes H^{0}(\mathbb{P}^{n}, \mathcal{O}(k'+\ell')) \to H^{0}(\mathbb{P}^{n}, \mathcal{O}(k+k'+\ell+\ell'))$ are surjective maps. The two vertical maps are also surjective since, as $k + \ell \geq 0$, $H^{1}(\mathbb{P}^{n}, S^{k-1} W(k-1+\ell)) = S^{k-1} W \otimes H^{1}(\mathbb{P}^{n}, \mathcal{O}(k-1+\ell)) = 0$.

Hence the bottom horizontal map is also surjective.

With the lemma, we can show that $\Lambda_{E}$ is generated by $U$. We separate the problem into three cases.

**Case 1 $i = j$.** By the lemma,

\[
H^{0}(\mathbb{P}^{n}, \mathcal{O}(1)) \otimes H^{0}(\mathbb{P}^{n}, T_{P_n}(-1)) \to H^{0}(\mathbb{P}^{n}, T_{P_n})
\]

is surjective, and hence so is the multiplication map

\[
\text{Hom}(\mathcal{O}(i), \mathcal{O}(i+1)) \otimes \text{Hom}(\mathcal{O}(i+1), T_{P_n}(-1)) \to \text{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes T_{P_n}).
\]
Again by the lemma, the map
\[ H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}) \otimes H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \rightarrow H^0(\mathbb{P}^n, S^{k+1} T_{\mathbb{P}^n}) \]
is surjective, and hence so is
\[ \text{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^k T_{\mathbb{P}^n}) \otimes \text{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes T_{\mathbb{P}^n}) \rightarrow \text{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^{k+1} T_{\mathbb{P}^n}). \]
This shows the vector subspace \( \text{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^k T_{\mathbb{P}^n}) \) is generated by \( U \).

**Case 2** \( i < j \). Surjectivity of
\[ H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}) \otimes H^0(\mathbb{P}^n, \mathcal{O}(j-i)) \rightarrow H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(j-i)) \]
implies surjectivity of
\[ \text{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \otimes \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \rightarrow \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}). \]
Since both \( \text{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \) and \( \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \) are generated by \( U \), so is \( \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \).

**Case 3** \( i > j \). Since
\[ \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}^\vee) = H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}^\vee(j-i)) = 0, \]
we may assume \( k \geq i - j \). If \( k = i - j \), surjectivity of
\[ H^0(\mathbb{P}^n, S^{k-1} T_{\mathbb{P}^n}(-k+1)) \otimes H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1)) \rightarrow H^0(\mathbb{P}^n, S^{k} T_{\mathbb{P}^n}(-k)) \]
implies surjectivity of
\[ \text{Hom}(\mathcal{O}(i), \mathcal{O}(j+1) \otimes S^k T_{\mathbb{P}^n}) \otimes \text{Hom}(\mathcal{O}(j+1), \mathcal{O}(j) \otimes T_{\mathbb{P}^n}) \rightarrow \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}). \]
Hence \( \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \) is generated by \( U \). For general \( k \geq i - j \), surjectivity of
\[ H^0(\mathbb{P}^n, S^{i-j} T_{\mathbb{P}^n}(j-i)) \otimes H^0(\mathbb{P}^n, S^{k-i+j} T_{\mathbb{P}^n}) \rightarrow H^0(\mathbb{P}^n, S^{k} T_{\mathbb{P}^n} \otimes \mathcal{O}(j-i)) \]
implies surjectivity of
\[ \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^{i-j} T_{\mathbb{P}^n}) \otimes \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^{k-i+j} T_{\mathbb{P}^n}) \rightarrow \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}). \]
This implies \( \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \) is generated by \( U \).

**Example 6.1.3.** Let us look at the special case \( X = \mathbb{P}^2 \) and \( V = T_{\mathbb{P}^2}^\vee \), which is Calabi–Yau 4. The tilting quiver becomes

```
\[
\begin{array}{c}
\bullet \quad \bullet \\
| & | \\
| & |
\end{array}
\]
```

with relations
\[
\begin{align*}
\sum_{j=0}^{2} a_j b_j &= 0 \\
\sum_{j=0}^{2} b_j a_j &= 0 \\
\sum_{j=0}^{2} c_j d_j &= 0 \\
\sum_{j=0}^{2} d_j c_j &= 0 \\
c_{j+1} a_j &= c_j a_{j+1} \\
b_{j+1} d_j &= b_{j+1} d_j \\
a_j b_k &= d_k c_j
\end{align*}
\]

The relations listed above are not all independent: the relations \( a_j b_k = d_k c_j \) and \( \sum_{j=0}^{2} a_j b_j = 0 \) together imply \( \sum_{j=0}^{2} d_j c_j = 0 \).
Example 6.1.4. Take $X = \mathbb{P}^n$, the full strongly exceptional collection $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ and $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m)$, where $0 \leq a_1 \leq \cdots \leq a_m$. Since

$$S^k V' = \bigoplus_{\sum k_m = k, k_m \geq 0}^{p} \mathcal{O} \left( \sum_{m=1}^{p} k_m a_m \right)$$

for all $\ell \geq 1$, $k \geq 0$ and $1 \leq i, j \leq n$,

the exceptional collection $\mathcal{E}$ is $V$-geometric:

$$\text{Ext}^\ell(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k V') = \bigoplus_{\sum k_m = k, k_m \geq 0}^{p} H^\ell(\mathbb{P}^n, \mathcal{O}(j - i + \sum_{m=1}^{p} k_m a_m)) = 0.$$ 

Since $a_m \geq 0$, $V'$ is globally generated. Hence by Corollary 5.2.9, we have a derived equivalence $D^b(Coh(V)) \cong D^b_f(\mathcal{A}^{gp}_V)$, where

$$\mathcal{A}_\mathcal{E} = \bigoplus_{i,j=0}^{n} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^* V')$$ 

and

$$= \bigoplus_{k_1, \ldots, k_m \geq 0}^{p} \bigoplus_{i,j=0}^{n} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes \left( \sum_{m=1}^{p} k_m a_m \right)).$$

Note that $\mathcal{A}_\mathcal{E}$ is $\mathbb{Z} \times \mathbb{Z}^p$-graded by $(j - i, k_1, \ldots, k_m)$. Now, let $q$ be such that $a_1 \leq \cdots \leq a_q \leq n < a_{q+1} \leq \cdots \leq a_p$. We claim that $\mathcal{A}_\mathcal{E}$ is generated as a $\bigoplus_{i=1}^{n} \text{Hom}(\mathcal{O}(i), \mathcal{O}(i))$ algebra by the vector subspace

$$U = \bigoplus_{i=1}^{n} \text{Hom}(\mathcal{O}(i-1), \mathcal{O}(i)) \oplus \bigoplus_{m=q+1}^{p} \text{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a_m)) \oplus \bigoplus_{m=1}^{q} \bigoplus_{j=a_m}^{n} \text{Hom}(\mathcal{O}(j), \mathcal{O}(j-a_m) \otimes \mathcal{O}(a_m)).$$

First, observe that the multiplication map,

$$\text{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \otimes \bigoplus_{m=1}^{p} \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a_m)) \otimes_{k_m} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes \mathcal{O}(\sum_{m=1}^{p} k_m a_m))$$

which corresponds to multiplication of homogeneous polynomials, is surjective.

Now, $\text{Hom}(\mathcal{O}(i), \mathcal{O}(j))$ is generated by $\bigoplus_{i=1}^{n} \text{Hom}(\mathcal{O}(i-1), \mathcal{O}(i))$. For $a_m \geq n$, the map

$$\text{Hom}(\mathcal{O}(j), \mathcal{O}(n)) \otimes \text{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a)) \rightarrow \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a_m))$$

is surjective, and for $a_m \leq n$, the map

$$\text{Hom}(\mathcal{O}(j), \mathcal{O}(j-a) \otimes \mathcal{O}(a)) \otimes \text{Hom}(\mathcal{O}(j-a), \mathcal{O}(a)) \rightarrow \text{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a))$$

is surjective. Hence $\mathcal{A}_\mathcal{E}$ is generated by $U$.

Example 6.1.5. Let us consider the special case $V = \mathcal{O}_{\mathbb{P}^n}(-a)$ over $\mathbb{P}^n$, where $a \geq 0$.

**Case 1** $a \geq n$. The tilting quiver is

![Diagram](image)

with relations $a_{i+1,j+1}a_{ij} = a_{i+1}a_{i+1,j+1}$ The vertex $v_i$ correspond to $\mathcal{O}(i)$, the arrows $a_{ij}$ corresponds to $\text{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$, and the arrows $b_i$ correspond to $\text{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a))$. 

Case 2 $0 \leq a \leq n$. The tilting quiver is

that is, $n+1$ arrows from $v_i$ to $v_{i+1}$ and one arrow from $v_i$ to $v_{i-a}$. The vertex $v_i$ correspond to $\mathcal{O}(i)$, the arrows $a_{ij}$ corresponds to $\text{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$, and the arrows $b_i$ correspond to $\text{Hom}(\mathcal{O}(i), \mathcal{O}(i-a) \oplus \mathcal{O}(a))$.

The general case $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m)$ is then given by the superposition of the tilting quivers corresponding to $\mathcal{O}(-a_1), \cdots, \mathcal{O}(-a_p)$.

Example 6.1.6. Consider the special case $V = \mathcal{O}_{P^n}(-n-1) = K_{P^n}$. The tilting quiver becomes

with relations $a_{i+1,k}a_{ij} = a_{i+1,j}a_{ik}$.

Example 6.1.7. Consider the Hirzebruch surface $\mathbb{F}_r = \mathbb{P}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(r))$. It is a toric surface defined by the complete fan in $\mathbb{R}^2$ given by $u_1 = (-1,r), u_2 = (0,1), u_3 = (1,0), u_4 = (0,-1)$. Denote by $D_i$ the divisor defined by $u_i$. Then $D_1$ and $D_3$ are both linearly equivalent to the divisor defined by the fiber of the projection $\mathbb{F}_r \rightarrow \mathbb{P}^1$. The divisor $D_2$ is the exceptional divisor $\mathbb{P}(\mathcal{O}(n))$ and $D_4$ is linearly equivalent to the divisor defined by $\mathbb{P}(\mathcal{O})$. We have the linear relation $\mathcal{O}(D_1) = \mathcal{O}(D_3)$ and $\mathcal{O}(D_2) = \mathcal{O}(D_4 - rD_3)$. The picard group Pic($\mathbb{F}_r$) is thus the free abelian group generated by $\mathcal{O}(D_1)$ and $\mathcal{O}(D_4)$. A line bundle $\mathcal{O}(a_3,a_4) = \mathcal{O}(a_3D_3 + a_4D_4)$ is globally generated (resp. ample) if and only if $a_i \geq 0$ (resp. $a_i > 0$) for $i = 3, 4$. From the generalized Euler sequence

we see that

$$K_{\mathbb{F}_r}^{-1} = \mathcal{O}(D_1 + D_2 + D_3 + D_4) = \mathcal{O}(2 - r, 2).$$

For all $s \geq -1$, $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(s+1,1), \mathcal{O}(s+2,1))$ is a strongly full exceptional sequence [33]. In particular, when $s = -1$, we have the strongly full exceptional sequence

$$\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(0,1), \mathcal{O}(1,1)).$$

We claim that $\mathcal{E}$ is geometric with respect to the vector bundle $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m, -b_m)$, where $a_m, b_m \geq 0$. In particular, it is geometric with respect to $K_{\mathbb{F}_r}, K_{\mathbb{F}_1}$ and $K_{\mathbb{F}_2}$. Indeed, the vanishing of higher Ext groups are reduced to the following vanishing of cohomology of line bundles.

Lemma 6.1.8. For $\ell \geq 1$, $H^\ell(\mathbb{F}_r, \mathcal{O}(a,b)) = 0$ whenever $a, b \geq -1$.

Proof. The case $a, b \geq 0$ follows from Demazure vanishing theorem [18, Theorem 9.2.3]. The case $-1 \leq a, b \leq 0$ follows from Batyrev-Borisov vanishing theorem [18, Theorem 9.2.7].
In section 5.4, we produced an 6.2 Koszul Algebras

Let \( S \Lambda \). This \( A \)

Hence

Take the vector bundle \( E \)

The set \( \mathcal{E} = \{ \Sigma^\lambda U^\nu : \lambda \in Y_{k,n} \} \) is a full strongly exceptional collection on \( G \). In the case \( G = \text{Gr}(2,4) \), the sequence \( (\mathcal{O}, U^\nu, \Sigma^{2,0} U^\nu, \Sigma^{1,1} U^\nu, \Sigma^{2,1} U^\nu, \Sigma^{2,2} U^\nu) = (\mathcal{O}, U^\nu, S^2 U^\nu, \mathcal{O}(1), U^\nu(1), \mathcal{O}(2)) \) is full and strongly exceptional, with tilting quiver

**Example 6.1.10.** Take the vector bundle \( V = \bigoplus_{i=1}^m \mathcal{O}(-a_i) \) over \( \text{Gr}(k,n) \), where \( a_i \geq 0 \). Since \( \mathcal{O}(1) \)
defines the plucker embedding, it is ample. Hence \( V^\nu \) is globally generated. To calculate the Ext group, we need

**Lemma 6.1.11.** If \( n-k \geq a_1 \geq \cdots \geq 0 \) and \( \beta_1 \geq \cdots \geq \beta_k \geq 0 \), then for all \( \ell \geq 1 \),

\[
\text{Ext}^\ell(\Sigma^\alpha U^\nu, \Sigma^\beta U^\nu) = 0.
\]

**Proof.** Let \( \bar{\alpha} = (\alpha_1 - a_n, \cdots, \alpha_1 - a_2, 0) \). Then \( \Sigma^\alpha U \otimes \Sigma^\beta U^\nu = \Sigma^\alpha U \otimes \Sigma^\beta U^\nu \otimes \mathcal{O}(-\alpha_1) \). By the Littlewood-Richardson rule, \( \Sigma^\alpha U \otimes \Sigma^\beta U^\nu \) can be decomposed into a direct sum of vector bundles of the form \( \Sigma^\mu U^\nu \), where \( \mu_i \geq \bar{\alpha}_i = \alpha_i - \alpha_{k+1-1} \). Now, for any such \( \Sigma^\mu U^\nu \), we have \( \Sigma^\nu U^\nu \otimes \mathcal{O}(-\alpha_1) = \Sigma^{(\nu_1 - \alpha_1, \cdots, \nu_n - \alpha_1)} U^\nu \). Since \( \nu_i - \alpha_i \geq -(n-k) \), by [38, Lemma 3.2],

\[
H^\ell(\text{Gr}(k,n), \Sigma^{(\nu_1 - \alpha_1, \cdots, \nu_n - \alpha_1)} U^\nu) = 0
\]

for all \( \ell \geq 1 \) and we get the required vanishing.

Then

\[
\text{Ext}^\ell(\Sigma^\alpha U^\nu, \Sigma^\beta U^\nu \otimes S^k V^\nu) = \bigoplus_{k_1, \cdots, k_n \geq 0} \text{Ext}^\ell(\Sigma^\alpha U^\nu, \Sigma^{(\beta_1 + \sum k_1 a_1, \cdots, \beta_k + \sum k_n a_n)} U^\nu) = 0.
\]

Hence \( \mathcal{E} = \{ \Sigma^\lambda : \lambda \in Y_{k,n} \} \) is V-geometric. In particular, it is compatible with the canonical bundle \( K_{\text{Gr}(k,n)} = \mathcal{O}(-n) \).

### 6.2 Koszul Algebras

In section 5.4, we produced an \( A_{\text{fin}} \)-structure on \( A = \text{Ext}_\Lambda^*(S, S) \) such that \( E(A) \) is a quasi-free model of \( \Lambda \). This \( A_{\text{fin}} \)-structure is not readily known in general. In this section, we introduce a class of algebras, called Koszul algebras, whose \( A_{\text{fin}} \)-structure on \( A \) are readily computable. A good introduction is given by Martín-Vega-Villa [56].

**Definition 6.2.1.** A \( \mathbb{Z} \)-graded algebra \( \Lambda \) over a semisimple algebra \( S = K^r \) is called *Koszul* if there is a linear graded projective resolution of \( S \), i.e., an exact sequence of \( \Lambda \)-modules

\[
\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to S \to 0,
\]

where each \( P_i \) is projective and concentrated in degree \( i \).
It is known that Koszul algebras are always quadratic, so Koszulness is in fact a rather restrictive property. Below are the main examples of vector bundles producing Koszul algebras we have in mind:

**Proposition 6.2.2.** Let $X$ be a smooth projective variety whose $K_X^{-1}$ is semi-ample. Let $E$ be a $K_X$-geometric full exceptional collection on $D^b(\text{Coh}(X))$. If length $E = \dim X + 1$, then $A_E$ is Koszul.

*Proof.* By Proposition 2.8.3, length $E = \text{rank} D^b(\text{Coh}(X))$. The rest follows from [15, Proposition 4.2].

**Remark 6.2.3.** Examples of varieties with full exceptional collection $E$ whose length $E = \dim X + 1$ include projective spaces, odd-dimensional quadrics [39] and certain Fano threefolds [57].

Koszul algebras are stable under taking tensor products and quotients.

**Proposition 6.2.4.**

1. If both $\Lambda$ and $\Lambda'$ are Koszul, then so is $\Lambda \otimes \Lambda'$.
2. $\Lambda^{\text{op}}$ is Koszul.
3. $\Lambda \# G$ is Koszul.
4. $\Lambda/G$ is Koszul.

*Proof.* For the first claim, one takes the total complex of the tensor product of the two linear graded resolutions. The proof for the second statement can be found in Woodcock’s paper [73]. For the third claim, one smashes the whole linear resolution of $S$ by $G$. Since tensoring by $\mathbb{K}G$ is an exact functor, this produces a linear resolution of $S \otimes \mathbb{K}G$. The last follows from Corollary 3.5.8 by applying the Morita functor in Theorem 3.5.2 to the linear resolution of that of $(\Lambda^{\text{op}} \# G)^{\text{op}}$ to get a linear resolution for $A/G$.

As a result, if vector bundles $V$ and $W$ produce Koszul algebras, then so does $V \boxtimes W$. If $V$ is equipped with a $G$-action making it $G$-equivariant, then its corresponding tilting algebra is also Koszul.

Here is the main proposition which makes the $A_{\text{fin}}$-structure on $\text{Ext}_{\Lambda}^*(S, S)$ simple.

**Proposition 6.2.5.** If $\Lambda$ is a Koszul algebra, then any $A_{\infty}$-structure on $\text{Ext}_{\Lambda}^*(S, S)$ which preserve the grading induced by $\Lambda$ is an ordinary algebra structure, i.e., $m_n = 0$ for $n \neq 2$.

*Proof.* Each component of $\text{Ext}_{\Lambda}^i(S, S)$ is a $\mathbb{Z}$-graded $\Lambda$-module. By the definition of Koszulness, $\text{Ext}_{\Lambda}^i(S, S)$ is concentrated in degree $i$, i.e., $\text{Ext}_{\Lambda}^i(S, S) = [\text{Ext}_{\Lambda}^i(S, S)]_i$. Since by assumption $m_n$ preserve the grading induced by $\Lambda$, it must map

$$m_n : [\text{Ext}_{\Lambda}^i(S, S)]_{i_1} \otimes \cdots \otimes [\text{Ext}_{\Lambda}^i(S, S)]_{i_n} \to [\text{Ext}_{\Lambda}^{i_1 + \cdots + i_n + 2 - n}(S, S)]_{i_1 + \cdots + i_n}.$$ 

Hence $m_n$ must vanish since the right hand side is zero unless $n = 2$.

Now, we can determine the $A_{\text{fin}}$-structure in Theorem 5.4.5.

**Theorem 6.2.6.** Let $X$ be a smooth projective variety and $\pi : V \to X$ be an anti-semiample vector bundle. Suppose $E = \{E_i\}_{i \in I}$ is a full $V$-geometric exceptional poset on $D^b(\text{Coh}(X))$ with dual exceptional poset $F = \{F_i\}_{i \in I^{op}}$. If the classical tilting algebra

$$A_E = \bigoplus_{i,j \in I} \text{Hom}(E_i, E_j \otimes S^*V^\vee)$$

is Koszul, then the $A_{\text{fin}}$-algebra

$$A_E' = \bigoplus_{i,j \in I} \bigoplus_{k=0}^{\text{rank} V} \text{Hom}^t - k(F_i, F_j \otimes \wedge^k V)$$

given in Theorem 5.4.5 is an ordinary algebra, i.e., $m_n = 0$ for $n \neq 2$, with $m_2$ given by composition of maps.
Proof. This follows from Theorem 5.4.5 and Proposition 6.2.5.

We can describe the algebra $A_{\mathcal{F}} \cong \text{Ext}_{\mathcal{F}}^*(S, S)$ more explicitly. Recall that we can write $\Lambda$ as a path algebra with relations $KQ/I$. If $\Lambda$ is Koszul, then $I$ is homogeneous and is generated by $I_2 = I \cap (KQ)_2$. Define a pairing $\langle - , - \rangle : (KQ)_2 \times (KQ^{op})_2 \to K$ by

$$\langle a, b^{op} \rangle = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

where $a$ and $b$ are paths of length 2 in $Q$. The orthogonal subspace of $I_2$ is defined to be

$$I_2^\perp = \{ b \in (KQ^{op})_2 : \langle a, b \rangle = 0 \text{ for all } a \in I_2 \}.$$ 

We have the following

**Theorem 6.2.7** ([29] Theorem 2.2). Let $KQ/I$ be a Koszul algebra. Then the Yoneda algebra $\text{Ext}_{\mathcal{F}}^*(S, S)$ of $\Lambda$ is isomorphic to $KQ^{op}/(I_2^\perp)$.

**Example 6.2.8.** Let $Q$ be the quiver with one vertex and $n$ loops. Then $KQ = K\langle x_1, \ldots, x_n \rangle$. Let $I$ be the ideal generated by elements in the form $x_i x_j - x_j x_i$ for all $1 \leq i, j \leq n$. Then $KQ/I = K[x_1, \ldots, x_n]$. The orthogonal subspace $I_2^{op}$ is spanned by elements in the form $x_i x_j + x_j x_i$ and $x_i^2$ for all $1 \leq i, j \leq n$. Hence $KQ^{op}/(I_2^{op}) = \bigwedge^n K^n$. The standard Koszul resolution

$$0 \to K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n] \otimes K^n \to \cdots \to K[x_1, \ldots, x_n] \otimes K^n \to K,$$

where the first $n$ maps are given by wedging with the vector $(x_1, \ldots, x_n)$ and the last map given by evaluation of polynomials at zero, shows that $K[x_1, \ldots, x_n]$ is Koszul. Hence $\text{Ext}_{KQ/I}^*(K, K) = \bigwedge^n K^n$. The $A_{\mathcal{F}}$-structure is given by $m_i = 0$ for $i \neq 2$, and $m_2$ by the Yoneda product, which is the wedge product on $\bigwedge^n K^n$.

### 6.3 DG-Tilting Quivers

In this section, we compute some examples illustrating Theorem 5.4.5 and Proposition 5.5.2. All formulae for cohomologies can be found in the Appendix A.

**Example 6.3.1.** Let $X = \text{Spec } K$ and $V = K^n$. Then $E = \mathcal{O}_X$ is a $V$-geometric exceptional collection on $\text{D}^b(\text{Coh} (\text{Spec } K))$. The classical tilting algebra is given by $\Lambda_{\mathcal{F}} = K[x_1, \ldots, x_n]$ and the classical tilting quiver is given by one vertex with $n$ loops $x_i$, with relations given by $x_i x_j = x_j x_i$. By Example 6.2.8, $A_{\mathcal{F}} = \bigwedge^l K^n$, with $m_2$ given by the wedge product and all other $m_n = 0$. Below is the quasi-free dg-quiver derived equivalent to $K^n$ for $n = 1, 2, 3$.

When $n = 1$, the quasi-free dg-quiver is given by

$$
\begin{array}{c}
\bullet \\
\circ \\
\end{array}
$$

with $\deg x = 0$ and $dx = 0$.

When $n = 2$, we have

$$
\begin{array}{c}
\bullet \\
\circ \\
\circ \\
\end{array}
$$

with $\deg x = \deg y = 0$, $\deg t = -1$, $dx = dy = 0$ and $dt = xy - yx$.

When $n = 3$, we have

$$
\begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
$$

with $\deg x = \deg y = \deg z = 0$, $\deg x^* = \deg y^* = \deg z^* = -1$, $\deg t = -2$ and $dx = dy = dz = 0$, $dx^* = yz - zy$, $dy^* = zx - xz$, $dz^* = xy - yx$, and $dt = [x, x^*] + [y, y^*] + [z, z^*]$. This dg-quiver has a superpotential given by $\Phi = xyz - xzy$. 

When $n = 4$, we have

\[
\begin{array}{c}
\text{•} \quad \text{(Diagram of edges)}
\end{array}
\]

where the black edges are of degree 0, red edges are of degree $-1$, blue edges are of degree $-2$ and brown edges are of degree $-3$. We will denote the four degree 0 (resp. degree $-2$) edges by $x_i$ (resp. by $x_i^*$), where $1 \leq i \leq 4$ and the six degree $-1$ edges by $r_{ij}$ where $1 \leq i < j < 4$. If $i > j$, by $r_{ij}$ we will mean $-r_{ji}$. The pairing of degree $-1$ edges is then given by

\[
(r_{ij}, r_{k\ell}) = \epsilon_{ij\ell} = \begin{cases} 
1 & \text{if } (i, j, k, \ell) \text{ is an even permutation of } (1, 2, 3, 4) \\
-1 & \text{if } (i, j, k, \ell) \text{ is an odd permutation of } (1, 2, 3, 4) \\
0 & \text{otherwise,}
\end{cases}
\]

with superpotential given by

\[
\Phi = \sum \epsilon_{ij\ell} (x_i x_j - x_j x_i) r_{k\ell}.
\]

**Example 6.3.2.** Consider $X = \mathbb{P}^1$ and $V = K_{\mathbb{P}^1}$ which is Calabi–Yau 2. Take the exceptional sequence $E = (\mathcal{O}, \mathcal{O}(1))$ which is $K_{\mathbb{P}^1}$-geometric. The classical tilting quiver is given by

\[
\begin{array}{c}
\text{•} \quad \text{(Quiver diagram)}
\end{array}
\]

with relations $f_j e_i = f_i e_j$ and $e_j f_i = e_i f_j$.

The corresponding $A_{\infty}$-category is given by

\[
\begin{align*}
\mathcal{A}^0(v_0, v_0) &= \text{Hom}(\mathcal{O}, \mathcal{O}) \oplus \text{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O} \otimes K_{\mathbb{P}^1}) \\
&= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-2)) \\
&= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 1 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}^0(v_0, v_1) &= \text{Hom}(\mathcal{O}(1), \mathcal{O}) \oplus \text{Hom}^{\ell-1}(\mathcal{O}(1), \mathcal{O} \otimes K_{\mathbb{P}^1}) \\
&= H^\ell(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-1)) \\
&= \begin{cases} 
\mathbb{C}^2 & \text{if } \ell = 1 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}^0(v_1, v_0) &= \text{Hom}(\mathcal{O}(1), \mathcal{O}(1)) \oplus \text{Hom}^{\ell-1}(\mathcal{O}(1), \mathcal{O} \otimes K_{\mathbb{P}^1}) \\
&= H^\ell(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-3)) \\
&= \begin{cases} 
\mathbb{C}^2 & \text{if } \ell = 1 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}^0(v_1, v_1) &= \text{Hom}(\mathcal{O}(1), \mathcal{O}(1)) \oplus \text{Hom}^{\ell-1}(\mathcal{O}(1), \mathcal{O}(1) \otimes K_{\mathbb{P}^1}) \\
&= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-2)) \\
&= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 1 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Thus the dg-tilting quiver is given by

\[
\begin{array}{c}
\text{•} \quad \text{(dg-tilting quiver diagram)}
\end{array}
\]
where black edges are of degree 0 and brown edges are of degree −1. Since $H^0$ of the dg-tilting quiver gives the classical tilting quiver, by a change of basis if necessary, we may assume the differential sends

$$d v^*_0 = f_1 e_0 - f_0 e_1, \quad dv^*_1 = e_1 f_0 - e_0 f_1 \quad \text{and} \quad de_i = df_i = 0.$$  

The $A_\infty$-structure is given by

$$m_2(v_0, f_1) = m_2(f_i, v_1) = f_i \quad m_2(v_1, e_i) = m_2(e_i, v_0) = e_i$$

$$m_2(f_1, e_0) = -m_2(f_0, e_1) = v_0^* \quad m_2(e_1, f_0) = -m_2(e_0, f_1) = v_1^*$$

$$m_2(v_0, v_0^*) = m_2(v_0^*, v_0) = v_0^* \quad m_2(v_1, v_1^*) = m_2(v_1^*, v_1) = v_1^*$$

and zero otherwise.

