Compact Manifolds with Holonomy Spin(7)

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Contents

Chapter 1

Introduction

In this chapter, we discuss the basic concepts of holonomy groups, and introduce the origin of our problem of finding compact Riemannian manifolds of holonomy Spin(7).

1.1 Holonomy Groups

Let M be a C^∞, connected and simply-connected n-dimensional oriented Riemannian manifold with Riemannian metric g. Let ∇ be the Levi-Civita connection associated to g. For $x, y \in M$, parallel transport along a smooth path γ between x and y using the connection ∇ gives rise to an isometry between T_xM and T_yM.

Definition 1.1.1 For $x \in M$, let $Hol(x, g)$ be the group of isometries of T_xM generated by parallel transport around piecewise smooth loops based at x. It's clear that $Hol(x, g)$ is conjugate to $Hol(y, g)$ for any $x, y \in M$. Therefore, it makes sense to define $Hol(g)$ to be the holonomy group of (M, g), where $Hol(g) \subset SO(n)$. $Hol(g)$ is defined up to conjugation in $SO(n)$.

In fact, the notion of holonomy group can be defined for much more general objects than Riemannian manifold, e.g. a principal bundle equipped with a connection. However, general connections don't have the nice property of being torsion-free as Levi-Civita connection of Riemannian metrics, which is essential in classification of possible holonomy groups. Throughout this article, we will only consider connected and simply-connected Riemannian manifolds.

Holonomy group is intimately connected to curvature by definition. In fact, the Lie algebra of $Hol(x, g) \subset SO(T_xM)$, $hol(x, g)$, is the subalgebra of $so(T_xM)$ generated by $\tau(\lambda)^{-1}R(\tau(\lambda)v, \tau(\lambda)w)\tau(\lambda)$ for all $v, w \in T_xM$, and $\tau(\lambda)$ a parallel transport along a path λ from $x \in M$ to all points $y \in M$. This is the Ambrose-Singer theorem. Furthermore, in analytic case, $hol(x, g)$ is generated by the set of all covariant derivatives of the curvature tensor evaluated at x. The Bianchi identities therefore impose various restrictive conditions on the holonomy group. In particular, if the holonomy representation on the tangent space is reducible, it has to be a direct sum representation.

The work of Elie Cartan pinpointed the beautiful connection between Riemannian geometry and the representation of Lie groups. For the curvature tensor R on (M, g), the covariant derivative ∇R is zero if and only if there exists $x \in M$ and a $(3, 1)$-type curvature-like tensor on $T_x M$ invariant under the holonomy representation. So, Ambrose-Singer implies that the holonomy representation of a manifold whose Riemann curvature tensor has vanishing covariant derivative must leave the curvature tensor invariant.

Definition 1.1.2 Let R be the Riemann curvature tensor of the manifold (M, g). If $DR = 0$, then M is called a symmetric space. Equivalently, symmetric spaces are complete Riemannian manifolds for which the geodesic symmetry around any point is a well-defined isometry.

The study of irreducible symmetric spaces, i.e. complete manifolds whose holonomy $Hol(g)$ is irreducible and $DR = 0$, is then reduced by Cartan to the study of irreducible linear representations of Lie groups, in particular, the classification of real forms of simple Lie algebras.

Berger's work led to the classification of holonomy for non-symmetric spaces.

Theorem 1.1.3 [4] *Let (M, g) be a non-symmetric simply-connected Riemannian manifold of dimension n whose holonomy representation is irreducible, then the holonomy group $Hol(g)$ is one of the following*

(i) $Hol(g) = SO(n)$

(ii) $n = 2m$ *and* $Hol(g) = U(m)$ *(Kähler) or* $SU(m)$ *(Ricci-flat and Kähler) for* $m \geq 2$

(iii) $n = 4m$ *and* $Hol(g) = Sp(m)$ *(hyperkähler) or* $Sp(1)Sp(m)$ *(quaternionic Kähler) for* $m \geq 2$

(iv) $n = 16$ *and* $Hol(g) = Spin(9)$

(v) $n = 8$ *and* $Hol(g) = Spin(7)$

(vi) $n = 7$ *and* $Hol(g) = G_2$

From the Bianchi identities and the Ambrose-Singer theorem, one can deduce that manifolds with holonomy contained in G_2 or Spin(7) are Ricci-flat [24] Proposition 12.5.

Since then, Alekseevskii [2] concluded that $Spin(9)$ can not occur as a non-symmetric holonomy group. Bryant [5] showed the local existence of metrics with holonomy G_2 and Spin(7). Bryant and Salamon [6], in addition, constructed complete metrics of holonomy G_2 and Spin(7). Recently, Joyce [11, 12, 13] found compact $7-$ and $8-$ manifolds of holonomy G_2 and Spin(7).

1.2 Basics of Spin(7)

In this article, we will only study compact 8-manifolds of the exceptional holonomy Spin(7). One should note that some techniques involved in the studying of compact manifolds of holonomy Spin(7) and G_2 can be quite similar.

Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$. Define a 4-form on $\mathbb{R}^8$ by

$$\begin{aligned}
\Omega_0 &= dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_6 + dx_1 \wedge dx_2 \wedge dx_7 \wedge dx_8 + dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\
&+ dx_3 \wedge dx_4 \wedge dx_7 \wedge dx_8 + dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_7 - dx_1 \wedge dx_3 \wedge dx_6 \wedge dx_8 \\
&- dx_2 \wedge dx_4 \wedge dx_5 \wedge dx_7 + dx_2 \wedge dx_4 \wedge dx_6 \wedge dx_8 - dx_1 \wedge dx_4 \wedge dx_5 \wedge dx_8 \\
&- dx_1 \wedge dx_4 \wedge dx_6 \wedge dx_7 - dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_8 - dx_2 \wedge dx_3 \wedge dx_6 \wedge dx_7 \\
&+ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8. \qquad (1.1)
\end{aligned}$$

The subgroup of $GL(8, \mathbb{R})$ preserving Ω_0 is exactly Spin(7), which is a compact, semisimple, 21-dimensional Lie group. It is a subgroup of $SO(8)$ and preserves the orientation and standard Euclidean metric on $\mathbb{R}^8$. It is clear that Ω_0 is self-dual under the Hodge $*$ operation. Spin(7)-structures on an 8-manifold M correspond bijectively to the sections of the subbundle of 4-forms on M, Ω, that are identified with Ω_0 under suitable isomorphism of $\mathbb{R}^8$ and the tangent spaces of M. Therefore, a Spin(7)- structure Ω on M induces a natural metric g on M via the inclusion of Spin(7) into $SO(8)$, as well as a 4-form Ω such that the tangent space of M admits an isomorphism with $\mathbb{R}^8$ identifying Ω and g with Ω_0 and the Euclidean standard metric on $\mathbb{R}^8$ respectively. By abuse of notation, we shall identify the Spin(7)-structure with its associated 4-form Ω.

The 4-form Ω_0 is the Cayley calibration on the octonions $\mathbb{O}$[10]. On a Riemannian manifold M, a closed exterior p-form ϕ with the property that $\phi|_\xi \leq vol_\xi$ for all oriented tangent p-plane ξ on M is called a calibration, M with a calibration is called a calibrated manifolds. Further, any p-dimensional submanifold N of M with $\phi|_N = vol_N$ is called a ϕ-submanifold, it is homologically volume minimizing, i.e. N has the minimal volume among all the submanifolds N' homologous to N and $\partial N = \partial N'$. For example, the complex submanifolds of a Kähler manifold are homologically volume minimizing. This calibration of complex Hermitian manifold by powers of Kähler form is classically known. Harvey and Lawson[10] gave new examples of calibrated geometries such as special Lagrangian calibration (real n-form defined on a 2n-dimensional manifold with holonomy contained in $SU(n)$.) and exceptional calibrations including Cayley calibration. As Cayley calibration is a four form defined on an 8-dimensional manifold with holonomy contained in Spin(7), and the construction of compact 8-manifolds of holonomy Spin(7) is a recent result due to Joyce, one might hope that Harvey and Lawson's work on calibrated geometry would be carried further in this new testing ground of compact holonomy Spin(7) manifolds.

An oriented 4-plane N' in $\wedge^4\mathbb{O}$ is with $\Omega_0(N') = 1$ is called a Cayley 4-plane, and an oriented 4-dimensional submanifold N of $\mathbb{O} \simeq \mathbb{R}^8$ is called a

Cayley submanifold if the tangent plane to N at each point is a Cayley 4-plane. The geometry of Cayley 4-folds in $\mathbb{O}$ is most intriguing, it is invariant under the 8-dimensional representation of Spin(7) as the form Ω is fixed by Spin(7). Cayley submanifolds are clearly volume minimizing. Further, every Cayley 4-fold carries a natural 21-dimensional family of anti-self-dual SU_2 Yang-Mills fields[10].

The k-forms on $\mathbb{R}^8$, $\wedge^k(\mathbb{R}^8)^*$, split into a direct sum of orthogonal representations of Spin(7). Let $\wedge^k_l$ denote the component of l-dimensional Spin(7) irreducible representation in $\wedge^k T^*M$. By [24] Proposition 12.5 and Hodge star operation, we have

Proposition 1.2.1 *Let M be an oriented 8-manifold with* Spin(7)*-structure. Then $\wedge^k T^*M$ is split into orthogonal components of irreducible* Spin(7) *representations as follows:* $\wedge^1 T^*M = \wedge^1_8$, $\wedge^2 T^*M = \wedge^2_7 \oplus \wedge^2_{21}$, $\wedge^3 T^*M = \wedge^3_8 \oplus \wedge^3_{48}$, $\wedge^4 T^*M = \wedge^4_+ T^*M \oplus \wedge^4_- T^*M$, $\wedge^4_+ T^*M = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}$, $\wedge^4_- T^*M = \wedge^4_{35}$, $\wedge^5 T^*M = \wedge^5_8 \oplus \wedge^5_{48}$, $\wedge^6 T^*M = \wedge^6_7 \oplus \wedge^6_{21}$, $\wedge^7 T^*M = \wedge^7_8$.

Let AM be the subbundle of $\wedge^4 T^*M$ of 4-forms which are identified with the canonical 4-form Ω_0 under some isomorphism of $\mathbb{R}^8$ and T_xM. The fibre of AM is $GL(8,\mathbb{R})/\mathrm{Spin}(7)$ of dimension $64 - 21 = 43$. So AM is of codimension 27 in $\wedge^4 T^*M$. Given the splitting in the proposition above, we see that $T_\Omega AM \simeq \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{35}$. Smooth sections of AM are called admissable 4-forms, and are essentially Spin(7)-structures on M.

By [5], g has holonomy contained in Spin(7) and Ω is the associated Spin(7)-structure if and only if Ω is torsion-free, i.e. $\nabla\Omega = 0$ on M, where ∇ is the Levi-Civita connection of g. By [24] Lemma 12.4, $\nabla\Omega = 0$ if and only if $d\Omega = 0$.

The technique employed by Joyce in proving the existence of torsion-free Spin(7)-structures Ω is both analytic and topological. It starts off with a Spin(7)-structure Ω on a compact 8-manifold with small torsion, constructed in the same spirit as the Kummer construction of metrics of holonomy $SU(2)$ on $K3$ surfaces, then the Ω is deformed to a one that is torsion-free via analytic means, and the associated metric is seen to be of holonomy Spin(7) by topological considerations, namely $\hat{A}$-genus of M.

Theorem 1.2.2 [13] *Let (M, g) be a compact, connected and simply connected 8-manifold with torsion-free* Spin(7)*-structure Ω. The $\hat{A}$-genus of M is*

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-. \tag{1.2}$$

The holonomy group $Hol(g)$ is determined by $\hat{A}(M)$ as follows

(i) $Hol(g) = \mathrm{Spin}(7)$ *if and only if* $\hat{A}(M) = 1$

(ii) $Hol(g) = SU(4)$ *if and only if* $\hat{A}(M) = 2$

(iii) $Hol(g) = Sp(2)$ *if and only if* $\hat{A}(M) = 3$

(iv) $Hol(g) = SU(2) \times SU(2)$ *if and only if* $\hat{A}(M) = 4$

Furthermore, if M has holonomy $\mathrm{Spin}(7)$*, the moduli space of metrics with holonomy* $\mathrm{Spin}(7)$ *on M up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension* $1 + b^4_-(M)$*.*

We will discuss the above in more detail in the next chapter.

1.3 Kummer Construction and Eguchi-Hanson Metric

Suppose M is a compact complex n-manifold with Kähler metric g and Kähler form ω. In the case $c_1 = 0$, as a special case of the Calabi conjecture, M admits a Ricci-flat Kähler metric with holonomy in $SU(n)$. $K3$ surfaces are by definition compact, simply-connected, complex surfaces with $c_1 = 0$. By Calabi-Yau, they admit a 58-dimensional family of metrics of holonomy $SU(2)$. However, although these metrics exist, they are so far not known explicitly and exceedingly difficult to describe.

Page [18] developed a good description of some of these Calabi-Yau metrics based on the Kummer construction. Let T^4 be a 4-torus with coordinates (x_1, x_2, x_3, x_4), with $x_i \in \mathbb{R}/\mathbb{Z}$. Let -1 act on T^4 by $(x_1, x_2, x_3, x_4) \longmapsto (-x_1, -x_2, -x_3, -x_4)$. $T^4/\{\pm 1\}$ becomes a singular 4-manifold with 16 singular points, $\{(x_1, x_2, x_3, x_4) | x_i \in \{0, \frac{1}{2}\}\}$. Regard T^4 as a complex manifold, and let Y be the blow-up of $T^4/\{\pm 1\}$ at the 16 singular points. As

$$b^0(T^4) = 1, b^1(T^4) = 4, b^2_+(T^4) = 3, b^2_-(T^4) = 3, b^3(T^4) = 4, b^4(T^4) = 1,$$

$$b^0(T^4/\{\pm 1\}) = 1, b^1(T^4/\{\pm 1\}) = 0, b^2_+(T^4/\{\pm 1\}) = 3, b^2_-(T^4/\{\pm 1\}) = 3,$$
$$b^3(T^4/\{\pm 1\}) = 0, b^4(T^4/\{\pm 1\}) = 1.$$

Blowing up at each singular point of $T^4/\{\pm 1\}$ changes the Betti numbers by adding 1 to b^2_- and nothing else. Hence,

$$b^0(Y) = 1, b^2_+(Y) = 3, b^2_-(Y) = 3 + 16 = 19, b^4(Y) = 1, b^1(Y) = b^3(Y) = 0.$$

It is well-known that Y is a $K3$ surface.

The blow-ups of these singular points are modelled on the Eguchi-Hanson spaces [8]. Let $\mathbb{C}^2$ have complex coordinates (z_1, z_2), and consider the action of $-1 : (z_1, z_2) \longmapsto (-z_1, -z_2)$. Let X be the blow-up of $\mathbb{C}^2/\{\pm 1\}$, X is then biholomorphic to $T^*\mathbb{CP}^1$, $\pi_1(X) = 1$, and $H^2(X, \mathbb{R}) = \mathbb{R}$. We shall find explicitly a hyperkähler structure (i.e. a metric with holonomy in $SU(2)$) on X. A hyperkähler 4-manifold is by definition Ricci-flat and self-dual, and its metric is Kähler with respect to each of the three anti-commuting complex structures. More explicitly, we can describe a hyperkähler structure on an oriented 4-manifold M by a triple $(\omega_1, \omega_2, \omega_3)$ of smooth, closed 2-forms on M. Given an oriented orthonormal basis of $T^*_x M$ $\{dx_1, dx_2, dx_3, dx_4\}$, we have

$$\omega_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4,$$

$$\omega_2 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4,$$

$$\omega_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3.$$

Now, the closed, holomorphic 2-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ descends to $\mathbb{C}^2/\{\pm 1\}$ and lifts to X in turn. We define closed 2-forms ω_2, ω_3 on X by $\omega_2 + i\omega_3 = dz_1 \wedge dz_2$, i.e. $\omega_2 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4$, $\omega_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3$ where z_1 and z_2 are represented by real coordinates (x_1, x_2) and (x_3, x_4) respectively.

Furthermore, the function $u = |z_1|^2 + |z_2|^2$ on $\mathbb{C}^2$ descends to $\mathbb{C}^2/\{\pm 1\}$ and lifts to X in turn. Let $t \geq 0$, and define a function f_t on X by

$$f_t = \sqrt{u^2 + t^4} + t^2 \log u - t^2 \log(\sqrt{u^2 + t^4} + t^2). \tag{1.3}$$

Then f_t extends to a Kähler potential for ω_1 on X. Namely, define a 2-form ω_t on X by $\omega_t = \frac{1}{2} i\partial\bar{\partial} f_t$. For $t > 0$, ω_t is the Kähler form of a Kähler metric on X. ω_t together with ω_2 and ω_3 defined above form the triplet of smooth closed 2-forms on X determining the hyperkähler structure on X and giving the so-called Eguchi-Hanson metric with holonomy $SU(2)$. When $t = 0$, h_0 is just the pullback to X of the Euclidean metric on $\mathbb{C}^2 \backslash \{\pm 1\}$. The smaller t is the closer h_t and h are. Furthermore, the Eguchi-Hanson metric is asymptotic to the flat metric h_0 at infinity. Thus Eguchi-Hanson space is an asymptotically locally Euclidean (ALE) space.

Page proposed to glue 16 copies of the Eguchi-Hanson space described above in small neighborhoods of the singular points of $T^4/\{\pm 1\}$ to produce a metric on the $K3$ surface which is approximately hyperkähler depending on how small t is.

An ALE space is a complete Riemannian manifold with one end modelled upon the end of $\mathbb{R}^n/G$, where G is a nontrivial finite subgroup of $SO(n)$ acting freely on $\mathbb{R}^n \backslash \{0\}$. Note that n must be even if G is to act fixed point freely, as any nontrivial element of $SO(n)$ has to act with fixed points for n odd.

In fact, later results [25][16] show that the $K3$ surface does admit hyperkähler metrics using Page's construction when t is small.

Chapter 2

Existence and Construction of Compact 8-manifolds with Holonomy Spin(7)

In this chapter, we discuss D. Joyce's work generalizing the Kummer construction to prove the existence and construct compact 8-manifolds with holonomy Spin(7).

2.1 Motivation

Page's construction of hyperkähler structure on $K3$ surfaces obtained by gluing 16 Eguchi-Hanson spaces to the singular neighborhoods of $T^4/\{\pm 1\}$ is generalized by Joyce to construct compact 8-manifolds by resolving T^8/G, where G is some finite group.

Given Page's construction of hyperkähler structure on the resolution $K3$ of $T^4/\{\pm 1\}$, we can produce immediately $T^4 \times K3$ as a resolution of $T^8/\mathbb{Z}_2 = T^4 \times (T^4/\{\pm 1\})$ and $K3 \times K3$ as a resolution of $T^8/\mathbb{Z}_2^2 = (T^4/\{\pm 1\}) \times (T^4/\{\pm 1\})$. These two compact manifolds have holonomy groups $\{1\} \times SU(2)$ and $SU(2) \times SU(2)$, both are subgroups of Spin(7). So $T^4 \times K3$ and $K3 \times K3$ both have torsion-free Spin(7)-structure. Similar to the Page's method, we can approximate torsion-free Spin(7)-structures on $T^4 \times K3$ and $K3 \times K3$ by piecing together the flat Spin(7)-structure on $T^8/\mathbb{Z}_2$ and $T^8/\mathbb{Z}_2^2$ with the Eguchi-Hanson metrics.

We shall then look for more general orbifolds T^8/G whose singularities are modelled upon the singularities of the above two examples. Resolving these singularities carefully, we might obtain resolutions that have Spin(7)-structure and holonomy Spin(7).

2.2 Resolution of Certain Orbifolds

As usual, let Ω be a flat Spin(7)-structure with zero torsion on T^8, then Ω gives rise to a flat metric on T^8. Let G be a finite subgroup of isometries of T^8 preserving Ω, i.e. $G \subset \mathrm{Spin}(7)$. For $g \in G$, let S_g be the image in T^8/G of the fixed points of g in T^8. and S be the union of S_g as $g \neq e$ ranges in G in T^8/G. Let X be the Eguchi-Hanson space described in Chapter 1. Namely, there is a surjective map, $\pi : X \to \mathbb{C}^2/\{\pm 1\}$. Let B^4_ζ be the open ball of radius ζ (ζ is sufficiently small) around 0 in $\mathbb{C}^2 \simeq \mathbb{R}^4$. Then there is an open subset U of X and a map $\pi : U \to B^4_\zeta/\{\pm 1\}$ which resolves the singularity of $B^4_\zeta/\{\pm 1\}$ at 0.

In [13] section 3.1, Joyce introduced five types of nice orbifold models T with singularities which have resolutions R, $\pi : R \to T$ as follows.

(i) $T = T^4 \times (B^4_\zeta/\{\pm 1\})$, $S = T^4 \times \{0\}$, and $R = T^4 \times U$, where the map $\pi : U \to B^4_\zeta/\{\pm 1\}$ gives rise to the resolving map $\pi : R \to T$.
The resolution fixes b^1, and increases b^2 by 1, b^3 by 4, b^4_+ by 3, and b^4_- by 3. This can be easily seen using the Kunneth formula.

(ii) $T = (T^4/\{\pm 1\}) \times (B^4_\zeta/\{\pm 1\})$, $S = (T^4/\{\pm 1\}) \times \{0\}$, and $R = (T^4/\{\pm 1\}) \times U$ with obvious resolving map π.
The resolution fixes b^1 and b^3, and increases b^2 by 1, b^4_+ by 3, and b^4_- by 3.

(iii) $T = (B^4_\zeta/\{\pm 1\}) \times (B^4_\zeta/\{\pm 1\})$, $S = \{0 \times 0\}$, and $R = U \times U$ with obvious resolving map π.
The resolution fixes b^1, b^2, b^3, and b^4_-, and increases b^4_+ by 1. Here, in the resolution, one effectivly put in $S^2 \times S^2$ in place of the singular point, which is the intersection of type ii and type v singularities. Obviously b^4 increases by 1, but b^2 remains fixed, because in resolving type ii and type v singularities, we will have already accounted for the increase of b^2 there.

(iv) $T = (T^4 \times (B^4_\zeta/\{\pm 1\}))/\langle\sigma\rangle$, where σ is an isometric involution of $T^4 \times (B^4_\zeta/\{\pm 1\})$ given by for example

$$\sigma : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \longmapsto (\frac{1}{2} + x_1, x_2, -x_3, -x_4, y_1, y_2, -y_3, -y_4)$$

and $S = (T^4 \times \{0\})/\langle\sigma\rangle$, $R = (T^4 \times U)/\langle\sigma\rangle$.
Then σ acts on T^4 by $(x_1, x_2, x_3, x_4) \longmapsto (\frac{1}{2} + x_1, x_2, -x_3, -x_4)$, and σ acts on U by an isometric involution of U that projects to the action $(y_1, y_2, y_3, y_4) \longmapsto (y_1, y_2, -y_3, -y_4)$ on $B^4_\zeta/\{\pm 1\}$, with obvious resolving map π.
There are two possible actions of σ on U. Let (z_1, z_2) be the complex coordinates on B^4_ζ giving U a complex structure: the involution $(z_1, z_2) \longmapsto (z_1, -z_2)$ induces a holomorphic involution of U; the involution $(z_1, z_2) \longmapsto (\bar{z_1}, \bar{z_2})$ induces an antiholomorphic involution of U. A rational curve $\mathbb{CP}^1$ in U generates $H_2(U, \mathbb{R})$. The first involution leaves the homology class generated by $\mathbb{CP}^1$ invariant, while the second involution changes its sign. Therefore, we get two topologically distinct resolutions R of T.

The first resolution fixes b^1, and increases b^2 by 1, b^3 by 2, b^4_+ by 1 and b^4_- by 1.
The second resolution fixes b^1 and b^2, and increases b^3 by 2, b^4_+ by 2 and b^4_- by 2.

(v) $T = (T^4/\{\pm 1\} \times B^4_\zeta/\{\pm 1\})/\langle\sigma\rangle$, σ is defined as above.
$S = (T^4/\{\pm 1\} \times \{0\})/\langle\sigma\rangle$, $R = ((T^4/\{\pm 1\}) \times U)/\langle\sigma\rangle$.
The two possible actions of σ on U give rise to two topologically distinct resolutions R, π.
The first resolution fixes b^1 and b^3, and increases b^2 by 1, b^4_+ by 1 and b^4_- by 1.
The second resolution fixes b^1, b^2, and b^3 , and increases b^4_+ by 2 and b^4_- by 2.

If all the singular points of T^8/G are of one of the five types above, we can then obtain a compact, non-singular 8-manifold M by resolving the singularities accordingly. For each singularity T, we remove the subset T from T^8/G and replace it with the corresponding resolution R just as in Page's resolution with Eguchi-Hanson spaces.

We should remark that the singular points of type (iii) are not isolated, rather they are transverse intersections of two submanifolds of singularities of type (ii) or (v). In resolving T^8/G, we must be careful to combine all three types of resolution for type (ii), (iii) and (v) singularities all at once.

One should also note that the resolution $\pi : U \to B^4_\zeta/\{\pm 1\}$ does not change the fundamental group. So $\pi_1(M) \simeq \pi_1(T^8/G)$.

2.3 Existence of Torsion Free $\mathrm{Spin}(7)$ Structures

Having obtained resolutions M of T^8/G, using analytic tools, Joyce in [13] section 4 then shows first, there exists Spin(7)-structures Ω on M with small torsion, and second, these Ω can be deformed to one that is torsion-free. The analytic techniques involved in showing the existence of torsion-free Spin(7)-structure are quite lengthy and sophisticated. We shall not discuss this here.

2.4 Topology of Compact Riemannian Manifolds with Holonomy $\mathrm{Spin}(7)$

Let M be a compact 8-manifold with metric g such that $Hol(g) \subset \mathrm{Spin}(7)$. Then M is a spin manifold. Let $\Delta = \Delta_+ \oplus \Delta_-$ be the spin bundle of M. We have on Δ Dirac operators D_+ and D_-, where

$$D_+ : C^\infty(\Delta_+) \to C^\infty(\Delta_-), \text{ and } D_- : C^\infty(\Delta_-) \to C^\infty(\Delta_+).$$

By the Atiyah-Singer index theorem,

$$ind(D_+) = dimKerD_+ - dimKerD_- = \hat{A}(M).$$

In particular, when $dim(M) = 8$, $45 \times 2^7 \hat{A}(M) = 7p_1(M)^2 - 4p_2(M)$.

Since $Hol(g) \subset$Spin(7), g must be Ricci-flat, and hence has vanishing scalar curvature. Lichnerowicz's Weitzenbock formula implies that any harmonic spinor must be constant, i.e. $KerD_\pm$ contain only constant positive and negative spinors. $Hol(g)$ determines the constant spinors, and therefore $ind(D_+)$. For any subgroup G of Spin(7), $KerD_\pm$ are simply the G-invariant subbundles of $\Delta_\pm$, so $ind(D_+)$ is the dimension of the G-invariant part of Δ_+ minus the dimension of the G -invariant part of Δ_-. Also [13] formula (69) gives

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-$$

Now given Berger's list and the fact that $Hol(g) \subset$ Spin(7), we see that $Hol(g)$ is either Spin(7), $SU(4), Sp(2)$, or $SU(2) \times SU(2)$. [26] gives the dimensions of the spaces of parallel spinors for various holonomy groups. Therefore, we can distinguish the holonomy groups of g by the $\hat{A}$-genus. Namely,

$$Hol(g) = \begin{cases} Spin(7) & \text{iff } \hat{A}(M) = 1 \\ SU(4) & \text{iff } \hat{A}(M) = 2 \\ Sp(2) & \text{iff } \hat{A}(M) = 3 \\ SU(2) \times SU(2) & \text{iff } \hat{A}(M) = 4 \end{cases}$$

Further, if a compact Riemannian 8-manifold M has holonomy Spin(7), it must be Ricci-flat; by the Cheeger-Gromoll splitting theorem, M must have finite fundamental group. Let $\tilde{M}$ be the universal cover of M, and d the degree of covering. Then $\hat{A}(M) = \hat{A}(\tilde{M})/d$. As $\hat{A}(\tilde{M})$ must be 1, we see that $d = 1$. So M is simply-connected.

Chapter 3

New Examples of Compact 8-Manifolds of Holonomy Spin(7)

In [13], Joyce found a series of examples of compact 8-manifolds with holonomy group Spin(7). The basic strategy is to find finite subgroups G acting on T^8 such that the singularities are of the 5 types described in section 2.3. Joyce found altogether at least 95 topologically distinct examples of compact 8-manifolds of holonomy Spin(7). We will carry out the search a little further, to find another class of examples.

In [13], the two families of group action on T^8 are $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^5$, their action on T^8 is chosen carefully to preserve the Spin(7)-structure, i.e. the associated 4-form Ω defined in chapter 1.

Let $(x_1, \ldots, x_8)$ be coordinates on $T^8 = \mathbb{R}^8/\mathbb{Z}^8$, where $x_i \in \mathbb{R}/\mathbb{Z}$.

Let α, β, γ, δ, ϵ, η be the involutions of T^8 defined by

$$\begin{aligned}
\alpha((x_1,\ldots,x_8)) &= (-x_1,-x_2,-x_3,-x_4,x_5,x_6,x_7,x_8)\\
\beta((x_1,\ldots,x_8)) &= (x_1,x_2,x_3,x_4,-x_5,-x_6,-x_7,-x_8)\\
\gamma((x_1,\ldots,x_8)) &= (c_1-x_1,c_2-x_2,x_3,x_4,c_5-x_5,c_6-x_6,x_7,x_8)\\
\delta((x_1,\ldots,x_8)) &= (d_1-x_1,x_2,d_3-x_3,x_4,d_5-x_5,x_6,d_7-x_7,x_8)\\
\epsilon((x_1,\ldots,x_8)) &= (c_1+x_1,c_2+x_2,x_3,\frac{1}{2}+x_4,c_5+x_5,c_6+x_6,x_7,\frac{1}{2}+x_8),
\end{aligned} \tag{3.1}$$

as in [13] section 3.2, where c_i and d_i take values in $\{0, \frac{1}{2}\}$. We will see later how η is defined. In addition, in order for the singular points to be of the specified types, one had to require that

$$\begin{array}{lll} (c_1, c_2) \neq (0,0) & (c_5, c_6) \neq (0,0) & (d_1, d_3) \neq (0,0) \\ (d_5, d_7) \neq (0,0) & (c_1, c_5) \neq (d_1, d_5) & (d_3, d_7) \neq (0,0) \end{array} \tag{3.2}$$

It is clear that all these six elements preserve the 4-form Ω which defines the Spin(7)-structure. Also $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \epsilon^2 = 1$, and all these elements mutually commute. Joyce worked with the abelian groups $\langle \alpha, \beta, \gamma, \delta \rangle$ and $\langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$. These groups are certainly automorphisms of T^8 preserving the Spin(7) structure Ω.

The 95 examples Joyce found using the actions defined above have the property that $b^3 = 4, 8$, or 16, and that $2b^2 + b^4 = 150, 158$ or 174 respectively. Furthermore, they all satisfy the relations that $b^4_+ = 103 - b^2 + b^3$ and $b^4_- = 39 - b^2 + b^3$.

Looking at the example more closely, we see that the examples arising from the group $\langle \alpha, \beta, \gamma, \delta \rangle$ all have $b^3 = 16$. In these examples, $S_{\alpha\beta} = S_\alpha \cap S_\beta$ is a set of 256 singular points, which yield the only 64 singularities of type (iii), S_α and S_β yield singularities of type (ii) or (v). The example arising from the group $\langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$ have $b^3 = 4$ or $b^3 = 8$. Here, $S_{\alpha\beta}$ and $S_{\alpha\beta\epsilon}$ each yield 32 singularities of type (iii). $S_{\alpha\beta} = S_\alpha \cap S_\beta$, and $S_{\alpha\beta\epsilon} = S_\gamma \cap S_{\alpha\beta\gamma\epsilon}$. S_α, S_β, S_γ, and $S_{\alpha\beta\gamma\epsilon}$ yield singularities of type (ii) or (v). Other singular sets, e.g. S_δ yield singularities of type (i) or (iv), which contribute to b^3.