**Example 6.3.3.** Consider $X = \mathbb{P}^1$ and $V = \mathcal{O}(-1)^{\oplus 2}$ which is Calabi–Yau 3. Take the exceptional sequence $E = (\mathcal{O}, \mathcal{O}(1))$ which is $V$-geometric. The classical tilting quiver is given by

$$
\begin{array}{c}
\bullet & \xrightarrow{e_0} & \bullet \\
\bullet & \xleftarrow{e_1} & \bullet \\
\bullet & \xrightarrow{f_0} & \bullet \\
\bullet & \xleftarrow{f_1} & \bullet \\
\end{array}
$$

with relations $e_1 f_1 e_0 = e_0 f_1 e_1$ and $f_1 e_1 f_0 = f_0 e_1 f_1$.

The corresponding $A_\infty$-category is given by

$$A(v_0, v_0) = \text{Hom}^f(\Omega(1)[1], \Omega(1)[1]) \oplus \text{Hom}^{-1}(\Omega(1)[1], \Omega(1)[1] \otimes V) \oplus \text{Hom}^{-2}(\Omega(1)[1], \Omega(1)[1] \otimes \wedge^2 V)$$

$$= H^f(\mathbb{P}^1, \mathcal{O}) \oplus H^{-1}(\mathbb{P}^1, \mathcal{O}(-1)^{\oplus 2}) \oplus H^{-2}(\mathbb{P}^1, \mathcal{O}(-2))$$

$$= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$A(v_0, v_1) = \text{Hom}^f(\Omega(1)[1], \mathcal{O}) \oplus \text{Hom}^{-1}(\Omega(1)[1], \mathcal{O} \otimes V) \oplus \text{Hom}^{-2}(\Omega(1)[1], \mathcal{O} \otimes \wedge^2 V)$$

$$= H^{-1}(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^{-2}(\mathbb{P}^1, \mathcal{O}(-1)^{\oplus 2}) \oplus H^{-3}(\mathbb{P}^1, \mathcal{O}(-1))$$

$$= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$A(v_1, v_0) = \text{Hom}^f(\mathcal{O}, \Omega(1)[1]) \oplus \text{Hom}^{-1}(\mathcal{O}, \Omega(1)[1] \otimes V) \oplus \text{Hom}^{-2}(\mathcal{O}, \Omega(1)[1] \otimes \wedge^2 V)$$

$$= H^{f+1}(\mathbb{P}^1, \mathcal{O}(-1)) \oplus H^f(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus 2}) \oplus H^{-1}(\mathbb{P}^1, \mathcal{O}(-3))$$

$$= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$A(v_1, v_1) = \text{Hom}^f(\mathcal{O}, \mathcal{O}) \oplus \text{Hom}^{-1}(\mathcal{O}, \mathcal{O} \otimes V) \oplus \text{Hom}^{-2}(\mathcal{O}, \mathcal{O} \otimes \wedge^2 V)$$

$$= H^f(\mathbb{P}^1, \mathcal{O}) \oplus H^{-1}(\mathbb{P}^1, \mathcal{O}(-1)^{\oplus 2}) \oplus H^{-2}(\mathbb{P}^1, \mathcal{O}(-2))$$

$$= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the dg-quiver is given by

$$
\begin{array}{c}
\bullet & \xrightarrow{e} & \bullet \\
\bullet & \xleftarrow{f} & \bullet \\
\bullet & \xrightarrow{f} & \bullet \\
\bullet & \xleftarrow{e} & \bullet \\
\end{array}
$$
Since $H^0$ of the dg-quiver gives back the classical tilting quiver, by a change of basis, we may assume
\[
d e_1 = f_0 e_0 f_1 - f_1 e_0 f_0, \quad d e_0 = f_1 e_1 f_0 - f_0 e_1 f_1,
\]
\[
d f_1 = e_1 f_0 e_0 - e_0 f_0 e_1, \quad d f_0 = e_0 f_1 e_1 - e_1 f_1 e_0.
\]
The superpotential is thus given by $\Phi = f_1 e_0 f_0 e_1 - f_0 e_1 f_1$.

Example 6.3.4. Consider $X = \mathbb{P}^2$, $V = K_{\mathbb{P}^2}$ which is Calabi–Yau 3, with exceptional sequence $E = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ which is $K_{\mathbb{P}^2}$-geometric. The classical tilting quiver is given by

\[
\begin{array}{c}
v_0 \\
\downarrow \downarrow a \\
v_1 \\
\downarrow b \\
v_2 \\
\end{array}
\]

with relations $b_{i+1} a_i = b_i a_{i+1}$, $c_{i+1} b_i = c_i b_{i+1}$, and $a_{i+1} c_i = a_i c_{i+1}$.

The corresponding $A_n$-category is given by
\[
\mathcal{A}^{\ell}(v_0, v_1) = \operatorname{Hom}^\ell(\mathcal{O}(2)[2], \mathcal{O}(2)[2]) \oplus \operatorname{Hom}^{\ell-1}(\mathcal{O}(2)[2], \mathcal{O}(2)[2] \otimes K_{\mathbb{P}^2})
\]
\[
= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-3))
\]
\[
= \begin{cases} 
H^0(\mathbb{P}^2, \mathcal{O}) & \text{if } \ell = 0 \\
H^2(\mathbb{P}^2, \mathcal{O}(-3)) & \text{if } \ell = 3 \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 3 \\
0 & \text{otherwise}
\end{cases}
\]

$\mathcal{A}^{\ell}(v_0, v_2) = \operatorname{Hom}^\ell(\mathcal{O}(2)[2], \mathcal{O}) \oplus \operatorname{Hom}^{\ell-1}(\mathcal{O}(2)[2], \mathcal{O} \otimes K_{\mathbb{P}^2})$
\[
= H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^{\ell-3}(\mathbb{P}^2, \mathcal{O}(-2))
\]
\[
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 2 \\
0 & \text{otherwise}
\end{cases}
\]

$\mathcal{A}^{\ell}(v_1, v_0) = \operatorname{Hom}^\ell(\mathcal{O}(1)[1], \mathcal{O}(2)[2]) \oplus \operatorname{Hom}^{\ell-1}(\mathcal{O}(1)[1], \mathcal{O}(2)[2] \otimes K_{\mathbb{P}^2})$
\[
= H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2)) \oplus H^{\ell}(\mathbb{P}^2, \mathcal{T}(-5))
\]
\[
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 2 \\
0 & \text{otherwise}
\end{cases}
\]
\[ \mathcal{A}^\ell(v_1, v_1) = \text{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) \oplus \text{Hom}^{\ell-1}(\Omega(1)[1], \Omega(1)[1] \otimes K_{\mathbb{P}^2}) \]
\[ = H^\ell(\mathbb{P}^2, T \otimes \Omega) \oplus H^{\ell-1}(\mathbb{P}^2, T \otimes \Omega(-3)) \]
\[ = \begin{cases} 
H^0(\mathbb{P}^2, T \otimes \Omega) & \text{if } \ell = 0 \\
H^2(\mathbb{P}^2, T \otimes \Omega(-3)) & \text{if } \ell = 3 \\
0 & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 3 \\
0 & \text{otherwise}. 
\end{cases} \]

\[ \mathcal{A}^\ell(v_1, v_2) = \text{Hom}^\ell(\Omega(1)[1], \mathcal{O}) \oplus \text{Hom}^{\ell-1}(\Omega(1)[1], \mathcal{O} \otimes K_{\mathbb{P}^2}) \]
\[ = H^{\ell-1}(\mathbb{P}^2, T(-1)) \oplus H^{\ell-2}(\mathbb{P}^2, T(-4)) \]
\[ = \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1 \\
0 & \text{otherwise}. 
\end{cases} \]

\[ \mathcal{A}^\ell(v_2, v_0) = \text{Hom}^\ell(\mathcal{O}, \Omega^2(2)[2]) \oplus \text{Hom}^{\ell-1}(\mathcal{O}, \Omega^2(2)[2] \otimes K_{\mathbb{P}^2}) \]
\[ = H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1)) \oplus H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-4)) \]
\[ = \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 2 \\
0 & \text{otherwise}. 
\end{cases} \]

The quiver with superpotential is given by

\[ (6.3.1) \]

where black edges are of degree 0, blue edges are of degree -1, brown loops are of degree -2. Since \( H^0 \) of the quiver must be isomorphic to the quiver with relation in (6.3.4), by a change of basis if necessary,
we may assume the differential $d$ sends
\[
da_i^* = b_{i+1}c_{i+2} - b_{i+2}c_{i+1} \quad \text{and} \quad db_i^* = c_{i+1}a_{i+2} - c_{i+2}a_{i+1} \quad \text{and} \quad dc_i^* = a_{i+1}b_{i+2} - a_{i+2}b_{i+1}.
\]
Thus the superpotential is given by $\Phi = \sum_{i=0}^{2} a_i(b_{i+1}c_{i+2} - b_{i+2}c_{i+1})$.

**Example 6.3.5.** Consider $V = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ over $\mathbb{P}^2$ which is Calabi–Yau 4. The classical tilting quiver is given by

\[
\begin{align*}
\begin{array}{c}
\bullet \quad a_0 & \quad \bullet \quad \omega \\
\downarrow & \quad \downarrow \\
a_1 & \quad a_2
\end{array}
\end{align*}
\]

with relations
\[
a_iu = vb_i \quad a_{i+1}w = a_iw b_{i+1} \\
b_{i+1}a_i = b_ia_{i+1} \quad wb_i = ua_iw.
\]

We calculate its $A_\infty$-category:
\[
A^f(v_0, v_0) = \text{Hom}^f(\Omega^2(2)[2], \Omega^2(2)[2]) \oplus \text{Hom}^{f-1}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes V) \\
\oplus \text{Hom}^{f-2}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes \Lambda^2 V) \\
= H^f(\mathbb{P}^2, \mathcal{O}) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{O}(-1) \oplus \mathcal{O}(-2)) \oplus H^{f-2}(\mathbb{P}^2, \mathcal{O}(-3)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 4 \\
0 & \text{otherwise}
\end{cases}
\]
\[
A^f(v_0, v_1) = \text{Hom}^f(\Omega^2(2)[2], \Omega(1)[1]) \oplus \text{Hom}^{f-1}(\Omega^2(2)[2], \Omega(1)[1] \otimes V) \\
\oplus \text{Hom}^{f-2}(\Omega^2(2)[2], \Omega(1)[1] \otimes \Lambda^2 V) \\
= H^f(\mathbb{P}^2, \Omega(2)) \oplus H^{f-1}(\mathbb{P}^2, \Omega \otimes \Omega(1)) \oplus H^{f-3}(\mathbb{P}^2, \Omega(-1)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1 \\
\mathbb{C} & \text{if } \ell = 3 \\
0 & \text{otherwise}
\end{cases}
\]
\[
A^f(v_0, v_2) = \text{Hom}^f(\Omega^2(2)[2], \mathcal{O}) \oplus \text{Hom}^{f-1}(\Omega^2(2)[2], \mathcal{O} \otimes V) \oplus \text{Hom}^{f-2}(\Omega^2(2)[2], \mathcal{O} \otimes \Lambda^2 V) \\
= H^f(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{O} \otimes \mathcal{O}(-1)) \oplus H^{f-4}(\mathbb{P}^2, \mathcal{O}(-2)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 2 \\
\mathbb{C} & \text{if } \ell = 3 \\
0 & \text{otherwise}
\end{cases}
\]
\[
A^f(v_1, v_0) = \text{Hom}^f(\Omega(1)[1], \Omega^2(2)[2]) \oplus \text{Hom}^{f-1}(\Omega(1)[1], \Omega^2(2)[2] \otimes V) \\
\oplus \text{Hom}^{f-2}(\Omega(1)[1], \Omega^2(2)[2] \otimes \Lambda^2 V) \\
= H^{f+1}(\mathbb{P}^2, \mathcal{T}(-2)) \oplus H^f(\mathbb{P}^2, \mathcal{T}(-3) \oplus \mathcal{T}(-4)) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{T}(-5)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 1 \\
\mathbb{C}^3 & \text{if } \ell = 3 \\
0 & \text{otherwise}
\end{cases}
\]
\[
A^f(v_1, v_1) = \text{Hom}^f(\Omega(1)[1], \Omega(1)[1]) \oplus \text{Hom}^{f-1}(\Omega(1)[1], \Omega(1)[1] \otimes V) \oplus \text{Hom}^{f-2}(\Omega(1)[1], \Omega(1)[1] \otimes \Lambda^2 V) \\
= H^f(\mathbb{P}^2, \mathcal{T} \otimes \mathcal{O}) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{T} \otimes \mathcal{O}(-1) \oplus \mathcal{T} \otimes \mathcal{O}(-2)) \oplus H^{f-2}(\mathbb{P}^2, \mathcal{T} \otimes \mathcal{O}(-3)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 4 \\
\mathbb{C}^6 & \text{if } \ell = 2 \\
0 & \text{otherwise}
\end{cases}
\]
\[ A^\ell(v_1, v_2) = \text{Hom}^\ell(\Omega(1)[1], \mathcal{O}) \oplus \text{Hom}^{\ell-1}(\Omega(1)[1], \mathcal{O} \otimes V) \oplus \text{Hom}^{\ell-2}(\Omega(1)[1], \mathcal{O} \otimes \wedge^2 V) \]
\[ = H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1)) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{T}(-2) \oplus \mathcal{T}(-3)) \oplus H^{\ell-3}(\mathbb{P}^2, \mathcal{T}(-4)) \]
\[ = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ \mathbb{C} & \text{if } \ell = 3 \\ 0 & \text{otherwise.} \end{cases} \]
\[ A^\ell(v_2, v_0) = \text{Hom}^\ell(\mathcal{O}, \Omega^2(2)[2]) \oplus \text{Hom}^{\ell-1}(\mathcal{O}, \Omega^2(2)[2] \otimes V) \oplus \text{Hom}^{\ell-2}(\mathcal{O}, \Omega^2(2)[2] \otimes \wedge^2 V) \]
\[ = H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-1)) \oplus H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-2) \oplus \mathcal{O}(-3)) \oplus H^{\ell+3}(\mathbb{P}^2, \mathcal{O}(-4)) \]
\[ = \begin{cases} \mathbb{C} & \text{if } \ell = 1 \\ \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]
\[ A^\ell(v_2, v_1) = \text{Hom}^\ell(\mathcal{O}, \Omega(1)[1]) \oplus \text{Hom}^{\ell-1}(\mathcal{O}, \Omega(1)[1] \otimes V) \oplus \text{Hom}^{\ell-2}(\mathcal{O}, \Omega(1)[1] \otimes \wedge^2 V) \]
\[ = H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^{\ell+2}(\mathbb{P}^2, \mathcal{O} \otimes \Omega(-1)) \oplus H^{\ell+3}(\mathbb{P}^2, \mathcal{O}(-2)) \]
\[ = \begin{cases} \mathbb{C} & \text{if } \ell = 1 \\ \mathbb{C}^3 & \text{if } \ell = 3 \\ 0 & \text{otherwise.} \end{cases} \]
\[ A^\ell(v_2, v_2) = \text{Hom}^\ell(\mathcal{O}, \mathcal{O}) \oplus \text{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O} \otimes V) \oplus \text{Hom}^{\ell-2}(\mathcal{O}, \mathcal{O} \otimes \wedge^2 V) \]
\[ = H^{\ell}(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-1) \oplus \mathcal{O}(-2)) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(-3)) \]
\[ = \begin{cases} \mathbb{C} & \text{if } \ell = 0, 4 \\ 0 & \text{otherwise.} \end{cases} \]

The underlying graded quiver of \( E(\mathcal{A}_\ell) \) is given by

where black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2 and brown edges are of degree -3. Since \( H^0(E(\mathcal{A}_\ell)) = \mathbb{A}_\ell \), by a linear change of basis if necessary, we may assume the differential \( d \) in \( E(\mathcal{A}_\ell) \) is given by

\[
\begin{align*}
dp_i &= a_i u - v b_i \\
dq_i &= a_{i+2} w b_{i+1} - a_{i+1} w b_{i+2} \\
dr_j &= w b_j v - u a_j w \\
ds_j &= b_{j+2} a_{j+1} - b_{j+1} a_{j+2},
\end{align*}
\]
where \( i \in \mathbb{Z}_3 \). Hence the only nonvanishing maps in the form \( m_n : \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^1 \to \mathcal{A}^2 \) are

\[
m_2(a_i, u) = p_i, \quad m_2(v, b_i) = -p_i, \quad m_3(a_{i+2}, w, b_{i+1}) = q_i, \quad m_3(a_{i+1}, w, b_{i+2}) = -q_i \]
\[
m_3(w, b_i, v) = r_i, \quad m_3(a_i, a_i, w) = -r_i, \quad m_2(b_{i+2}, a_{i+1}) = s_i, \quad m_2(b_{i+1}, a_{i+2}) = -s_i.
\]

Next, we would like to calculate the cyclic structure \( \langle -,- \rangle : \mathcal{A}^2 \otimes \mathcal{A}^2 \to \mathbb{C} \). Recall that the pairing is cyclic: \( \langle m_n(e_1, \ldots, e_n), e_{n+1} \rangle = (-1)^{i(i+1)} m_n(e_1, m_n(e_2, \ldots, e_{n+1})) \). We have

\[
\langle p_i, q_j \rangle = \langle m_2(a_i, u), m_3(a_{j+2}, w, b_{j+1}) \rangle = \langle a_i, m_2(u, m_3(a_{j+2}, w, b_{j+1})) \rangle = -\langle a_i, m_2(r_{j+2}, b_{j+1}) \rangle
\]
\[
= -\langle m_2(a_i, r_{j+2}), b_{j+1} \rangle = \langle b_{j+1}, m_2(a_i, r_{j+2}) \rangle = \langle m_2(b_{j+1}, a_i), r_{j+2} \rangle = \delta_{ij} (s_{i+2}, r_{i+2}).
\]
In particular, this implies \( \langle p_i, q_i \rangle = \langle s_i, r_i \rangle \) for all \( i \), and
\[
\langle p_i, p_j \rangle = \langle m_2(a_i, u), m_2(a_j, u) \rangle = \langle a_i, m_2(u, m_2(a_j, u)) \rangle = \langle a_i, m_2(u, a_j) \rangle = 0.
\]
\[
\langle q_i, q_j \rangle = \langle m_3(a_{i+2}, w, b_{i+1}), m_3(a_{j+2}, w, b_{j+1}) \rangle = \pm \langle a_{i+2}, m_3(w, b_{i+1}, m_3(a_{j+2}, w, b_{j+1})) \rangle.
\]
By the \( A_{\infty} \)-relation for \( n = 5 \), we get
\[
m_3(w, b_{i+1}, m_3(a_{j+2}, w, b_{j+1})) = \pm m_4(w, m_2(b_{i+1}, a_{j+2}), w, b_{j+1})
\]
since all other terms vanish. Then
\[
\langle q_i, q_j \rangle = \pm \langle a_{i+2}, m_4(w, m_2(b_{i+1}, a_{j+2}), w, b_{j+1}) \rangle = \pm \langle m_2(b_{i+1}, a_{j+2}), m_4(w, b_{j+1}, a_{i+2}, w) \rangle = 0.
\]
This shows that the only (possibly) nonvanishing pairings are \( \langle p_i, q_i \rangle = \langle s_i, r_i \rangle \). Since the pairing is nondegenerate, we may normalize this number to 1. Thus the superpotential is given by
\[
\Phi = (a_{i+2}w b_{i+1} - a_{i+1}w b_{i+2}) p_i + (a_iw - vb_i) q_i + (b_{i+2}a_{i+1} - b_{i+1}a_{i+2}) r_i + (wb_i v - ua_i w) s_i.
\]

**Example 6.3.6.** Consider \( X = \mathbb{P}^2 \) and \( V = T_{\mathbb{P}^2}^\vee \) which is Calabi–Yau 4. The classical quiver is given by

![Quiver Diagram]

with relations
\[
\begin{align*}
\sum_{j=0}^{2} a_j b_j &= 0 \\
\sum_{j=0}^{2} b_j a_j &= 0 \\
\sum_{j=0}^{2} c_j d_j &= 0 \\
\sum_{j=0}^{2} d_j c_j &= 0 \\
c_{j+1} a_j &= c_j a_{j+1} \\
b_{j+1} d_j &= b_j d_{j+1} \\
a_j b_k &= d_k c_j
\end{align*}
\]

The relations listed above are not all independent: the relation \( a_j b_k = d_k c_j \) and \( \sum_{j=0}^{2} a_j b_j = 0 \) together implies \( \sum_{j=0}^{2} d_j c_j = 0 \).

Next, we would like to calculate \( \mathcal{A}_E \). Recall the dual sequence to \( \mathcal{E} \) is given by
\[
\mathcal{F} = (\Omega^2(2)[2], \Omega(1)[1], \mathcal{O}) = (\mathcal{O}(-1)[2], \Omega(1)[1], \mathcal{O}).
\]
We have
\[
\begin{align*}
\mathcal{A}^\ell(v_0, v_0) &= \text{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) \oplus \text{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes T_{\mathbb{P}^2}^\vee) \\
&\quad \oplus \text{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes \wedge^2 T_{\mathbb{P}^2}^\vee) \\
&= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \Omega) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(-3)) \\
&= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 2, 4 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}^\ell(v_0, v_1) &= \text{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) \oplus \text{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega(1)[1] \otimes T_{\mathbb{P}^2}^\vee) \\
&\quad \oplus \text{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega(1)[1] \otimes \wedge^2 T_{\mathbb{P}^2}^\vee) \\
&= H^{\ell-1}(\mathbb{P}^2, \Omega(2)) \oplus H^{\ell-2}(\mathbb{P}^2, \Omega(2) \otimes \Omega(2)) \oplus H^{\ell-3}(\mathbb{P}^2, \Omega(-1)) \\
&= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1, 3 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]
\( \mathcal{A}(v_0, v_2) = \text{Hom}^f(\Omega^2(2)[2], \mathcal{O}) \oplus \text{Hom}^{f-1}(\Omega^2(2)[2], \mathcal{O} \otimes T_{p_2}^*) \oplus \text{Hom}^{f-2}(\Omega^2(2)[2], \mathcal{O} \otimes \wedge^2 T_{p_2}^*) \\
= H^{f-2}(\mathbb{P}^2, \Omega(1)) \oplus H^{f-3}(\mathbb{P}^2, \Omega(1)) \oplus H^{f-4}(\mathbb{P}^2, \mathcal{O}(-2)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 2 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_1, v_0) = \text{Hom}^f(\Omega(1)[1], \Omega^2(2)[2]) \oplus \text{Hom}^{f-1}(\Omega(1)[1], \Omega^2(2)[2] \otimes T_{p_2}^*) \\
\oplus \text{Hom}^{f-2}(\Omega(1)[1], \Omega^2(2)[2] \otimes \wedge^2 T_{p_2}^*) \\
= H^{f+1}(\mathbb{P}^2, \mathcal{T}(-2)) \oplus H^f(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-2)) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{T}(-5)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1, 3 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_1, v_1) = \text{Hom}^f(\Omega(1)[1], \Omega(1)[1]) \oplus \text{Hom}^{f-1}(\Omega(1)[1], \Omega(1)[1] \otimes T_{p_2}^*) \\
\oplus \text{Hom}^{f-2}(\Omega(1)[1], \Omega(1)[1] \otimes \wedge^2 T_{p_2}^*) \\
= H^f(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \oplus H^{f-2}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 4 \\
\mathbb{C}^{10} & \text{if } \ell = 2 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_1, v_2) = \text{Hom}^f(\Omega(1)[1], \mathcal{O}) \oplus \text{Hom}^{f-1}(\Omega(1)[1], \mathcal{O} \otimes T_{p_2}^*) \oplus \text{Hom}^{f-2}(\Omega(1)[1], \mathcal{O} \otimes \wedge^2 T_{p_2}^*) \\
= H^{f-1}(\mathbb{P}^2, \mathcal{T}(-1)) \oplus H^{f-2}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1)) \oplus H^{f-3}(\mathbb{P}^2, \mathcal{T}(-4)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1, 3 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_2, v_0) = \text{Hom}^f(\mathcal{O}, \Omega^2(2)[2]) \oplus \text{Hom}^{f-1}(\mathcal{O}, \Omega^2(2)[2] \otimes T^\vee \mathbb{P}^2) \oplus \text{Hom}^{f-2}(\mathcal{O}, \Omega^2(2)[2] \otimes \wedge^2 T^\vee \mathbb{P}^2) \\
= H^{f+2}(\mathbb{P}^2, \mathcal{O}(-1)) \oplus H^{f+1}(\mathbb{P}^2, \mathcal{O}(-1)) \oplus H^f(\mathbb{P}^2, \mathcal{O}(-4)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 2 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_2, v_1) = \text{Hom}^f(\mathcal{O}, \Omega(1)[1]) \oplus \text{Hom}^{f-1}(\mathcal{O}, \Omega(1)[1] \otimes T_{p_2}^*) \oplus \text{Hom}^{f-2}(\mathcal{O}, \Omega(1)[1] \otimes \wedge^2 T_{p_2}^*) \\
= H^{f+1}(\mathbb{P}^2, \Omega(1)) \oplus H^f(\mathbb{P}^2, \Omega \otimes \Omega(1)) \oplus H^{f-1}(\mathbb{P}^2, \Omega(-2)) \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \ell = 1, 3 \\
0 & \text{otherwise.} 
\end{cases} 
\)

\( \mathcal{A}(v_2, v_2) = \text{Hom}^f(\mathcal{O}, \mathcal{O}) \oplus \text{Hom}^{f-1}(\mathcal{O} \otimes T^\vee \mathbb{P}^2) \oplus \text{Hom}^{f-2}(\mathcal{O} \otimes \wedge^2 T^\vee \mathbb{P}^2) \\
= H^f(\mathbb{P}^2, \mathcal{O}) \oplus H^{f-1}(\mathbb{P}^2, \mathcal{O}) \oplus H^{f-2}(\mathbb{P}^2, \mathcal{O}(-3)) \\
= \begin{cases} 
\mathbb{C} & \text{if } \ell = 0, 2, 4 \\
0 & \text{otherwise.} 
\end{cases} 
\)

The underlying graded quiver of the dg-quiver \( \mathcal{E}(A) \) is thus given by
notation, we will denote an edge \( e \) in the quiver and its corresponding element in the \( A_\infty \)-category \( \mathcal{A}_E \) by the same symbol \( e \), instead of the more correct \( e' \). Since \( T^\mathcal{N}_N \) is noncompact 4-Calabi–Yau, \( \mathcal{A}_E \) has a cyclic structure of degree 4. By a linear change of basis if necessary, we may assume

\[
m_2(a^*, a) = v_0^*, \quad m_2(b^*, b) = m_2(c^*, c) = v_1^*, \quad m_2(d^*, d) = v_2^*.
\]

Since \( H^0(E(\mathcal{A}_E)) = \Lambda_E \), by a linear change of basis if necessary, we may assume the differential \( d \) in \( E(\mathcal{A}_E) \) sends

\[
d \rho_i = c_{i+2} a_{i+1} - c_{i+1} a_{i+2}, \quad d \sigma_i = b_{i+2} d_{i+1} - b_{i+1} d_{i+2}
\]

\[
dr = \sum_{i=0}^2 b_i a_i, \quad dt = \sum_{i=0}^2 a_i b_i + d_i c_i, \quad ds = \sum_{i=0}^2 c_i d_i \quad ds_{ij} = a_i b_j - d_j c_i.
\]

Hence the only nonvanishing maps in the form \( m_n : \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^1 \to \mathcal{A}^2 \) are

\[
m_2(c_{i+2}, a_{i+1}) = \rho_i, \quad m_2(c_{i+1}, a_{i+2}) = -\rho_i,
\]

\[
m_2(b_{i+2}, d_{i+1}) = \sigma_i, \quad m_2(b_{i+1}, d_{i+2}) = -\sigma_i,
\]

\[
m_2(b_i, a_i) = r, \quad m_2(c_i, d_i) = s,
\]

\[
m_2(a_i, b_j) = \tau_{ij} + \rho_i + \delta_{ij} t, \quad m_2(d_j, c_i) = -\tau_{ij} + \delta_{ij} t
\]

where \( \delta_{ij} \) is the Kronecker delta, i.e., \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

We would like to calculate the cyclic structure \( (-,-) : \mathcal{A}^2 \otimes \mathcal{A}^2 \to \mathbb{C} \). Recall that the pairing is cyclic: \( (m_n(e_1, \ldots, e_n), e_{n+1}) = (-1)^{(e_1+1)e_n} (e_1, m_n(e_2, \ldots, e_{n+1})) \). If \( i \neq j \),

\[
2(t, \tau_{ij}) = \langle m_2(a_k, b_k) + m_2(d_k, c_k), \tau_{ij} \rangle
\]

\[
= \langle m_2(a_k, b_k), m_2(a_i, b_j) \rangle - \langle m_2(d_k, c_k), m_2(d_j, c_i) \rangle
\]

\[
= \langle a_k, m_2(a_i, b_j) - m_2(d_i, c_i) \rangle = 0
\]

by choosing \( k \neq i \) and \( k \neq j \). Also,

\[
4(t, \tau_{ij}) = \langle m_2(a_k, b_k) + m_2(d_k, c_k), m_2(a_i, b_j) - m_2(d_i, c_i) \rangle
\]

\[
= \langle m_2(a_k, b_k), m_2(a_i, b_j) \rangle + \langle m_2(d_k, c_k), m_2(a_i, b_j) \rangle
\]

\[
- \langle m_2(a_k, b_k), m_2(d_i, c_i) \rangle - \langle m_2(d_k, c_k), m_2(d_i, c_i) \rangle
\]

\[
= \left\{ \begin{array}{ll}
\langle a_k, m_2(r, b_j) \rangle - \langle d_i, m_2(s, c_i) \rangle & \text{if } k = i \\
\langle d_i, m_2(r, b_j) \rangle - \langle a_i, m_2(\rho_{i+2}, c_i) \rangle & \text{if } k = i + 1
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\langle -r, m_2(\rho_{i+2}, a_i) \rangle + \langle s, m_2(c_i, d_i) \rangle & \text{if } k = i \\
\langle -\rho_{i+2}, m_2(b_i, d_{i+1}) \rangle + \langle \sigma_{i+2}, m_2(c_i, a_{i+1}) \rangle & \text{if } k = i + 1
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\langle -r, r \rangle + \langle s, s \rangle & \text{if } k = i \\
\langle -\rho_{i+2}, \sigma_{i+2} \rangle + \langle \sigma_{i+2}, \rho_{i+2} \rangle & \text{if } k = i + 1
\end{array} \right.
\]

Since \( k \) is arbitrary, \( \langle t, \tau_{ij} \rangle = 0 \) for all \( i, j \) and \( (r, r) = \langle s, s \rangle \).

For \( i \neq j \) and \( k \neq l \),

\[
\langle \tau_{ij}, \tau_{kl} \rangle = \langle m_2(a_i, b_j), m_2(a_k, b_k) \rangle - \langle a_k, \delta_{jk} m_2(r, b_j) \rangle = -\delta_{jk} \langle r, m_2(b_j, a_i) \rangle = -\delta_{jk} \delta_{il} \langle r, r \rangle
\]

For \( i \neq j \),

\[
2(\tau_{ij}, \tau_{kk}) = \langle \tau_{ij}, m_2(a_k, b_k) - m_2(d_k, c_k) \rangle = \langle m_2(a_i, b_j), m_2(a_k, b_k) \rangle + \langle m_2(d_j, c_i), m_2(d_k, c_k) \rangle
\]

\[
= \delta_{jk} \langle a_i, m_2(r, b_j) \rangle + \delta_{ik} \langle d_j, m_2(s, c_k) \rangle = -\delta_{jk} \delta_{ik} \langle r, r \rangle - \delta_{jk} \delta_{ik} \langle s, s \rangle = -2\delta_{jk} \delta_{ik} \langle r, r \rangle = 0
\]
since \(i \neq j\). We have
\[
4(t_t, \tau_{ji}) = \langle m_2(a_t, b_t) - m_2(d_i, c_i), m_2(a_j, b_j) - m_2(d_j, c_j) \rangle \\
= \langle m_2(a_i, b_i), m_2(a_j, b_j) \rangle - \langle m_2(d_i, c_i), m_2(d_j, c_j) \rangle \\
= \langle a_i, m_2(b_i, a_j) \rangle - \langle d_i, m_2(c_i, a_j) \rangle \\
= \langle a_i, m_2(b_i, a_j) \rangle - \langle d_i, m_2(c_i, a_j) \rangle \\
= \left\{ \begin{array}{ll}
-\langle m_2(b_i, a_i), m_2(b_i, a_j) \rangle - \langle m_2(c_i, d_i), m_2(c_i, d_i) \rangle & \text{if } j = i \\
\langle m_2(c_i, a_{i+1}), m_2(b_{i+1}, d_i) \rangle + \langle m_2(b_i, a_{i+1}), m_2(c_{i+1}, a_i) \rangle & \text{if } j = i + 1 \\
\langle m_2(c_i, a_{i-1}), m_2(b_{i-1}, d_i) \rangle + \langle m_2(b_i, a_{i-1}), m_2(c_{i-1}, a_i) \rangle & \text{if } j = i - 1 \\
\end{array} \right.
\]
\[
4(t_t, \rho_{i+1, \sigma_{i+2}}) = \langle m_2(c_{i+2}, a_{i+1}), m_2(b_{j+2}, d_{j+1}) \rangle \\
= \langle c_{i+2}, m_2(a_{i+1}, m_2(b_{j+2}, d_{j+1})) \rangle \\
= \left\{ \begin{array}{ll}
\langle c_{i+2}, m_2(\tau_{i+1,i+2}, d_{i+1}) \rangle & \text{if } i = j \\
\langle c_{i+2}, m_2(\tau_{i+1,i+2}, d_{i+1}) \rangle & \text{if } i = j - 1 \\
\langle c_{j}, m_2(\tau_{j+2,j+2}, t, d_{i+1}) \rangle & \text{if } i = j + 1 \\
\end{array} \right.
\]
\[
4(t_t, \rho_{i+1, \sigma_{i+2}}) = \langle m_2(c_{i+2}, a_{i+1}), m_2(b_{j+2}, d_{j+1}) \rangle \\
= \left\{ \begin{array}{ll}
\langle c_{i+2}, m_2(\tau_{i+1,i+2}, d_{i+1}) \rangle & \text{if } i = j \\
\langle c_{i+2}, m_2(\tau_{i+1,i+2}, d_{i+1}) \rangle & \text{if } i = j - 1 \\
\langle c_{j}, m_2(\tau_{j+2,j+2}, t, d_{i+1}) \rangle & \text{if } i = j + 1 \\
\end{array} \right.
\]

Summarizing, we conclude
\[
\langle r, r \rangle = \langle s, s \rangle = -2(t_t, t_t) = -\langle \rho_t, \sigma_i \rangle = -\langle \sigma_i, \rho_t \rangle,
\]
\[
4(t_t, \tau_{ij}) = -\langle r, r \rangle \quad \text{for } i \neq j,
\]
\[
\tau_{ii} = \langle t_t, t_t \rangle, \quad \tau_{ii+1} = \langle t_t, t_{ii+1} \rangle = \langle t_{ii+1}, t_i \rangle = -\langle t_t, t_t \rangle
\]
and all other pairings are zero. Since the pairing is nondegenerate, we conclude \(\langle t_t, t_t \rangle \neq 0\). Normalizing if necessary, we may assume \(\langle t_t, t_t \rangle = 1\). Then the pairing is given by
\[
\langle t_t, t_t \rangle = 1, \quad \langle r, r \rangle = \langle s, s \rangle = -2, \quad \langle \rho_t, \sigma_i \rangle = \langle \sigma_i, \rho_t \rangle = 2
\]
\[
\langle \tau_{ij}, \tau_{ji} \rangle = 2, \quad \text{for } i \neq j, \quad \langle \tau_{ii}, \tau_{ii} \rangle = 1, \quad \langle \tau_{ii}, \tau_{ii+1} \rangle = \langle \tau_{ii+1}, \tau_{ii} \rangle = -1
\]
and all other pairings are zero. Thus the superpotential is given by

\[
\Phi = \sum_{i=0}^{2} \left( a_i b_i + d_i c_i \right) t - 2 (b_i a_i) r - 2 (c_i d_i) s + 2 (c_{i+2} a_{i+1} - c_{i+1} a_{i+2}) \sigma_i + 2 (b_{i+2} d_{i+1} - b_{i+1} d_{i+2}) \rho_i + 2 (a_i b_{i+1} - d_{i+1} c_i) \tau_{i,i+1} + 2 (a_i b_{i+1} - d_{i+1} c_i) \tau_{i,i-1} + (a_i b_i - d_i c_i) (\tau_{i} - \tau_{i+1,i+1} - \tau_{i-1,i-1}) \right).
\]

The remaining examples are calculated using the same method. We will skip the calculations and simply state the answers.

**Example 6.3.7.** Take \( X = \mathbb{P}^1 \times \mathbb{P}^1 \), \( V = K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}(-2, -2) \), which is Calabi–Yau 3, and \( \mathcal{E} = (\mathcal{O}, \mathcal{O}(0, 1), \mathcal{O}(1, 0), \mathcal{O}(1, 1)) \). The classical tilting quiver is given by

![Diagram](image)

with relations

\[
e_{i,j+1} b_j = e_{i,j} b_{j+1}, \quad e_{i+1,j} d_i = e_{i,j} d_{i+1},
\]

\[
c_i e_{i+1} = c_{i+1} e_i, \quad a_i e_{i+1} = a_{i+1} e_i,
\]

where the four edges \( e \) are indexed by subscript \( e_{ij} \), edges \( a, b, c, d \) indexed by \( a_i, b_i, c_i, d_i \), with \( 0 \leq i, j \leq 1 \).

The quiver with superpotential is given by

![Diagram](image)

with superpotential

\[
\Phi = e_{00}(b_0 a_0 - d_0 c_0) + e_{01}(b_1 a_0 - d_0 c_1) + e_{10}(b_0 a_1 - d_1 c_0) + e_{11}(b_1 a_1 - d_1 c_1).
\]

**Example 6.3.8.** Take \( X = \mathbb{P}^2 \times \mathbb{P}^1 \), \( V = K_{\mathbb{P}^2 \times \mathbb{P}^1} \) which is Calabi–Yau 4, with exceptional sequence \( \mathcal{E} = (\mathcal{O}, \mathcal{O}(0, 1), \mathcal{O}(1, 0), \mathcal{O}(1, 1), \mathcal{O}(2, 0), \mathcal{O}(2, 1)) \). The classical tilting quiver is given by

![Diagram](image)

with relations

\[
b_{i+1} a_{i+2} = b_{i+2} a_{i+1}, \quad d_{i+1} c_{i+2} = d_{i+2} c_{i+1}, \quad e_{0} h_{i+1} = e_{1} h_{0}, \quad d_{i} f_{a} = g_{a} b_{i}, \quad h_{i+1} a_{i+2} = h_{i+2} a_{i+1}, \quad a_{i+1} h_{i+2} = a_{i+2} h_{i+1} a.
\]
where $0 \leq i \leq 2$ and $0 \leq \alpha \leq 1$.

The dg-quiver is given by

$$
\begin{align*}
\Phi &= \sum_{i=0}^{2} r_i (c_0 h_{i+1} - c_1 h_{i+2}) + \rho_i (d_{i+1} c_{i+2} - d_{i+2} c_{i+1}) + s_i (h_{i+1} a_{i+2} - h_{i+2} a_{i+1}) + \sigma_i (b_{i+1} a_{i+2} - b_{i+2} a_{i+1}) \\
+ \sum_{\alpha=0}^{1} \sum_{i=0}^{1} (-1)^{\alpha+1} p_{i+1, \alpha+1} (h_{i+1, \alpha+1} d_{i+2} - h_{i+2, \alpha+1} d_{i+1}) + (-1)^{\alpha+1} \phi_{i+1, \alpha+1} (c_{i+1} e_{\alpha} - f_{\alpha} a_{i+1}) \\
+ \sum_{\alpha=0}^{1} \sum_{i=0}^{1} (-1)^{\alpha+1} q_{i+1, \alpha+1} (a_{i+1} h_{i+2, \alpha+1} - a_{i+2} h_{i+1, \alpha+1}) + (-1)^{\alpha+1} \psi_{i+1, \alpha+1} (d_{i+1} f_{\alpha} - g_{\alpha} b_{i+1}).
\end{align*}
$$

6.4 Product Construction

In this section, we compute some examples illustrating the product construction given in Proposition 5.7.3. The first example is of course the simplest case:

**Example 6.4.1.** Take $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, both regarded as vector bundle over the point $\text{Spec } \mathbb{C}$. By Example 6.3.1, the $A_\infty$-category corresponding to $V$ and $W$ are respectively given by $\wedge^\bullet V$ and $\wedge^\bullet W$ with wedge product as the only nonvanishing $A_\infty$-structure. The $A_\infty$-category corresponding to $V \times W \cong V \oplus W = \mathbb{C}^{n+m}$ is given by $\wedge^\bullet (V \oplus W)$. This verifies our product construction since

$$
\wedge^\bullet (V \oplus W) \cong \wedge^\bullet V \otimes \wedge^\bullet W
$$

as algebras.
**Example 6.4.2.** Take \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( V = 0 \). This example can be viewed as the product of the zero vector bundle over \( \mathbb{P}^1 \) with itself. Take the exceptional sequence \( \mathcal{E} = (\mathcal{O}, \mathcal{O}(1)) \) on \( D^b(\text{Coh}(\mathbb{P}^1)) \). The product sequence is then \( \mathcal{E} \boxtimes \mathcal{E} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1)) \) which is strong and full. The dg-tilting quiver for \( \mathbb{P}^1 \) is given by

\[
\begin{array}{c}
\bullet & \overset{e}{\longrightarrow} & \bullet
\end{array}
\]

The corresponding \( A_\infty \)-category \( \mathcal{A} \) for \( \mathbb{P}^1 \) is given by

\[
\mathcal{A}^0(u_0, u_0) = \text{span} \mathbb{C}\{u_0\}, \quad \mathcal{A}^0(u_1, u_1) = \text{span} \mathbb{C}\{u_1\}, \quad \mathcal{A}^1(u_0, u_1) = \text{span} \mathbb{C}\{e_0, e_1\}.
\]

and zero otherwise; with the only nonvanishing \( A_\infty \)-structure given by

\[
m_2(u_1, e_i) = -m_2(e_i, u_1) = e_i = r_{ij}, \quad m_2(u_1, u_1) = u_i.
\]

We will use the following naming scheme:

\[
v_{ij} = u_i \otimes u_j, \quad a_i = e_i \otimes u_0, \quad b_i = u_1 \otimes e_i, \quad c_i = u_0 \otimes e_i, \quad d_i = e_i \otimes u_1, \quad r_{ij} = e_i \otimes e_j
\]

The product dg-tilting quiver, i.e., the dg-tilting quiver for \( \mathbb{P}^1 \times \mathbb{P}^1 \), is thus given by

\[
\begin{array}{c}
v_{00} & \overset{d}{\longrightarrow} & v_{11}
\end{array}
\]

where black edges are of degree 0, red edges are of degree \(-1\). Since \( \mathcal{A} \) has only nonvanishing \( m_2 \), so is \( \mathcal{A} \otimes \mathcal{A} \). The only nonvanishing \( m_2 : (\mathcal{A} \otimes \mathcal{A})^1 \otimes (\mathcal{A} \otimes \mathcal{A})^1 \to (\mathcal{A} \otimes \mathcal{A})^2 \) is given by

\[
m_2(b_j, a_i) = -m_2(e_i, u_1) \otimes m_2(u_1, e_j) = e_i \otimes e_j = -r_{ij},
\]

\[
m_2(d_i, c_j) = m_2(e_i, u_0) \otimes m_2(u_1, e_j) = e_i \otimes e_j = r_{ij}.
\]

Thus the differential \( d \) is given by

\[
\text{dr}_{ij} = d_i c_j - b_j a_i \quad \text{and} \quad \text{da}_i = db_i = dc_i = dd_i = 0.
\]

The classical tilting quiver is given by

\[
\begin{array}{c}
v_{00} & \overset{d}{\longrightarrow} & v_{11}
\end{array}
\]

with relations \( b_j a_i = d_i c_j \).

**Example 6.4.3.** Consider \( X = \mathbb{P}^1 \) and \( V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \) which is Calabi–Yau 4. Then \( V \) can be viewed as the product of the trivial bundle \( \mathbb{C}^2 \to \text{Spec} \mathbb{C} \) over a point and the canonical bundle \( K_{\mathbb{P}^1} \to \mathbb{P}^1 \) over the projective line. Take the exceptional sequences \( \mathcal{O}_{\text{Spec} \mathbb{C}} \) on \( D^b(\text{Coh}(\mathbb{P}^1)) \) and \( (\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)) \) on \( D^b(\text{Coh}(\mathbb{P}^1)) \). Recall from Examples 6.3.1 and 6.3.2 that the quivers with superpotential derived equivalent to \( \mathbb{C} \) and \( K_{\mathbb{P}^1} \) are respectively

\[
\begin{array}{c}
\bullet & \overset{u}{\longrightarrow} & \bullet
\end{array}
\]

with \( \deg x_0 = \deg x_1 = 0, \deg u^* = -1, dx_i = 0 \) and \( du^* = x_1 x_0 - x_0 x_1 \); and

\[
\begin{array}{c}
v_0 & \overset{\varepsilon}{\longrightarrow} & v_1
\end{array}
\]

with \( \deg x_0 = 0, \deg x_1 = 0, \deg u^* = -1, dx_i = 0 \) and \( du^* = x_1 x_0 - x_0 x_1 \); and
where black edges are of degree 0 and brown edges are of degree −1, with differential given by

\[ dc_0^* = f_1c_0 - f_0c_1, \quad dc_1^* = e_1f_0 - e_0f_1 \quad \text{and} \quad dc_i = df_i = 0. \]

Using the \( A_\infty \)-categories corresponding to \( \mathbb{C}^2 \) and \( K_{\mathbb{P}^2} \) given in Examples 6.3.1 and 6.3.2, we calculate the tensor product category \( \mathcal{A} \). We will use the following naming scheme for simplicity:

\[
\begin{align*}
    w_i &= u \otimes v_i, \quad c_i = x_i \otimes v_0, \quad d_i = x_i \otimes v_1, \quad a_i = u \otimes c_i, \quad b_i = u \otimes f_i, \\
    r_0 &= u^* \otimes v_0, \quad r_1 = u \otimes v_0^*, \quad s_0 = u^* \otimes v_1, \quad s_1 = u \otimes v_1^*, \quad p_{ij} = x_i \otimes c_j, \quad q_{ij} = x_i \otimes f_j \\
    a_i^* &= u^* \otimes f_{i+1}, \quad b_i^* = u^* \otimes c_{i+1}, \quad c_i^* = x_{i+1} \otimes v_0^*, \quad d_i^* = x_{i+1} \otimes v_1^*, \quad w_i^* = u^* \otimes v_i^*.
\end{align*}
\]

Thus the dg-tilting quiver is given by

where black edges are of degree 0, red of degree −1, blue of degree −2 and brown of degree −3.

Since \( m_n = 0 \) for all \( n \neq 2 \) for both \( A_\infty \)-category, \( m_n \) of their tensor product also vanishes for \( n \neq 2 \).

The action of \( m_2 : \mathcal{A}^1 \otimes \mathcal{A}^1 \rightarrow \mathcal{A}^2 \) is given by

\[
\begin{align*}
    m_2(c_0, c_0) &= m_2(x_0, x_0) \otimes m_2(v_0, v_0) = u^* \otimes v_0 = r_0, \\
    m_2(c_0, c_1) &= m_2(x_0, x_1) \otimes m_2(v_0, v_0) = -u^* \otimes v_0 = -r_0, \\
    m_2(b_0, a_1) &= m_2(u, u) \otimes m_2(f_0, c_1) = -u \otimes v_0^* = -r_1, \\
    m_2(b_1, a_0) &= m_2(u, u) \otimes m_2(f_1, e_0) = u \otimes v_0^* = r_1, \\
    m_2(b_1, d_0) &= m_2(x_1, x_0) \otimes m_2(v_1, v_1) = u^* \otimes v_1 = s_0, \\
    m_2(d_0, d_1) &= m_2(x_0, x_1) \otimes m_2(v_1, v_1) = -u^* \otimes v_1 = -s_0, \\
    m_2(a_0, b_1) &= m_2(u, u) \otimes m_2(e_0, f_1) = -u \otimes v_1^* = -s_1, \\
    m_2(a_1, b_0) &= m_2(u, u) \otimes m_2(e_1, f_0) = u \otimes v_1^* = s_1, \\
    m_2(a_1, c_1) &= -m_2(u, x_j) \otimes m_2(e_1, v_0) = -x_j \otimes e_i = -p_{ji}, \\
    m_2(d_j, a_i) &= m_2(x_j, u) \otimes m_2(v_1, c_i) = x_j \otimes e_i = p_{ji}, \\
    m_2(c_i, b_j) &= m_2(x_i, u) \otimes m_2(v_0, f_j) = x_i \otimes f_j = q_{ij}, \\
    m_2(b_j, d_i) &= -m_2(u, x_i) \otimes m_2(f_j, v_1) = -x_i \otimes f_j = -q_{ij},
\end{align*}
\]

and zero otherwise. Hence the differential in the dg-quiver sends

\[
\begin{align*}
    dr_0 &= c_1c_0 - c_0c_1, \quad dr_1 = b_1a_0 - b_0a_1, \\
    ds_0 &= d_1d_0 - d_0d_1, \quad ds_1 = a_1b_0 - a_0b_1, \\
    dp_{ij} &= d_ia_j - a_jc_i, \quad dq_{ij} = c_ib_j - b_jd_i.
\end{align*}
\]

The cyclic structure on the tensor product \( \langle -, - \rangle : \mathcal{A}^2 \otimes \mathcal{A}^2 \rightarrow \mathbb{C} \) is given by

\[
\langle p_{ij}, q_{kl} \rangle = \langle q_{kl}, p_{ij} \rangle = \langle x_k \otimes f_\ell, x_i \otimes c_j \rangle = -\langle x_k, x_i \rangle \langle f_\ell, c_j \rangle
\]

\[
\begin{cases}
    1 & \text{if } (i, j, k, \ell) = (1, 0, 0, 1) \text{ or } (0, 1, 1, 0) \\
    -1 & \text{if } (i, j, k, \ell) = (0, 0, 1, 1) \text{ or } (1, 1, 0, 0) \\
    0 & \text{otherwise.}
\end{cases}
\]
Now, we construct a quasi-free resolution of the quiver. Observe that
\[(r_1, r_0) = (r_0, r_1) = (u^*, u)(v_0, v_0^*) = 1, \quad (s_1, s_0) = (s_0, s_1) = (u^*, u)(v_1, v_1^*) = 1\]
\[(r_0, r_0) = (u^*, u^*)(v_0, v_0) = 0, \quad (s_0, s_0) = (u^*, u^*)(v_1, v_1) = 0\]
and zero otherwise.

Hence the superpotential is given by

\[
\Phi = r_0(b_1a_0 - b_0a_1) + r_1(c_1c_0 - c_0c_1) + s_0(a_2b_0 - a_0b_1) + s_1(d_1d_0 - d_0d_1)
\]
\[
+ p_{01}(c_1b_0 - b_0d_1) + q_{10}(d_0a_1 - a_1c_0) + p_{10}(c_0b_1 - b_1d_0) + q_{01}(d_1a_0 - a_0c_1)
\]
\[- p_{10}(c_1b_1 - b_1d_1) - q_{11}(d_1a_1 - a_1c_1) - p_{11}(c_0b_0 - b_0d_0) - q_{00}(d_0a_0 - a_0c_0).
\]

Taking $H^0$ of the dg-quiver, we see that the classical tilting quiver is given by

\[
\begin{array}{c}
\circ \quad \circ \\
\bullet \quad \bullet \\
\circ \quad \circ
\end{array}
\begin{array}{c}
a_0 \rightarrow \\
b_0 \rightarrow \\
b_1 \rightarrow
\end{array}
\begin{array}{c}
\circ \quad \circ \\
\bullet \quad \bullet \\
\circ \quad \circ
\end{array}
\begin{array}{c}
d_1 \\
d_0 \\
d_0
\end{array}
\]

with relations

\[
a_0b_1 = a_1b_0 \quad d_0d_1 = d_1d_0
\]
\[
b_0a_1 = b_1a_0 \quad c_0c_1 = c_1c_0
\]
\[
a_i c_j = d_j a_i \quad c_i b_j = b_j d_i,
\]

where $0 \leq i, j \leq 1$.

The remaining examples are calculated using the same method. We will skip the calculations and simply state the answers.