Given the above observation, we might try to add another generator η to form group $G = \mathbb{Z}_2^6$, so that the singular sets would contain $S_{\alpha\beta\eta}$ and $S_{\alpha\beta\epsilon\eta}$ in addition to $S_{\alpha\beta}$ and $S_{\alpha\beta\epsilon}$. Each of these four sets would yield 16 singularities of type (iii). It also turns out that none of the singular sets arising from $\langle \alpha, \beta, \gamma, \delta, \epsilon, \eta \rangle$ yields singularities of type (i) or (iv), hence b^3 must be 0.

To define η, we look at how $S_{\alpha\beta\eta}$ and $S_{\alpha\beta\epsilon\eta}$ could arise from singular sets which contain type (ii) or (v) singularities. There are two possibilities. Firstly, $S_{\alpha\beta\eta} = S_\beta \cap S_{\alpha\eta}$, and $S_{\alpha\beta\epsilon\eta} = S_\delta \cap S_{\alpha\beta\delta\epsilon\eta}$. Second possibility is that $S_{\alpha\beta\eta} = S_\delta \cap S_{\alpha\beta\delta\eta}$, and $S_{\alpha\beta\epsilon\eta} = S_\beta \cap S_{\alpha\epsilon\eta}$.

In the case $S_{\alpha\beta\eta} = S_\beta \cap S_{\alpha\eta}$ and $S_{\alpha\beta\epsilon\eta} = S_\delta \cap S_{\alpha\beta\delta\epsilon\eta}$, we deduce that η must be of the following form

$$\eta((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)) = (e_1 + x_1, e_2 + x_2, e_3 + x_3, e_4 + x_4, x_5, x_6, x_7, x_8) \tag{3.3}$$

where $e_i \in \{0, \frac{1}{2}\}$ for $i \in \{1, 2, 3, 4\}$. Furthermore, we need to impose further constraints in addition to the ones in Lemma 3.2.1 and Lemma 3.3.1 of [13]. In order for $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, $S_{\alpha\beta\eta}$, and $S_{\alpha\beta\epsilon\eta}$ to be singularities of type (iii) which we can resolve using the method in [13], we need $S_{\alpha\beta}$ to be fixed by α and β only, $S_{\alpha\beta\epsilon}$ to be fixed by γ and $\alpha\beta\gamma\epsilon$ only, $S_{\alpha\beta\eta}$ to be fixed by β and $\alpha\eta$ only, and $S_{\alpha\beta\epsilon\eta}$ to be fixed by δ and $\alpha\beta\delta\epsilon\eta$ only. Thus $S_{\alpha\beta}, S_{\alpha\beta\epsilon}, S_{\alpha\beta\eta}, S_{\alpha\beta\epsilon\eta}$ each contributes 16 singularities of type (iii). From these, we see that

$$d_1 = c_1 + e_1, d_3 = e_3 = \frac{1}{2}, d_5 = c_5 = \frac{1}{2}, d_7 = 0,$$

$$e_1 = \frac{1}{2}, \text{ and } (c_1 \neq e_1 \text{ or } c_2 \neq e_2).$$

In order that the only singular sets possible are S_α, S_β, S_γ, S_δ, $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, $S_{\alpha\beta\eta}$, $S_{\alpha\beta\epsilon\eta}$, $S_{\alpha\eta}$, $S_{\alpha\beta\gamma\epsilon}$, and $S_{\alpha\beta\delta\epsilon\eta}$, we must have in addition that

$$d_1 + e_1 = \frac{1}{2}, e_4 = \frac{1}{2}, c_2 + e_2 \neq 0, \text{ and } e_2 = \frac{1}{2}.$$

Hence

$$(c_1, c_2, c_5) = (\frac{1}{2}, 0, \frac{1}{2}), (d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0),$$
$$(e_1, e_2, e_3, e_4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), c_6 = 0 \text{ or } \frac{1}{2}.$$

In the case $S_{\alpha\beta\eta} = S_\delta \cap S_{\alpha\beta\delta\eta}$ and $S_{\alpha\beta\epsilon\eta} = S_\beta \cap S_{\alpha\epsilon\eta}$, we deduce that η must be of the following form

$$\eta((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)) = (d_1{+}x_1, e_2{+}x_2, d_3{+}x_3, e_4{+}x_4, c_5{+}x_5, c_6{+}x_6, x_7, \frac{1}{2}{+}x_8), \tag{3.4}$$

where $e_i \in \{0, \frac{1}{2}\}$ for $i \in \{2, 4\}$. Again, we need to impose further constraints in addition to the ones in [13] Lemma 3.2.1 and Lemma 3.3.1 as in the previous case. In order that the $S_{\alpha\beta}$, $S_{\alpha\beta\gamma}$, $S_{\alpha\beta\eta}$, and $S_{\alpha\beta\epsilon\eta}$ are singularities of the right type, we must have

$$c_5 = d_5 = \frac{1}{2}, d_7 = 0, d_3 = \frac{1}{2}, \text{ and } c_1 \neq d_1.$$

In order that the only singular sets possible are S_α, S_β, S_γ, S_δ, $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, $S_{\alpha\beta\eta}$, $S_{\alpha\beta\epsilon\eta}$, $S_{\alpha\epsilon\eta}$, $S_{\alpha\beta\gamma\epsilon}$, and $S_{\alpha\beta\delta\eta}$ and nothing else, the following must also hold:

$$e_2 = \frac{1}{2}, e_4 = 0, c_2 = 0, c_1 = \frac{1}{2}.$$

Hence

$$(c_1, c_2, c_5) = (\frac{1}{2}, 0, \frac{1}{2}), (d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0),$$
$$(e_2, e_4) = (\frac{1}{2}, 0), c_6 = 0 \text{ or } \frac{1}{2}.$$

The reason that we only allow the 11 elements specified above in G to have fixed points is that if more elements in G have fixed point set, namely, 16 T^4's, the singular sets will intersect with each other in some T^2's. But we don't know yet how to resolve this within holonomy Spin(7).

From above, we found two classes of orbifolds $T^8/\mathbb{Z}_2^6$ such that using the resolutions discussed in [13] they become compact 8-manifolds with holonomy Spin(7) .

Example 1 Consider the first possibility of η defined by equation (3.3). Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$, $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0)$, and $(e_1, e_2, e_3, e_4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

(i) The fixed point set of α F_α consists of 16 T^4's. β acts as -1 on each T^4 but fixes the set F_α, $\langle\gamma,\delta,\epsilon,\eta\rangle$ acts freely on F_α. So S_α contains 1 singularity of type (ii).

(ii) The fixed point set of β F_β consists of 16 T^4's. α fixes the set F_β, and acts as -1 on each T^4. η, $\gamma\delta$ fixes F_β, and acts as fixed point free involution σ on each T^4. $\langle\gamma,\epsilon\rangle$ acts freely on F_β. So S_β contains 4 singularities of type (v).

(iii) The fixed point set of γ F_γ consists of 16 T^4's. $\alpha\beta\epsilon$ fixes F_γ, and acts as -1 on each T^4. $\alpha\delta$ fixes F_γ, and acts as fixed point free involution σ on each T^4. $\langle\alpha,\beta,\eta\rangle$ acts freely on F_γ. So S_γ contains 2 singularities of type (v).

(iv) The fixed point set of δ F_δ consists of 16 T^4's. $\alpha\beta\epsilon\eta$ fixes F_δ, and acts as -1 on each T^4. $\beta\gamma\epsilon$ fxes F_δ, and acts as fixed point free involution σ on each T^4. $\langle\alpha,\beta,\gamma\rangle$ acts freely on F_δ. So S_δ contains 2 singularities of type (v).

(v) The fixed point set of $\alpha\beta$ $F_{\alpha\beta}$ consists of 256 singular points. α fixes $F_{\alpha\beta}$, and $\langle\gamma,\delta,\epsilon,\eta\rangle$ acts freely on each point. So $S_{\alpha\beta}$ contains 16 singularities of type (iii).

(vi) The fixed point set of $\alpha\beta\epsilon$ $F_{\alpha\beta\epsilon}$ consists of 256 singular points. γ fixes $F_{\alpha\beta\epsilon}$, and $\langle\alpha,\beta,\delta,\eta\rangle$ acts freely on each point. So $S_{\alpha\beta\epsilon}$ contains 16 singularities of type (iii).

(vii) The fixed point set of $\alpha\beta\eta$ $F_{\alpha\beta\eta}$ consists of 256 singular points. β fixes $F_{\alpha\beta\eta}$, and $\langle\alpha,\gamma,\delta,\epsilon\rangle$ acts freely on each point. So $S_{\alpha\beta\eta}$ contains 16 singularities of type (iii).

($viii$) The fixed point set of $\alpha\beta\epsilon\eta$ $F_{\alpha\beta\epsilon\eta}$ consists of 256 singular points. δ fixes $F_{\alpha\beta\epsilon\eta}$, and $\langle\alpha,\beta,\gamma,\epsilon\rangle$ acts freely on each point. So $S_{\alpha\beta\epsilon\eta}$ contains 16 singularities of type (iii).

(ix) The fixed point set of $\alpha\eta$ $F_{\alpha\eta}$ consists of 16 T^4's. β fixes $F_{\alpha\eta}$, and acts as -1 on each T^4. $\langle\alpha,\gamma,\delta,\epsilon\rangle$ acts freely on $F_{\alpha\eta}$. So $S_{\alpha\eta}$ contains 1 singularity of type (ii).

(x) The fixed point set of $\alpha\beta\gamma\epsilon$ $F_{\alpha\beta\gamma\epsilon}$ consists of 16 T^4's. γ fixes $F_{\alpha\beta\gamma\epsilon}$, and acts as -1 on each T^4. $\alpha\delta\eta$ fxes $F_{\alpha\beta\gamma\epsilon}$, and acts as fixed point free involution σ on each T^4. $\langle\alpha,\beta,\delta\rangle$ acts freely on $F_{\alpha\beta\gamma\epsilon}$. So $S_{\alpha\beta\gamma\epsilon}$ contains 2 singularities of type (v).

(xi) The fixed point set of $\alpha\beta\delta\epsilon\eta$ $F_{\alpha\beta\delta\epsilon\eta}$ consists of 16 T^4's. δ fixes $F_{\alpha\beta\delta\epsilon\eta}$, and acts as -1 on each T^4. $\alpha\gamma$ fxes $F_{\alpha\beta\delta\epsilon\eta}$, and acts as fixed point free involution σ on each T^4. $\langle\alpha,\beta,\epsilon\rangle$ acts freely on $F_{\alpha\beta\delta\epsilon\eta}$. So $S_{\alpha\beta\delta\epsilon\eta}$ contains 2 singularities of type (v).

So we have altogether 2 singularities of type (ii), 64 singularities of type (iii), and 12 singularities of type (v). Let k_1 singularities in S_β have the resolution of the first type, and $4-k_1$ singularities in S_β have the resolution of the second type; let k_2 singularities in S_γ have the resolution of the first type, and $2-k_2$ singularities in S_γ have the resolution of the second type; let k_3 singularities in

S_δ have the resolution of the first type, and $2-k_3$ singularities in S_δ have the resolution of the second type; let k_4 singularities in $S_{\alpha\beta\gamma\epsilon}$ have the resolution of the first type, and $2-k_4$ singularities in $S_{\alpha\beta\gamma\epsilon}$ have the resolution of the second type; let k_5 singularities in $S_{\alpha\beta\delta\epsilon\eta}$ have the resolution of the first type, and $2-k_5$ singularities in $S_{\alpha\beta\delta\epsilon\eta}$ have the resolution of the second type. Let $k = k_1 + k_2 + k_3 + k_4 + k_5$.

Computing Betti numbers according to the rule in section 2.2, we have

$$b^0 = 1, b^1 = 0, b^2 = 2 + k, b^3 = 0, b^4_+ = 101 - k, b^4_- = 37 - k$$

Looking at the intersection product on the cohomology of M,

$$\bigcap_{i,j} : H^i(M, \mathbb{R}) \times H^j(M, \mathbb{R}) \to H^{i+j}(M, \mathbb{R})$$

we discover that for k_2, k_3, k_4, k_5, the group of permutations on the indices generated by $(24), (35)$, and $(23)(45)$ leaves invariant the properties of $\bigcap_{i,j}$. In fact, let $\chi(x) = 1$ if $x \neq 0$ and 0 if $x = 0$, then

$$dim(Im \bigcap_{2,2}) = 2 + \sum_{i=1}^{5} \chi(k_i) + 4\chi(k_1)k_1 + \chi(k_2)\chi(k_4)k_2k_4 + \chi(k_3)\chi(k_5)k_3k_5$$

The numbers of manifolds distinguishable from $dim(Im \bigcap_{2,2})$ for $0 \leq k \leq 12$ are listed as follows:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	5	8	13	15	17	15	13	8	5	2	1

Example 2 Consider the second case when η is defined by (3.4). Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$, $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0)$, and $(e_2, e_4) = (\frac{1}{2}, 0)$. We have similar to Example 1, the following categorization of singular sets:

(i) S_α contains 1 singularity of type (ii).

(ii) S_β contains 4 singularities of type (v).

(iii) S_γ contains 2 singularities of type (v).

(iv) S_δ contains 2 singularities of type (v).

(v) $S_{\alpha\beta}$ contains 16 singularities of type (iii).

(vi) $S_{\alpha\beta\epsilon}$ contains 16 singularities of type (iii).

(vii) $S_{\alpha\beta\eta}$ contains 16 singularities of type (iii).

(viii) $S_{\alpha\beta\epsilon\eta}$ contains 16 singularities of type (iii).

(ix) $S_{\alpha\epsilon\eta}$ contains 2 singularities of type (v).

(x) $S_{\alpha\beta\gamma\epsilon}$ contains 2 singularities of type (v).

(xi) $S_{\alpha\beta\delta\eta}$ contains 2 singularities of type (v).

So we have altogether 1 singularity of type (ii), 64 singularities of type (iii), and 14 singularities of type (v). Let j_1 singularities in S_β have the resolution of the first type, and $4-j_1$ singularities in S_β have the resolution of the second type; j_2 singularities in S_γ have the resolution of the first type, and $2-j_2$ singularities in S_γ have the resolution of the second type; j_3 singularities in S_δ have the resolution of the first type, and $2-j_3$ singularities in S_δ have the resolution of the second type; j_4 singularities in $S_{\alpha\epsilon\eta}$ have the resolution of the first type, and $2-j_4$ singularities in $S_{\alpha\epsilon\eta}$ have the resolution of the second type; j_5 singularities in $S_{\alpha\beta\gamma\epsilon}$ have the resolution of the first type, and $2-j_5$ singularities in $S_{\alpha\beta\gamma\epsilon}$ have the resolution of the second type; j_6 singularities in $S_{\alpha\beta\delta\eta}$ have the resolution of the first type, and $2-j_6$ singularities in $S_{\alpha\beta\delta\eta}$ have the resolution of the second type. Let $j=j_1+j_2+j_3+j_4+j_5+j_6$.

Computing Betti numbers according to the rule in section 2.2, we have

$$b^0=1, b^1=0, b^2=1+j, b^3=0, b^4_+=102-j, b^4_-=38-j$$

Again, by looking at the intersection product on the cohomology of M, we discover that the group of permutations on the indices of j_2, j_3, j_5, and j_6 generated by $(23), (56)$, and $(25)(36)$ leaves the properties of $\bigcap_{2,2}$ invariant. In fact,

$$dim(Im\bigcap_{2,2}) = 1+\sum_{i=1}^{6}\chi(j_i)+2\chi(j_1)j_1+\chi(j_1)\chi(j_4)j_1j_4+\chi(j_2)\chi(j_5)j_2j_5+\chi(j_3)\chi(j_6)j_3j_6$$

The numbers of manifolds distinguishable from $dim(Im\bigcap_{2,2})$ for $0\le k\le 14$ are listed as follows:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	3	8	15	26	36	45	47	45	36	26	15	8	3	1

The manifolds constructed from these two examples may not all be topologically distinct. So far, we have not yet been able to find all the necessary topological invariants to distinguish them.

All the known examples thus far have Euler characteristic 144, equivalently, $b^4_+=103-b^2+b^3$, and $b^4_-=39-b^2+b^3$. This might due the particular Eguchi-Hanson construction we use in deriving the known examples. There may exist other interesting relations or equalities for the Betti numbers of compact 8-manifolds of holonomy Spin(7). We also notice that all known examples of Calabi-Yau manifolds of holonomy $SU(4)$ have Euler characteristics divisible by 24, eg. 24, 48, 72, and the Euler characteristic of the known Spin(7) manifolds is 144, divisible by 24 again.