**Example 6.4.4.** Consider the vector bundle $O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-3)$ which is Calabi–Yau 4. The classical tilting quiver is given by

\[
\begin{array}{c}
\circ \quad \circ \\
\bullet \quad \bullet \\
\circ \quad \circ
\end{array}
\begin{array}{c}
u_1 \\
u_2 \\
u_0
\end{array}
\]

with relations

\[
b_i+1 a_i = b_i a_{i+1} \quad c_i u_2 = u_0 c_i
\]
\[
c_i b_i = c_i b_{i+1} \quad a_i u_0 = u_1 a_i
\]
\[
a_{i+1} c_i = a_i c_{i+1} \quad b_i u_1 = u_2 b_i.
\]

Now, we construct a quasi-free resolution of the quiver. Observe that $O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-3) = \mathbb{C} \times O_{\mathbb{P}^2}(-3)$. Hence the dg-quiver for $O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-3)$ is the product of dg-quivers $\mathbb{C}$ and $O_{\mathbb{P}^2}(-3)$ in Examples 6.3.1 and 6.5.1, which is given by
where black edges are of degree 0, red edges are of degree $-1$, blue edges are of degree $-2$ and brown edges are of degree $-3$. The differential sends

$$
\begin{align*}
&d \tau_i = b_{i+2} a_{i+1} - b_{i+1} a_{i+2} & d t_i = c_i u_2 - u_0 c_i \\
&d p_i = c_{i+2} b_{i+1} - c_{i+2} b_i & d r_i = a_i u_0 - u_1 a_i \\
&d \sigma_i = a_{i+2} c_{i+1} - a_{i+1} c_{i+2} & d s_i = b_i u_1 - u_2 b_i
\end{align*}
$$

and the symmetric pairing on degree $-1$ edges is given by

$$
\langle \tau_i, t_i \rangle = \langle s_i, \sigma_i \rangle = \langle p_i, r_i \rangle = 1.
$$

The superpotential is given by

$$
\Phi = \sum_{i=1}^{3} \left( (b_{i+2} a_{i+1} - b_{i+1} a_{i+2}) t_i + (c_i u_2 - u_0 c_i) \tau_i \\
+ (c_{i+2} b_{i+1} - c_{i+2} b_i) r_i + (a_i u_0 - u_1 a_i) p_i \\
+ (a_{i+2} c_{i+1} - a_{i+1} c_{i+2}) s_i + (b_i u_1 - u_2 b_i) \sigma_i \right).
$$

**Example 6.4.5.** Take $\mathbb{P}^1 \times \mathbb{P}^1$, $V = T_{\mathbb{P}^1 \times \mathbb{P}^1}^*$ which is Calabi–Yau 4. We can view $V$ as the product of canonical bundle on $\mathbb{P}^1$ with itself. Take the exceptional sequence $E = (\mathcal{O}, \mathcal{O}(1))$ on $D^b(\text{Coh}(\mathbb{P}^1))$. Then $E \boxtimes = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1))$. The tilting quiver is

![Tilting Quiver](image)

with relations

$$
\begin{align*}
b_j a_i &= d_i c_j, & g_j b_i &= c_i f_j, \\
c_j c_i &= h_i b_j, & f_j d_i &= a_i g_j, \\
a_j c_i &= a_i c_j, & c_j a_i &= c_i a_j, \\
b_j f_i &= b_i f_j, & f_j b_i &= f_i b_j, \\
c_j g_i &= c_i g_j, & g_j c_i &= g_i c_j, \\
d_j h_i &= d_i h_j, & h_j d_i &= h_i d_j,
\end{align*}
$$
where $0 \leq i, j \leq 1$. The dg-tilting quiver is given by

$$
\begin{align*}
\frac{dp_{ij} = b_j a_i - d_i c_j}{dr_{ij} = c_j e_i - h_j b_j} \quad \frac{dq_{ij} = g_j h_i - e_i f_j}{ds_{ij} = f_j d_i - a_i g_j} \\
\frac{dw_0 = e_1 a_1 - e_0 a_1}{dw_{00} = f_1 d_0 - a_0 g_1} \quad \frac{dw_{01} = c_1 g_0 - c_0 g_1}{dw_{01} = h_1 d_0 - h_0 d_1} \\
\frac{dw_{10} = f_1 b_0 - f_0 b_1}{dw_{10} = a_1 c_0 - a_0 c_1} \quad \frac{dw_{11} = d_1 h_0 - d_0 h_1}{dw_{11} = b_1 f_0 - b_0 f_1}
\end{align*}
$$

with differential given by

$$
\begin{align*}
dp_{ij} &= b_j a_i - d_i c_j \\
dq_{ij} &= g_j h_i - e_i f_j \\
dr_{ij} &= c_j e_i - h_j b_j \\
ds_{ij} &= f_j d_i - a_i g_j \\
dw_0 &= e_1 a_1 - e_0 a_1 \\
dw_{00} &= f_1 d_0 - a_0 g_1 \\
dw_{01} &= c_1 g_0 - c_0 g_1 \\
dw_{10} &= h_1 d_0 - h_0 d_1 \\
dw_{11} &= b_1 f_0 - b_0 f_1
\end{align*}
$$

The pairing on degree $-1$ edges is given by

$$
\langle p_{i,i+1}, q_{i+1,i} \rangle = \langle r_{i,i+1}, s_{i+1,i} \rangle = \langle u_{ij}, w_{ij} \rangle = 1,
$$

and zero otherwise, where we treat $i \in \mathbb{Z}/2\mathbb{Z}$. Thus the superpotential is given by

$$
\Phi = u_{00}(g_1 c_0 - g_0 c_1) + u_{00}(c_1 a_0 - e_0 a_1) + u_{01}(h_1 d_0 - h_0 d_1) + u_{01}(e_1 g_0 - c_0 g_1) + u_{10}(a_1 c_0 - a_0 c_1) + u_{10}(f_1 b_0 - f_0 b_1) + u_{11}(b_1 f_0 - b_0 f_1) + u_{11}(d_1 h_0 - d_0 h_1)
$$

$$
+ \sum_{i=0}^{1} \left( p_{i,i+1}(g_i h_{i+1} - e_{i+1} f_i) + q_{i+1,i}(b_{i+1} a_i - d_i c_{i+1}) + r_{i,i+1}(f_{i+1} d_i - a_{i+1} g_i) + s_{i+1,i}(c_{i+1} e_i - h_i b_{i+1}) \right)
$$

$$
- \sum_{i=0}^{1} \left( p_{i,i+1}(g_i h_{i+1} - e_{i+1} f_i) + q_{i+1,i+1}(b_i a_i - d_i c_i) + r_{i,i}(f_{i+1} d_i - a_{i+1} g_{i+1}) + s_{i+1,i+1}(c_i e_i - h_i b_i) \right).
$$

**Example 6.4.6.** Take $X = \mathbb{P}^2 \times \mathbb{P}^1$ and $V = 0$. It can be view as the product of the zero vector bundle on $\mathbb{P}^2$ and the zero vector bundle on $\mathbb{P}^1$. Take $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ an exceptional sequence on $\text{D}^b(\text{Coh}(\mathbb{P}^2))$ and $\mathcal{F} = (\mathcal{O}, \mathcal{O}(1))$ an exceptional sequence on $\text{D}^b(\text{Coh}(\mathbb{P}^1))$. Then $\mathcal{E} \boxtimes \mathcal{F} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,0), \mathcal{O}(2,1))$. The dg-tilting quiver is given by
where black edges are of degree 0, red edges are of degree $-1$, blue edges are of degree $-2$. We will index edges $p,q,u$ by $p_{ai},q_{ai},u_{ai}$, where $0 \leq i \leq 2$ and $0 \leq a \leq 1$. The differential is given by

$$
d e\alpha = df\alpha = dg\alpha = 0, \quad d a_i = db_i = dc_i = dd_i = 0, \\
d q_{ai} = d f\alpha - g a_i,
$$

$$
d r_i = d_{i+1}c_{i+2} - d_{i+2}c_{i+1}, \quad d s_i = b_{i+1}a_{i+2} - b_{i+2}a_{i+1},
$$

$$
d u_{ai} = d_{i+1}p_{ai+2} - d_{i+2}p_{ai+1} + g_{ai}a_{i+2} - g_{ai}a_{i+1} + g_{ai}s_i - r_i e\alpha.
$$

The classical tilting quiver is thus given by

![Diagram](image)

with relations

$$
b_i a_i = b_i a_j, \quad d_j c_i = d_i c_j, \\
c_i e\alpha = f a_i, \quad d_i f a_i = g a_i b_i,
$$

where $0 \leq i, j \leq 2$ and $0 \leq \alpha, \beta \leq 1$.

### 6.5 Quotient Construction

In this section, we compute some examples illustrating the quotient construction given by Proposition 5.6.5.

**Example 6.5.1.** Consider $V = \mathcal{O}_{\mathbb{P}^n}(-n-1) = K_{\mathbb{P}^n}$ and $X = \mathbb{P}^n$. By Example 6.1.6, the classical tilting quiver is

![Diagram](image)

with relations $a_{i+1} k a_{i+1} = a_{i+1} j a_{i+1}$. Note that this quiver with relations has path algebra isomorphic to $\mathbb{C}[x_0, \ldots, x_n] \# Z_{n+1}$, with $\mathbb{Z}_{n+1}$ acting on each variable by multiplying $\omega^{-1}$, where $\omega = e^{2\pi i/n}$ is an $n$-th root of unity. Hence its corresponding $A_{\infty}$-algebra is $A_{\infty}$-isomorphic to $\bigoplus_{\rho,\sigma \in \text{Int}(\mathbb{Z}_{n+1})} \text{Hom}_{\mathbb{Z}_{n+1}}(\rho, \Lambda^* \mathbb{K}^{n+1} \otimes \sigma)$, which is cyclic since $\Lambda^* \mathbb{K}^{n+1}$ is. This corresponds to the fact that $K_{\mathbb{P}^n}$ is Calabi-Yau. This example is an incarnation of the McKay correspondence as given by Bridgeland, King and Reid [14]: $K_{\mathbb{P}^n} \to \mathbb{K}^{n+1}/\mathbb{Z}_{n+1}$ is a crepant resolution and we have $D^b(\text{Coh}(K_{\mathbb{P}^n})) \cong D^b(\text{Coh}(\mathbb{Z}_{n+1}(\mathbb{K}^{n+1})))$.

The group $\mathbb{Z}_{n+1}$ has $n + 1$ irreducible representations $\rho_0, \ldots, \rho_n$, where $\rho_i$ is the one dimensional representation on which $\mathbb{Z}_{n+1}$ acts by multiplication by $\omega^i$. We will denote by $c_i$ a basis for $\rho_i$. We will also choose a basis $u_1, \ldots, u_{n+1}$ for $\mathbb{K}^{n+1}$. For any $0 \leq \ell \leq n + 1$, we have isomorphisms $\rho^{\otimes(n+1)}\cong \Lambda^\ell \mathbb{K}^{n+1} \otimes \rho_{j+\ell}$ via the sum of the following maps:

$$
x_{i_1}^{(j)} \cdot \cdots \cdot x_{i_\ell}^{(j)} : \rho_j \to \Lambda^\ell \mathbb{K}^{n+1} \otimes \rho_{j+\ell}, \quad c_j \mapsto u_{i_1} \wedge \cdots \wedge u_{i_\ell} \otimes c_{j+\ell}
$$

where $1 \leq i_1 < i_2 < \cdots < i_{\ell+1} \leq n + 1$.

Hence the dg-quiver of $K_{\mathbb{P}^n}$ has $n + 1$ vertices $v_0, \ldots, v_{n+1}$ and $(n+1)$ arrows of degree $1 - \ell$ from vertex $v_j$ to $v_{j+\ell}$, where we think of $j \in \mathbb{Z}_{n+1}$. If we extend the definition of $x_{i_1}^{(j)} \cdots x_{i_\ell}^{(j)}$ to all $1 \leq i_k \leq n + 1$ by setting $x_{i_1}^{(j)} \cdots x_{i_\ell}^{(j)} = (\text{sgn } \alpha)x_{i_{\alpha(1)}}^{(j)} \cdots x_{i_{\alpha(\ell)}}^{(j)}$ where $\alpha$ is the permutation such that $i_{\alpha(1)} < \cdots < i_{\alpha(\ell)}$ if all
the indices $i_k$ are distinct and setting $x_{i_1 \ldots i_\ell} = 0$ if two indices $i_k$ are equal, then we can describe the $A_\infty$-structure by

$$m_2(x_{i_1 \ldots i_{k_1} \ldots k_\ell}, x_{i_1 \ldots i_{k_1' \ldots k_\ell'}}) = x_{i_1 \ldots i_{k_1} \ldots k_\ell}$$

and all other $m_n$’s are zero.

When $n = 2$, we recover the quiver with superpotential in (6.3.1):

![Quiver with superpotential](image)

where black edges are of degree 0, blue edges are of degree $-1$, brown loops are of degree $-2$, with superpotential $\Phi = \sum a_i(b_{i+1}c_{i+2} - b_{i+2}c_{i+1})$.

When $n = 3$, the dg-tilting quiver is given by

![Quiver for n=3](image)

Here, the black edges are of degree 0, red edges of degree $-1$, blue edges of degree $-2$ and brown edges of degree $-3$. We will label the red edges by $(i, j)$, where $0 \leq i < j \leq 3$. The differential in the dg-quiver sends

$$dp_{ij} = b_i a_j - b_j a_i, \quad dq_{ij} = d_i c_j - d_j c_i$$

$$dr_{ij} = c_i b_j - c_j b_i, \quad ds_{ij} = a_i d_j - a_j d_i.$$

The symmetric pairing between the red (degree $-1$) edges is given by

$$\langle p_{ij}, q_{k\ell} \rangle = \langle r_{ij}, s_{k\ell} \rangle = \epsilon_{ij k\ell} = \begin{cases} 
1 & \text{if } (i, j, k, \ell) \text{ is an even permutation of (1,2,3,4)} \\
-1 & \text{if } (i, j, k, \ell) \text{ is an odd permutation of (1,2,3,4)} \\
0 & \text{otherwise.}
\end{cases}$$

The superpotential is given by

$$\Phi = \sum_{i,j,k,\ell} \epsilon_{ij k\ell} \left( q_{ij}(b_i a_\ell - b_\ell a_i) + p_{ij}(d_i c_\ell - d_\ell c_i) + s_{ij}(c_i b_\ell - c_\ell b_i) + r_{ij}(a_i d_\ell - a_\ell d_i) \right).$$

**Example 6.5.2.** Suppose $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{P}^2$ by multiplication of $e^{4\pi i/3}$ on homogeneous coordinate. This $\mathbb{Z}_3$-action lifts to $\mathcal{O}(1)$ and turn $\mathcal{O}(1)$ into a $\mathbb{Z}_3$-bundle which will be denoted by $E$. Let $p_k$ be the irreducible representation of $\mathbb{Z}_3$ corresponding to multiplication of $e^{2\pi ik/3}$. The sequence $\mathcal{E} = (\mathcal{O}, E, E^{\oplus 2})$ is a sequence of $\mathbb{Z}_3$-sheaves whose underlying sequence form an full strong exceptional sequence. To calculate the corresponding $A_\infty$-category, we need the following
Lemma 6.5.3.

\[
H^f(\mathbb{P}^2, \mathcal{O}) = \begin{cases} 
\rho_0 & \text{if } \ell = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \mathcal{O}(1)) = \begin{cases} 
\rho_2^{\oplus 3} & \text{if } \ell = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \mathcal{O}(2)) = \begin{cases} 
\rho_1^{\oplus 3} & \text{if } \ell = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \omega(-1)) = \begin{cases} 
\rho_1^{\oplus 3} & \text{if } \ell = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \omega(-2)) = \begin{cases} 
\rho_2^{\oplus 3} & \text{if } \ell = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \omega(-3)) = \begin{cases} 
\rho_0 & \text{if } \ell = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \omega(-4)) = \begin{cases} 
\rho_1^{\oplus 3} & \text{if } \ell = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

\[
H^f(\mathbb{P}^2, \omega(-5)) = \begin{cases} 
\rho_2^{\oplus 3} & \text{if } \ell = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. It suffices to show the lemma for the nonvanishing cases. Since \(\mathcal{O}\) is trivial \(\mathbb{Z}_3\)-equivariantly, \(H^0(\mathbb{P}^2, \mathcal{O}) = \rho_0\). By Serre duality, \(H^2(\mathbb{P}^2, \mathcal{O}(-3)) \cong H^0(\mathbb{P}^2, \mathcal{O})^{-} = \rho_0\). By the definition of the \(\mathbb{Z}_3\)-action on \(\mathbb{P}^2\), \(H^0(\mathbb{P}^2, \mathcal{O}(1)) = \rho_2^{\oplus 3}\). By Serre duality, \(H^2(\mathbb{P}^2, \mathcal{O}(-4)) = H^0(\mathbb{P}^2, \mathcal{O}(1))^{-} = (\rho_2^{\oplus 3})^{-} = \rho_1^{\oplus 3}\).

Since \(\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee)\), we have the short exact sequence

\[
0 \to \mathcal{O}(2) \to \mathcal{O}(1) \otimes H^0(\mathbb{P}^2, \mathcal{O}(1)) \to \mathcal{O}(2) \to 0,
\]

and hence the exact sequence

\[
0 \to H^0(\mathbb{P}^2, \mathcal{O}(2)) \to H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{O}(1)) \to H^0(\mathbb{P}^2, \mathcal{O}(2)) \to 0.
\]

By the isomorphisms

\[
H^0(\mathbb{P}^2, \mathcal{O}(2)) \cong S^2 H^0(\mathbb{P}^2, \mathcal{O}(1)) \text{ and } H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{O}(1)) \cong S^2 H^0(\mathbb{P}^2, \mathcal{O}(1)) \oplus \wedge^2 H^0(\mathbb{P}^2, \mathcal{O}(1)),
\]

we conclude \(H^0(\mathbb{P}^2, \mathcal{O}(2)) \cong \wedge^2 H^0(\mathbb{P}^2, \mathcal{O}(1)) \cong \rho_1^{\oplus 3}\). By Serre duality, we have

\[
H^0(\mathbb{P}^2, \omega(-5)) = \rho_2^{\oplus 3}.
\]

From the short exact sequence

\[
0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes H^0(\mathbb{P}^2, \mathcal{O}(1))^{-} \to \omega(-1) \to 0
\]

we conclude \(H^0(\mathbb{P}^2, \omega(-1)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))^{-} \cong \rho_1^{\oplus 3}\). By Serre duality, we have

\[
H^2(\mathbb{P}^2, \omega(-2)) \cong H^0(\mathbb{P}^2, \omega(-1))^{-} \cong \rho_2^{\oplus 3}.
\]

From the short exact sequence

\[
0 \to \omega \to \mathcal{O}(1) \otimes H^0(\mathbb{P}^2, \mathcal{O}(1)) \to \omega \to 0,
\]
we have an isomorphism $H^1(\mathbb{P}^2, \Omega) \equiv H^0(\mathbb{P}^2, \mathcal{O}) = \rho_0$. By Serre duality, we have

$$H^1(\mathbb{P}^2, \mathcal{T}(-3)) \equiv H^0(\mathbb{P}^2, \mathcal{O})^\vee = \rho_0.$$ Twisting the Euler sequence with $\Omega(k)$, we have a short exact sequence

$$0 \to \Omega(k) \to H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee \otimes \Omega(k + 1) \to \mathcal{T} \otimes \Omega(k) \to 0.$$ We have isomorphisms $H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \equiv H^1(\mathbb{P}^2, \mathcal{O}) = \rho_0$ and

$$H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1)) \equiv H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee \otimes H^1(\mathbb{P}^2, \mathcal{O}) = \rho_1^{\otimes 3} \otimes \rho_0 = \rho_1^{\otimes 3}.$$ By Serre duality, we have

$$H^2(\mathbb{P}^2, \mathcal{T} \otimes (-2)) = H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega)^\vee = \rho_0 \quad \text{and} \quad H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1)) = H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1))^\vee = \rho_2^{\otimes 3}.$$}

We calculate its $A_{\infty}$-category:

$$\mathcal{A}^f(v_{0\alpha}, v_{0\beta}) = \text{Hom}_C(\mathcal{O}(2)[2] \otimes \rho_0, \mathcal{O}(2)[2] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^f(\mathbb{P}^2, \mathcal{O})^G \begin{cases} C & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^f(v_{0\alpha}, v_{1\beta}) = \text{Hom}_C(\mathcal{O}(2)[2] \otimes \rho_0, \mathcal{O}(1)[1] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f-1}(\mathbb{P}^2, \mathcal{O}(2))^G \begin{cases} C^3 & \text{if } \alpha - 1 = \beta \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^f(v_{0\alpha}, v_{2\beta}) = \text{Hom}_C(\mathcal{O}(2)[2] \otimes \rho_0, \mathcal{O} \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f-2}(\mathbb{P}^2, \mathcal{O}(1))^G \begin{cases} C^3 & \text{if } \alpha + 1 = \beta \text{ and } \ell = 2 \\ C & \text{if } \alpha = \beta \text{ and } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^f(v_{1\alpha}, v_{0\beta}) = \text{Hom}_C(\mathcal{O}(1)[1] \otimes \rho_0, \mathcal{O}(2)[2] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f+1}(\mathbb{P}^2, \mathcal{T}(-2))^G = 0.$$}

$$\mathcal{A}^f(v_{1\alpha}, v_{1\beta}) = \text{Hom}_C(\mathcal{O}(1)[1] \otimes \rho_0, \mathcal{O}(1)[1] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f}(\mathbb{P}^2, \mathcal{T} \otimes \Omega)^G \begin{cases} C & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^f(v_{1\alpha}, v_{2\beta}) = \text{Hom}_C(\mathcal{O}(1)[1] \otimes \rho_0, \mathcal{O} \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f-1}(\mathbb{P}^2, \mathcal{T}(-1))^G \begin{cases} C^3 & \text{if } \beta = \alpha - 1 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^f(v_{2\alpha}, v_{0\beta}) = \text{Hom}_C(\mathcal{O} \otimes \rho_0, \mathcal{O}(2)[2] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f+2}(\mathbb{P}^2, \mathcal{O}(1))^G = 0.$$}

$$\mathcal{A}^f(v_{2\alpha}, v_{1\beta}) = \text{Hom}_C(\mathcal{O} \otimes \rho_0, \mathcal{O}(1)[1] \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f+1}(\mathbb{P}^2, \mathcal{O}(1))^G = 0.$$}

$$\mathcal{A}^f(v_{2\alpha}, v_{2\beta}) = \text{Hom}_C(\mathcal{O} \otimes \rho_0, \mathcal{O} \otimes \rho_0) = \text{Hom}(\rho_0, \rho_0) \otimes H^{f}(\mathbb{P}^2, \mathcal{O})^G \begin{cases} C & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$
with differential $d s_{a, i} = b_{a - 1, i + 1} a_{a, i + 2} - b_{a - 1, i + 2} a_{a, i + 1}$. Here the vertex $v_{ij}$ corresponds to the bundle $E_{ij} \otimes \rho_j$. Note that this quiver is a disjoint union of three Beilinson quivers of $\mathbb{P}^2$ (Example 5.2.12).

This corresponds to the fact that $\text{Coh} \mathbb{Z}_3(\mathbb{P}^2) = \text{Coh}(\mathbb{P}^2)^{\otimes 3}$ since $\mathbb{Z}_3$ is acting trivially on $\mathbb{P}^2$.

**Example 6.5.4.** Following Example 6.5.2, we take the vector bundle $V = E^\vee \otimes E^\vee \otimes 2$, which is Calabi–Yau 4. Since the underlying bundle of $E$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$. This example is the $\mathbb{Z}_3$-equivariant version of Example 6.3.5. Since the underlying sequence of $E$ is compatible with $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, the $\mathbb{Z}_3$-exceptional sequence $E \otimes \text{Irr}(G)$ is $\mathbb{Z}_3$-compatible with $V$. Using Lemma 6.5.3, we calculate its $A_\infty$-category:

\[
\mathcal{A}'(v_{0\alpha}, v_{0\beta}) = \text{Hom}^f_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) \oplus \text{Hom}^{f-1}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^1(2)[2] \otimes \rho_\beta \otimes V) \\
\oplus \text{Hom}^{f-2}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes \wedge^2 V) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^f(\mathbb{P}^2, \mathcal{O})]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-1}(\mathbb{P}^2, \mathcal{O}(-1) \oplus \mathcal{O}(-2))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-2}(\mathbb{P}^2, \mathcal{O}(-3))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0, 4 \\
0 & \text{otherwise.} 
\end{cases}
\]

\[
\mathcal{A}'(v_{0\alpha}, v_{1\beta}) = \text{Hom}^f_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^1(1)[1] \otimes \rho_\beta) \oplus \text{Hom}^{f-1}_G(\Omega^2(2)[2] \rho_\alpha, \Omega^1(1)[1] \otimes \rho_\beta \otimes V) \\
\oplus \text{Hom}^{f-2}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^1(1)[1] \otimes \rho_\beta \otimes \wedge^2 V) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-1}(\mathbb{P}^2, \mathcal{O}(2))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-3}(\mathbb{P}^2, \mathcal{O}(-1))]^G \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \alpha - 1 = \beta \text{ and } \ell = 1 \\
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 3 \\
0 & \text{otherwise.} 
\end{cases}
\]

\[
\mathcal{A}'(v_{0\alpha}, v_{2\beta}) = \text{Hom}^f_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) \oplus \text{Hom}^{f-1}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \otimes V) \\
\oplus \text{Hom}^{f-2}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \wedge^2 V) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-3}(\mathbb{P}^2, \mathcal{O}(-1))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{f-4}(\mathbb{P}^2, \mathcal{O}(-2))]^G \\
= \begin{cases} 
\mathbb{C}^3 & \text{if } \alpha + 1 = \beta \text{ and } \ell = 2 \\
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 3 \\
0 & \text{otherwise.} 
\end{cases}
\]
\[ \mathcal{A}^\ell(v_{1\alpha}, v_{0\beta}) = \text{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) \oplus \text{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes \mathcal{V}) \\
\oplus \text{Hom}_G^{\ell-2}(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes \wedge^2 \mathcal{V}) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{T}(-3) \oplus \mathcal{T}(-4))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-5))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 1 \\
\mathbb{C}^3 & \text{if } \alpha + 1 = \beta \text{ and } \ell = 3 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \mathcal{A}^\ell(v_{1\alpha}, v_{1\beta}) = \text{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(1)[1] \otimes \rho_\beta) \oplus \text{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(1)[1] \otimes \rho_\beta \otimes \mathcal{V}) \\
\oplus \text{Hom}_G^{\ell-2}(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(1)[1] \otimes \rho_\beta \otimes \wedge^2 \mathcal{V}) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{T} \oplus \Omega)]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1) \oplus \mathcal{T} \otimes \Omega(-2))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0, 4 \\
\mathbb{C}^3 & \text{if } \ell = 2 \text{ and } \beta = \alpha + 1 \text{ or } \beta = \alpha - 1 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \mathcal{A}^\ell(v_{2\alpha}, v_{2\beta}) = \text{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) \oplus \text{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \otimes \mathcal{V}) \\
\oplus \text{Hom}_G^{\ell-2}(\Omega(1)[1] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \otimes \wedge^2 \mathcal{V}) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{T}(-2) \oplus \mathcal{T}(-3))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-3}(\mathbb{P}^2, \mathcal{T}(-4))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \beta = \alpha - 1 \text{ and } \ell = 1 \\
\mathbb{C} \text{ or } \mathbb{C}^3 & \text{if } \alpha = \beta \text{ and } \ell = 3 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \mathcal{A}^\ell(v_{2\alpha}, v_{0\beta}) = \text{Hom}_G^\ell(\mathcal{O} \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) \oplus \text{Hom}_G^{\ell-1}(\mathcal{O} \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes \mathcal{V}) \\
\oplus \text{Hom}_G^{\ell-2}(\mathcal{O} \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes \wedge^2 \mathcal{V}) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-2) \oplus \mathcal{O}(-3))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{O}(-4))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 1 \\
\mathbb{C}^3 & \text{if } \beta = \alpha - 1 \text{ and } \ell = 2 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \mathcal{A}^\ell(v_{2\alpha}, v_{1\beta}) = \text{Hom}_G^\ell(\mathcal{O} \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta) \oplus \text{Hom}_G^{\ell-1}(\mathcal{O} \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta \otimes \mathcal{V}) \\
\oplus \text{Hom}_G^{\ell-2}(\mathcal{O} \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta \otimes \wedge^2 \mathcal{V}) \\
= [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(1))]^G \oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{O} \otimes \Omega(-1))]^G \\
\oplus [\text{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-2))]^G \\
= \begin{cases} 
\mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 1 \\
\mathbb{C}^3 & \text{if } \beta = \alpha + 1 \text{ and } \ell = 3 \\
0 & \text{otherwise.} 
\end{cases} \]
\[ A'(v_{2\alpha}, v_{2\beta}) = \text{Hom}_{G}^{\ell}(\mathcal{O} \otimes \rho_{\alpha}, \mathcal{O} \otimes \rho_{\beta}) \oplus \text{Hom}_{G}^{\ell-1}(\mathcal{O} \otimes \rho_{\alpha}, \mathcal{O} \otimes \rho_{\beta} \otimes V) \]

\[ \oplus \text{Hom}_{G}^{\ell-2}(\mathcal{O} \otimes \rho_{\alpha}, \mathcal{O} \otimes \rho_{\beta} \otimes \wedge^{2}V) \]

\[ = [\text{Hom}(\rho_{\alpha}, \rho_{\beta}) \otimes H^{\ell}(\mathbb{P}^{2}, \mathcal{O})]^{G} \oplus [\text{Hom}(\rho_{\alpha}, \rho_{\beta}) \otimes H^{\ell-1}(\mathbb{P}^{2}, \mathcal{O}(-1) \otimes \mathcal{O}(-2))]^{G} \]

\[ \oplus [\text{Hom}(\rho_{\alpha}, \rho_{\beta}) \otimes H^{\ell-2}(\mathbb{P}^{2}, \mathcal{O}(-3))]^{G} \]

\[ = \begin{cases} \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0, 4 \\ 0 & \text{otherwise}. \end{cases} \]

The tilting quiver is

where the top and bottom row are identified. The differential sends

\[ ds_{\alpha,i} = b_{a-1,i+1}a_{\alpha,i+2} - b_{a-1,i+2}a_{\alpha,i+1} \quad dr_{\alpha,i} = w_{\alpha-1}b_{i}v_{\alpha} - u_{\alpha-1}b_{i}w_{\alpha} \]

\[ dp_{\alpha,i} = a_{\alpha,i}u_{\alpha} - v_{\alpha-1}b_{\alpha,i} \quad dq_{\alpha,i} = a_{\alpha-1,i+2}w_{\alpha-1}b_{\alpha,i+1} - a_{\alpha-1,i+1}w_{\alpha-1}b_{\alpha,i+2}. \]

The pairing on degree -1 edges is given by

\[ \langle s_{\alpha,i}, r_{\alpha+1,i} \rangle = \langle p_{\alpha,i}, q_{\alpha-1,i} \rangle = 1 \]

and zero otherwise. Thus the superpotential is given by

\[ \Phi = \sum_{\alpha,i=0}^{2} (a_{\alpha-2,i+2}w_{\alpha-2}b_{\alpha-1,i+1} - a_{\alpha-2,i+1}w_{\alpha-2}b_{\alpha-1,i+2}) p_{\alpha,i} + (a_{\alpha+1,i}u_{\alpha+1} - v_{\alpha}b_{\alpha+1,i}) q_{\alpha,i} \]

\[ + (b_{\alpha-2,i+1}a_{\alpha-1,i+2} - b_{\alpha-2,i+2}a_{\alpha-1,i+1}) r_{\alpha,i} + (w_{\alpha}b_{i}v_{\alpha+1} - u_{\alpha}b_{i}w_{\alpha+1}) s_{\alpha,i}. \]
Example 6.5.5. This example is essentially the dg version of Example 5.5 in Bocklandt, Schedler and Wemyss [7]. Let $G = \langle a, b | a^7 = b^3 = 1, ab^{-1} = b^{-1}a^4 \rangle$. This group has five irreducible representations:

1. $\rho_0$ the trivial one dimensional representation.
2. $\rho_1$ the one dimensional representation where $a$ acts as identity and $b$ by multiplication by $\eta = e^{2\pi i/3}$.
3. $\rho_2$ the one dimensional representation where $a$ acts as identity and $b$ by multiplication by $\eta^2 = e^{4\pi i/3}$.
4. $\rho_3$ the three dimensional representation with
   
   $a = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,

   where $\omega = e^{2\pi i/7}$.
5. $\rho_4$ the three dimensional representation with
   
   $a = \begin{pmatrix} \omega^6 & 0 & 0 \\ 0 & \omega^5 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,

   where $\omega = e^{2\pi i/7}$.

Now we suppose $G$ acts on $\mathbb{C}^3$ through $\rho_4$. Denote $\mathbb{C}^3$ with this $G$-action by $V$. Then $V = \rho_4$, $\wedge^2 V = \rho_3$ and $\wedge^3 V = \rho_0$. We would like to calculate its quiver by the quotient construction. We have decompositions

$\rho_4 \otimes \rho_0 = \rho_4$ 
$\rho_4 \otimes \rho_1 = \rho_4$ 
$\rho_4 \otimes \rho_2 = \rho_4$ 
$\rho_4 \otimes \rho_3 = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4$ 
$\rho_4 \otimes \rho_4 = \rho_3 \oplus \rho_4$ 

We then have

$\mathcal{A}/G^\ell (\rho_i, \rho_j) = \text{Hom}_G(\rho_i, \wedge^\ell V \otimes \rho_j) =$

$$
\begin{cases}
\mathbb{C} & \text{if } i = j \text{ and } \ell = 0, 3 \\
\mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\
\mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\
\mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\
\mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\
\mathbb{C}^2 & \text{if } i = 3, j = 4 \text{ and } \ell = 1 \\
\mathbb{C}^2 & \text{if } i = 4, j = 3 \text{ and } \ell = 2 \\
0 & \text{otherwise.}
\end{cases}
$$
The underlying graded quiver is then given by

Next we would like to calculate the superpotential. For this, we need the following explicit isomorphisms of representations. Denote by $c_i$ a basis element for the 1-dimensional representation $\rho_i$, where $0 \leq i \leq 2$, let $u_1, u_2, u_3$ be the standard basis for $\rho_3$, and $v_1, v_2, v_3$ be the standard basis for $\rho_4$. Then $\rho_4 \cong \rho_4 \otimes \rho_i$ for $0 \leq i \leq 2$ via the $G$-equivariant map

$$x_{4i} : \rho_4 \to \rho_4 \otimes \rho_i, \quad v_j \mapsto \eta^{-i} v_j \otimes c_i.$$  

Meanwhile, $\rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \cong \rho_4 \otimes \rho_3$ via the sum of the $G$-equivariant maps

$$x_{i3} : \rho_i \to \rho_4 \otimes \rho_3, \quad c_i \mapsto \sum_{j=1}^3 \eta^j v_j \otimes u_j, \quad \text{for } 0 \leq i \leq 2.$$  

$$x_{3i} : \rho_3 \to \rho_4 \otimes \rho_i, \quad u_j \mapsto v_j \otimes u_{j+1}, \quad \text{where } j \in \mathbb{Z}/3\mathbb{Z},$$  

$$x_{43} : \rho_4 \to \rho_4 \otimes \rho_3, \quad v_j \mapsto v_{j+1} \otimes u_j, \quad \text{where } j \in \mathbb{Z}/3\mathbb{Z}.$$  

Also, $\rho_3 \oplus \rho_4 \oplus \rho_4 \cong \rho_4 \otimes \rho_4$ via the sum of the $G$-equivariant maps

$$x_{44}^{34} : \rho_3 \to \rho_4 \otimes \rho_4, \quad u_i \mapsto v_{i+1} \otimes v_{i-1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z},$$  

$$x_{44}^{33} : \rho_4 \to \rho_4 \otimes \rho_4, \quad u_i \mapsto v_{i-1} \otimes v_{i+1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z},$$  

$$x_{44}^{34} : \rho_4 \to \rho_4 \otimes \rho_4, \quad v_i \mapsto v_{i-1} \otimes v_{i-1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z}.$$  

Since $V^\bullet$ is $A_2$, i.e., $m_n = 0$ for $n \geq 3$, $\text{Hom}_G(\rho_i, \rho_j \otimes V^\bullet)$ is also $A_2$. Thus the superpotential on the quiver is cubic.

$$m_2(x_{03}, x_{40}) : \rho_4 \to \wedge^2 V \otimes \rho_4, \quad v_j \mapsto v_j \otimes c_0 \mapsto \sum_{k=1}^3 v_j \otimes v_k \otimes u_k \mapsto \sum_{k=1}^3 v_j \wedge v_k \otimes u_k$$

$$m_2(x_{34}, m_2(x_{03}, x_{40})) = \left( v_j \mapsto \sum_{k=1}^3 v_j \wedge v_k \otimes u_k \mapsto \sum_{k=1}^3 v_j \wedge v_k \wedge v_{k+1} \otimes v_{k-1} = v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j \right)$$

Thus

$$\langle x_{34}^1, m_2(x_{03}, x_{40}) \rangle = \text{tr} (v_j \mapsto (v_j, v_{j+1} + v_{j+2}) v_j) = v_j) = 3$$

$$m_2(x_{34}^2, m_2(x_{03}, x_{40})) = \left( v_j \mapsto \sum_{k=1}^3 v_j \wedge v_k \otimes u_k \mapsto \sum_{k=1}^3 v_j \wedge v_k \wedge v_{k-1} \otimes v_{k+1} = v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j \right)$$
Thus

\[
\langle x^2_{34}, m_2(x_{23}, x_{43}) \rangle = \text{tr}(v_j \mapsto \langle v_j, v_{j-1} \wedge v_{j-2} \rangle v_j = -v_j) = -3
\]

\[
m_2(x_{13}, x_{24}) : \rho_4 \rightarrow \wedge^2 V \otimes \rho_3, \quad (v_j \mapsto \eta^{-j} v_j \otimes c_1 \mapsto \sum_{k=1}^{3} \eta^{k-j} v_j \wedge v_k \otimes u_k)
\]

\[
m_2(x^1_{34}, m_2(x_{13}, x_{24})) = \left( v_j \mapsto \sum_{k=1}^{3} \eta^{k-j} v_j \wedge v_k \otimes u_k \mapsto \eta v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j \right)
\]

\[
\langle x^1_{34}, m_2(x_{13}, x_{24}) \rangle = \text{tr}(v_j \mapsto \eta v_j) = 3 \eta
\]

\[
m_2(x^2_{34}, m_2(x_{13}, x_{24})) = \left( v_j \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k \mapsto \eta^2 v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j \right)
\]

\[
\langle x^2_{34}, m_2(x_{13}, x_{24}) \rangle = \text{tr}(v_j \mapsto -\eta^{-1} v_j) = -3 \eta^{-1} = -3 \eta^2
\]

\[
m_2(x_{23}, x_{42}) : \rho_2 \rightarrow \wedge^2 V \otimes \rho_3, \quad v_j \mapsto \eta^{-2j} v_j \otimes c_2 \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k
\]

\[
m_2(x^1_{34}, m_2(x_{23}, x_{42})) = \left( v_j \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k \mapsto \eta^2 v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j \right)
\]

\[
\langle x^1_{34}, m_2(x_{23}, x_{42}) \rangle = \text{tr}(v_j \mapsto \eta^2 v_j) = 3 \eta^2
\]

\[
m_2(x^2_{34}, m_2(x_{23}, x_{42})) = \left( v_j \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k \mapsto \eta^{-2} v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j \right)
\]

\[
\langle x^2_{34}, m_2(x_{23}, x_{42}) \rangle = \text{tr}(v_j \mapsto -\eta^{-1} v_j) = -3 \eta^{-1} = -3 \eta^2
\]

\[
m_2(x_{33}, x_{43}) : \rho_4 \rightarrow \wedge^2 V \otimes \rho_3, \quad (v_j \mapsto v_{j+1} \otimes u_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1})
\]

\[
m_2(x^1_{34}, m_2(x_{33}, x_{43})) = \left( v_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1} \mapsto v_{j+1} \wedge v_j \wedge v_{j+2} \otimes v_j \right)
\]

\[
\langle x^1_{34}, m_2(x_{33}, x_{43}) \rangle = \text{tr}(v_j \mapsto -v_j) = -3
\]

\[
m_2(x^2_{34}, m_2(x_{33}, x_{43})) = \left( v_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1} \mapsto v_{j+1} \wedge v_j \wedge v_{j+2} \otimes v_j \right)
\]

\[
\langle x^2_{34}, m_2(x_{33}, x_{43}) \rangle = 0
\]

\[
m_2(x_{43}, x_{34}) : \rho_4 \rightarrow \wedge^2 V \otimes \rho_3, \quad (v_j \mapsto v_{j-1} \otimes v_{j-1} \mapsto v_{j-1} \wedge v_j \otimes u_{j-1})
\]

\[
m_2(x^1_{34}, m_2(x_{43}, x_{34})) = \left( v_j \mapsto v_{j-1} \wedge v_j \otimes u_{j-1} \mapsto v_{j-1} \wedge v_j \wedge v_{j-2} \otimes v_j \right)
\]

\[
\langle x^1_{34}, m_2(x_{43}, x_{34}) \rangle = 0
\]

\[
m_2(x^2_{34}, m_2(x_{43}, x_{34})) = \left( v_j \mapsto v_{j-1} \wedge v_j \otimes u_{j-1} \mapsto v_{j-1} \wedge v_j \wedge v_{j-2} \otimes v_j \right)
\]

\[
\langle x^2_{34}, m_2(x_{43}, x_{34}) \rangle = 3
\]

\[
m_2(x_{33}, x_{33}) : \rho_3 \rightarrow \wedge^2 V \otimes \rho_3, \quad (u_j \mapsto v_j \otimes u_{j+1} \mapsto v_j \wedge v_{j+1} \otimes u_{j+2})
\]

\[
m_2(x_{33}, m_2(x_{33}, x_{33})) = \left( v_j \mapsto v_j \wedge v_{j+1} \otimes u_{j+2} \mapsto v_j \wedge v_{j+1} \wedge v_{j+2} \otimes u_j \right)
\]

\[
\langle x_{33}, m_2(x_{33}, x_{33}) \rangle = \text{tr}(u_j \mapsto u_j) = 3
\]

\[
m_2(x_{44}, x_{44}) : \rho_4 \rightarrow \wedge^2 V \otimes \rho_4, \quad (v_j \mapsto v_{j-1} \otimes v_{j-1} \mapsto v_{j-1} \wedge v_{j-2} \otimes v_{j-2})
\]

\[
m_2(x_{44}, m_2(x_{44}, x_{44})) = \left( v_j \mapsto v_{j-1} \wedge v_{j-2} \otimes v_{j-2} \mapsto v_{j-1} \wedge v_{j-2} \wedge v_j \otimes v_j \right)
\]

\[
\langle x_{44}, m_2(x_{44}, x_{44}) \rangle = \text{tr}(v_j \mapsto -v_j) = -3
\]
Therefore the superpotential is given by
\[
\Phi = 3x_{34}^1x_{03}x_{40} - 3x_{34}^2x_{03}x_{40} + 3\eta x_{34}^1x_{13}x_{41} - 3\eta^2 x_{34}^2x_{13}x_{41} + 3\eta^2 x_{34}^1x_{23}x_{42} - 3\eta x_{34}^2x_{23}x_{42} - 3x_{34}^1x_{33}x_{43} + 3x_{34}^2x_{33}x_{43} + x_{33}x_{33}x_{33} - x_{44}x_{44}x_{44} = 3x_{03}x_{40}(x_{34}^1 - x_{34}^2) + 3\eta x_{13}x_{41}(x_{34}^1 - \eta x_{34}^2) + 3\eta x_{23}x_{42}(\eta^2 x_{34}^1 - x_{34}^2) - 3x_{34}^1x_{33}x_{43} + 3x_{34}^2x_{33}x_{43} + x_{33}x_{33}x_{33} - x_{44}x_{44}x_{44}.
\]

The classical quiver is thus given by the $H^0$ of the dg-quiver:

![Quiver diagram]

with relation
\[
\begin{aligned}
x_{40}x_{34}^1 &= x_{40}x_{34}^2 & x_{41}x_{34}^1 &= \eta x_{41}x_{34}^2 & \eta x_{42}x_{34}^1 &= x_{42}x_{34}^2 \\
x_{34}^1x_{03} &= x_{34}^1x_{03} & x_{34}^1x_{13} &= x_{34}^1x_{13} & x_{34}^1x_{23} &= x_{34}^1x_{23} \\
x_{33}x_{33} &= x_{33}x_{33} & x_{44}x_{44} &= x_{44}x_{44} & x_{03}x_{40} + \eta x_{13}x_{41} + \eta^2 x_{23}x_{42} &= x_{33}x_{43} & x_{03}x_{40} + \eta x_{13}x_{41} + \eta x_{23}x_{42} &= x_{43}x_{44}
\end{aligned}
\]

**Example 6.5.6.** The action of $G$ on $\mathbb{C}^3$ induce an action of $G$ on $\mathbb{P}^2$ and an equivariant action on $O(1)$. Take the exceptional sequence $\mathcal{E} = (O, O(1), O(2))$ on $D^b(Coh(\mathbb{P}^2))$ and take $V = O(-3) = K_{\mathbb{P}^2}$, which is the canonical bundle on $\mathbb{P}^2$ equipped with an $G$-equivariant action.

**Lemma 6.5.7.**

\[
\begin{align*}
H^f(\mathbb{P}^2, O(1)) &= \begin{cases} 
\rho_4 & \text{if } \ell = 0 \\
0 & \text{otherwise.} \end{cases} & H^f(\mathbb{P}^2, O(-4)) &= \begin{cases} 
\rho_4 & \text{if } \ell = 2 \\
0 & \text{otherwise.} \end{cases} \\
H^f(\mathbb{P}^2, \Omega(2)) &= \begin{cases} 
\rho_4 & \text{if } \ell = 0 \\
0 & \text{otherwise.} \end{cases} & H^f(\mathbb{P}^2, T(-5)) &= \begin{cases} 
\rho_3 & \text{if } \ell = 2 \\
0 & \text{otherwise.} \end{cases} \\
H^f(\mathbb{P}^2, T(-1)) &= \begin{cases} 
\rho_4 & \text{if } \ell = 0 \\
0 & \text{otherwise.} \end{cases} & H^f(\mathbb{P}^2, \Omega(-2)) &= \begin{cases} 
\rho_3 & \text{if } \ell = 2 \\
0 & \text{otherwise.} \end{cases}
\end{align*}
\]

**Proof.** It suffices to show the lemma for all the nontrivial cases. Since $G$ acts on $\mathbb{P}^2$ through $V = \rho_4$,

\[
H^0(\mathbb{P}^2, O(1)) = \rho_4^\vee = \rho_3.
\]

By Serre duality,

\[
H^2(\mathbb{P}^2, O(-4)) \cong H^0(\mathbb{P}^2, O(1))^\vee = \rho_4^\vee = \rho_4.
\]

From the short exact sequence

\[
0 \to \Omega(2) \to O(1) \otimes V^\vee \to O(2) \to 0,
\]

we have an exact sequence

\[
0 \to H^0(\mathbb{P}^2, \Omega(2)) \to H^0(\mathbb{P}^2, O(1)) \otimes V^\vee \to H^0(\mathbb{P}^2, O(2)) \to 0.
\]
Since $H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes V^\vee \cong \rho_5 \otimes \rho_5$, $H^0(\mathbb{P}^2, \mathcal{O}(2)) = S^2V^\vee = S^2\rho_5$, and $\rho_3 \otimes \rho_3 \cong \wedge^2\rho_3 \oplus S^2\rho_3$, we conclude $H^0(\mathbb{P}^2, \mathcal{O}(2)) \cong \wedge^2\rho_3 \cong \rho_4$. By Serre duality, we have
\[ H^2(\mathbb{P}^2, T(-5)) \cong H^0(\mathbb{P}^2, \mathcal{O}(2))^\vee \cong \rho_3. \]

From the short exact sequence
\[ 0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes V \to T(-1) \to 0, \]
we conclude
\[ H^0(\mathbb{P}^2, T(-1)) \cong V \cong \rho_4. \]
By Serre duality,
\[ H^2(\mathbb{P}^2, \mathcal{O}(-2)) \cong H^0(\mathbb{P}^2, T(-1))^\vee \cong \rho_3. \]

\[ (A/G)^f(v_{0i}, v_{1j}) = \text{Hom}_G^f(\Omega^2(2)[2] \otimes \rho_1, \mathcal{O}(1)[1] \otimes \rho_2) \oplus \text{Hom}_G^{f-1}(\Omega^2(2)[2] \otimes \rho_1, \mathcal{O}(1)[1] \otimes \rho_2 \otimes K_{\mathbb{P}^2}) \]
\[ = [\text{Hom}(\rho_1, \rho_2) \otimes H^f(\mathbb{P}^2, \Omega(2))]^G \oplus [\text{Hom}(\rho_1, \rho_2) \otimes H^{f-2}(\mathbb{P}^2, \Omega(-1))]^G \]
\[ = \begin{cases} \text{Hom}_G(\rho_1, \rho_4 \otimes \rho_2) & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \]
\[ = \begin{cases} \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3, j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \]

\[ (A/G)^f(v_{0i}, v_{2j}) = \text{Hom}_G^f(\Omega^2(2)[2] \otimes \rho_1, \mathcal{O} \otimes \rho_2) \oplus \text{Hom}_G^{f-1}(\Omega^2(2)[2] \otimes \rho_1, \mathcal{O} \otimes \rho_2 \otimes K_{\mathbb{P}^2}) \]
\[ = [\text{Hom}(\rho_1, \rho_2) \otimes H^f(\mathbb{P}^2, \Omega(1))]^G \oplus [\text{Hom}(\rho_1, \rho_2) \otimes H^{f-2}(\mathbb{P}^2, \mathcal{O}(-2))]^G \]
\[ = \begin{cases} \text{Hom}_G(\rho_1, \rho_3 \otimes \rho_2) & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]
\[ = \begin{cases} \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4, j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]

\[ (A/G)^f(v_{1i}, v_{0j}) = \text{Hom}_G^f(\Omega(1)[1] \otimes \rho_1, \Omega^2(2)[2] \otimes \rho_2) \oplus \text{Hom}_G^{f-1}(\Omega(1)[1] \otimes \rho_1, \Omega^2(2)[2] \otimes \rho_2 \otimes K_{\mathbb{P}^2}) \]
\[ = [\text{Hom}(\rho_1, \rho_2) \otimes H^{f+1}(\mathbb{P}^2, T(-2))]^G \oplus [\text{Hom}(\rho_1, \rho_2) \otimes H^{f}(\mathbb{P}^2, T(-5))]^G \]
\[ = \begin{cases} \text{Hom}_G(\rho_1, \rho_3 \otimes \rho_2) & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]
\[ = \begin{cases} \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4, j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]
\( (A/G)\ell(v_1, v_2) = \text{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_i, O \otimes \rho_j) \oplus \text{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_i, O \otimes \rho_j \otimes K_{\mathbb{P}^2}) \)
\[= \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell-1}(\mathbb{P}^2, T(-1)) \right]^G \oplus \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell-2}(\mathbb{P}^2, T(-4)) \right]^G \]
\[= \begin{cases} \text{Hom}_G(\rho_i, \rho_j) & \text{if } \ell = 1 \\ \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3, j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \]

\( (A/G)\ell(v_2, v_0) = \text{Hom}_G^\ell(O \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j) \oplus \text{Hom}_G^{\ell-1}(O \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \)
\[= \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell+2}(\mathbb{P}^2, O(-1)) \right]^G \oplus \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, O(-4)) \right]^G \]
\[= \begin{cases} \text{Hom}_G(\rho_i, \rho_j) & \text{if } \ell = 1 \\ \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3, j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \]

\( (A/G)\ell(v_2, v_1) = \text{Hom}_G^\ell(O \otimes \rho_i, \Omega(1)[1] \otimes \rho_j) \oplus \text{Hom}_G^{\ell-1}(O \otimes \rho_i, \Omega(1)[1] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \)
\[= \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, \Omega(1)) \right]^G \oplus \left[ \text{Hom}(\rho_i, \rho_j) \otimes H^{\ell}(\mathbb{P}^2, \Omega(-2)) \right]^G \]
\[= \begin{cases} \text{Hom}_G(\rho_i, \rho_j) & \text{if } \ell = 2 \\ \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4, j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \]
The dg-quiver is given by

\[
\begin{align*}
\Phi &= \sum_{i \in \mathbb{Z}} 3x_{i0, (i+1)3} x_{(i+2)4, i0}(x_{(i+1)3,(i+2)4}^1 - x_{(i+1)3,(i+2)4}^2) \\
&\quad + 3\eta x_{1, (i+1)3} x_{(i+2)4, 01}(x_{(i+1)3,(i+2)4}^1 - \eta x_{(i+1)3,(i+2)4}^2) \\
&\quad + 3\eta x_{2, (i+1)3} x_{(i+2)4, 02}(\eta x_{(i+1)3,(i+2)4}^1 - x_{(i+1)3,(i+2)4}^2) \\
&\quad - 3x_{3,(i+1)4} x_{(i+2)3, 3} x_{(i+1)4,(i+2)3} + 3x_{3,(i+1)4} x_{(i+2)3, 3} x_{(i+1)4,(i+2)3} + x_{3,(i+1)4} x_{(i+2)3, 3} x_{(i+1)4,(i+2)3} - x_{4,(i+1)4} x_{(i+2)3, 4} x_{(i+1)4,(i+2)3}
\end{align*}
\]

The classical quiver is given by

where the vertices on the left and that on the right in the same row are identified. We will use the following scheme to label the edges: the black (degree 0) edges going from vertex \( v_{ij} \) to \( v_{kl} \) will be denoted by \( x_{ij,kl} \). In case there are two arrows, we will denote by \( x_{ij,kl}^1, x_{ij,kl}^2 \). The blue (degree \(-1\)) arrows going from \( v_{kl} \) to \( v_{ij} \) is denoted by \( x_{ij,kl}^3 \). The brown (degree \(-2\)) loops at vertex \( v_{ij} \) will be denoted by \( v_{ij}^\ast \).
where the vertices on the left and that on the right in the same row are identified, with relations

\[
\begin{align*}
    x_{(i+2)4, (i+1)0} x_{(i+1)3, i4} &= x_{(i+2)4, (i+1)1} x_{(i+1)3, i4} = \eta x_{(i+2)4, (i+1)1} x_{(i+1)3, i4} \\
    \eta^2 x_{(i+2)4, (i+1)2} x_{(i+1)3, i4} &= x_{(i+2)4, (i+1)2} x_{(i+1)3, i4} = x_{(i+2)4, (i+1)3} x_{(i+1)0, i3} = \eta^2 x_{(i+2)4, (i+1)3} x_{(i+1)0, i3} \\
    x_{(i+2)3, (i+1)4} x_{(i+1)1, i3} &= \eta x_{(i+2)3, (i+1)4} x_{(i+1)1, i3} = \eta x_{(i+2)3, (i+1)4} x_{(i+1)1, i3} \\
    x_{(i+2)3, (i+1)3} x_{(i+1)3, i3} &= x_{(i+2)4, (i+1)3} x_{(i+1)3, i3} = x_{(i+2)4, (i+1)4} x_{(i+1)3, i3} = \eta x_{(i+2)4, (i+1)4} x_{(i+1)3, i3} \\
    x_{(i+2)0, (i+1)3} x_{(i+1)4, i0} + \eta x_{(i+2)1, (i+1)3} x_{(i+1)4, i1} + \eta^2 x_{(i+2)2, (i+1)3} x_{(i+1)4, i2} &= x_{(i+2)0, (i+1)3} x_{(i+1)4, i0} + \eta x_{(i+2)1, (i+1)3} x_{(i+1)4, i1} + \eta x_{(i+2)2, (i+1)3} x_{(i+1)4, i2} = x_{(i+2)0, (i+1)3} x_{(i+1)4, i0}.
\end{align*}
\]
Chapter 7

Shifted Symplectic Structures on Moduli Spaces

This chapter contains some unfinished work which aims to make a connection between quivers with superpotential and the recent work on shifted symplectic structures by Pantev, Toën, Vaquié and Vezzosi [58], and Ben-Bassat, Brav, Bussi and Joyce [4].

Section 7.1 reviews the theory on derived algebraic geometry developed by Toën and Vezzosi [64, 65, 66] and Pantev, Toën, Vaquié and Vezzosi [58].

Section 7.2 develops the Lie algebra cohomology for dg-modules by modifying the usual Lie algebra cohomology theory.

Section 7.3 defines the $G$-invariant de Rham complex of on a derived scheme $\text{Spec } R$ by using the Lie algebra cohomology developed in Section 7.2. We conjecture that the $G$-invariant de Rham complex should describe forms and closed forms on the quotient stack $[\text{Spec } R/G]$ and outline a strategy of proof.

Section 7.4 describes the moduli space of representations of quiver with superpotential, and outline a strategy of proof on showing the existence of a shifted symplectic structure which is in a standard Darboux form by using the $G$-invariant de Rham complex introduced in Section 7.3.

7.1 Derived Schemes

The section is an outline of the theory of derived algebraic geometry needed to state the results in the next section, and is essentially a summary of Brav, Bussi and Joyce [13, §3]. For our purpose, we would not need the general definition of derived stacks. The main point here is that an affine derived scheme is essentially a commutative dg-algebra, in other words, there is a functor

$$\text{Spec } : \{\text{commutative dg-algebra}\}^{\text{op}} \to \{\text{derived stacks}\}.$$  

For any derived Artin stack $X$, Toen and Vezzosi constructed a triangulated category $L_{qcoh}(X)$ with a $t$-structure whose heart is the category of quasi-coherent sheaves on $X$ and defined a cotangent complex $\mathbb{L}_X$ in $L_{qcoh}(X)$. If $f : X \to Y$ is a morphism of derived Artin stacks, they constructed a morphism $\mathbb{L}_f : f^*\mathbb{L}_Y \to \mathbb{L}_X$ in $L_{qcoh}(X)$ and a relative cotangent complex $\mathbb{L}_{X/Y}$ which fits into a distinguished triangle

$$f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y} \to f^*\mathbb{L}_Y[1].$$

In the case when $X$ is an affine derived scheme, i.e., $X \cong \text{Spec } R$ for some commutative dg-algebra $R$, we have a derived equivalence $L_{qcoh}(X) \cong D(R\text{-mod})$ which identifies $\mathbb{L}_X \cong \mathbb{L}_R$. If $R$ is further assumed to be quasi-free, then the Kähler differential $\Omega_R$ gives a model for $\mathbb{L}_R$. Next, we introduce the notion of $p$-forms, closed $p$-forms, and symplectic forms on a affine derived scheme defined by Toën and Vezzosi, reinterpreted in the case for quasi-free affine derived scheme as per Brav, Bussi and Joyce [13, §5].
The grading on the dga $R$ induces a grading on $\Omega^*_R$, and we denote by $(\Omega^*_R)^k$ the $k$-th piece. The de Rham algebra of $R$ is defined to be a doubled graded algebra

$$\text{DR}(A) = \bigwedge^\cdot \Omega^1_R = \bigoplus_{p,k \in \mathbb{Z}} (\bigwedge^p \Omega^1_R)^k,$$

where each summand $\bigwedge^p \Omega^1_R$ is of ‘form degree’ $p$. There are two differential on the de Rham algebra: the differential induced by the differential on the dg-algebra $R$

$$d : (\bigwedge^p \Omega^1_R)^k \rightarrow (\bigwedge^p \Omega^1_R)^{k+1}$$

and the de Rham differential $d_{\text{dR}}$

$$d_{\text{dR}} : (\bigwedge^p \Omega^1_R)^k \rightarrow (\bigwedge^{p+1} \Omega^1_R)^k$$

**Definition 7.1.1.** A $p$-form of degree $k$ on Spec $R$ is an element in $H^k(\bigwedge^p \Omega^*_A, d)$. In other words, a $p$-form of degree $k$ can be represented by an element $\omega \in (\bigwedge^p \Omega^1_R)^k$ with $d\omega = 0$. Two such representatives $\omega$ and $\omega'$ are equivalent if there exists $\alpha \in (\bigwedge^p \Omega^1_R)^{k-1}$ such that $\omega - \omega' = d\alpha$.