We have exhausted all possible orbifolds of T^8 by actions of the groups $\mathbb{Z}_2^4$, $\mathbb{Z}_2^5$, and $\mathbb{Z}_2^6$, whose generators are very carefully chosen to preserve Ω and to generate good singularities of the five types which can be resolved using Eguchi-Hanson spaces. Furthermore, by looking at the 14 4-forms in Ω that need to be preserved, we can't produce more orbifolds of T^8 whose resolution would be of holonomy Spin(7) by expanding the abelian group G with more translational involution or even non-fixed-point-free involutions. We haven't checked several other possible actions which are essentially isometric to the ones already considered, our guess is that they don't generate anything new.

Chapter 4

Finite Subgroups of Spin(7) acting on $\mathbb{R}^8$

Our method of constructing compact 8-manifolds with Spin(7) structure has been resolving orbifolds using the Eguchi-Hanson space. The Eguchi-Hanson space is an example of an ALE space. One direction to generalize and look for further examples of compact 8-manifolds with holonomy Spin(7) would be to generalize Eguchi-Hanson. In other words, we want to look for some finite subgroup $G \subset SO(l)$, such that $\mathbb{R}^l/G$ has a resolution $\pi : X \to \mathbb{R}^l/G$, where X is a complete Riemannian manifold with one end asymptotic to $\mathbb{R}^l/G$. Such a space X is called an asymptotically locally Euclidean manifold. Such an ALE space has a metric g which is asymptotic to the Euclidean metric on $\mathbb{R}^l/G$. Namely, the resolution $\pi : X \to \mathbb{R}^l/G$ is continuous and surjective, smooth and injective except at the origin, and $\pi^{-1}(0)$ is connected and simply-connected, also a finite union of compact submanifolds of X. π induces a diffeomorphism from $X\backslash\pi^{-1}(0)$ to $(\mathbb{R}^l\backslash\{0\})/G$. g approximates the Euclidean metric h on $\mathbb{R}^l/G$ means that

$$\pi_*(g) - h = O(r^{-k}), \partial\pi_*(g) = O(r^{-k-1}), \partial^2\pi_*(g) = O(r^{-k-2})$$

for $k \geq 2$ an integer, r large, r being the distance from the origin in $\mathbb{R}^l/G$, and ∂ the flat connection on $\mathbb{R}^l/G$.

Suppose the singularities of an orbifold are of the type $\mathbb{R}^l/G$, let the resolution of these singularities be obtained by replacing the singular points with the ALE space X described above. The holonomy group of X must then be a subgroup of the holonomy group of the manifold obtained by resolving the orbifold in such a fashion. ALE spaces obtained from $\mathbb{R}^4/G$ and $\mathbb{R}^6/G$ have been studied and well understood, one can consult for example [15] [21][22]. Therefore, in order to further generalize the Kummer construction to obtain compact 8-manifolds with holonomy Spin(7) from orbifolds, it is a good first step to find out what the finite subgroups of Spin(7) are, which act linearly without fixed points on $\mathbb{R}^8\backslash\{0\}$.

4.1 Zassenhaus' Classification

Here, we want to find finite subgroups $G \subset \text{Spin}(7)$, s.t. G acts on $\mathbb{R}^8\backslash\{0\}$ linearly and without fixed points.

First, we want to establish the fact that G has the structure of Frobenius complement[7][19].

Definition 4.1.1 [19] Let G be a group acting transitively on a finite set A, assume that G_a, the stabilizer of $a \in A$ in G is non-trivial. Since G acts transitively on A, all the G_a's are conjugates as a ranges in A. Assume also that G_a is semi-regular, i.e. the stabilizer of a and b for $a \neq b$ in A, $G_{a,b}$ is $\{e\}$, then G is called a **Frobenius group**, and G_a is called a **Frobenius complement**.

Zassenhaus has completely determined the structure of Frobenius complements.

Theorem 4.1.2 *[19](Theorem 18.1) Let G be a Frobenius complement, and let p, q be distinct primes. Then*

(i) *G contains no subgroup of type (p,p), i.e. a direct product of two cyclic groups of order p.*

(ii) *Every subgroup of G of order pq is cyclic.*

(iii) *If $|G|$ is even, then G contains a unique element of order 2 which is therefore central.*

(iv) *If $p > 2$, then the Sylow p-subgroups of G are cyclic. If $p = 2$, the Sylow 2-subgroups of G are cyclic or quaternion.*

(v) *G has a faithful irreducible $\bar{\mathbb{Q}}$-representation ρ such that for any non-trivial element $g \in G$, $\rho(g)$ has no eigenvalue equal to 1.*

Theorem 4.1.3 Zassenhaus*[19](Theorem 18.2) If G is a solvable Frobenius complement, then G has a normal subgroup G_0, s.t. G/G_0 is isomorphic to a subgroup of S_4 and G_0 is a Z-group (i.e. all its Sylow subgroups are cyclic). Further, we can write $G_0 = \langle x, y | x^n = 1, y^m = 1, x^{-1}yx = y^r\rangle$, where $(r-1, m) = (n, m) = 1$, and $r^{n/n'} \equiv 1$ mod m, and n' = product of distinct prime factors in n.*

Theorem 4.1.4 Zassenhaus*[19](Theorem 18.6) If G is a non-solvable Frobenius complement, then G has a normal subgroup G_0 with $[G : G_0] = 1$ or 2, and $G_0 = SL(2, 5) \times M$, where M is a Z-group of order coprime to $2, 3$ and 5.*

In order to show that our finite group $G \subset \mathrm{Spin}(7)$, and more generally, $GL(8)$, with a fixed point free representation $\rho : G \to GL_n(\mathbb{C})$ is a Frobenius complement, we must pass down to a representation over a finite field, as the concept of Frobenius complement only makes sense for finite action.

First we can replace the $\mathbb{C}$ by $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, and then replace $\bar{\mathbb{Q}}$ by some number field K with $[K : \mathbb{Q}] < \infty$. Finally, we can replace K by the localization of the ring of algebraic integers of K at a prime ideal. We get a representation $\tilde{\rho} : G \to GL_n(\mathbb{F}_q)$ over a finite field of finite characteristic p. For almost all prime p, this representation induced is fixed-point free.

Let $\tilde{G} = \mathbb{F}_q^n \rtimes G$, where for $(v, g), (v', g') \in \tilde{G}$, $(v, g)(v', g') = (v + gv', gg')$.

$\tilde{G}$ acts on $\mathbb{F}_q^n$ via $(v, g)(w) = v + gw$. This is clearly a transitive action - just let $g = e$, $(v, e)(w) = v + w$. $\tilde{G}_0$, the stabilizer of $0 \in \mathbb{F}_q^n$ in $\tilde{G}$, is just G. $\tilde{G}_{0,w}$, the stablizer of 0 and $w \in \mathbb{F}_q^n$ in $\tilde{G}$, is $\{g \in G | gw = w\}$, which must be $\{e\}$, since G acts without fixed points.

Therefore, by abuse of notation our $G = \tilde{G}_0$ is a Frobenius complement.

4.2 G as a Non-Solvable Frobenius Complement

Let's first assume that G is a non-solvable Frobenius complement. So by Theorem 4.1.3, G has a normal subgroup G_0 of index 1 or 2, s.t. $G_0 = SL(2, 5) \times M$, where M is a Z-group of order coprime to $2, 3$ and 5.

Proposition 4.2.1 *[19](Lemma 12.8, Proposition 12.11) If M is a Z-group, then M is solvable. Furthermore, $M = \langle x, y | x^n = 1, y^m = 1, x^{-1}yx = y^r\rangle$, and $(r - 1, m) = (n, m) = 1$, $r^n \equiv 1 \mod m$.*

Proposition 4.2.2 *If M is a Z-group of order coprime to 30, and M has a fixed point free irreducible complex representation of dimension $d \leq 4$, then M must be cyclic.*

Proof. By the previous proposition, $M = \langle x, y | x^n = 1, y^m = 1, x^{-1}yx = y^r\rangle$, and $(r - 1, m) = (n, m) = 1$, $r^n \equiv 1 \bmod m$. Consider the normal subgroup of M, $\langle y\rangle$, generated by y. y has a d-dimensional representation inherited from M, and it can be diagonalized to

$$\begin{bmatrix} \xi^{e_1} & & & \\ & \xi^{e_2} & & \\ & & \ddots & \\ & & & \xi^{e_d} \end{bmatrix}$$

where ξ is a primitive m-th root of unity. Since the d-dimensional irreducible representation of M is fixed point free, all of the eigenvalues of y must be primitive m-th root of unity, i.e. $(e_i, m) = 1$ for all i. WLOG, let $e_1 = 1$.

Since $x^{-1}yx = y^r$, amongst $\{\xi, \xi^r, \xi^{r^2}, \cdots, \xi^{r^d}\}$, at least two are equal. So $r^i \equiv 1$ mod m for $i \leq d$.

Since n divides the order of M, and the order of M is coprime to 30, so 2 and 3 can not divide n. For $j \in \{1, 2, \cdots, d\}$, where $d \leq 4$, $(j, n) = 1$. In particular, $\exists \lambda, \mu$ s.t. $1 = \lambda i + \mu n$. Hence $r \equiv r^{\lambda i + \mu n} = (r^i)^\lambda (r^n)^\mu \equiv 1$ mod m. As $r^i \equiv 1$ mod m, and $r^n \equiv 1$ mod m, we have $r \equiv 1$ mod m. So, M is abelian. But $(n, m) = 1$, so M must be cyclic. ∎

Since M is cyclic, the Frobenius-Schur indicator of any irreducible representation χ of M of order > 2, $S_\chi(M) = \frac{1}{|M|} \sum_{g \in M} \chi(g^2) = 0$, as χ must be 1-dimensional. So, M has no real representations, and its irreducible characters can't be defined over $\mathbb{R}$ by the Frobenius-Schur criterion, namely

$$S_\rho = \begin{cases} 1 & \text{the character of } \rho \text{ is defined over } \mathbb{R} \text{ and } \rho \text{ can be realized in } \mathbb{R} \\ -1 & \text{the character of } \rho \text{ is defined over } \mathbb{R} \text{ and } \rho \text{ can't be realized in } \mathbb{R} \\ 0 & \text{the character of } \rho \text{ can't be defined over } \mathbb{R} \end{cases}$$

Now we look at the group $SL(2,5)$. We observe that $A_5 \simeq PSL(2,5)$. From the character table of a group, we can tell whether each irreducible representation can be defined over reals from the Frobenius-Schur indicator.

number	1	1	30	20	20	12	12	12	12
order	1	2	4	3	6	5	5	10	10
	1	1	1	1	1	1	1	1	1
	3	3	-1	0	0	$-\alpha$	$-\beta$	$-\alpha$	$-\beta$
	3	3	-1	0	0	$-\beta$	$-\alpha$	$-\beta$	$-\alpha$
	4	4	0	1	1	-1	-1	-1	-1
	5	5	1	-1	-1	0	0	0	0
ϕ_1	2	-2	0	-1	1	α	β	$-\alpha$	$-\beta$
ϕ_2	2	-2	0	-1	1	β	α	$-\beta$	$-\alpha$
	4	-4	0	1	-1	-1	-1	1	1
	6	-6	0	0	0	1	1	-1	-1

Character Table of $SL(2,5)$

where $\alpha = e^{2\pi\sqrt{-1}/5} + e^{-2\pi\sqrt{-1}/5}$, $\beta = e^{4\pi\sqrt{-1}/5} + e^{-4\pi\sqrt{-1}/5}$.

We can also tell from the character table whether an irreducible representation is fixed point free or not. For example, let e_j be the number of eigenvalues of an element of order 3 which equal $e^{2\pi\sqrt{-1}j/3}$ for $j = 0, 1, 2$ in a chosen irreducible representation. Then $\sum_{j=0}^{2} e_j$ is the dimension of the irreducible representation. The cyclotomic field $\mathbb{Q}[e^{2\pi\sqrt{-1}/3}]$ has degree 2 over $\mathbb{Q}$. So the trace (i.e. the sum of conjugate elements) of $e^{2\pi\sqrt{-1}j/3}$ is -1 if $j \neq 0$ and 2 if $j = 0$. Hence $2e_0 - (e_1 + e_2) = Trace$ (character of the element in the chosen conjugacy class). This irreducible representation is fixed point free if and only if $e_0 = 0$.

From the character table of $SL(2,5)$, we see that the only irreducible representations of $SL(2,5)$ that is fixed point free are the 2 2-dimensional complex irreducible representations ϕ_1 and ϕ_2 which have characters over $\mathbb{R}$ but can't themselves be defined over reals.

As ϕ_1 and ϕ_2 can't be defined over $\mathbb{R}$, by taking $2\phi_1$, $2\phi_2$, or $\phi_1 + \phi_2$, we see $SL(2,5)$ has fixed point free embedding in $SU(4,\mathbb{C}) \subseteq SO(8,\mathbb{R})$.

Given that our group G is a non-solvable Frobenius complement, G has a normal subgroup G_0 of index 1 or 2, s.t. $G_0 = SL(2,5) \times M$, where M is a cyclic group of order, say, m. We know that $SL(2,5)$ has 2 2- dimensional fixed point free irreducible complex representations ϕ_1 and ϕ_2 which are only defined over $\mathbb{C}$. Given an irreducible representation χ_i of $M = C_m$, we can construct $\rho = \phi_{i_1} \otimes \chi_1 \oplus \phi_{i_2} \otimes \chi_2$ as a representation of G_0 in $U(4)$, where $i_1, i_2 = 1$ or 2. By making $\chi_2 = \chi_1^{-1}$, ρ becomes a representation of G_0 in $SU(4)$, as we see from the character table that the representations ϕ_i all have determinant 1. Since $SU(4) \subset Spin(7) \subset SO(8,\mathbb{R})$, $U(4) \cap Spin(7) = SU(4)$, $\rho = \phi_{i_1} \otimes \chi \oplus \phi_{i_2} \otimes \chi^{-1}$ for $i_1, i_2 = 1$ or 2 gives an embedding of G_0 in Spin(7), here χ is an irreducible representation of $M = C_m$. Using Frobenius-Schur, we see that $\phi_i \otimes \chi$ can only be defined over $\mathbb{C}$.

As $G_0 = SL(2,5) \times C_m$ is a normal subgroup of G of index 1 or 2. We see that G must be generated by G_0 and an involution $\tau = \tau_1 \times \tau_2$, i.e. $G = \{(h,\alpha,\beta) \in H \times D_{m_1} \times C_{m_2} | sgn(h) = sgn(\alpha)\}$ for $[G : G_0] = 2$, $m = m_1 m_2$. Here, $H = \tilde{S_5} = \langle SL(2,5), \begin{bmatrix} 0 & 3\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix} \rangle$. The involution τ_1 acts on $SL(2,5)$ by conjugation of $SL(2,5)$ by $\begin{bmatrix} 0 & 3\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$, τ_2 acts on C_{m_1} as an involution and on C_{m_2} as an identity, such that $D_{m_1} = \langle x^2 = y^{m_1} = 1 | xyx^{-1} = y^{-1} \rangle$. $\tau^2 = (-1,1,1)$. We have on $H \times D_{m_1}$ a canonical sign function such that $sgn(H/SL(2,5)) \simeq \{\pm 1\}$, $sgn(D_{m_1}/C_{m_1}) \simeq \{\pm 1\}$.

Fixed point free representations of G in $SU(4)$ arise as induced representations from G_0. It is easy to see that since the only fixed point free representations of G_0 in $SU(4)$ are $\phi_i \otimes \chi$ for $i = 1, 2$, which are 2 complex dimension, for the representation of G induced from ϕ_i to be in $SU(4)$, m_2 has to be 1, i.e. $m_1 = m$.