**Definition 7.1.2.** A closed $p$-form is an element in $H^k(\prod_{i \geq 0} \bigwedge^{p+i} \Omega^1_R[-i], d + d_{\text{dR}})$. In other words, a closed $p$-form of degree $k$ can be represented by an sequence $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ with $\omega^i \in (\bigwedge^{p+i} \Omega^1_A)^{k-i}$ for $i = 0, 1, 2, \ldots$ satisfying the equations

$$d\omega^0 = 0 \quad \text{in } (\bigwedge^p \Omega^1_R)^{k+1}, \text{ and}$$

$$d_{\text{dR}}\omega^i + d\omega^{i+1} = 0 \quad \text{in } (\bigwedge^{p+i+1} \Omega^1_R)^{k-i} \text{ for all } i \geq 0.$$

Two such representations $\omega, \omega'$ are equivalent if there exists $\alpha = (\alpha^0, \alpha^1, \ldots)$ with $\alpha^i \in (\bigwedge^{p+i} \Omega^1_R)^{k-i-1}$ satisfying

$$\omega^0 - \omega'^0 = d\alpha^0 \quad \text{in } (\bigwedge^p \Omega^1_R)^k, \text{ and}$$

$$\omega^{i+1} - \omega'^{i+1} = d_{\text{dR}}\alpha^i + d\alpha^{i+1} \quad \text{in } (\bigwedge^{p+i+1} \Omega^1_R)^{k-i-1} \text{ for all } i \geq 0.$$

**Definition 7.1.3.** A closed 2-form $\omega = (\omega^0, \omega^1, \ldots)$ of degree $k$ is called $k$-shifted symplectic if $\omega^0$ is a nondegenerate 2-form of degree $k$.

### 7.2 Lie Algebra Cohomology

This section develops the Lie algebra cohomology theory on dg-modules and is essentially an adaptation of the usual Lie algebra cohomology discussed, for instance, in Weibel [72, Chapter 7].

Let $G$ be a linear algebraic group of finite type over $\mathbb{K}$ and $(R, d_R)$ be a dg-algebra over $\mathbb{K}$ together with a $G$-action which is compatible with the dg-algebra structure in the sense that

$$\text{deg}(gm) = \text{deg} m,$$

$$g(rr') = (gr)(gr'), \text{ and}$$

$$d_R(gr) = g(d_Rr)$$

for any $g \in G, r, r' \in R$. Let $M$ be an $R$-module together with a $G$-action which is compatible with the dg-module structure in the sense that

$$\text{deg} gm = \text{deg} m,$$

$$g(rm) = (gr)(gm), \text{ and}$$

$$d_M(gm) = g(d_Mm)$$

where $d_M$ is the differential on $M$.
for any $g \in G$, $r \in R$, $m \in M$. Then the Lie algebra $g$ also acts on $M$ as a derivation (Lie derivative) which satisfies

\begin{align*}
    d_M(x \cdot m) &= x \cdot d_M(m) \\
    x \cdot (rm) &= (x \cdot r)m + r(x \cdot m)
\end{align*}

for any $x \in g$, $r \in R$ and $m \in M$. We will introduce the Chevalley–Eilenberg double-complex $\text{Hom}_K(\wedge^\bullet g, M)$ with two anticommuting differentials

\begin{align*}
    \delta_{M+} : \text{Hom}_K(\wedge^n g, M) &\to \text{Hom}_K(\wedge^{n+1} g, M) \\
    (\delta_{M+} f)(x_1, \ldots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\
    &\quad + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1})
\end{align*}

and

\begin{align*}
    \delta_{M-} : \text{Hom}_K(\wedge^n g, M') &\to \text{Hom}_K(\wedge^n g, M^{\prime+1}) \\
    f &\mapsto (-1)^n d_M \circ f.
\end{align*}

**Proposition 7.2.1.** $(\text{Hom}_K(\wedge^\bullet g, M), \delta_{M+}, \delta_{M-})$ is double complex.
Proof. First, we show that $\delta_r^2 = 0$.

\[
\begin{align*}
(\delta_r^2 M) f(x_1, \ldots, x_{n+2}) &= \sum_{i=1}^{n+2} (-1)^{i+1} f(x_i, x_1, \ldots, x_{i-1}, \ldots, x_{n+2}) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{n+2}) \\
&= \sum_{j<i} (-1)^{i+1+j+1} x_i x_j f(x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{j<k<i} (-1)^{i+1+j+k} x_i x_j f([x_j, x_k], x_1, \ldots, x_{i-1}, \ldots, x_{k-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{j<k<l<i} (-1)^{i+1+j+k+l-1} x_i x_j f([x_j, x_k, x_l], x_1, \ldots, x_{i-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&\quad + \sum_{i<j<k+l} (-1)^{i+j+k+l+1} [x_i, x_j, x_k, x_l, x_1, \ldots, x_{i-1}, \ldots, x_{j-1}, \ldots, x_{k-1}, \ldots, x_{l-1}, \ldots, x_{n+2}) \\
&= 0,
\end{align*}
\]

where we have used the Jacobi identity, antisymmetry of $f$, and that $x_i(x_j m) - x_j(x_i m) = [x_i, x_j] m$ for any $x_i, x_j \in g$ and $m \in M$ in the last step.
Next, we would like to show $d$ and $\delta$ anticommute. For any $f \in \text{Hom}(\wedge^n g, M)$ and $x_i \in g$,

$$(\delta_M - \delta_{M+} f)(x_1, \ldots, x_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1+i+1} d_M (x_i f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}))$$

$$+ \sum_{i<j} (-1)^{n+1+i+j} d_M (f([x_i, x_j], x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}))$$

$$= - \sum_{i=1}^{n+1} (-1)^{i+1+n} x_i d_M (f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}))$$

$$- \sum_{i<j} (-1)^{i+j+n} d_M (f([x_i, x_j], x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}))$$

$$= - (\delta_{M+} \delta_M - f)(x_1, \ldots, x_{n+1})$$

Hence $\delta_M - \delta_{M+} + \delta_{M+} \delta_M = 0$. The last condition $\delta^2_M = 0$ follows immediately from $d^2_M = 0$ and hence $(\text{Hom}_K(\wedge^* g, M), \delta_{M+}, \delta_{M-})$ is double complex. \hfill $\blacksquare$

The total complex of the double complex $(\text{Hom}_K(\wedge^* g, M), \delta_{M+}, \delta_{M-})$ computes the Lie algebra cohomology $H^*(g, M)$. The total differential of the double complex will be denoted by $\delta_M$, i.e., $\delta_M = \delta_{M+} + \delta_{M-}$.

### 7.3 $G$-invariant de Rham Complex

This section aims to define the notion of $G$-invariant de Rham complex on a derived scheme $\text{Spec } R$, where $R$ is a quasi-free dg-algebra equipped with a $G$-action. Towards the end of this section, we outline a strategy to prove that the $G$-invariant de Rham complex should describe forms and closed forms on the quotient stack $[\text{Spec } R/G]$.

**$G$-invariant sections.** A generalized $G$-invariant section of degree $k$ of the module $M$ is an element in the $G$-invariant section $[\alpha] \in H^k(\pi, M)$. A generalized $G$-invariant section $[\alpha]$ is thus represented by an element $\alpha$ of degree $k$ in $(\text{Hom}(\wedge^* g, M), \delta_M)$, henceforth called the complex of $G$-invariant sections of $M$, which satisfies $\delta_M \alpha = (\delta_{M+} - \delta_{M-}) \alpha = 0$. If we decompose $\alpha$ into homogeneous terms and write $\alpha = \alpha^0 + \ldots + \alpha^{\dim g}$, where $\alpha^i \in \text{Hom}_K(\wedge^* g, M^{k-i})$, then the condition $\delta_M \alpha = 0$ becomes the system of equations

$$\delta_M - \alpha^0 = 0 \quad \text{in } \text{Hom}_K(\wedge^0 g, M^{k+1}) = M^{k+1}, \text{ and}$$

$$\delta_M + \alpha^i - \delta_M - \alpha^{i+1} = 0 \quad \text{in } \text{Hom}_K(\wedge^{i+1} g, M^{k-i}) \text{ for } i = 0, 1, \ldots, \dim g.$$

From these equations, we can see that in the case when $M$ is concentrated in degree 0 and when $k = 0$,

$$H^0(\pi, M) = M^0 = \{m \in M : a \cdot m = 0 \text{ for all } a \in g\},$$

i.e., a generalized $G$-invariant section is an honest $G$-invariant section.

**$G$-invariant functions.** When $M = R$, we can endow an associative product structure on the vector space $\text{Hom}(\wedge^* g, R)$, turning it into a dg-algebra, and regard it as the complex of $G$-invariant functions on $\text{Spec } R$. To construct such a product, recall that there is a coproduct structure $\Delta$ on $\wedge^* g$, which is the dual of the wedge product on $\wedge^* g^n$, defined by

$$\Delta : \wedge^* g \to \wedge^* g \otimes \wedge^* g,$$

$$\Delta(x_1 \wedge \ldots \wedge x_n) = \sum_{k=0}^n \sum_{\sigma \in S_{n-k}} (\text{sgn } \sigma)(x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \wedge \ldots \wedge x_{\sigma(n+k)}),$$

where $S_n$ is the symmetric group on $n$ elements.
where $\text{Sh}_{k,n-k}$ denote the set of all $(k, n-k)$-shuffles. We then define a product on $\text{Hom}(\wedge^*_g, R) \otimes \text{Hom}(\wedge^*_g, R) \to \text{Hom}(\wedge^*_g, R)$

$$f \cdot g : \wedge^*_g \xrightarrow{\Delta} \wedge^*_g \otimes \wedge^*_g \xrightarrow{\otimes g} R \otimes R \to R,$$

where $R \otimes R \to R$ is the product on $R$. The product so defined is associative since the product on $R$ is associative, and the coproduct $\Delta$ is coassociative. Explicitly, when $f \in \text{Hom}(\wedge^*_g, R^j)$ and $g \in \text{Hom}(\wedge^*_g, R^i)$, their product $f \cdot g \in \text{Hom}(\wedge^{i+j}_g, R^{i+j})$ is defined by

$$(f \cdot g)(x_1, \ldots, x_{i+k}) = (-1)^{jk} \sum_{\sigma \in \text{Sh}_{i,k}} (\text{sgn} \sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k)})).$$

**Proposition 7.3.1.** $(\text{Hom}(\wedge^*_g, R), \delta_R)$ is a dg-algebra.

**Proof.** It remains to check the Leibniz’s rule $\delta_R(f \cdot g) = (\delta_R f) \cdot g + (-1)^{i+j} f \cdot (\delta_R g)$. Since $\text{Hom}(\wedge^*_g, R)$ is generated by $\text{Hom}(g, R)$ as an algebra, it suffices to show Leibniz’s rule holds when $i = 1$, for the general case follows from induction on $i$.

$$(\delta_R f \cdot g)(x_1, \ldots, x_{k+2})$$

$$= \sum_{u=1}^{k+2} (-1)^{u+1} x_u ((f \cdot g)(x_1, \ldots, x_{u-1}, x_{u+1}, \ldots, x_{k+2}))$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v} (f \cdot g)([x_u, x_v], x_1, \ldots, x_{u-1}, \ldots, x_{v+1}, \ldots, x_{k+2})$$

$$= (-1)^{jk} \left( \sum_{u=1}^{k+2} \sum_{v=1 \leq u < v \leq k+2} x_u (-1)^{u+v+1} f(x_v) g(x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2}) \right)$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v} f([x_u, x_v]) g(x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v+w} f(x_w) g([x_u, x_v], x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v+w-1} f(x_w) g([x_u, x_v], x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v+w} f(x_w) g([x_u, x_v], x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2})$$

$$(\delta_R f \cdot g)(x_1, \ldots, x_{k+2})$$

$$= (-1)^{jk} \left( \sum_{1 \leq u < v \leq k+2} (-1)^{u+v-1} (\delta_R f)(x_u, x_v) g(x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2}) \right)$$

$$= (-1)^{jk} \left( \sum_{1 \leq u < v \leq k+2} (-1)^{u+v-1} (x_u f(x_v) - x_v f(x_u) - f([x_u, x_v]) g(x_1, \ldots, x_{u-1}, \ldots, x_v, \ldots, x_{k+2})) \right)$$
$$(f \cdot (\delta R + g))(x_1, \ldots, x_{k+2})$$

$$= (-1)^{j(k+1)} \sum_{u=1}^{k+2} (-1)^{u-1} f(x_u)(\delta R + g)(x_1, \ldots, x_u, \ldots, x_{k+2})$$

$$= (-1)^{j(k+1)} \left( \sum_{u=1}^{k+2} (-1)^{u+v} f(x_u)x_v g(x_1, \ldots, \hat{x}_u, \ldots, x_v, \ldots, x_{k+2}) \right)$$

$$+ \sum_{u=1}^{k+2} \sum_{u<v} (-1)^{u+v-1} f(x_u)x_v g(x_1, \ldots, \hat{x}_u, \hat{x}_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq v < w \leq k+2} (-1)^{u+v-w-1} f(x_u)g([x_v, x_w], x_1, \ldots, \hat{x}_u, \ldots, \hat{x}_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq u < v \leq k+2} (-1)^{u+v+w} f(x_u)g([x_v, x_u], x_1, \ldots, \hat{x}_u, \ldots, \hat{x}_v, \ldots, x_{k+2})$$

$$+ \sum_{1 \leq v < w \leq k+2} (-1)^{u+v-w} f(x_u)g([x_v, x_u], x_1, \ldots, \hat{x}_u, \ldots, \hat{x}_v, \ldots, x_{k+2})$$

Hence $\delta R + (f \cdot g) = (\delta R + f) \cdot g + (-1)^{j+1} f \cdot (\delta R + g)$. On the other hand,

$$(\delta_R - (f \cdot g))(x_1, \ldots, x_{k+1})$$

$$= (-1)^{j+1+k} \sum_{u=1}^{k+1} (-1)^{u-1} d_R f(x_u)g(x_1, \ldots, \hat{x}_u, \ldots, x_{k+1})$$

$$= (-1)^{j+1+k} \sum_{u=1}^{k+1} (-1)^{u-1} ((d_R f(x_u))g(x_1, \ldots, \hat{x}_u, \ldots, x_{k+1}) + (-1)^j f(x_u)d_R(g(x_1, \ldots, \hat{x}_u, \ldots, x_{k+1})))$$

$$= -((d_R \circ f)\cdot g)(x_1, \ldots, x_{k+2}) + (-1)^{j+k+1}(f \cdot (d_R \circ g))(x_1, \ldots, x_{k+1})$$

$$= ((\delta_R - f)\cdot g)(x_1, \ldots, x_{k+1}) + (-1)^{j+1}(f \cdot (\delta_R - g))(x_1, \ldots, x_{k+1}).$$

Hence Leibniz’s rule holds.

If $R$ is graded commutative and we endow a grading on $\text{Hom}(\wedge^i g, R)$ by declaring $f \in \text{Hom}(\wedge^i g, R^j)$ has total degree $i + j$ then the product on $\text{Hom}(\wedge^i g, R)$ is also graded commutative, i.e.,

$$f \cdot g = (-1)^{(i+j)(k+\ell)} g \cdot f,$$

since the coproduct $\Delta$ is cocommutative and the product on $R$ is graded commutative.

The complex of $G$-invariant sections of $M$ also acquire a structure of $\text{Hom}(\wedge^* g, M)$-module through the map

$$\text{Hom}(\wedge^* g, R) \otimes \text{Hom}(\wedge^* g, M) \to \text{Hom}(\wedge^* g, M),$$

$$f \cdot m : \wedge^* g \xrightarrow{\Delta} \wedge^* g \otimes \wedge^* g \xrightarrow{f \otimes m} R \otimes M \to M,$$

where the map $R \otimes M \to M$ is given by the $R$-module structure on $M$.

**G-invariant Kähler differentials.** Next, we would like to describe the Kähler differentials of the dg-algebra $\text{Hom}(\wedge^* g, R)$, i.e., the complex of $G$-invariant functions. If we regard $G$-invariant functions on $R$ as “functions” on the stack $[\text{Spec } R/G]$, then $G$-invariant Kähler differentials should correspond to $\text{Kähler differentials on the stack } [\text{Spec } R/G]$.

Denote the $G$-Kähler differentials on $R$ by $\Omega_R^G$. Recall that since $R$ is a $g$-module and $g$ acts on $R$ by derivation, the map $R \to \text{Hom}(g, R)$ sending $r \mapsto (a \mapsto a \cdot r)$ is a derivation which factorizes to a $R$-linear map $\alpha : \Omega_R^G \to \text{Hom}(g, R)$ by the universal property of $\Omega_R^G$.

Both $R$-modules $\Omega_R^G$ and $\text{Hom}(g, R)$ have a natural $g$ action which turns $\alpha$ into a $g$-module morphism: $g$ acts on the $\Omega_R^G$ component by

$$x \cdot d_R r = d_R (x \cdot r)$$
and on the \( \text{Hom}(g, R) \) component by enforcing the Leibniz rule:

\[
(x \cdot f)(y) = x \cdot (f(y)) - f([x, y]).
\]

Both \( g \)-action commutes with the internal differential on the two \( R \)-modules, and \( \alpha \) is a \( g \)-module morphism since for all \( x, y \in g \) and \( r \in R \), we have

\[
\alpha(x \cdot d_{dr}r)(y) - (x \cdot \alpha(d_{dr}r))(y) = y \cdot (x \cdot r) - x \cdot (y \cdot r) + [x, y] \cdot r = 0.
\]

The atlas \( \varphi : \text{Spec } R \to [\text{Spec } R/G] \) is a principal \( G \)-bundle. Hence the relative cotangent complex is given by \( L_{[\text{Spec } R/G]} \cong g^\vee \otimes R \cong \text{Hom}(g, R) \). We have a distinguished triangle in \( D(R \text{-mod}) \)

\[
\varphi^* L_{[\text{Spec } R/G]} \to L_R \xrightarrow{\varphi} \text{Hom}(g, R) \to \varphi^* L_{[\text{Spec } R/G]}[1].
\]

Since \( R \) is assumed to be quasi-free, the Kähler differentials \( \Omega^*_R \) gives a model for \( L_R \). Thus a model for \( \varphi^* L_{[\text{Spec } R/G]} \) is given by \( \text{cone} \alpha[-1] \). Recall the cone of the map \( \alpha[-1] \) is given by the \( R \)-module cone \( \alpha[-1] = \Omega^*_R \oplus \text{Hom}(g, R)[-1] \) together with the differential

\[
d_{\text{cone}} = \begin{pmatrix} d_{\Omega_R} & 0 \\ \alpha & -d_{\text{Hom}(g, R)} \end{pmatrix},
\]

where \( d_{\text{Hom}(g, R)} \) is the differential on \( \text{Hom}(g, R) \) which maps \( f \mapsto d_R \circ f \). The direct sum \( g \)-module structure on \( \text{cone} \alpha[-1] \) is compatible with the internal differential \( d_{\text{cone}} \) since \( d_{\Omega_R} \), \( d_{\text{Hom}(g, R)} \) and \( \alpha \) all commute with \( g \)-action. We are going to construct a universal derivation \( d_{dr} : \text{Hom}(\wedge^* g, R) \to \text{Hom}(\wedge^* g, \text{cone} \alpha[-1]) \). Consider the following two maps

\[
d_{dr^+} : \text{Hom}(\wedge^* g, R) \to \text{Hom}(\wedge^* g, \Omega^*_R)
\]

\[
f \mapsto (-1)^|f| d_{dr} \circ f,
\]

\[
d_{dr^-} : \text{Hom}(\wedge^* g, R) \to \text{Hom}(\wedge^{*-1} g, \text{Hom}(g, R)[-1]),
\]

\[
(d_{dr^-} f)(x_1, \ldots, x_{r-1}) = (x \mapsto -f(x, x_1, \ldots, x_{r-1})).
\]

The direct sum of these two maps defines a map

\[
d_{dr} = d_{dr^+} \oplus d_{dr^-} : \text{Hom}(\wedge^* g, R) \to \text{Hom}(\wedge^* g, \text{cone} \alpha[-1])
\]

of degree 1 satisfying the Leibnitz's rule which graded-commutes with the internal differentials:

**Proposition 7.3.2.** The map \( d_{dr} : \text{Hom}(\wedge^* g, R) \to \text{Hom}(\wedge^* g, \text{cone} \alpha[-1]) \) is a derivation which satisfies \( d_{dr} \delta R = -\delta_{\text{cone}} d_{dr} \).

**Proof.** First, we check Leibnitz's rule:

\[
(d_{dr^-}(f \cdot g))(x_1, \ldots, x_{i+k-1})
\]

\[
= (x_0 \mapsto -(f \cdot g)(x_0, x_1, \ldots, x_{i+k-1}))
\]

\[
= (x_0 \mapsto -(1)^j \sum_{\sigma \in S_{h_{i,k}}} (\text{sgn } \sigma) f(x_{\sigma(0)}, \ldots, x_{\sigma(i-1)}, g(x_{\sigma(i)}, \ldots, x_{\sigma(i+k-1)}))
\]

\[
= (x_0 \mapsto -(1)^j \sum_{\sigma \in S_{h_{i,k}}} (\text{sgn } \sigma) f(x_0, x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, g(x_{\sigma(i)}, \ldots, x_{\sigma(i+k-1)}))
\]

\[
+ (x_0 \mapsto -(1)^{i+j} \sum_{\sigma \in S_{h_{i,k}}} (\text{sgn } \sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)} g(x_0, x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k-1)}))
\]

\[
= ((d_{dr^-} f) \cdot g)(x_1, \ldots, x_{i+k}) + (-1)^{i+j} (f \cdot (d_{dr^-} g))(x_1, \ldots, x_{i+k}),
\]
\[(d_{dR} + (f \cdot g))(x_1, \ldots, x_{i+k})\]
\[= (-1)^{i+k+j} d_{dR} \left( \sum_{\sigma \in Sh_{i,k}} (\text{sgn } \sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)} g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k)}) \right)\]
\[= (-1)^{i+k+j} \sum_{\sigma \in Sh_{i,k}} (\text{sgn } \sigma) (d_{dR} f(x_{\sigma(1)}, \ldots, x_{\sigma(i)})) g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k)})\]
\[+ (-1)^{i+k+j+k} \sum_{\sigma \in Sh_{i,k}} (\text{sgn } \sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)} d_{dR} (g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k)}))\]
\[= ((d_{dR} + f) \cdot g)(x_1, \ldots, x_{i+k}) + (-1)^{i+j} (f \cdot (d_{dR} + g))(x_1, \ldots, x_{i+k}).\]

Hence \(d_{dR}\) is a derivation. The equation \(d_{dR} \delta_R = -\delta_{\text{cone}} d_{dR}\) amounts to three equations:

\[d_{dR} + \delta_{R+} = -\delta_{\text{cone}} + d_{dR+},\]
\[d_{dR} + \delta_{R-} + d_{dR} - \delta_{R+} = -\delta_{\text{cone}} + d_{dR-} + \delta_{\text{cone}} - d_{dR+},\]
\[d_{dR} - \delta_{R-} = -\delta_{\text{cone}} - d_{dR-}.\]

Given \(f \in \text{Hom}(\wedge^\ell \mathfrak{g}, R),\)

\[(d_{dR} + \delta_{R+} f)(x_1, \ldots, x_{\ell+1})\]
\[= \sum_{i=1}^{\ell+1} (-1)^{i+1+\ell+1} d_{dR} (x_i (f(x_1, \ldots, x_i, \ldots, x_{\ell+1}))\]
\[+ \sum_{i<j} (-1)^{i+j+\ell+1} d_{dR} (f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{\ell+1}))\]
\[= - \sum_{i=1}^{\ell+1} (-1)^{i+1} x_i ((d_{dR} + f)(x_1, \ldots, \hat{x}_i, \ldots, x_{\ell+1}))\]
\[+ \sum_{i<j} (-1)^{i+j+\ell} (d_{dR} + f)([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{\ell+1})\]
\[= -(-1)^{\ell+1} d_{dR} + f)(x_1, \ldots, x_{\ell+1}),\]
\[(d_{dR} - \delta_{R-} f)(x_1, \ldots, x_{\ell-1}) = - (x \mapsto (\delta_{R-} f)(x, x_1, \ldots, x_{\ell-1}))\]
\[= - (x \mapsto (\delta_{R-} f)(x, x_1, \ldots, x_{\ell-1}))\]
\[= - (x \mapsto (-1)^{\ell} d_{\text{cone}} (x \mapsto f(x, x_1, \ldots, x_{\ell-1}))\]
\[= -(-1)^{\ell+1} d_{\text{cone}} (d_{dR} - f)(x_1, \ldots, x_{\ell-1}))\]
\[= -\delta_{\text{cone}} - d_{dR} - f)(x_1, \ldots, x_{\ell-1}),\]
\[(d_{dR} + \delta_{R-} f)(x_1, \ldots, x_{\ell}) = (d_{dR} \circ d_{R} \circ f)(x_1, \ldots, x_{\ell}),\]
\[(d_{dR} - \delta_{R+} f)(x_1, \ldots, x_{\ell}) = - (x_{0} \mapsto (\delta_{R+} f)(x_0, x_1, \ldots, x_{\ell}))\]
\[= - \left( x_{0} \mapsto \sum_{i=0}^{\ell} (-1)^i x_i f(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_{\ell}) + \sum_{0 \leq i < j \leq \ell} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{\ell}) \right).\]
On the other hand,
- \((\partial_{\text{cone}} + d_{\text{dR}} \cdot f)(x_1, \ldots, x_\ell)\)
- \(-\sum_{i=1}^{\ell} (-1)^{i+1} x_i ((d_{\text{dR}} \cdot f)(x_1, \ldots, x_i, \ldots, x_\ell))\)
- \(-\sum_{1 \leq i < j \leq \ell} (-1)^{i+j} (d_{\text{dR}} \cdot f)([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_\ell)\)
- \(= \sum_{i=1}^{\ell} (-1)^{i+1} x_i (x_0 \mapsto f(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_\ell))\)
- \(+ \left( x_0 \mapsto \sum_{1 \leq i < j \leq \ell} (-1)^{i+j} f(x_0, [x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_\ell) \right)\)
- \(= \sum_{i=1}^{\ell} (-1)^{i+1} (x_0 \mapsto x_i f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell) - f([x_i, x_0], x_1, \ldots, \hat{x}_i, \ldots, x_\ell))\)
- \(-\left( x_0 \mapsto \sum_{1 \leq i < j \leq \ell} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_\ell) \right)\)
- \(-\left( x_0 \mapsto \sum_{i=1}^{\ell} (-1)^{i+1} x_i f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell) + \sum_{0 \leq i < j \leq \ell} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_\ell) \right)\).

Hence \(d_{\text{dR}} + \delta_{\text{R-}} + d_{\text{dR}} - \delta_{\text{R+}} = - (\partial_{\text{cone}} + d_{\text{dR}} - \partial_{\text{cone}} - d_{\text{dR}})\).

**Proposition 7.3.3.** The Kähler differentials \((\Omega_{\text{Hom}(\wedge^\bullet g, R)}, d_{\Omega})\) of the dg-algebra \(\text{Hom}(\wedge^\bullet g, R)\) is isomorphic to \((\text{Hom}(\wedge^\bullet g, \text{cone} \, \alpha[-1]), \partial_{\text{cone}})\).

**Proof.** Let \(d'_{\text{dR}} : \text{Hom}(\wedge^\bullet g, R) \to \text{Hom}(\wedge^\bullet g, R)\) denote the universal derivation. Since

\[ \text{Hom}(\wedge^\bullet g, R) = \wedge^\bullet g^\vee \otimes R \quad \text{and} \quad \Omega_{\text{Hom}(\wedge^\bullet g, R)} = \Omega_{\wedge^\bullet g^\vee \otimes R} = \wedge^\bullet g^\vee \otimes \wedge^\bullet g^\vee \otimes \Omega_R \]

as a \(\text{Hom}(\wedge^\bullet g, R)\)-module, explicitly, for

\[ f \in \text{Hom}(g, K) \subseteq \text{Hom}(g, R) \quad \text{and} \quad r \in R \cong \text{Hom}(K, R) \subseteq \text{Hom}(\wedge^\bullet g, R), \]

the isomorphism

\[ \Phi : \Omega^1_{\text{Hom}(\wedge^\bullet g, R)} \cong \text{Hom}(\wedge^\bullet g, \Omega_R \otimes \text{Hom}(g, R)[-1]) \]

sends

\[ d'_{\text{dR}} f \mapsto (k \mapsto k f) \in \text{Hom}(K, \text{Hom}(g, R)[-1]) \quad \text{and} \quad d'_{\text{dR}} r \mapsto (k \mapsto k d_{\text{dR}} r) \in \text{Hom}(K, \Omega_R). \]

Since \(\text{Hom}(\wedge^\bullet g, R)\) is generated as an algebra by elements of the form

\[ f \in \text{Hom}(g, K) \subseteq \text{Hom}(g, R) \quad \text{and} \quad r \in R \cong \text{Hom}(K, R) \subseteq \text{Hom}(\wedge^\bullet g, R), \]

the square

\[ \begin{array}{ccc}
\text{Hom}(\wedge^\bullet g, R) & \xrightarrow{d_{\text{dR}}} & \Omega^1_{\text{Hom}(\wedge^\bullet g, R)} \\
\downarrow & & \downarrow \Phi \\
\text{Hom}(\wedge^\bullet g, R) & \xrightarrow{d_{\text{dR}}} & \text{Hom}(\wedge^\bullet g, \text{cone} \, \alpha[-1])
\end{array} \]
forms. We define $d_{\text{dR}}: \Omega^k \rightarrow \Omega^{k+1}$, $\partial = \delta_{\text{con}}$, where here $\delta_{\text{con}}$ is a slideshow in your slide. We will call $(\Omega^k, \delta_{\text{con}})$ the complex of $G$-invariant 1-forms. Taking the $n$-th wedge product, we get the complex of $G$-invariant $n$-forms

$$\text{Hom}(\wedge^n \Omega [-1]) = \bigoplus_{j+k=n} \text{Hom}(\wedge^j \Omega^1, \wedge^k \Omega^1)$$

**G-invariant de Rham complex.** We would like to extend the de Rham differential to all $G$-invariant forms. We define

$$d_{\text{dR}+}: \text{Hom}(\wedge^l \Omega^1, \text{Hom}(S^l \Omega^1, \wedge^{k+1} \Omega^1_R)[-j]) \longrightarrow \text{Hom}(\wedge^l \Omega^1, \text{Hom}(S^l \Omega^1, \wedge^k \Omega^1_R)[-j])$$

$$d_{\text{dR}-}: \text{Hom}(\wedge^l \Omega^1, \text{Hom}(S^l \Omega^1, \wedge^{k+1} \Omega^1_R)[-j]) \longrightarrow \text{Hom}(\wedge^{l-1} \Omega^1, \text{Hom}(S^{l-1} \Omega^1, \wedge^k \Omega^1_R)[-j-1])$$

$$f \mapsto (-1)^f d_{\text{dR}} \circ f$$

where here $d_{\text{dR}}: \wedge^k \Omega^1_R \rightarrow \wedge^{k+1} \Omega^1_R$ is the de Rham differential on the complex $\wedge^* \Omega^1_R$.

**Proposition 7.3.4.** $(\Omega^k, \text{Hom}(S^* \Omega^1, \wedge^k \Omega^1_R) [-j]), d_{\text{dR}+}, d_{\text{dR}-}$ is a double complex.

**Proof.** $d_{\text{dR}+}^2 = 0$ since $d_{\text{dR}-}^2 = 0$ on $\wedge^* \Omega^1_R$. Given $f \in \text{Hom}(\wedge^l \Omega^1, \text{Hom}(S^l \Omega^1, \wedge^k \Omega^1_R)[-j])$,

$$(d_{\text{dR}+}f)(x_1, \ldots, x_{l-1}, y_1, \ldots, y_{j+2}) = \sum_{u=1}^{j+2} (d_{\text{dR}-}f)(y_u, x_1, \ldots, x_{l-2}, y_1, \ldots, y_u, \ldots, y_{j+2})$$

$$= \sum_{1 \leq u < v \leq j+2} f(y_u, y_v, x_1, \ldots, x_{l-2}, y_1, \ldots, y_u, \ldots, y_v, \ldots, y_{j+2})$$

$$+ \sum_{1 \leq v < w \leq j+2} f(y_v, y_w, x_1, \ldots, x_{l-2}, y_1, \ldots, y_v, \ldots, y_w, \ldots, y_{j+2})$$

$$= 0$$

since $f$ is antisymmetric in the first two variables.

$$(d_{\text{dR}+}d_{\text{dR}-}f)(x_1, \ldots, x_{l-1}, y_1, \ldots, y_{j+1}) = (-1)^l d_{\text{dR}} \left( \sum_{i=1}^{j+1} f(y_i, x_1, \ldots, x_{l-1}, y_1, \ldots, y_{j+1}) \right)$$

$$= - \left( \sum_{i=1}^{j+1} (-1)^{l-i} d_{\text{dR}}(f(y_i, x_1, \ldots, x_{l-1}, y_1, \ldots, y_{j+1})) \right)$$

$$= -(d_{\text{dR}-}d_{\text{dR}+}f)(x_1, \ldots, x_{l-1}, y_1, \ldots, y_{j+1})$$

Hence the $d_{\text{dR}+}$ and $d_{\text{dR}-}$ anticommute.
The total complex of
\[(\text{Hom}(\wedge^\bullet g, \wedge^\bullet \text{cone} \alpha [-1]), d_{dR+}, d_{dR-}) = (\text{Hom}(\wedge^\bullet g, \wedge^\bullet \Omega^1_{R^-}[-\tau]), d_{dR+}, d_{dR-})\]
is called the \textit{G-invariant de Rham complex} and the total differential \(d_{dR}\) is called the de Rham differential.

There is a wedge product on the de Rham complex
\[\wedge : \text{Hom}(\wedge^\bullet g, \wedge^\bullet \text{cone} \alpha [-1]) \otimes \text{Hom}(\wedge^\bullet g, \wedge^\bullet \text{cone} \alpha [-1]) \rightarrow \text{Hom}(\wedge^\bullet g, \wedge^\bullet \text{cone} \alpha [-1])\]
defined by
\[f \wedge g : \wedge^\bullet g \xrightarrow{\Delta'} \wedge^\bullet g \otimes \wedge^\bullet g \xrightarrow{\otimes g} \wedge^\bullet \text{cone} \alpha [-1] \otimes \wedge^\bullet \text{cone} \alpha [-1] \xrightarrow{\wedge} \wedge^\bullet \text{cone} \alpha [-1].\]

Under the isomorphism
\[\wedge^n \text{cone} \alpha [-1] = \bigoplus_{j+k=n} \text{Hom}(S^j g, \wedge^n \Omega^1_{R^-})[-j],\]
the wedge product on \(\wedge^\bullet \text{cone} \alpha [-1]\) is given by
\[\text{Hom}(S^j g, \wedge^n \Omega^1_{R^-})[-j] \otimes \text{Hom}(S^k g, \wedge^m \Omega^1_{R^-})[-\ell] \rightarrow \text{Hom}(S^{j+k} g, \wedge^{k+m} \Omega^1_{R^-})[-j-\ell] \tag{1}\]
\[f \wedge g : S^{j+k} g \otimes S^k g \xrightarrow{\otimes g} \wedge^{k+m} \Omega^1_{R^-} \xrightarrow{\Delta} \wedge^{k+m} \Omega^1_{R^-},\]
where \(\Delta'\) is the coproduct on \(S^\bullet g\). Hence, when
\[f \in \text{Hom}(\wedge^\bullet g \otimes S^j g, \wedge^k \Omega^1_{R^-}[-j]) \cong \text{Hom}(\wedge^\bullet g, \text{Hom}(S^j g, \wedge^k \Omega^1_{R^-})[-j]) \subseteq \text{Hom}(\wedge^\bullet g, \wedge^{j+k} \Omega^1_{R^-})\]
and
\[g \in \text{Hom}(\wedge^\bullet g \otimes S^m g, \wedge^m \Omega^1_{R^-}[-m]) \subseteq \text{Hom}(\wedge^\bullet g, \text{Hom}(S^m g, \wedge^m \Omega^1_{R^-})[-m]) \subseteq \text{Hom}(\wedge^\bullet g, \wedge^{m+n} \Omega^1_{R^-})\]
their wedge product \(f \wedge g \in \text{Hom}(\wedge^{i+\ell} g \otimes S^{j+m} g, (\wedge^\bullet \Omega^1_{R^-})^{k+n}[-j - m])\) is explicitly given by
\[(f \wedge g)(x_1, \ldots, x_{i+\ell}, y_1, \ldots, y_{j+m}) = (-1)^{i\ell} \sum_{\sigma \in S_h, \ell} \sum_{\tau \in S_{h,m}} \text{sgn} \sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)})
\wedge g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m)}).
\]
If we endow a grading on \(\text{Hom}(\wedge^\bullet g, \wedge^\bullet \text{cone} \alpha [-1])\) by declaring
\[f \in \text{Hom}(\wedge^\bullet g \otimes S^j g, (\wedge^\bullet \Omega^1_{R^-})^k[-j]) \subseteq \text{Hom}(\wedge^\bullet g, (\wedge^\bullet \text{cone} \alpha [-1])^j_k)
\]
and
\[g \in \text{Hom}(\wedge^\bullet g \otimes S^m g, (\wedge^\bullet \Omega^1_{R^-})^m[-m]) \subseteq \text{Hom}(\wedge^\bullet g, (\wedge^\bullet \text{cone} \alpha [-1])^m_m)
\]
have total degree \(i + 2j + k + \ell + 2m + n\) respectively, then the wedge product is graded commutative: \(f \wedge g = (-1)^{(i+\ell)k}(i+\ell+2m+n)g \wedge f\).

\textbf{Proposition 7.3.5.} The de Rham differential satisfies the Leibniz’s rule
\[d_{dR}(f \wedge g) = (d_{dR}f) \wedge g + (-1)^{i+\ell+2j+k} f \wedge (d_{dR}g).\]

\textit{Proof.}
\[(d_{dR} - (f \wedge g))(x_1, \ldots, x_{i+\ell-1}, y_1, \ldots, y_{j+m+1})
= \sum_{i=1}^{j+m+1} (f \wedge g)(y_i, x_1, \ldots, x_{i+\ell-1}, y_1, \ldots, y_{j+m+1})
= (-1)^{i\ell} \sum_{\sigma \in S_h, \ell} \sum_{\tau \in A} \text{sgn} \sigma f(y_{\tau(1)}, x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, y_{\tau(2)}, \ldots, y_{\tau(j+1)})
\wedge g(x_{\sigma(i)}, \ldots, x_{\sigma(i+k-1)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)})
+ (-1)^{i\ell} \sum_{\sigma \in S_h, \ell} \sum_{\tau \in B} (-1)^i \text{sgn} \sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)})
\wedge g(y_{\tau(j+1)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(i+k-1)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)}).\]
where

\[ A = \{ \tau \in S_{j+m+1} : \tau(2) < \cdots < \tau(j+1) \text{ and } \tau(j+2) < \cdots < \tau(j+m+1) \}, \]

\[ B = \{ \tau \in S_{j+m+1} : \tau(1) < \cdots < \tau(j) \text{ and } \tau(j+2) < \cdots < \tau(j+m+1) \}, \]

with \( S_{j+m+1} \) denoting the symmetric group on \( j+m+1 \) letters.

\[
((d_{DR-f}) \wedge g)(x_1, \ldots, x_{i+\ell-1}, y_1, \ldots, y_{j+m+1}) \\
= (-1)^{k\ell} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m+1}} \text{sgn}(\sigma)(d_{DR-f})(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j+1)}) \\
\quad \wedge g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)}) \\
= (-1)^{k\ell} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m+1}} \text{sgn}(\sigma) \sum_{\ell=1}^{j+1} f(y_{\tau(u)}, x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j+1)}) \\
\quad \wedge g(x_{\sigma(i)}, \ldots, x_{\sigma(i+\ell-1)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)}) \\
= (-1)^{k\ell} \sum_{\sigma \in Sh_{i-1}, \tau \in A} \text{sgn}(\sigma) f(y_{\tau(1)}, x_{\sigma(1)}, \ldots, x_{\sigma(i+\ell)}, y_{\tau(2)}, \ldots, y_{\tau(j+1)}) \\
\quad \wedge g(x_{\sigma(i)}, \ldots, x_{\sigma(i+\ell-1)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)})
\]

\[
\text{(f} \wedge (d_{DR-g})(x_1, \ldots, x_{i+\ell-1}, y_1, \ldots, y_{j+m+1}) \\
= (-1)^{k(\ell-1)} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m+1}} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)}) \\
\quad \wedge (d_{DR-g})(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell-1)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m+1)}) \\
= (-1)^{k(\ell-1)} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m+1}} \sum_{u=j+1}^{j+m+1} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)}) \\
\quad \wedge g(y_{\tau(u)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell-1)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m+1)}) \\
= (-1)^{k(\ell-1)} \sum_{\sigma \in Sh_{i-1}, \tau \in B} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)}) \\
\quad \wedge g(y_{\tau(j+1)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell-1)}, y_{\tau(j+2)}, \ldots, y_{\tau(j+m+1)})
\]

Hence \( d_{DR-f} \wedge g = (d_{DR-f}) \wedge g + (-1)^{i+2j+k} f \wedge (d_{DR-g}). \)

Also,

\[
(d_{DR+}(f \wedge g))(x_1, \ldots, x_{i+\ell}, y_1, \ldots, y_{j+m}) \\
= (-1)^{i+\ell} d_{DR+}((-1)^{k\ell} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m}} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)})) \\
\quad \wedge g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m)}) \\
= (-1)^{i+\ell+k\ell} d_{DR+} \left( \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m}} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)}) \right) \\
\quad \wedge g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m)}) \\
+ (-1)^{i+\ell+k+k\ell} \sum_{\sigma \in Sh_{i-1}, \tau \in Sh_{j+m}} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(j)})) \\
\quad \wedge d_{DR+} \left( g(x_{\sigma(i+1)}, \ldots, x_{\sigma(i+\ell)}, y_{\tau(j+1)}, \ldots, y_{\tau(j+m)}) \right)
\]

Hence we have the Leibniz’s rule.
The internal differential $\delta_{\text{cone}}$ on $\text{Hom}(\wedge^\bullet \mathfrak{g}, \wedge \text{cone} \alpha[-1])$ also extends to the $G$-invariant de Rham complex $\text{Hom}(\wedge^\bullet \mathfrak{g}, \wedge \text{cone} \alpha[-1])$ by imposing the Leibniz’s rule

$$\delta_{\text{cone}} (x \wedge y) = (\delta_{\text{cone}} x) \wedge y + (-1)^{|x|} x \wedge \delta_{\text{cone}} y.$$ 

**Proposition 7.3.6.** The de Rham differential $d_{\text{dR}}$ commutes with the internal differential of the de Rham complex, i.e., $\delta_{\text{cone}} d_{\text{dR}} + d_{\text{dR}} \delta_{\text{cone}} = 0$.

**Proof.** This follows from Leibniz’s rule and Proposition 7.3.3. \hfill \blacksquare

We arrive at the following definition, motivated by Pantev, Toën, Vaquié and Vezzosi [58].

**Definition 7.3.7 (G-invariant de Rham complex).** The double complex

$$\text{Hom}(\wedge^\bullet \mathfrak{g}, \wedge \text{cone} \alpha[-1]), \delta_{\text{cone}}, d_{\text{dR}})$$

is called the $G$-invariant de Rham differential complex on $\text{Spec } R$. A $G$-invariant differential p-form of degree $k$ on $\text{Spec } R$ is an element in the cohomology group $H^k(\mathfrak{g}, \wedge^p \text{cone} \alpha[-1], \delta_{\text{cone}})$. In other words, it is represented by an element

$$\omega^k \in \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^p \text{cone} \alpha[-1])^{k-i}) \text{ satisfying } \delta_{\text{cone}} \omega^k = 0.$$ 

Two such representations $\omega^k, \omega'^k$ are equivalent if there exists $\alpha^k$ such that $\omega^k - \omega'^k = \delta_{\text{cone}} \alpha^k$.

A $G$-invariant closed differential p-form of degree $k$ is an element in the cohomology

$$H^k(\mathfrak{g}, \prod_{j \geq 0} (\wedge^{p+j} \text{cone} \alpha[-1])[-j], \delta_{\text{cone}} + d_{\text{dR}}).$$

In other words, it is represented by a sequence $\omega = (\omega^0, \omega^1, \ldots)$, where

$$\omega^j \in \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^{p+j} \text{cone} \alpha[-1])^{k-i-j})$$

satisfying $(\delta_{\text{cone}} + d_{\text{dR}}) \omega = 0$. Equivalently, the sequence $\omega$ satisfies the system of equations

$$\begin{align*}
\delta_{\text{cone}} \omega^0 &= 0 \quad \text{in } \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^p \text{cone} \alpha[-1])^{k-i+1}), \\
d_{\text{dR}} \omega^k + \delta_{\text{cone}} \omega^{k+1} &= 0 \quad \text{in } \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^{p+j} \text{cone} \alpha[-1])^{k-i-j+1}) \text{ for all } k \geq 0.
\end{align*}$$

Two such representations are equivalent if there exists $\alpha = (\alpha^0, \alpha^1, \ldots)$ such that

$$\begin{align*}
\omega^0 - \omega'^0 &= \delta_{\text{cone}} \alpha^0 \quad \text{in } \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^{p+j} \text{cone} \alpha[-1])^{k-i-j}), \\
\omega^{k+1} - \omega'^{k+1} &= d_{\text{dR}} \alpha^k + \delta_{\text{cone}} \alpha^{k+1} \quad \text{in } \bigoplus_{i=0}^{\dim \mathfrak{g}} \text{Hom}(\wedge^i \mathfrak{g}, (\wedge^{p+j+1} \text{cone} \alpha[-1])^{k-i-j+1}) \text{ for all } i \geq 0.
\end{align*}$$

A $G$-invariant closed 2-form $\omega = (\omega^0, \omega^1, \ldots)$ of degree $k$ is a $G$-invariant $k$-shifted symplectic form if $\omega^0$ is a nondegenerate $G$-invariant 2-form of degree $k$. 

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Relations between $G$-invariant forms and forms on quotient stacks. We conjecture the $G$-invariant forms and $G$-invariant closed forms in Definition 7.3.7 are equivalent to forms and closed forms on the stack $[\text{Spec } R/G]$ in the sense of Pantev, Toën, Vaquié and Vezzosi [58] (c.f. Section 7.1, Definitions 7.1.1 and 7.1.2). A possible way of showing this equivalence is to use the language of simplicial schemes on the stack $[\text{Spec } G]$ in invariant forms and forms on quotient stacks.

We conjecture the equivalence is to use the language of simplicial schemes in the sense of Pridham [59] to describe derived stacks. Presumably the quotient stack $[\text{Spec } R/G]$ can be described as the simplicial scheme

$$\xymatrix{ \cdots \ar[r] & G^n \times \text{Spec } R \ar[r] & G \times G \ar[r] & \text{Spec } R \ar[r] & \text{Spec } (\text{Spec } R / G) }$$

where

$$\partial_i (g_1, \ldots, g_n, x) = \begin{cases} (g_2, \ldots, g_n, x) & \text{if } i = 0, \\ (g_1, \ldots, g_{i+1}, \ldots, g_n, x) & \text{if } 1 \leq i \leq n - 1, \\ (g_1, \ldots, g_{n-1}, g_n x) & \text{if } i = n, \text{ and} \end{cases}$$

$$\sigma_i (g_1, \ldots, g_n, x) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n, x).$$

Pridham [59, Definition 7.7] then gives a chain complex which represents the cotangent complex $\mathcal{L}_{\text{Spec } R/G}$ which we expect to be quasi-isomorphic to the $G$-invariant de Rham complex in Definition 7.3.7.

7.4 Moduli Spaces of Representations of Quivers with Superpotential

In this section, we study the moduli space of representations of a quiver with superpotential, and outline a strategy of proof of the existence of a shifted symplectic form which is in a standard Darboux form on the moduli space.

Let $Q$ be a dg-quiver. A dg-representation of $Q$ consists of chain complexes $W_v$ for each vertex $v$ in $Q$ and linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ of degree $i$ for each degree $i$ edge $e$ in $Q$. Let $S$ denote the $K$-algebra spanned by the vertices of $Q$. Then a dg-representation of $Q$ is the same as a chain complex $(W, d_W)$ over $S$ together with a morphism $KQ \to \text{End}^\bullet_S(V)$ of dg-algebras over $S$, where the differential on $\text{End}^\bullet_S(W)$ is given by

$$df = d_W f - (-1)^{\deg f} f d_W.$$

Thus given a fixed chain complex $(W, d_W)$ over $S$, we can write down a moduli functor for dg-representations of $Q$:

$$\text{Rep}_W(Q) : \{\text{commutative dg-algebras}\} \to \{\text{Sets}\}, \quad C \mapsto \text{Hom}_{\text{dg}/S}(KQ, \text{End}^\bullet_S(W) \otimes C).$$

Berest, Khachatryan and Ramadoss proved in [6, Theorem 2.1] that this functor is representable. In the special case when $Q$ is quasi-free and the chain complex $W$ is concentrated in degree 0 with dimension vector $d$, the commutative dg-algebra $R$ representing $\text{Rep}_W(Q)$ can be explicitly described: the underlying graded algebra of $R$ is the coordinate algebra of the graded vector space $[6, \text{Theorem 2.8}]:$

$$\bigoplus_{e \text{ edges in } Q} \text{Hom}(W_{t(e)}, W_{h(e)})[\deg e].$$

In other words, for each edge $e$ in $Q$, there is an associated $d_{t(e)} \times d_{h(e)}$ matrix $M_e = (e_{ij})$, and

$$R = \bigotimes_{e \text{ edges in } Q} K[M_e] = \bigotimes_{e \text{ edges in } Q} K[e_{ij}],$$

where $\bigotimes$ indicates tensor product over $K$. This description is useful for computations and theoretical considerations.
where each $e_{ij}$ has degree $\text{deg} e$. If $p = e_1 \cdots e_n$ is a path in $Q$, we write $M_p = M_{e_1} \cdots M_{e_n}$ where the product on the right hand side is the matrix product. For general $p \in KQ$, we extend bilinearly as usual to define $M_p$. Then the differential on $R$ can be described as

$$dM_e = M_{de}.$$ 

Note also that the differential on $R$ defines a cohomological vector field on the graded vector space $\bigoplus_{\text{edges in } Q} \text{Hom}(W_t(e), W_h(e))[\text{deg } e]$, and turns it into a dg-manifold.

The moduli stack of dg-representations of $Q$ concentrated in degree 0 and with dimension vector $d = (d(v))$ is then given by

$$\text{Rep}_d(Q) = \left[ \left( \bigoplus_{\text{edges in } Q} \text{Hom}(W_t(e), W_h(e))[\text{deg } e], d \right) / \prod_{v \text{ vertices in } Q} \text{GL}(d(v)) \right],$$

where $\text{GL}(d(v))$ acts on each vector space $W_v$ by conjugation. Notice here that although representations are concentrated in degree 0, the higher degree edges do not act trivially in the moduli functor, hence the higher degree edges are not redundant. For simplicity, we will denote the product group $\prod_{v \text{ vertices in } Q} \text{GL}(d(v))$ by $G$ and write elements in $G$ in the form $g = (g_v)$. In local coordinates, the $G$-action on $R$ is described by

$$g \cdot M_e = g_{h(e)}^{-1} M_e g_{t(e)}.$$

The associated $\mathfrak{g}$-action on $R$ is then given by, for any $\xi = (\xi_v) \in \prod \mathfrak{gl}(d(v)) = \mathfrak{g}$,

$$\xi \cdot M_e = -\xi_{h(e)} M_e + M_e \xi_{t(e)}.$$

Now, suppose our dg-quiver admits $Q$ a superpotential $\Phi$. Then Pantev, Toen, Vaquié and Vezzosi [58, p.9-10] claimed without proof that $\text{Rep}_d(Q)$ admits a shifted symplectic structure. We outline a strategy of proof by explicitly writing down a $G$-invariant shifted symplectic form. Recall that a quiver with superpotential $(Q, \Phi)$ of dimension $m$ has a dg-subquiver $\tilde{Q}$ such that $K\tilde{Q}$ has a noncommutative symplectic 2-form and that $Q$ can be constructed from $\tilde{Q}$ by adding a degree $1 - m$ loop on each vertex on $\tilde{Q}$. Then $\tilde{Q}$ correspond to a dg-subalgebra

$$\tilde{R} = \bigotimes_{\text{edges in } \tilde{Q}} K[M_e] \xrightarrow{i} R,$$

and we have a diagram

$$\text{Spec } \tilde{R} \xrightarrow{i = \text{Spec } i} \text{Spec } R \xrightarrow{\varphi} \text{Rep}_d(Q).$$

Let us choose some good models for the cotangent complexes for these three spaces.

Models for cotangent complexes. The atlas $\text{Spec } R \xrightarrow{\varphi} \text{Rep}_d(Q)$ is a principal $G$-bundle. Hence the relative cotangent complex $L_{\text{Spec } R/\text{Rep}_d(Q)} \cong \mathfrak{g}^\vee \otimes R$. We thus have a distinguished triangle in $\text{D}(R\text{-mod})$

$$\varphi^* L_{\text{Rep}_d(Q)} \rightarrow L_R \xrightarrow{\alpha} \mathfrak{g}^\vee \otimes R \rightarrow \varphi^* L_{\text{Rep}_d(Q)}[1].$$

On the other hand, we also have a distinguished triangle

$$i^* L_{\tilde{R}} \rightarrow L_R \rightarrow L_{R/\tilde{R}} \rightarrow i^* L_{\tilde{R}}[1].$$
Since $Q$ and $\tilde{Q}$ are both dg-quivers, both $R$ and $\tilde{R}$ are quasi-free as commutative dg-algebras. Hence the Kähler differentials $\Omega^1_R$ (resp. $\Omega^1_{\tilde{R}}$) gives a model for $L_R$ (resp. $L_{\tilde{R}}$). For each edge $e$ in $Q$, we will denote the matrix of 1-forms $d_{\text{DR}}M_e = (d_{\text{DR}}e_{ij})$.

The cone of $\alpha[-1] : \Omega^1_R[-1] \to g^\vee \otimes R[-1]$ then gives a model for $\varphi^*L_{\text{Rep}_d(Q)}$. Recall that

$$\text{cone}(\alpha[-1]) = \Omega^1_R \oplus (g^\vee \otimes R[-1]), \text{ with differential } d_{\text{cone}(\alpha[-1])} = \begin{pmatrix} d\Omega^1_R & 0 \\ \alpha & \text{id} \otimes dR[-1] \end{pmatrix}.$$ 

The map $\alpha$ comes from the $G$-action on $R$ as follows: The $G$-action on $R$ induces a linear map $g \otimes R \to R$, $\xi \otimes r \mapsto \xi \cdot r$ which satisfies Leibniz rule $\xi \cdot (r_1 r_2) = (\xi \cdot r_1) r_2 + r_1 (\xi \cdot r_2)$. The dual map $R \to g^\vee \otimes R \cong \text{Hom}_G(g, R)$, $r \mapsto (\xi \mapsto \xi \cdot r)$ is a derivation, hence it factorizes into a linear map $\alpha : \Omega^1_R \to g^\vee \otimes R$. In local coordinates, the map $\alpha$ is given by

$$d_{\text{DR}}M_e \mapsto -a_{h(e)}Me + Mea_{t(e)},$$

where for any vertex $v$, the $(i,j)$-th element $(a_v)_{ij}$ of the matrix $a_v$ denotes the linear map in $\text{gl}d(v))$ which projects a $d(v) \times d(v)$ matrix to its $(i,j)$-th element.