From the character table of $\tilde{S_5}$ below, we see that it only has one fixed point free irreducible representation ϕ which restricts to ϕ_1 or ϕ_2, representation of $SL(2,5)$.

number	1	1	30	20	20	24	24	20	30	30	20	20
order	1	2	4	3	6	5	10	4	8	8	6	6
	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
	6	6	-2	0	0	1	1	0	0	0	0	0
	4	4	0	1	1	-1	-1	2	0	0	-1	-1
	4	4	0	1	1	-1	-1	-2	0	0	1	1
	5	5	1	-1	-1	0	0	1	-1	-1	1	1
	5	5	1	-1	-1	0	0	-1	1	1	-1	-1
ϕ	4	-4	0	-2	2	-1	1	0	0	0	0	0
	4	-4	0	1	-1	-1	1	0	0	0	$\sqrt{-3}$	$-\sqrt{-3}$
	4	-4	0	1	-1	-1	1	0	0	0	$-\sqrt{-3}$	$\sqrt{-3}$
	6	-6	0	0	0	1	-1	0	$\sqrt{-2}$	$-\sqrt{-2}$	0	0
	6	-6	0	0	0	1	-1	0	$-\sqrt{-2}$	$\sqrt{-2}$	0	0

The induced representation $Ind_{G_0}^{G}(\phi_i \otimes \chi)$ is irreducible since $\tau_1(\phi_i) \neq \phi_i$, and $\tau_2(\chi) \neq \chi$. WLOG, let $\rho = \phi_1 \otimes \chi$ be the representation of G_0, The representation $\tilde{\rho}$ of G is as follows:

For $g \in G_0$, $\tilde{\rho}(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho(\tau(g)) \end{bmatrix} = \begin{bmatrix} \phi_1 \otimes \chi(g) & 0 \\ 0 & \phi_2 \otimes \chi^{-1}(g) \end{bmatrix}$.

For $g = \begin{bmatrix} 0 & 3\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix} \times x \in G \backslash G_0$, $\tilde{\rho}(g) = \begin{bmatrix} 0 & 1 \\ \rho(-1 \times 1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

In summary, we have

Theorem 4.2.3 *If G is a non-solvable Frobenius complement that has a representation $\rho : G \to Spin(7) \to SO(8)$ which is fixed point free, then ρ factors through $SU(4) \subset \mathrm{Spin}(7)$, and ρ as a representation of G is one of the following:*

(i) $G = SL(2,5) \times C_m$ *where* $(m, 30) = 1$. $\rho = \phi_i \otimes \chi \oplus \phi_j \otimes \chi^{-1}$ *for* $i, j = 1, 2$. ϕ_1 *and* ϕ_2 *are the two 2-dimensional irreducible complex representations of* $SL(2,5)$, χ *is a faithful character of* C_m.

(ii) $G = \{(h, \alpha) \in H \times D_m | sgn(h) = sgn(\alpha)\}$, $(m, 30) = 1$. $G_0 = SL(2,5) \times C_m$. $\rho = Ind_{G_0}^{G}(\phi_1 \otimes \chi)$, ϕ_1, χ *are as in the above case.* $H = \langle SL(2,5), \begin{bmatrix} 0 & 3\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix} \rangle \simeq \tilde{S_5}$.

Note that $\tilde{S_5}$ is a special case of the second possibility for G.

4.3 G as a Solvable Frobenius Complement

Now let's assume that G is a solvable Frobenius complement. By Theorem 4.1.3, G has a normal subgroup G_0 such that G/G_0 is isomorphic to a subgroup of S_4 and G_0 is a Z-group, $G_0 = \langle x, y | x^n = 1, y^m = 1, x^{-1}yx = y^r \rangle$, where $(r-1, m) = (n, m) = 1$, and $r^{n/n'} \equiv 1$ mod m, and n' = product of distinct prime factors in n.

If $2, 3, 5, 7$ do not divide $\frac{n}{n'}$, and G_0 has an 8-dimensional real representation, then a similar argument to the proof for Proposition 4.2.2 implies that G_0 must be cyclic.

G_0 is not cyclic if r is not 1 mod m in the presentation of G_0. Suppose $r^2 \equiv 1$ mod m, and $r \neq 1$ mod m. Then by presentation of G_0, $2|\frac{n}{n'}$, so $4|n$, $(2, m) = 1$, i.e. m is odd. $x^{-1}yx = y^r$. In this case, we find one class of G_0 which can have an 4-dimensional fixed point free complex representation which can't be realized over the reals. Now suppose $r^4 \equiv 1(m)$, and 4 is the least positive integer with this property. Then we find one class of G_0 which can have 4-dimensional fixed point free complex representation which can't be realized over the reals. In the case $r^8 \equiv 1(m)$, and 8 is the least positive integer with this property, we have that G_0 has an 8-dimensional complex fixed point free irreducible representation which can't be realized over reals, so G_0 can't be represented in $SO(8)$. If $r^k \equiv 1$ mod m for $k \neq 2, 4, 8$, it can be shown easily that G_0 can't have fixed point free representation in $SU(4)$. Therefore G_0 is one of the following:

(i) $G_0 = \langle x, y | x^n = y^m = 1, x^{-1}yx = y^r \rangle$ where $(n, m) = (r-1, m) = 1$, $r^2 \equiv 1(m)$ implies $r = m - 1$. $2|\frac{n}{n'}$, so $4|n$. G_0 has a 2-dimensional fixed point free complex irreducible representation which can't be realized over the reals, as follows.

$$y = \begin{bmatrix} \xi & 0 \\ 0 & \xi^r \end{bmatrix}, x = \begin{bmatrix} 0 & \omega \\ 1 & 0 \end{bmatrix},$$

where ξ is a primitive m-th root of 1 and ω is a primitive $\frac{n}{2}$-th root of 1. Since we want the direct sum of two copies of this representation to be in $SU(4)$, so $det(x) = -\omega = \pm 1$, $4|n$, so $\omega = -1$ and $n = 4$. Using the Frobenius-Schur indicator, we find that this representation can't be realized over reals.

(ii) $G_0 = \langle x, y | x^n = y^m = 1, x^{-1}yx = y^r \rangle$ where $(n, m) = (r-1, m) = 1$, and $r^4 \equiv 1(m)$. $4|\frac{n}{n'}$, so $8|n$. G_0 has a 4-dimensional fixed point free complex irreducible representation which can't be realized over the reals, as follows.

$$y = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^r & 0 & 0 \\ 0 & 0 & \xi^{r^2} & 0 \\ 0 & 0 & 0 & \xi^{r^3} \end{bmatrix}, x = \begin{bmatrix} 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where ξ is a primitive m-th root of 1 and ω is a primitive $\frac{n}{4}$-th root of 1. For this representation to be in $SU(4)$, $det(x) = -\omega = 1$, so $\omega = -1$ and $n = 8$. Using the Frobenius-Schur indicator, we find that this representation can't be realized over the reals.

These are the only possibilities for G_0 with the property that G_0 has a fixed point free real 8-dimensional irreducible representation.

As G_0 is a normal subgroup of G such that G/G_0 is isomorphic to a subgroup of S_4, to obtain G, it's helpful to look at $Aut(G_0)$.

For the first possibility, where $G_0 = \langle x, y | x^4 = y^m = 1, x^{-1}yx = y^r \rangle$, and $(4, m) = (r-1, m) = 1$, $r^2 \equiv 1(m)$ and $r = m-1$. For $\sigma \in Aut(G_0)$, suppose that $\sigma(x) = x^\alpha y^\beta$, and $\sigma(y) = y^\gamma$, where $\alpha \in (\mathbb{Z}/4\mathbb{Z})^*$, $\beta \in (\mathbb{Z}/m\mathbb{Z})$, and $\gamma \in (\mathbb{Z}/m\mathbb{Z})^*$, then $Aut(G_0) = (\mathbb{Z}/4\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/m\mathbb{Z})^*$, where $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha\alpha', \beta + \beta'\gamma, \gamma\gamma')$. $Inn(G_0)$, the group of inner automorphisms of G_0 generated by conjugation by elements of G_0 forms the subgroup of $Aut(G_0)$ generated by $(1, 2b, r^a)$ corresponding to the action of conjugation by $x^a y^b$. So $Out(G_0) = \{1, 3\} \times \{0\} \rtimes ((\mathbb{Z}/m\mathbb{Z})^*/\{1, r\})$.

For the case $G_0 = \langle x, y | x^8 = y^m = 1, x^{-1}yx = y^r \rangle$ where $(8, m) = (r-1, m) = 1$, $r^4 \equiv 1(m)$. G_0 has a 4-dimensional irreducible fixed point free complex representation. We see that $Aut(G_0) = ((\mathbb{Z}/8\mathbb{Z})^* \bigcap \{4k+1 | k \geq 0\}) \times (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/m\mathbb{Z})^*$, where $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha\alpha', \beta + \beta'\gamma, \gamma\gamma')$, $Inn(G_0)$ is generated by $(1, b(1-r), r^a)$ corresponding to the conjugation by $x^a y^b$, so $Out(G_0) = \{1, 5\} \times \{0\} \rtimes ((\mathbb{Z}/m\mathbb{Z})^*/\{1, r, r^2, r^3\})$.

The possible subgroups of S_4 are $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, Klein 4-group K, S_3, D_8, A_4, and S_4.

If G_0 is cyclic of order m, then $Aut(G_0) = (\mathbb{Z}/m\mathbb{Z})^*$, so $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and the Klein 4-group K can embed in $Aut(G_0)$ as a subgroup. If G_0 is one of the two possibilities described above generated by two elements, i.e. $Aut(G_0) = (\mathbb{Z}/4\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/m\mathbb{Z})^*$, or $Aut(G_0) = ((\mathbb{Z}/8\mathbb{Z})^* \cap \{4k + 1 | k \geq 0\}) \times (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/m\mathbb{Z})^*$. Again, it turns out that $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, Klein 4-group are the only groups that can embed in $Aut(G_0)$.

Now, since $G/G_0 \simeq \{e\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ or K, to obtain G, we just need to induce the irreducible representation from G_0 by elements of $Out(G_0)$. We obtain the following possible G for $[G : G_0] \neq 1$ such that G has a fixed point free representation in $\mathbb{R}^8$:

(i) $G_0 \simeq C_m$, $G/G_0 \simeq \mathbb{Z}_2$, then $G = D_{m_1} \times C_{m_2}$ where $m_1 m_2 = m$. Since G is a Frobenius complement, it has no subgroup of type (p, p), so $(m_1, m_2) \neq 1$ and $(m_2, 2) = 1$.

(ii) $G_0 \simeq C_m$, $G/G_0 \simeq \mathbb{Z}_2$, for the largest k such that $2^k | m$, if $k \geq 2$, then $G = Q_{k+1} \times C_{m'}$, where $m' = \frac{m}{2^k}$, Q_{k+1} is the generalized quaternion group of order 2^{k+1}.

(iii) $G_0 \simeq C_m$, $G/G_0 \simeq K$, then $G = D_{m_1} \times D_{m_2} \times C_{m_3}$, where $m_1 m_2 m_3 = m$. Since G contains a subgroup of type (p, p), it's not a Frobenius complement

(a Frobenius complement has no subgroup of type (p,p)). So this G is excluded.

(iv) $G_0 \simeq C_m$, $G/G_0 \simeq K$, for the largest k such that $2^k|m$, if $k \geq 4$, let $k = k_1 + k_2$ such that $k_1, k_2 \geq 2$, then $G = Q_{k_1+1} \times Q_{k_2+1} \times C_{m'}$, where $m' = \frac{m}{2^k}$. As G contains a subgroup of type $(2,2)$, it's not a Frobenius complement, so it's excluded.

(v) $G_0 \simeq C_m$, $G/G_0 \simeq K$, for the largest k such that $2^k|m$, if $k \geq 2$, let $k = k_1 + k_2$ such that $k_1 \geq 2$, then $G = Q_{k_1+1} \times D_{k_2} \times C_{m'}$, where $m' = \frac{m}{2^k}$. Again, G contains a subgroup of type $(2,2)$, so it's excluded here as well.

(vi) $G_0 = \langle x, y | x^4 = y^m = 1, x^{-1}yx = y^r \rangle$ where $(4,m) = (r-1,m) = 1$, $r^2 \equiv 1(m)$ implies $r = m-1$. G_0 has a 2-dimensional fixed point free complex irreducible representation ρ which can't be realized over the reals as follows.

$$\rho(y) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^r \end{bmatrix}, \rho(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where ξ is a primitive m-th root of 1.

Given $\sigma = (\alpha, \beta, \gamma) \in Out(G_0) = (\mathbb{Z}/4\mathbb{Z})^* \times \{0\} \rtimes ((\mathbb{Z}/m\mathbb{Z})^*/\{1, r\})$, we first note that $\sigma^2 = (\alpha^2, \beta(1+\gamma), \gamma^2)$ must belong to $Inn(G_0) = \{1\} \times (\mathbb{Z}/m\mathbb{Z}) \rtimes \{1, r = m-1\})$. So $\alpha^2 \equiv 1(4)$ and $\gamma^2 \equiv \pm 1$ mod m.

Let $G = \langle G_0, \sigma \rangle$, G has induced representation $\tilde{\rho}$ s.t.

$$\tilde{\rho}(x) = \begin{bmatrix} \rho(x) & 0 \\ 0 & \rho(\sigma(x)) \end{bmatrix} = \begin{bmatrix} \rho(x) & 0 \\ 0 & \rho(x^\alpha) \end{bmatrix}$$

$$\tilde{\rho}(y) = \begin{bmatrix} \rho(y) & 0 \\ 0 & \rho(\sigma(y)) \end{bmatrix} = \begin{bmatrix} \rho(y) & 0 \\ 0 & \rho(y^\gamma) \end{bmatrix}$$

$$\tilde{\rho}(\sigma) = \begin{bmatrix} 0 & \rho(\sigma^2) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \rho(x^\delta) \\ 1 & 0 \end{bmatrix}$$

where δ is even (resp. odd) if $\gamma^2 = 1(resp. -1)(m)$.

It turns out that the problem of finding finite subgroups of $\mathbb{R}^l$ which act without fixed points has investigated in [27](part III) with regard to the problem of spherical space form. This is communicated to us by the thesis examiners. The classification of fixed point free groups in [27] is more detailed and general than what is presented here, but less explicit as a trade-off of generality. The techniques Wolf used also involve much group and represetation theoretic results. We have checked that Wolf's results indeed coincide with ours. The two classes of non-solvable fixed point free finite subgroups of Spin(7) and their representations correspond to Wolf's type V and VI groups and their representations. The presentation and representation of non-cyclic fixed point free finite solvable subgroups of Spin(7) presented in this section correspond to those of Wolf's type I and II groups. Wolf has in addition type III and IV groups which are of odd order, they don't appear in our classification of fixed point free finite subgroups with possible 8-dimensional real representation, because these groups must have even order (as one can see from the groups analysis earlier in the chapter)!

Chapter 5

Finite Subgroups of Spin(7) acting on T^8

In [13], Spin(7) manifolds are constructed by resolving T^8/G for various finite subgroups G of Spin(7). It is a natural to determine all finite groups of Spin(7) which act on T^8 and such that the resolution of the orbifold would be a manifold of holonomy Spin(7). The classification of such groups is still quite daunting at this stage; furthermore, even if we were able to obtain an exhaustive list of subgroups of Spin(7) which preserve the lattice defining T^8, there still remain the greater problem of resolving the quotient singularities. Here, we can only scratch the surface of these problems.

5.1 Analysis on Elements of Spin(7) fixing T^8

For $g \in G \subset \mathrm{Spin}(7)$ where g fixes the lattice determining the torus, we see that some conjugate of g by elements of $GL(8,\mathbb{Q})$ must be an element of $SL(8,\mathbb{Z})$. Thus the characteristic polynomial of g must have integral coefficients and $det(g) = 1$. Suppose the order of g is n, then for g's characteristic polynomial $g(t)$ to be in $\mathbb{Z}[t]$ with $deg(g) = 8$ and $det(g) = g(0) = 1$, the minimal polynomial (i.e. the integral polynomial with g as a root of minimal degree) must divide $g(t)$. The degree of minimal polynomial of g is at least $\phi(n)$, the Euler ϕ function which counts the number of integers mod n which are coprime to n. Since the minimal polynomial divides $g(t)$ of degree 8, $\phi(n) \leq 8$. Calculating $\phi(n)$ for integers n, we see that the order of g must lie in the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 24\}$.