The $g$-module structure on cone $\alpha[-1]$ can be described as follows: Let $\xi = (\xi_v) \in \prod \text{gl}(d(v)) = g$. Then $g$ acts on cone $\alpha[-1] = \Omega^1_R \oplus g^\vee \otimes R[-1]$ by

$$\xi \cdot d_{\text{DR}}M_e = -\xi_{h(e)}(d_{\text{DR}}Me) + (d_{\text{DR}}Me)\xi_{t(e)}$$

$$\xi \cdot a_v = -\xi_v a_v + a_v \xi_v.$$

**Symplectic form on** $\text{Spec } \tilde{R}$. The noncommutative symplectic form $\omega = \sum_{x,y \text{ edges in } \tilde{Q}} \langle x, y \rangle d_{\text{DR}}x d_{\text{DR}}y$ on $K\tilde{Q}$ induces a shifted symplectic form $\omega_R = (\omega^0_R, 0, \ldots)$ on $\text{Spec } \tilde{R}$ by

$$\omega^0_R = \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} \langle x', y' \rangle \text{tr}(d_{\text{DR}}M_x d_{\text{DR}}M_y)$$

$$= \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} \langle x', y' \rangle \sum_{ij} (d_{\text{DR}}x_{ij} d_{\text{DR}}y_{ij}),$$

where $d_{\text{DR}}M_x$ denotes the matrix of 1-form $(d_{\text{DR}}x_{ij})$.

The superpotential $\Phi$ on $Q$ also induces a function $\Phi_R = \text{tr}(M\Phi)$ which satisfy the equation

$$i_{\Phi}^* \omega^0_R = d_{\text{DR}}\Phi_R.$$

With these models for the cotangent complexes, we can proceed to write down some symplectic forms.

**Symplectic form on** $L_{\text{Rep}_d(Q)}$. Recall $G = \prod_{v \text{ vertices in } Q} \text{GL}(d(v))$, where $d(v)$ is the dimension of the vector space at the vertex $v$, and $\text{gl}(d(v))$ is given by all $d(v) \times d(v)$ matrices. Let $(a_v)_{ij}$ denote the linear map in $\text{Hom}(\text{gl}(d(v)), R)$ which maps a $d(v) \times d(v)$ matrix to its $(i,j)$-th element. Let $a_v$ denote a matrix whose $(i,j)$-th element is given by $(a_v)_{ij}$. Define

$$\omega^0_{\varphi^*L_{\text{Rep}_d(Q)}} = \omega^0_{\tilde{R}} + \sum_{v \text{ vertices in } Q} \text{tr}(a_v d_{\text{DR}}M_v)$$

$$= \frac{1}{2} \sum_{x, y \text{ edges in } \tilde{Q}} \langle x', y' \rangle \text{tr}(d_{\text{DR}}M_x d_{\text{DR}}M_y) + \sum_{v \text{ vertices in } Q} \text{tr}(a_v d_{\text{DR}}M_v).$$
Chapter 7. Shifted Symplectic Structures on Moduli Spaces

Hence

\[ \text{tr}(a_v d_{\Omega} M_v) \]

is then a shifted symplectic structure on \( \text{Rep} \).

Assuming our definition of \( G \)-invariant symplectic form is Definition 7.3.7 is equivalent to the definition of symplectic forms on \( \mathbb{L}_{\text{Rep}(Q)} \) in the sense of Panet, Toën, Vaquié and Vezzosi [58]. \( \omega_{\varphi^1}^{\text{Rep}(Q)} \) is then a shifted symplectic structure on \( \text{Rep}_d(Q) \). Observe that the 2-form \( \omega_{\varphi^1}^{\text{Rep}(Q)} \) on \( \text{Rep}_d(Q) \) and \( \omega_0^V \) on \( \text{Spec} \) are equal only by a term. A natural guess would be that this extra term would drop out and the two 2-forms would be equal when both are pulled back to \( \text{Spec} \). The shifted symplectic structure would then be in the Darboux form described by Ben-Bassat, Brav, Bussi and Joyce [4, Theorem 2.10] as follows:

**Conjecture 7.4.1** (Shifted symplectic structure). *The moduli space \( \text{Rep}_d(Q) \) of representations of a quiver with superpotential \( Q \) of dimension \( m \) has a \((2 - m)\)-shifted symplectic structure \( \omega_{\varphi^1}^{\text{Rep}(Q)} \) such
that $\varphi^*(\omega_{\varphi^{*}L_{\text{rep}}(Q)}) = i^*\omega_{\tilde{R}}$, where $i : \text{Spec} \tilde{R} \to \text{Spec} R$ is the inclusion map. Moreover, Spec $R$ and $\omega$ are in a standard Darboux form in the sense of Ben-Bassat, Brav, Bussi and Joyce [4, Theorem 2.10] described as follows:

1. The degree 0 part $R_0$ is a smooth algebra of dimension $\sum_{\text{degree 0 edges in } Q} d(t(e))d(h(e))$ generated by the entries of the matrices $M_e$, where $\deg e = 0$, and that entries in $d_{\text{dR}}M_e$ form a basis of $\Omega^1_{\tilde{R}^0}$ over $R^0$.

2. $R$ is freely generated over $R^0$ by the entries of the matrices $M_e$ with $\deg e = -1, \ldots, 1 - m$, and

   $(i^*\omega_{\tilde{R}})^0 = i^*(\omega_R^0) = \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} (x^\vee, y^\vee)\text{tr}(d_{\text{dR}}M_x d_{\text{dR}}M_y)$.

3. The superpotential on $Q$ induces a function $\Phi_R = \text{tr}(M_\Phi)$ which satisfies the equation known as the classical master equation $\{\Phi_R, \Phi_R\} = 0$, with differential $d$ on $\tilde{R}$ given by

   $da_{ij} = (-1)^{1+|a|+m(a,b)}\partial_{b_{ij}}\Phi_R$,

   where $a_{ij}$ denotes the $(i,j)$-th entry of the matrix $M_a$ for any edge $a$ in $\tilde{Q}$.  

Chapter 8

Future Directions

This chapter discusses several future research directions.

One possible direction is to generalize Theorem 5.4.5 by removing the vanishing condition (or equivalently the assumption that the exceptional poset $E$ is $V$-geometric):

**Conjecture 8.0.2.** Let $X$ be a smooth variety and $\pi : V \to X$ be a vector bundle. Let $E$ be an exceptional poset on $D^b(\text{Coh}(X))$ with dual exceptional poset $F$. Then there is a natural $A_{\text{fin}}$-structure on $A_E = \bigoplus_{i,j \in I} \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}^{\ell-k}(F_i, F_j \otimes \wedge^k V)$,

making it a finite dimensional $A_{\text{fin}}$-algebra augmented over $S$, such that $D^b(\text{Coh}(V)) \cong \text{Per}(E(A_E))$.

In fact, in Chapter 5, although not explicitly stated, we have already proven a derived equivalence $D^b(\text{Coh}(X)) \cong \text{Per}(E(\text{Ext}^\bullet_{A}(S,S)))$. However, we do not know if $\text{Ext}^\bullet_{A}(S,S) = A_E$ (see Remark 5.4.2).

In another direction, we might be able to generalize the quotient construction (Theorem 5.6.5) by relaxing the assumption that each object in the exceptional sequence admits a $G$-linearization. First, we outline a way to generalize the quotient construction for $A_{\infty}$-categories in Section 3 by allowing $G$ to permute the objects in $\mathcal{A}$. Let $G$ be a finite group and $\mathcal{A}$ be an $A_{\infty}$-category with a finite set of objects. Suppose $G$ acts on $\text{Obj}(\mathcal{A})$ and for each $g \in G$, the action $g : \mathcal{A}(u,v) \to \mathcal{A}(g \cdot u, g \cdot v)$ is a strict $A_{\infty}$-isomorphism. Then $\text{Obj}(\mathcal{A})$ is partitioned into $G$-orbits $\mathcal{A}_{u} = \{E_{u_1}, \ldots, E_{u_k}\}$ where $u = (u_1, \ldots, u_k)$. Denote by $\text{Stab}(u)$ the stabilizer group of the orbit $\mathcal{A}_u$. A candidate for the quotient $A_{\infty}$-category $\mathcal{A}/G$ would be given by

$$\text{Obj}((\mathcal{A}/G)) = \{([u], \rho) : [u] \text{ is a } G\text{-orbit in } \text{Obj}(\mathcal{A}) \text{ and } \rho \text{ is a representation of } \text{Stab}(u)\},$$

with $A_{\infty}$-structures induced from that of $\mathcal{A}$ in a way similar to that described in Section 3. With this, suppose $H$ is a normal subgroup of $G$. Then we should also have $(\mathcal{A}/H)/(G/H) \cong \mathcal{A}/G$ strictly $A_{\infty}$-isomorphic. On the geometry side, we assume that the finite group $G$ permutes the objects in the exceptional poset $E$ and partitions $E$ into disjoint $G$-orbits $E_{u} = \{E_{u_1}, \ldots, E_{u_k}\}$ where $u = (u_1, \ldots, u_k)$. Let $E_u = E_{u_1} \oplus \cdots \oplus E_{u_k}$. Then $E_u$ is equipped with a natural $G$-linearization. Let

$$E/G = \{E_u \otimes \rho : E_u \text{ is a } G\text{-orbit of } E \text{ and } \rho \text{ is a representation of } \text{Stab}(E_u)\}.$$ 

Then we should be able to define on $E/G$ a partial order by declaring

$$E_u \prec E_v \text{ if and only if } \text{Hom}_G(E_u, E_v) \neq 0 \text{ and } u \neq v,$$

making $E/G$ an exceptional poset on $D^b(\text{Coh}^G(X))$. We should be able to prove:
Conjecture 8.0.3. Let $G$ be a finite group, $X$ be a smooth variety with $G$ acting by automorphisms, and $\pi : V \to X$ an equivariant vector bundle. Let $\mathcal{E}$ be an exceptional poset on $D^b(\text{Coh}(X))$ and suppose the action of $G$ on $X$ induces a permutation on the objects in $\mathcal{E}$. Then there is an equivalence $D^b(\text{Coh}^G(V)) \cong \text{Per}(E(A_{\mathcal{E}}/G))$. Furthermore, when $V$ is anti-semiample and $\text{Hom}(E_i, E_j \otimes S^*V^*) = 0$, then the above equivalence becomes $D^b(\text{Coh}^G(V)) \cong D^b(E(A_{\mathcal{E}}/G))$.

Thus in this situation, if we know the dg-quiver derived equivalent to $D^b(\text{Coh}(V))$, we would also know the dg-quiver derived equivalent to $D^b(\text{Coh}^G(V))$ by applying the quotient construction. Moreover, this quotient construction could be factorized: if there is a normal subgroup $H$ in $G$, we can compute the quotient by first taking quotient by $H$, followed by taking quotient by $G/H$.

One might also try to remove the vanishing condition for the product construction (Theorem 5.7.3). To do this, recall that in Theorem 5.2.4, we have constructed from the exceptional poset $\mathcal{E}$ a dg-algebra

$$R_{\mathcal{E}} = \mathcal{H}om(I_E, I_E \otimes S^*V^*),$$

where $E = \bigoplus E_i$ and $I_E$ is an injective resolution for $E$. In the product situation, one starts with two exceptional posets $\mathcal{E}$ and $\mathcal{F}$ and ends up with two dg-algebras $R_{\mathcal{E}}$ and $R_{\mathcal{F}}$. For the product construction to work without the vanishing condition, one needs to prove $R_{\mathcal{E}} \otimes R_{\mathcal{F}} \cong R_{\mathcal{E} \boxtimes \mathcal{F}}$ as a dg-algebra. One possible way to do this is to show that $I_{E \boxtimes F}$ is an injective resolution for $E \boxtimes F$. However, we do not know how to do this.

The above directions are more or less straightforward generalizations on the results of this thesis. A more ambitious direction is perhaps the following. Pantev, Toën, Vasić and Vezzosi has shown in [58, Theorem 2.9] that if we have an $n$-shifted symplectic derived stack $(X, \omega)$ together with Lagrangians $f_i : L_i \to X$, for $i = 1, 2$, then the derived fiber product $L_1 \times_X L_2$ has an $(n-1)$-shifted symplectic structure. Suppose Conjecture 7.4.1 is true, then quivers with superpotential correspond to shifted symplectic derived stacks via the moduli construction. One might then ask if one can define a similar notion of “Lagrangian” and “fiber product” in the quiver picture, which should be in the following form: Let $(Q, \Phi)$ be a quiver with superpotential of dimension $n$. A Lagrangian should be given by a quiver $Q_i$ with some additional structures encoding homotopy information, together with a morphism of dg-algebras $f_i : \mathbb{K}Q \to \mathbb{K}Q_i$, satisfying some compatibility conditions. The “fiber product” $Q_1 \times_{(Q, \Phi)} Q_2$ should be a quiver with superpotential of dimension $n-1$, whose path algebra is given by the “tensor product” $\mathbb{K}Q_1 \otimes_{\mathbb{K}Q} \mathbb{K}Q_2$. All these constructions, after taking moduli, should yield their corresponding counterparts in the derived stack picture.
Appendix A

Some Cohomological Formulae

This chapter computes some cohomological formulae for computing examples in Chapter 6.

A.1 Cohomologies of Tangent and Cotangent Sheaf of \( \mathbb{P}^n \)

In this section, we calculate some sheaf cohomologies for the tangent sheaf \( T \) and cotangent sheaf \( \Omega \) of \( \mathbb{P}^n = \mathbb{P}(V) \), where \( V \) is an \((n + 1)\)-dimensional vector space. We will write \( T^p = \wedge^p T \) and \( \Omega^p = \wedge^p \Omega \).

Lemma A.1.1. Let \( 0 \to L \to U \to W \to 0 \) be a sequence of vector bundles with \( \text{rank } L = 1 \). Then for \( p \geq 1 \) we have an exact sequence

\[
0 \to L \otimes \bigwedge^{p-1} W \to \bigwedge^p U \to \bigwedge^p W \to 0.
\] (A.1.1)

Proof. Taking wedge product of the exact sequence, and since \( L \) is a line bundle, we have an exact sequence

\[
\cdots \to L^{\otimes 2} \otimes \bigwedge^{p-2} U \to L \otimes \bigwedge^{p-1} U \to \bigwedge^p U \to \bigwedge^p W \to 0.
\]

We claim that this exact sequence factorizes into the short exact sequence (A.1.1). We induct on \( p \). The case \( p = 1 \) holds tautologically. For general \( p \), by induction assumption and twisting with \( L \), we have an exact sequence

\[
0 \to L^{\otimes 2} \otimes \bigwedge^{p-2} W \to L \otimes \bigwedge^{p-1} U \to \bigwedge^p U \to \bigwedge^p W \to 0.
\]

The map \( L \otimes \bigwedge^{p-1} U \to \bigwedge^p U \) vanishes on \( L^{\otimes 2} \otimes \bigwedge^{p-2} W \) and thus descends to form an exact sequence

\[
L \otimes \bigwedge^{p-1} W \to \bigwedge^p U \to \bigwedge^p W \to 0.
\]

To show the above exact sequence is also exact on the left, it suffices to show

\[
\ker \left( \bigwedge^p U \to \bigwedge^p W \right) = \text{coker} \left( L^{\otimes 2} \otimes \bigwedge^{p-2} U \to L \otimes \bigwedge^{p-1} U \right) = L \otimes \bigwedge^{p-1} W.
\]

This follows since by induction assumption

\[
\text{coker} \left( L \otimes \bigwedge^{p-2} U \to \bigwedge^p U \right) = \bigwedge^{p-1} W.
\]
Proposition A.1.2. The following sequence of vector bundles on \( \mathbb{P}^n \) is exact:

\[
0 \to T^{p-1}(k) \to \bigwedge^p V \otimes \mathcal{O}(p + k) \to T^p(k) \to 0.
\]

Proof. Apply Lemma A.1.1 to the Euler exact sequence \( 0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to T \to 0 \).

Lemma A.1.3. For \( 2 \leq q \leq n - 1 \),

\[
H^q(\mathbb{P}^n, T^p(k)) = H^{q-1}(\mathbb{P}^n, T^{p+1}(k)).
\]

Proof. We have an exact sequence

\[
\bigwedge^p V \otimes H^{q-1}(\mathbb{P}^n, \mathcal{O}(p + k)) \to H^{q-1}(\mathbb{P}^n, T^p(k)) \to H^q(\mathbb{P}^n, T^{p-1}(k)) \to \bigwedge^p V \otimes H^q(\mathbb{P}^n, \mathcal{O}(p + k)).
\]

For \( 2 \leq q \leq n - 1 \), the flanking terms vanish and we have the isomorphism as desired.

Lemma A.1.4. For \( n \geq 2 \) and \( p \neq n - 1 \), \( H^1(\mathbb{P}^n, T^p(k)) = 0 \) for all \( k \).

Proof. We may assume \( 0 \leq p \leq n \). For \( 0 \leq p \leq n - 2 \), applying Lemma A.1.3 repeatedly, we obtain

\[
H^1(\mathbb{P}^n, T^p(k)) = H^{p+1}(\mathbb{P}^n, \mathcal{O}(k)) = 0.
\]

For \( p = n - 1 \),

\[
H^1(\mathbb{P}^n, T^n(k)) = H^1(\mathbb{P}^n, \mathcal{O}(n + 1 + k)) = 0.
\]

Lemma A.1.5. For \( n \geq 2 \),

\[
h^0(\mathbb{P}^n, T^p(k)) = \begin{cases} 
\binom{n+k}{p} & \text{if } 0 \leq p \leq n, k \geq -p, \\
\binom{n+k+p+1}{p} & \text{if } p = n, k = -n-1, \\
1 & \text{otherwise}.
\end{cases}
\]

Proof. For \( p = 0 \),

\[
h^0(\mathbb{P}^n, T^p(k)) = \begin{cases} 
\binom{n+k}{k} & \text{if } k \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

For \( p = n \),

\[
h^0(\mathbb{P}^n, T^n(k)) = h^0(\mathbb{P}^n, \mathcal{O}(n + 1 + k)) = \begin{cases} 
\binom{2n+1+k}{n} & \text{if } k \geq -n-1 \\
0 & \text{otherwise}.
\end{cases}
\]

For \( 1 \leq p \leq n - 1 \), \( H^1(\mathbb{P}^n, T^{p-1}(k)) = 0 \) by Lemma A.1.4. Hence we have short exact sequences

\[
0 \to H^0(\mathbb{P}^n, T^{p-1}(k)) \to \bigwedge^p V \otimes H^0(\mathbb{P}^n, \mathcal{O}(p + k)) \to H^0(\mathbb{P}^n, T^p(k)) \to 0.
\]

If \( k < -p \), we have \( H^0(\mathbb{P}^n, T^p(k)) = 0 \). If \( k \geq -p \), we use the equation

\[
h^0(\mathbb{P}^n, T^p(k)) = \left( \dim \bigwedge^p V \right) h^0(\mathbb{P}^n, \mathcal{O}(p + k)) - h^0(\mathbb{P}^n, T^{p-1}(k))
\]

\[
= \left( n + 1 \right) \binom{n + p + k}{p + k} - h^0(\mathbb{P}^n, T^{p-1}(k))
\]

to induct on \( p \) and obtain the desired formula.
Lemma A.1.6. For $n \geq 2$,

$$H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) = \begin{cases} \mathbb{C} & \text{if } k = -n - 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We have short exact sequences

$$0 \to H^0(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) \to \bigwedge^n V \otimes H^0(\mathbb{P}^n, \mathcal{O}(n + k)) \to H^0(\mathbb{P}^n, \mathcal{T}^n(k)) \to H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) \to 0.$$

If $k < -n$, we have $H^0(\mathbb{P}^n, \mathcal{O}(n + k)) = 0$ and hence

$$H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) = H^0(\mathbb{P}^n, \mathcal{T}^n(k)) = H^0(\mathbb{P}^n, \mathcal{O}(n + 1 + k)) = \begin{cases} \mathbb{C} & \text{if } k = -n - 1 \\ 0 & \text{if } k < -n - 1. \end{cases}$$

If $k \geq -n$, using Lemma A.1.5, we have

$$h^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) = h^0(\mathbb{P}^n, \mathcal{T}^n(k)) - \left( \dim \bigwedge^n V \right) h^0(\mathbb{P}^n, \mathcal{O}(n + k)) + h^0(\mathbb{P}^n, \mathcal{T}^{n-1}(k))$$

$$= \left( \frac{2n + 1 + k}{n} \right) - \left( \frac{n + 1}{n} \right)^2(2n + k) + \left( \frac{n + k}{n - 1 + k} \right)^2(2n + k)$$

$$= 0.$$

Lemma A.1.7. For $1 \leq q \leq n - 1$,

$$H^q(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \mathbb{C} & \text{if } p + q = n \text{ and } k = -n - 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This holds by Lemma A.1.3, A.1.4 and A.1.6. ■

Lemma A.1.8.

$$h^n(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \binom{-n-k-2}{-p} \binom{-k-p-1}{n-p} & \text{if } 0 \leq p \leq n, k \leq -n - p - 2 \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By the vanishing $H^{n-2}(\mathbb{P}^n, \mathcal{O}(p + k)) = 0$ and $H^{n+1}(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) = 0$ (since $\dim \mathbb{P}^n = n$), we have exact sequences

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{T}^p(k)) \to H^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \to \bigwedge^p V \otimes H^n(\mathbb{P}^n, \mathcal{O}(p + k)) \to H^n(\mathbb{P}^n, \mathcal{T}^p(k)) \to 0.$$

If $k \geq -n - p$, $H^n(\mathbb{P}^n, \mathcal{O}(p + k)) = 0$, hence $H^n(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$.

If $k \leq -n - 1 - p$, then $H^{n-1}(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$. We then have the equation

$$h^n(\mathbb{P}^n, \mathcal{T}^p(k)) = \left( \dim \bigwedge^p V \right) h^n(\mathbb{P}^n, \mathcal{O}(p + k)) - h^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k))$$

$$= \left( \frac{n + 1}{p} \right) \left( \frac{-p - k - 1}{-p - k - 1 - n} \right) h^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k))$$

and induction on $p$ gives us the desired formula. ■

Putting Lemmas A.1.5, A.1.7 and A.1.8 together, we obtain
Proposition A.1.9.

\[ h^q(\mathbb{P}^n, T^p(k)) = \begin{cases} 
\binom{n+k}{p+k} & \text{if } q = 0, 0 \leq p \leq n, k \geq -p, \\
\binom{n+k+p+1}{p} & \text{if } p + q = n, 0 \leq p, q \leq n, k = -n - 1, \\
1 & \text{if } q = n, 0 \leq p \leq n, k \leq -n - p - 2, \\
0 & \text{otherwise.} 
\end{cases} \]

By the isomorphism \( T^p \cong \mathcal{T}^n \otimes \Omega^{n-p} \cong \Omega^{n-p}(n + 1) \), one can calculate the cohomologies of exterior powers of cotangent sheaf of \( \mathbb{P}^n \).

Proposition A.1.10.

\[ h^q(\mathbb{P}^n, \Omega^p(k)) = \begin{cases} 
\binom{n+k-p}{k} & \text{if } q = 0, 0 \leq p \leq n, k \geq p + 1, \\
\binom{n+k+p}{k} & \text{if } 0 \leq p = q \leq n, k = 0, \\
1 & \text{if } q = n, 0 \leq p \leq n, k \leq p - n - 1, \\
0 & \text{otherwise.} 
\end{cases} \]

### A.2 Cohomologies of Tangent and Cotangent Sheaf of \( \mathbb{P}^2 \)

In this section, we specialize to \( \mathbb{P}^2 \). Let \( V \) be a three dimensional vector space over \( \mathbb{C} \). Denote by \( \Omega \) and \( \mathcal{T} \) the cotangent and tangent sheaf on \( \mathbb{P}(V) = \mathbb{P}^2 \). From the last section, we have the formulæ

\[ h^q(\mathbb{P}^2, \mathcal{O}(k)) = \begin{cases} 
\frac{1}{2}(k+1)(k+2) & \text{if } q = 0 \text{ and } k \geq 0 \\
\frac{1}{2}(k+1)(k+2) & \text{if } q = 2 \text{ and } k \leq -3 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ h^q(\mathbb{P}^2, \mathcal{T}(k)) = \begin{cases} 
(k+2)(k+4) & \text{if } q = 0 \text{ and } k \geq -1 \\
1 & \text{if } q = 1 \text{ and } k = -3 \\
(k+2)(k+4) & \text{if } q = 2 \text{ and } k \leq -5 \\
0 & \text{otherwise.} 
\end{cases} \]

\[ h^q(\mathbb{P}^2, \Omega(k)) = \begin{cases} 
(k+1)(k+1) & \text{if } q = 0 \text{ and } k \geq 2 \\
1 & \text{if } q = 1 \text{ and } k = 0 \\
(k+1)(k+1) & \text{if } q = 2 \text{ and } k \leq -2 \\
0 & \text{otherwise.} 
\end{cases} \]

Proposition A.2.1.

\[ h^q(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = \begin{cases} 
2k^2 + 6k + 1 & \text{if } q = 0 \text{ and } k \geq 0 \\
3 & \text{if } q = 1 \text{ and } k = -1 \text{ or } k = -2 \\
2k^2 + 6k + 1 & \text{if } q = 2 \text{ and } k \leq -3 \\
0 & \text{otherwise.} 
\end{cases} \]

**Proof.** We have a long exact sequence

\[ 0 \rightarrow H^0(\mathbb{P}^2, \Omega(k)) \rightarrow V \otimes H^0(\mathbb{P}^2, \Omega(k+1)) \rightarrow H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \rightarrow H^1(\mathbb{P}^2, \Omega(k)) \rightarrow V \otimes H^1(\mathbb{P}^2, \Omega(k+1)) \rightarrow H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \rightarrow \cdots \]

\[ \cdots \rightarrow \mathcal{H}^2(\mathbb{P}^2, \mathcal{T}(k)) \rightarrow V \otimes \mathcal{H}^2(\mathbb{P}^2, \Omega(k+1)) \rightarrow \mathcal{H}^2(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \rightarrow 0. \]

For \( k \geq 1 \), from the vanishing \( H^i(\mathbb{P}^2, \Omega(k)) = H^i(\mathbb{P}^2, \Omega(k+1)) = 0 \), we immediately have the vanishing \( H^i(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = 0 \) for \( i \geq 1 \). Also, we have

\[ h^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = 3h^0(\mathbb{P}^2, \Omega(k+1)) - h^0(\mathbb{P}^2, \Omega(k)) = 2k^2 + 6k + 1. \]
For $k = 0$, from $H^i(P^2, \Omega(1)) = 0$ for all $i$, we have

$$H^i(P^2, T \otimes \Omega) = H^{i+1}(P^2, \Omega) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $k = -1$, from $H^i(P^2, \Omega(-1)) = 0$ for all $i$, we have

$$H^i(P^2, T \otimes \Omega(-1)) = V \otimes H^i(P^2, \Omega) = \begin{cases} \mathbb{C}^3 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The rest follows from Serre duality

$$h^i(P^2, T \otimes \Omega(k)) = h^{2-i}(P^2, \Omega \otimes T(-k) \otimes \mathcal{O}(-3)) = h^{2-i}(P^2, T \otimes \Omega(-3-k)).$$

\[\square\]

Using the isomorphism $T \cong \Omega(3)$, we also have

$$h^q(P^2, \Omega \otimes \Omega(k)) = h^q(P^2, T \otimes \Omega(k-3)) = \begin{cases} 2k^2 + 6k + 1 & \text{if } q = 0 \text{ and } k \geq 3 \\ 3 & \text{if } q = 1 \text{ and } k = 1 \text{ or } k = 2 \\ 2k^2 + 6k + 1 & \text{if } q = 2 \text{ and } k \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Twisting and dualizing the Euler sequence again, we get short exact sequences

$$0 \to T \otimes \Omega \otimes \Omega(k) \to V^* \otimes T \otimes \Omega(k-1) \to T \otimes \Omega(k) \to 0$$

$$0 \to \Omega \otimes \Omega(k) \to V \otimes \Omega \otimes \Omega(k+1) \to T \otimes \Omega \otimes \Omega(k) \to 0.$$ 

**Proposition A.2.2.**

$$h^q(P^2, T \otimes \Omega \otimes \Omega) = \begin{cases} \mathbb{C}^{10} & \text{if } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** This follows from the long exact sequence

\begin{align*}
0 & \to H^0(P^2, \Omega \otimes \Omega) \to V \otimes H^0(P^2, \Omega \otimes \Omega(1)) \to H^0(P^2, T \otimes \Omega \otimes \Omega) \\
& \to H^1(P^2, \Omega \otimes \Omega) \to V \otimes H^1(P^2, \Omega \otimes \Omega(1)) \to H^1(P^2, T \otimes \Omega \otimes \Omega) \\
& \to H^2(P^2, \Omega \otimes \Omega) \to V \otimes H^2(P^2, \Omega \otimes \Omega(1)) \to H^2(P^2, T \otimes \Omega \otimes \Omega) \to 0
\end{align*}

\[\square\]
Bibliography

Bibliography