Let M be the diagonal matrix with entries $\omega_1, \cdots, \omega_{\phi(n)/2}$ on the diagonal, where ω_i is a primitive n-th root of 1 for all i, such that $M \in U(\phi(n)/2)$ if $\phi(n) < 8$ and $M \in SU(4)$ if $\phi(n) = 8$. Let g be an element in $SL(\phi(n),\mathbb{Z})$ conjugate to M (more precisely, $M \oplus \bar{M}$), then, the eigenvalues of g (i.e. $\omega_i, \bar{\omega}_i$

for all i) are all the roots of the minimal polynomial of the primitive n-th root of 1.

Consider the $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in \mathbb{Z}[t]$, using elementary linear algebra, we see that the characteristic polynomial for the matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

is simply $f(t)$.

Therefore, for M defined above to be conjugate to some element in $SL(\phi(n), \mathbb{Z})$, we see that either ω or ω^{-1} for each ω a primitive n-th root of 1 must appear as one of the ω_i's on the diagonal of M. This significantly reduces the number of possible representations of the cyclic group generated by $M \in U(\phi(n)/2)$ for $\phi(n) < 8$ (respectively $M \in SU(4)$ for $\phi(n) = 8$) which is conjugate to some element of $SL(\phi(n), \mathbb{Z})$ fixing the torus lattice.

In particular, for each $n \in S$, let ω be a primitive n-th root of one,

$$M = \begin{bmatrix} \omega^{e_1} & 0 & 0 & 0 \\ 0 & \omega^{e_2} & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \omega^{e_{\frac{\phi(n)}{2}}} \end{bmatrix},$$

the above discussions reveal that $(e_1, e_2, \cdots, e_{\phi(n)/2})$ must be the following for different n's.

(*i*) $n = 2$, $(e_1) = (1)$.
(*ii*) $n = 3$, $(e_1) = (1)$ or (2).
(*iii*) $n = 4$, $(e_1) = (1)$ or (3).
(*iv*) $n = 5$, $(e_1, e_2) = (1, 2)$, $(3, 4)$, $(1, 3)$, or $(2, 4)$.
(*v*) $n = 6$, $(e_1) = (1)$ or (5).
(*vi*) $n = 7$, $(e_1, e_2, e_3) = (1, 2, 4)$,$(3, 5, 6)$, $(1, 2, 3)$, $(4, 5, 6)$, $(1, 3, 5)$, $(2, 4, 6)$, $(2, 3, 6)$ or $(1, 4, 5)$.
(*vii*) $n = 8$, $(e_1, e_2) = (1, 3)$, $(1, 5)$, $(3, 7)$ or $(5, 7)$.
(*viii*) $n = 9$, $(e_1, e_2, e_3) = (1, 2, 4)$, $(5, 7, 8)$, $(1, 2, 5)$, $(4, 7, 8)$, $(1, 4, 7)$, $(2, 5, 8)$, $(2, 4, 8)$ or $(1, 5, 7)$.
(*ix*) $n = 10$, $(e_1, e_2) = (1, 3)$, $(1, 7)$, $(3, 9)$ or $(7, 9)$.
(*x*) $n = 12$, $(e_1, e_2) = (1, 5)$, $(1, 7)$, $(5, 11)$ or $(7, 11)$.
(*xi*) $n = 14$, $(e_1, e_2, e_3) = (1, 3, 5)$, $(9, 11, 13)$, $(1, 3, 9)$, $(5, 11, 13)$, $(1, 5, 11)$, $(3, 9, 13)$, $(3, 5, 13)$ or $(1, 9, 11)$.

(xii) $n = 15$, $(e_1, e_2, e_3, e_4) = (1, 2, 4, 8)$ or $(7, 11, 13, 14)$.

(xiii) $n = 16$, $(e_1, e_2, e_3, e_4) = (1, 3, 5, 7)$ or $(9, 11, 13, 15)$.

(xiv) $n = 18$, $(e_1, e_2, e_3) = (1, 5, 7)$, $(11, 13, 17)$, $(1, 5, 11)$, $(7, 13, 17)$, $(1, 7, 13)$, $(5, 11, 17)$, $(5, 7, 17)$ or $(1, 11, 13)$.

(xv) $n = 20$, $(e_1, e_2, e_3, e_4) = (1, 3, 7, 9)$ or $(11, 13, 17, 19)$.

(xvi) $n = 24$, $(e_1, e_2, e_3, e_4) = (1, 5, 7, 11)$ or $(13, 17, 19, 23)$.

We only require $M \in U(\phi(n)/2)$ rather than $SU(\phi(n)/2)$ for $\phi(n) < 8$ because taking an appropriate direct sum of those M's we can get a subgroup of $SU(4)$ to act on $\mathbb{C}^4$ as we wished. In what follows, by using the M's described above as building blocks, we give an exhaustive list of finite cyclic subgroups G of $SU(4)$ which have diagonal action on $\mathbb{C}^4$ such that it fixes the lattice defining the torus T^8.

(i) $G = Z_2$, the diagonal G action is generated by $M = (-1, -1, -1, -1)$ or $(-1, -1, 1, 1)$.

(ii) $G = Z_3$, the diagonal G action is generated by $M = (\omega, \omega^2, \omega, \omega^2)$, $(\omega, \omega, \omega, 1)$, $(\omega, \omega^2, 1, 1)$, or $(\omega^2, \omega^2, \omega^2, 1)$, where $\omega = 1^{\frac{1}{3}}$.

(iii) $G = Z_4$, the diagonal G action is generated by $M = (\omega, \omega^3, \omega, \omega^3)$, $(\omega, \omega, -1, 1)$, $(\omega, \omega^3, 1, 1)$, $(\omega, \omega, \omega, \omega)$, $(\omega, \omega^3, -1, -1)$, $(\omega^3, \omega^3, -1, 1)$, or $(\omega^3, \omega^3, \omega^3, \omega^3)$, where $\omega = 1^{\frac{1}{4}}$.

(iv) $G = Z_5$, the diagonal G action is generated by $M = (\omega, \omega^2, \omega^3, \omega^4)$, where $\omega = 1^{\frac{1}{5}}$.

(v) $G = Z_6$, the diagonal G action is generated by $M = (\omega, \omega^5, \omega, \omega^5)$, $(\omega, \omega, \nu^2, 1)$, $(\omega^5, \omega^5, \nu, 1)$, $(\omega^5, \omega^5, \nu^2, \nu^2)$, $(\omega, \omega, \nu, \nu)$, $(\omega, \omega^5, -1, -1)$, $(\omega, \omega, \omega, -1)$, $(\omega^5, \omega^5, \omega^5, -1)$, $(\omega, \omega^5, 1, 1)$, $(\omega, \omega^5, \nu, \nu^2)$, $(\omega, \nu, -1, 1)$, $(\omega^5, \nu^2, -1, 1)$, or $(\nu, \nu^2, -1, -1)$, where $\omega = 1^{\frac{1}{6}}$ and $\nu = 1^{\frac{1}{3}}$.

(vi) $G = Z_7$, the diagonal G action is generated by $M = (\omega, \omega^2, \omega^4, 1)$ or $(\omega^3, \omega^5, \omega^6, 1)$, where $\omega = 1^{\frac{1}{7}}$.

(vii) $G = Z_8$, the diagonal G action is generated by $M = (\omega, \omega^3, \omega^5, \omega^7)$, $(\omega, \omega^3, \omega, \omega^3)$, $(\omega^5, \omega^7, \omega^5, \omega^7)$, $(\omega, \omega^3, \delta, \delta)$, $(\omega^5, \omega^7, \delta^3, \delta^3)$, $(\omega, \omega^3, -1, 1)$, $(\omega^5, \omega^7, -1, 1)$, $(\omega, \omega^3, \delta^3, \delta^3)$, $(\omega^5, \omega^7, \delta, \delta)$, $(\omega, \omega^5, \delta^3, -1)$, $(\omega^3, \omega^7, \delta, -1)$, $(\omega, \omega^5, \delta, 1)$, or $(\omega^3, \omega^7, \delta^3, 1)$, where $\omega = 1^{\frac{1}{8}}$ and $\delta = 1^{\frac{1}{4}}$.

(viii) $G = Z_9$, the diagonal G action is generated by $M = (\omega^1, \omega^4, \omega^7, \delta^2)$ or $(\omega^2, \omega^5, \omega^8, \delta)$, where $\omega = 1^{\frac{1}{9}}$ and $\delta = 1^{\frac{1}{3}}$.

(ix) $G = Z_{10}$, the diagonal G action is generated by $M = (\omega, \omega^3, \omega^7, \omega^9)$, $(\omega, \omega^3, \delta, \delta^2)$, $(\omega^7, \omega^9, \delta^3, \delta^4)$, $(\omega, \omega^7, \delta^2, \delta^4)$, or $(\omega^3, \omega^9, \delta, \delta^3)$, where $\omega = 1^{\frac{1}{10}}$, and $\delta = 1^{\frac{1}{5}}$.

(x) $G = Z_{12}$, the diagonal G action is generated by $M = (\omega, \omega^5, \omega^7, \omega^{11})$, $(\omega, \omega^5, \omega, \omega^5)$, $(\omega^7, \omega^{11}, \omega^7, \omega^{11})$, $(\omega, \omega^5, -1, 1)$, $(\omega^7, \omega^{11}, -1, 1)$, $(\omega, \omega^5, \delta, \delta)$, $(\omega^7, \omega^{11}, \delta^3, \delta^3)$, $(\omega, \omega^5, \nu, \eta)$, $(\omega^7, \omega^{11}, \nu^2, \eta^5)$, $(\omega, \omega^7, \nu, 1)$, $(\omega^5, \omega^{11}, \nu^2, 1)$,

$(\omega, \omega^7, \nu^2, \nu^2)$, $(\omega^5, \omega^{11}, \nu, \nu)$, $(\omega, \omega^7, \eta, \eta)$, $(\omega^5, \omega^{11}, \eta^5, \eta^5)$, $(\eta, \eta^5, \delta, \delta^3)$, or $(\delta, \delta^3, \nu, \nu^2)$, where $\omega = 1^{\frac{1}{12}}$, $\delta = 1^{\frac{1}{4}}$, $\nu = 1^{\frac{1}{3}}$, $\eta = 1^{\frac{1}{6}}$.

(xi) $G = Z_{14}$, the diagonal G action is generated by $M = (\omega^3, \omega^5, \omega^{13}, -1)$ or $(\omega, \omega^9, \omega^{11}, -1)$, where $\omega = 1^{\frac{1}{14}}$.

(xii) $G = Z_{15}$, the diagonal G action is generated by $M = (\omega, \omega^2, \omega^4, \omega^8)$ or $(\omega^7, \omega^{11}, \omega^{13}, \omega^{14})$, where $\omega = 1^{\frac{1}{15}}$.

(xiii) $G = Z_{16}$, the diagonal G action is generated by $M = (\omega, \omega^3, \omega^5, \omega^7)$ or $(\omega^9, \omega^{11}, \omega^{13}, \omega^{15})$, where $\omega = 1^{\frac{1}{16}}$.

(xiv) $G = Z_{18}$, the diagonal G action is generated by $M = (\omega, \omega^5, \delta^2, 1)$, $(\omega^{13}, \omega^{17}, \delta, 1)$, $(\omega, \omega^7, \omega^{13}, \eta^5)$, or $(\omega^5, \omega^{11}, \omega^{17}, \eta)$, where $\omega = 1^{\frac{1}{18}}$, $\eta = 1^{\frac{1}{6}}$, and $\delta = 1^{\frac{1}{3}}$.

(xv) $G = Z_{20}$, the diagonal G action is generated by $M = (\omega, \omega^3, \omega^7, \omega^9)$ or $(\omega^{11}, \omega^{17}, \omega^{13}, \omega^{19})$, where $\omega = 1^{\frac{1}{20}}$.

(xvi) $G = Z_{24}$, the diagonal G action is generated by $M = (\omega, \omega^5, \omega^7, \omega^{11})$, $(\omega^{13}, \omega^{17}, \omega^{19}, \omega^{23})$, (ν, ν^5, μ, μ^3), $(\nu^7, \nu^{11}, \mu^5, \mu^7)$, $(\nu, \nu^5, \mu^5, \mu^7)$, or $(\nu^7, \nu^{11}, \mu, \mu^3)$, where $\omega = 1^{\frac{1}{24}}$, $\mu = 1^{\frac{1}{12}}$, and $\mu = 1^{\frac{1}{8}}$.

5.2 Resolution of Singularities using Toric Geometry

Having obtained the representation of finite cyclic subgroups G of $SU(4)$ on $\mathbb{C}^4$ which fixes some integral lattice defining T^8, we hope to look at the problem of resolving $\mathbb{C}^4/G$.

While resolving cyclic quotient singularity, e.g. $\mathbb{C}^4/Z_n$, seems to be a very natural problem, it is nevertheless still not completely solved as of yet. However, quotient singularities of $\mathbb{C}^2$ and $\mathbb{C}^3$ by finite subgroups of $SU(2)$ and $SU(3)$ have been studied extensively and resolved using toric geometry.

Here, we give a brief discussion of singularities and their resolutions based on [20].

Definition 5.2.1 A variety X (which we will assume to be normal and quasi-projective and defined over $\mathbb{C}$) has canonical singularities if some positive integral multiple of the Weil divisor rK_X is Cartier; furthermore, if $f : Y \to X$ is a resolution of X, and $\{E_i\}$ a family of exceptional prime divisors of f, then $rK_Y = f^*(rK_X) + \sum a_i E_i$, where $a_i \geq 0$. If all the a_i's are greater than 0, then, X is said to have terminal singularities.

Using toric geometry, there is a even more explicit criterion for canonical and terminal singularities. In particular, in the case of cyclic quotient singularities, i.e. $\mathbb{C}^n/Z_m$, where the generator of the group acts by the diagonal matrix with entries $\omega^{e_1}, \cdots, \omega^{e_n}$, ω a primitive m-th root of 1, we say that $\mathbb{C}^n/Z_m$ is a cyclic quotient singularity of type $\frac{1}{m}(e_1, \cdots, e_n)$. This singularity is said to be terminal (resp. canonical) if and only if all the weights (to be discussed in the next chapter) defined by $\alpha_k = \frac{1}{m}\sum_j \overline{ke_j}$ are > 1 (resp. ≥ 1) for $k = 1, 2, \cdots, m-1$, here $\overline{ke_j}$ denote the integer modulo m.

With this criterion, we see immediately that for $n \in S$ defined at the beginning of this chapter, the cyclic quotient singularities $\mathbb{C}^4/Z_n$ must be terminal for $n = 5 \in S$ and can be non-terminal for $n \in \{2, 3, 4, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 24\} \subset S$. Here, $\mathbb{C}^4/Z_n$ is terminal if in all the representations of Z_n in $SU(4)$ which are conjugate to some element in $SL(8, Z)$ (i.e. fixing some integral lattice defining T^8), all elements have weight greater than 1. We remark that for $n = 9$ or $n = 14$, even though the generator M described in the last section has weight > 1, there is some non-generator in the group Z_n with weight 1 making $\mathbb{C}^4/Z_n$ non-terminal. We shall see later on by explicit examples why terminal singularities are terminal using this criterion.

Definition 5.2.2 A birational morphism $f : Y \to X$ between normal varieties is called crepant (or minimal) if $K_Y = f^*K_X$. A crepant partial resolution of a variety X with canonical singularities is the one which pulls out only the exceptional divisors E_i with $a_i = 0$.

Unfortunately, crepant partial resolutions are not unique, so the concept of minimal resolution is not defined. Algebraic geometers have studied and classified canonical singularities for $\mathbb{C}^2/G$ and $\mathbb{C}^3/G$ for G a finite abelian group. However, for dimension greater than 3, no such classification work has been pursued due to its difficulty. As our main goal is to study resolutions related to Spin(7)-manifolds obtained by resolving T^8/G, we shall only hope to investigate what is needed for this problem and nothing more.

Example The most classical example is surface singularities. One can show that surface canonical singularities are the nonsingular points and the so called Du Val surface singularities, which are the hypersurface singularities given by the equations

(*i*) $A_n : x^2 + y^2 + z^{n+1} = 0$ for $n \geq 1$.

(*ii*) $D_n : x^2 + y^2 z + z^{n-1} = 0$ for $n \geq 4$.

(*iii*) $E_6 : x^2 + y^3 + z^4 = 0$.

(*iv*) $E_7 : x^2 + y^3 + yz^3 = 0$.

(*v*) $E_8 : x^2 + y^3 + z^5 = 0$.

They are classical, since they were classfied by Klein in the 19th century in his work on invariant theory of regular solids in $\mathbb{R}^3$, and these singularities are called A-D-E type singularities due to Klein. Each of these hypersurface singularities admit a resolution $f : Y \to X$, such that $K_y = f^*K_X$, and the exceptional locus of f is a collection of (-2)-curves (i.e. $\mathbb{P}^1$) whose intersection matrix corresponds to its associated Dynkin diagram.

For 3-folds $\mathbb{C}^3/G$ with canonical singularities, it can be shown that there exists a crepant partial resolution $f : Y \to X$ such that Y only has terminal singularities, which have been classified by algebraic geometers.

There are some general notions of quotient singularities which one encounters in the literature that we summarize here briefly. If G is a finite subgroup of $GL(n, \mathbb{C})$ acting on $\mathbb{C}^n$, Cartan proved that singularities of $\mathbb{C}^n/G$ are normal. In particular, the singular sets of $\mathbb{C}^n/G$ have codimension at least 2. Usually, one's only interested in the case that $\mathbb{C}^n/G$ is Gorenstein, which is a technical algebraic geometry notion that the dualizing sheaf of $\mathbb{C}^n/G$ is trivial. Watanabe established that $\mathbb{C}^n/G$ is Gorenstein if and only if $G \subset SL(n, \mathbb{C})$. As our groups are all finite subgroups of $SU(4)$, we can assume from now on that all our singularities are Gorenstein.

The importance of crepant resolution is that the resolution Y of X has trivial canonical bundle if X has trivial canonical bundle. From results on Calabi-Yau space M, i.e. compact complex manifold admitting Kähler-Einstein metric with zero Ricci curvature, we know that $c_1(M) = 0$, furthermore, the canonical bundle has finite order in the Picard group of M. As we are looking for Spin(7) space by resolving T^8/G, since $SU(4) = U(4) \cap$ Spin(7), by Calabi-Yau, the canonical bundle is trivial (i.e. $c_1(Y) = 0$) if and only if there is a crepant resolution of T^8/G, so locally, we want $\mathbb{C}^4/Z_n$ for the action classified in the previous section to have crepant resolution. Since our $X = T^8/G$ has trivial canonical bundle, for its resolution Y to have trivial canonical bundle K_Y where $K_Y = f^*K_X + \sum a_i E_i$, then X certainly can't have terminal singularities.

Toric geometry gives an explicit recipe for determining whether $\mathbb{C}^n/G$ has a smooth crepant resolution with trivial canonical bundle.

Notations: Let $\mathbb{R}^n$ be the vector space with standard basis $\{e^1, \cdots, e^n\}$, its dual basis is $\{f_1, \cdots, f_n\}$. Suppose G is a finite diagonal group acting on the vector space $\mathbb{C}^n$ such that the singular set of each nontrivial element in G has codim at least 2. Further, suppose G acts freely on $(\mathbb{C}^*)^n$, then the quotient $(\mathbb{C}^*)^n/G$ has the structure of an algebraic n-torus T defined over $\mathbb{C}$, and $\mathbb{C}^n/G$ becomes a toric variety with the following structure[21]. Let $N = Hom(\mathbb{C}^*, T)$ be the group of 1-parameter subgroups of T, $M = Hom(T, \mathbb{C}^*)$ be the group of characters of T. M and N are free abelian groups of rank n and there is a canonical nonsingular pairing $M \times N \to \mathbb{Z}$. Define the maps

$$exp : \mathbb{R}^n \to \mathbb{C}^n, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi\sqrt{-1}x_1} \\ \vdots \\ e^{2\pi\sqrt{-1}x_n} \end{pmatrix},$$

$$tr : \mathbb{R}^n \to \mathbb{R}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^{n} x_i.$$

In our case for $\mathbb{C}^n/G$, N is the lattice generated by $exp^{-1}(g)$ for $g \in G$ in $\mathbb{R}^n$. N contains the standard lattice $\sum_{i=1}^n \mathbb{Z}e^i$, and in fact $N/\sum_{i=1}^n \mathbb{Z}e^i$ is isomorphic to G. As a toric variety, $\mathbb{C}^n/G$ is isomorphic to the affine variety X_C, where C is the cone $\{\sum_{i=1}^n x_i e^i | x_i \geq 0\}$ in $\mathbb{R}^n$. The T-invariant divisors D_i are in one-to-one correspondence with all the 1-dimensional faces $\mathbb{R}_+ e^i$ of C of $1 \leq i \leq n$. Let $\Delta = \{x \in \mathbb{R}^n | tr(x) = 1\} \cap C$. A desingularization Y of X_C corresponds to subdividing the cone C into a fan, i.e. a set of cones σ_i, such that $\cup \sigma_i = C$ and each σ_i is generated by a subset of a $\mathbb{Z}$-basis of N. Furthermore, the canonical bundle of the desingularization Y is trivial if and only if there exists a $u \in M$ such that $< u, v_i >= -1$ for all the v_i's, which are the first integral lattice point on the edges of the subdivided C[21][9]. In other words, the canonical bundle is trivial if and only if each σ_i in the fan obtained from subdivision is a simplicial cone generated by vectors in $\Delta \cap N$.

Now we can see how the singularities classified in the previous section can be resolved in algebraic geometry using rather combinatorial toric geometric tools. Here, we are only interested in finite diagonal group G acting on $\mathbb{C}^n$ such that G acts freely on on $(\mathbb{C}^*)^4$. So, some of the singularity types which have at least one eigenvalue 1 are not studied here. In fact, as we remarked before, singularities of the form $\mathbb{C}^2/G$ and $\mathbb{C}^3/G$ for finite groups $G \subset SU(2)$ or $SU(3)$ respectively have been studied and resolved completely[21][17]. In what follows, we let a, b, c, d denote the points in $\mathbb{C}^4$ of coordinates $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ and $(0,0,0,1)$ respectively. Let d_k be the number of k-dimensional cones in a fan Δ. If the toric variety $X(\Delta)$ is nonsingular and projective, then the Betti numbers b_i of $X(\Delta)$ are determined by

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_{n-i},$$

$b_j = 0$ for j odd. [9]

(i) $G = Z_2$, generator g acts diagonally by $(-1,-1,-1,-1)$. $N = \langle e_1, \cdots, e_4, \frac{1}{2}(e_1 + \cdots + e_4)\rangle$, $M = \{\sum_{i=1}^4 \lambda_i f_i | \lambda_i \in \mathbb{Z}, \sum_{i=1}^4 \lambda_i \equiv 0(2)\}$. $C = \{\sum_{i=1}^4 x_i e^i | x_i \geq 0\}$. Since $\Delta \cap N = \{e_1, \cdots, e_4\}$, and it doesn't form a $\mathbb{Z}$-basis for N, so $\mathbb{C}^4/G$ can't be resolved. In fact, as the weight of all elements in G is 2, $\mathbb{C}^4/G$ is a terminal singularity which can't be resolved.

(ii) $G = Z_3$, generator g acts diagonally by $(\omega, \omega^2, \omega, \omega^2)$, where $\omega = 1^{\frac{1}{3}}$. $N = \langle e_1, \cdots, e_4, \frac{1}{3}e_1 + 2e_2 + e_3 + 2e_4\rangle$, $M = \{\sum_{i=1}^4 \lambda_i f_i | \lambda_i \in \mathbb{Z}, \lambda_1 + \lambda_3 \equiv \lambda_2 + \lambda_4(3)\}$. $\Delta \cap N = \{e_1, \cdots, e_4\}$, and it doesn't form a $\mathbb{Z}$-basis for N. Again, all elements of G have weight 2, $\mathbb{C}^4/G$ is a terminal singularity which can't be resolved.

(iii) $G = Z_4$, generator g acts diagonally by $(\omega, \omega, \omega, \omega)$, where $\omega = 1^{\frac{1}{4}}$. $N = \langle e_1, \cdots, e_4, \frac{1}{4}(e_1 + \cdots + e_4)\rangle$, $M = \{\sum_{i=1}^4 \lambda_i f_i | \lambda_i \in \mathbb{Z}, \sum_{i=1}^4 \lambda_i \equiv 0(4)\}$. Let $u = -(f_1 + \cdots + f_4)$, then adding the vertex $e = \frac{1}{4}(e_1 + \cdots + e_4)$, we can subdivide C into a fan of 4 simplicial 4-cones $abce, abde, acde, bcde$, each generated by a $\mathbb{Z}$-basis of N, and $< u, v >= -1$ for all the minimal integral edges in this fan. $d_0 = 1, d_1 = 5, d_2 = 10, d_3 = 10, d_4 = 4$. Calculating the Betti number using the formula, we see that $b_i = 1$ for $i = 2, 4, 6, 8$ and $b_i = 0$ otherwise. We note finally that $\mathbb{C}^4/G$ is not a terminal singularity for the g action, as weight of g is 1; the singularity can be resolved. However, T^8/G can not be resolved since it has $\mathbb{R}^8/\pm 1$ singularities which are not resolvable. Note however, if g acts by $(\omega, \omega^3, -1, -1)$ or $(\omega, \omega^3, \omega, \omega^3)$, then $\mathbb{C}^4/G$ is a terminal singularity with no resolution.

(iv) $G = Z_5$, the generator g acts diagonally by $(\omega, \omega^2, \omega^3, \omega^4)$, where $\omega = 1^{\frac{1}{5}}$. Since the singularity $\mathbb{C}^4/G$ is terminal, $\Delta \cap N = \{e_1, \cdots, e_4\}$, the singularity is not resolvable.

(v) $G = Z_6$, generator g acts diagonally by $(\omega, \omega, \nu, \nu)$, or $(\omega, \omega, \omega, -1)$, where $\omega = 1^{\frac{1}{6}}$ and $\nu = 1^{\frac{1}{3}}$, then the singularity $\mathbb{C}^4/G$ is non- terminal and has a resolution. In the first case, let $e = \frac{1}{6}(1, 1, 2, 2)$, and $f = \frac{1}{2}(1, 1, 0, 0)$, then C can be partitioned into 6 simplicial 4-cones, acde, bcde, acef, adef, bcef, bdef, each generated by a $\mathbb{Z}$ basis of N. In the second case, let $e = \frac{1}{6}(1, 1, 1, 3)$, and $f = \frac{1}{3}(1, 1, 1, 0)$, then C can be partitioned into 6 simplicial 4-cones, abde, acde, bcde, acef, bcef, abef, each generated by a $\mathbb{Z}$ basis of N. In both cases, $d_0 = 1, d_1 = 6, d_2 = 14, d_3 = 15, d_4 = 6$, and $b_0 = 0, b_2 = 1, b_4 = 2, b_6 = 2, b_8 = 1$. But T^8/G is not resolvable since it has singularities of type $\mathbb{C}^4/\mathbb{Z}_2$ or $\mathbb{C}^4/\mathbb{Z}_3$ which are not resolvable. For other possible representation of G, $\mathbb{C}^4/G$ is terminal singularity with no resolution.

(vi) $G = Z_7$, there is no action of $G \subset SU(4)$ on $\mathbb{C}^4$ such that it's free on $(\mathbb{C}^*)^4$. G can act on $\mathbb{C}^3$ with 0 as the only singularity, and this singularity is easily desingularized[17] [21].

(vii) $G = Z_8$, generator g acts diagonally by $(\omega, \omega^3, \omega, \omega^3)$ or $(\omega, \omega^3, \delta, \delta)$, where $\omega = 1^{\frac{1}{8}}$ and $\delta = 1^{\frac{1}{4}}$, the singularity $\mathbb{C}^4/G$ is non-terminal, and it is resolvable in the first case but non-resolvable in the second case using toric geometry. In the first case, let $e = \frac{1}{8}(1, 1, 3, 3)$, and $f = \frac{1}{8}(3, 3, 1, 1)$, then C can be partitioned into 8 simplicial 4-cones, abcf, abdf, acde, bcde, acef, adef, bcef, bdef, each generated by a $\mathbb{Z}$ basis of N. Here, $d_0 = 1, d_1 = 6, d_2 = 15, d_3 = 18, d_4 = 8$, and $b_0 = 0, b_2 = 2, b_4 = 3, b_6 = 2, b_8 = 1$. In the second case, $e = \frac{1}{8}(1, 2, 2, 3)$, and $f = \frac{1}{2}(1, 0, 0, 1)$, there are only 6 possible 4-cones that are generated by some $\mathbb{Z}$ basis of N, i.e. bcde, abef, acef, bcef, bdef, cdef, and they don't partition C. T^8/G is not resolvable since it has singularities which are not resolvable. For other possible representation of G, $\mathbb{C}^4/G$ is terminal singularity with no resolution.

(viii) $G = Z_9$, the only possible action of $G \subset SU(4)$ on $\mathbb{C}^4$ is generated by $(\omega, \omega^4, \omega^7, \omega^6)$, where $\omega = 1^{\frac{1}{9}}$. Although $\mathbb{C}^4/G$ is a non-terminal singularity since g^3 has weight 1, the 4 simplicial cones in the fan generated by the

elements $\{e_1, \cdots, e_4, \frac{1}{3}(e_1+e_2+e_3)\}$ are not generated by a subset of the $\mathbb{Z}$-basis of $N = \langle e_1, \cdots, e_4, \frac{1}{9}(e_1+4e_2+7e_3+6e_4)\rangle$. So $\mathbb{C}^4/G$ is not resolvable.

(ix) $G = Z_{10}$, generator g acts diagonally by $(\omega, \omega^3, \delta, \delta^2)$, where $\omega = 1^{\frac{1}{10}}$ and $\delta = 1^{\frac{1}{5}}$, the singularity $\mathbb{C}^4/G$ is non-terminal and non-resolvable using toric geometry. Let $e = \frac{1}{10}(1, 2, 3, 4)$, and $f = \frac{1}{2}(1, 0, 1, 0)$, there are only 4 possible 4-simplices generated by a $\mathbb{Z}$ basis of N, bcde, adef, bdef, cdef, and they don't partition C. A fortiori, T^8/G is not resolvable. For other possible representation of G, $\mathbb{C}^4/G$ is terminal singularity with no resolution.

(x) $G = Z_{12}$, generator g acts diagonally by $(\omega, \omega^5, \omega, \omega^5)$, or $(\omega, \omega^5, \delta, \delta)$, or $(\omega, \omega^5, \nu, \eta)$, or $(\omega, \omega^7, \eta, \eta)$, where $\omega = 1^{\frac{1}{12}}$, $\delta = 1^{\frac{1}{4}}$, $\nu = 1^{\frac{1}{3}}$, and $\eta = 1^{\frac{1}{6}}$. Then $\mathbb{C}^4/G$ is a non-terminal singularity, we have checked that it is only resolvable using toric geometry in the first representation of G but not the other ones. In the first representation, let $e = \frac{1}{12}(1, 1, 5, 5), f = \frac{1}{4}(1, 1, 1, 1)$, and $g = \frac{1}{12}(5, 5, 1, 1)$, then C can be partitioned into 12 4-simplices, acde, bcde, abcg, abdg, acef, bcef, adef, bdef, acfg, bcfg, adfg, bdfg, each generated by a $\mathbb{Z}$ basis of N. Here, $d_0 = 1, d_1 = 7, d_2 = 20, d_3 = 26, d_4 = 12$, and $b_0 = 0, b_2 = 3, b_4 = 5, b_6 = 3, b_8 = 1$. In the next three fixed point free representation of G in $\mathbb{C}^4$, we have checked that there isn't a partition of C by simplicial 4-cones. Finally, T^8/G is not resolvable. For other possible representation of G, $\mathbb{C}^4/G$ is terminal singularity with no resolution.

(xi) $G = Z_{14}$, the possible action of $G \subset SU(4)$ on $\mathbb{C}^4$ is generated by $(\omega^3, \omega^5, \omega^{13}, -1)$, where $\omega = 1^{\frac{1}{14}}$. Although $\mathbb{C}^4/G$ is a non-terminal singularity since g^6 has weight 1. The 4 simplicial cones in the fan generated by the elements $\{e_1, \cdots, e_4, \frac{1}{14}(4e_1 + 2e_2 + 8e_3)\}$ are not generated by a subset of the $\mathbb{Z}$-basis of $N = \langle e_1, \cdots, e_4, \frac{1}{14}(3e_1 + 5e_2 + 13e_3 + 7e_4)\rangle$. So $\mathbb{C}^4/G$ is not resolvable.

(xii) $G = Z_{15}$, generator g acts diagonally by $(\omega, \omega^2, \omega^4, \omega^8)$, where $\omega = 1^{\frac{1}{15}}$. $\mathbb{C}^4/G$ is a non-terminal singularity resolvable using toric geometry. Let $e = \frac{1}{15}(1, 2, 4, 8), f = \frac{1}{15}(2, 4, 8, 1), g = \frac{1}{15}(4, 8, 1, 2)$, and $h = \frac{1}{15}(8, 1, 2, 4)$, then C can be partitioned into 15 4-simplical cones, abcf, acdh, bcde, abdg, bcef, bdeg, cdeh, acfh, abfg, adgh, afgh, befg, cefh, degh, and efgh, each generated by a $\mathbb{Z}$ basis of N. Here, $d_0 = 1, d_1 = 8, d_2 = 24, d_3 = 32, d_4 = 15$, and $b_0 = 0, b_2 = 4, b_4 = 6, b_6 = 4, b_8 = 1$. Finally, T^8/G is not resolvable as it has singularities of non-resolvable type.

(xiii) $G = Z_{16}$, generator g acts diagonally by $(\omega, \omega^3, \omega^5, \omega^7)$, where $\omega = 1^{\frac{1}{16}}$. $\mathbb{C}^4/G$ is a non-terminal singularity non-resolvable using toric geometry. It's not resolvable, as for $e = \frac{1}{16}(1, 3, 5, 7)$ and $f = \frac{1}{16}(7, 5, 3, 1)$, there are only 4 4-simplicial cones bcde, abcf, cdef, and abef, that are generated by a $\mathbb{Z}$ basis of N and they are not enough to partition C. A fortiori, T^8/G is not resolvable.

(xiv) $G = Z_{18}$, the possible action of $G \subset SU(4)$ on $\mathbb{C}^4$ is generated by $(\omega, \omega^7, \omega^{13},$

ω^{15}), where $\omega = 1^{\frac{1}{18}}$. Although $\mathbb{C}^4/G$ is a non-terminal singularit y since g^3 and g^6 both have weight 1. The simplicial cones in the fan generated by the elements $\{e_1, \cdots, e_4, \frac{1}{6}(e_1+e_2+e_3+3e_4), \frac{1}{3}(e_1+e_2+e_3)\}$ are not generated by a subset of the $\mathbb{Z}$-basis of $N = \langle e_1, \cdots, e_4, \frac{1}{18}(e_1 + 7e_2 + 13e_3 + 15e_4)\rangle$. So $\mathbb{C}^4/G$ is not resolvable.

(xv) $G = Z_{20}$, generator g acts diagonally by $(\omega, \omega^3, \omega^7, \omega^9)$, where $\omega = 1^{\frac{1}{20}}$. $\mathbb{C}^4/G$ is a non-terminal singularity but not resolvable using toric geometry. The vertices determined by weight 1 elements of G are $e = \frac{1}{20}(1, 3, 7, 9)$, $f = \frac{1}{20}(3, 9, 1, 7)$, $g = \frac{1}{20}(7, 1, 9, 3)$, and $h = \frac{1}{20}(9, 7, 3, 1)$. There are 36 simplicial 4-cones generated by $\mathbb{Z}$ basis of N from the 8 vertices $\{a, b, c, d, e, f, g, h\}$, but there aren't 20 of the 36 which will partition C. A fortiori, T^8/G is not resolvable.

(xvi) $G = Z_{24}$, generator g acts diagonally by $(\omega, \omega^5, \omega^7, \omega^{11})$ or (ν, ν^5, μ, μ^3), where $\omega = 1^{\frac{1}{24}}$, $\nu = 1^{\frac{1}{12}}$, and $\mu = 1^{\frac{1}{8}}$. $\mathbb{C}^4/G$ is a non-terminal singularity but not resolvable using toric geometry in either case. Since, there are not sufficiently many 4-simplicial cones generated by a $\mathbb{Z}$ basis of N to partition C. A fortiori, T^8/G is not resolvable.

Observation 5.2.3 *We note that of all the examples studied above, if $\mathbb{C}^4/\mathbb{Z}_n$ is resolvable, then the sum of the even Betti numbers (which is also the Euler characteristic) of the resolution is the number of conjugacy classes in $\mathbb{Z}_n$, which is n.*

Recently, Batyrev and Kontsevich announced that they can prove the physicists' orbifold Euler characteristic formula (to be discussed in the next chapter) holds for the resolution of $\mathbb{C}^l/G$ provided that the resolution is crepant. And indeed, our observation verifies this.

From these examples of element of $SU(4)$ acting on T^8 fixing the torus lattice, we see that the singularities $\mathbb{C}^4/Z_n$ (Z_n acting fixed point freely on $(\mathbb{C}^*)^4$)) is resolvable using toric geometry for $n \in \{4, 6, 8, 12, 15\}$ for certain representation of Z_n in $SU(4)$. We observe that even though $\mathbb{C}^4/Z_n$ is a non-terminal singularity for $n \in \{9, 14, 18\}$, $\mathbb{C}^4/Z_n$ can't be resolved (the generator has weight > 1), and neither for $n \in \{10, 16, 20, 24\}$ (even though the generator has weight 1, since C can't be properly partitioned). Furthermore, $T^8/\mathbb{Z}_n$ is not resolvable for any n, since it will always have singularities of the type $\mathbb{C}^4/\mathbb{Z}_m$ for some m which is terminal. Therefore, in order for us to produce more examples of Spin(7) manifolds from torus orbifold actions, we need to go down in dimension and use T^6/G or T^4/G instead.

Chapter 6

Orbifold Formulas and Topological Obstruction

For a compact manifold M and with a finite group G action, there is a well-known physicists' Euler characteristic formula in string theory for orbifold M/G which admits a Calabi-Yau metric:

$$\chi(M,G)=\frac{1}{|G|}\sum_{g,h\in G,gh=hg}\chi(M^g\cap M^h).$$

In [3], physicists have since come up with string-theoretic formulae for Hodge numbers of smooth compact Kähler complex n-manifold M with a finite group G-action. Suppose M has a G-invariant volume form. Let $C(g)$ = centralizer of g in G. For $x\in M^g$, the eigenvalues of g in the holomorphic tangent space T_xM are roots of unity: $e^{2\pi\sqrt{-1}\alpha_1},\cdots,e^{2\pi\sqrt{-1}\alpha_n}$, where $0\le\alpha_j<1$. M^g can be written as a disjoint union of $C(g)$-orbits of connected components $M_1^g,\cdots,M_{r_g}^g$. Define $F_i(g)$, the fermion shift number, to be $\sum_{1\le j\le n}\alpha_j$ on M_i^g. Let $h_{C(g)}^{p,q}(M_i^g)$ be the dimension of the $C(g)$-invariant subspace of $H^{p,q}(M_i^g)$. Let

$$h_g^{p,q}(M,G)=\sum_{i=1}^{r_g}h_{C(g)}^{p-F_i(g),q-F_i(g)}(M_i^g)$$

The resolution of the orbifold M/G are predicted to have the Hodge numbers

$$h^{p,q}(M,G)=\sum_{\{g\}}h_g^{p,q}(M,G) \tag{6.1}$$

summing over the conjugacy classes of G.

[3] also conjectured a strong McKay correspondence for ALE spaces.

Conjecture 6.0.4 *(Strong McKay Correspondence)[3] Let $G \subset SL(n, \mathbb{C})$ be a finite group. Assume that $\mathbb{C}^n/G$ admits a smooth crepant desingularization $\pi : \hat{M} \to \mathbb{C}^n/G$, where $F = \pi^{-1}(0)$, then*

$$b^i(F, \mathbb{C}) = b^i(\hat{M}) = \#\{\textit{conjugacy classes } \{g\} \subset G, \textit{ such that } wt(g) = \frac{i}{2}\} \tag{6.2}$$

where $wt(g)$ is defined as $\sum_{i=1}^{n} \alpha_i$, for $g \in G$ with eigenvalues $e^{2\pi\sqrt{-1}\alpha_i}$ for $1 \le i \le n$, $\alpha_i \in \mathbb{Q} \cap [0, 1)$.

In this chapter, we shall give orbifold formulas for certain linear combinations of Betti numbers of Spin(7)-manifolds based on the results of [3], these formulae are true for $SU(4)$ manifolds, and make sense for Spin(7)-manifolds. We also use the topogical data derived to give a criterion for what orbifold constructions are possible to generate manifolds of Spin(7)-structure.

6.1 Betti Numbers for Compact Spin(7)-Manifolds

In chapter 3, we showed how to construct Spin(7)-manifolds by resolving T^8/G, for a finite group G. In resolving the singularities, we noticed that singularities of type iv and type v admit two different resolutions, hence it is not possible to predict the Betti number of Spin(7) manifolds obtained by resolving an orbifold. On the other hand, for all known examples, we find that $\sum b^{\text{even}}$, $\sum b^{\text{odd}}$, and $b^4_+ - b^4_-$ are indenpendent of the resolution chosen. Based on string theory and McKay conjectures, Batyrev-Dais[3] developed formulae to predict Hodge numbers for smooth compact Kähler manifolds equipped with a finite group action. Here, we can adapt their conjectures and ideas to predict formulae for the three linear combinations of Betti number for examples of Spin(7)-manifolds constructed using orbifold construction.

Suppose we have a connected 8-manifold M with a finite group action G such that $\hat{M/G}$, the resolution of M/G has holonomy in Spin(7). For $g \in G$, the possible real dimensions for M^g are $0, 2, 4$, or 8. $dim(M^g)$ has to be even, otherwise, $g \in SO(k)$ for some k odd would act fixed point freely, which is impossible. $dim(M^g) \neq 6$, else $g \in SO(2)$, but this is not possible in $Spin(7)$.

(*i*) $dim(M^g) = 8$, then $g = 1$. So $F_i(g) = 0$.

(*ii*) $dim(M^g) = 4$, we see that $0 < F_i(g) < 2$, so $F_i(g) = 1$. Since $Spin(7) \cap SO(4) = SU(2)$, the normal bundle to M^g has an $SU(2)$-structure. There is an $SU(2)$ family of $SU(2)$-structures on M^g, thus the splitting of H^i into $H^{p,q}$ is non-canonical. So $H^{p,q}$ makes no sense in this context. However, $F_i(g)$ is well-defined to be 1.

(*iii*) $dim(M^g) = 2$, since M is not complex, M^g is not a complex curve, thus has no canonical orientation. Further, the normal bundle of M^g has $Spin(7) \cap SO(6) = SU(3)$ structure. A choice of orientation gives 2 $SU(3)$- structures, equivalent under $(z_1, z_2, z_3) \to (\bar{z_1}, \bar{z_2}, \bar{z_3})$. We see that $0 < F_i(g) < 3$. The two $SU(3)$ structures give $F_i(g) = 1$ or 2 respectively. So $F_i(g)$ is not determined uniquely.

(*iv*) $dim(M^g) = 0$, $0 < F_i(g) < 4$. Here $(z_1, z_2, z_3, z_4) \to (\bar{z_1}, \bar{z_2}, \bar{z_3}, \bar{z_4})$ on the normal bundle of a fixed point sends $F_i(g)$ to $4 - F_i(g)$. If $F_i(g) = 2$, then it's well-defined. However, if $F_i(g) = 1$, then it's undetermined. The two $SU(4)$ structures give $F_i(g) = 1$ or 3.

From the above discussion and the physicists' orbifold Hodge number formula, and the fact that the Fermion shift $F_i(g)$ doesn't affect the even or odd part of cohomology, let r_g be the number of $C(g)$-orbits of connected components of M^g, we predict that

$$\sum b^{\text{even}} = \sum_{k=0}^{4} \sum_{\{g\} \text{ conjugacy classes}} \sum_{i=1}^{r_g} dim((H^{2k-2F_i(g)}(M_i^g))^{C(g)}) \quad (6.3)$$

$$\sum b^{\text{odd}} = \sum_{0}^{3} \sum_{\{g\} \text{ conjugacy classes}} \sum_{i=1}^{r_g} dim((H^{2k+1-2F_i(g)}(M_i^g))^{C(g)}) \quad (6.4)$$

These orbifold formulae for the sum of odd and even Betti numbers derived from the orbifold Hodge number formula is certainly true for $SU(4)$-manifolds, and it makes sense for Spin(7)-manifolds. Indeed, the quantities predicted by these formulae are the same as the ones computed in all known examples of compact Spin(7)-manifolds obtained by resolving T^8/G.

In particular, we see that we can predict b^3 and χ for Spin(7)-manifolds.

Another important topological invariant for Spin(7)-manifolds is the signature, i.e. $\sigma = b^4_+(M, G) - b^4_-(M, G)$. We see from Chapter 1, that $\wedge^4_+ T^*M = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}$, and $\wedge^4_- T^*M = \wedge^4_{35}$. From [24], we see that on Kähler manifolds

$$b^4_+ = h^{40} + h^{22} + h^{04} + h^{42} + h^{24} - h^{33} + h^{44} \quad (6.5)$$

$$b^4_- = h^{31} + h^{13} - h^{20} - h^{02} + h^{11} - h^{00} \quad (6.6)$$

We observe that $b^4_+ - b^4_- = \sum_{p+q \text{ even}} (-1)^{pq} h^{pq}$. Again, we rely on the orbifold formula for Hodge numbers to determine contributions to $b^4_+(M, G) - b^4_-(M, G)$ arising from singularities in M/G.

(*i*) $dim M^g = 8$, i.e. $g = 1$, $b^4_{+,g}(M, G) - b^4_{-,g}(M, G)$ is $dim(H^4_+(M)^G) - dim(H^4_-(M)^G)$.

(*ii*) $dim M^g = 4$, then $F_i(g) = 1$, $b^4_{+,g}(M, G) = \sum_{j=1}^{r_g} (h^{11}_{C(g)}(M^g_j) - h^{22}_{C(g)}(M^g_j))$, and $b^4_{-,g}(M, G) = \sum_{j=1}^{r_g} (h^{20}_{C(g)}(M^g_j) + h^{02}_{C(g)}(M^g_j) + h^{00}_{C(g)}(M^g_j))$. So, $b^4_{+,g}(M, G) - b^4_{-,g}(M, G)$ is just $\sum_{j=1}^{r_g} (dim(H^2_-(M^g_j))^{C(g)} - dim(H^2_+(M^g_j))^{C(g)})$.

(iii) $dimM^g = 2$, then $F_i(g)$ is either 1 or 2. If $F_i(g) = 1$, we have $b^4_{+,g}(M,G) = \sum_{j=1}^{r_g} h^{11}_{C(g)}(M^g_j)$, and $b^4_{-,g}(M,G) = \sum_{j=1}^{r_g} h^{00}_{C(g)}(M^g_j)$. If $F_i(g) = 2$, we have $b^4_{+,g}(M,G) = \sum_{j=1}^{r_g} h^{00}_{C(g)}(M^g_j) - h^{11}_{C(g)}(M^g_j)$, and $b^4_{-,g}(M,G) = 0$. In either case, $b^4_{+,g}(M,G) - b^4_{-,g}(M,G)$ contributes 0.

(iv) $dimM^g = 0$, then either $F_i(g) = 2$, or $F_i(g) = 1$ or 3. If $F_i(g) = 2$, $b^4_{+,g}(M,G) - b^4_{-,g}(M,G) = \sum_{j=1}^{r_g} h^{00}_{C(g)}(M^g_j) = r_g$. If $F_i(g) = 1$ or 3, then $b^4_{+,g}(M,G) - b^4_{-,g}(M,G) = \sum_{j=1}^{r_g} -h^{00}_{C(g)}(M^g_j) = -r_g$.

Putting these together, we can compute the signature for $\hat{M/G}$. Indeed, the signatures predicted via these calculations correspond exactly to the results in all known examples of compact Spin(7)-manifolds, i.e. the signature is always 64.

For all known examples of Spin(7) manifolds, we can work out their Euler characteristics. We see that χ for all known Spin(7)-manifolds are 144. So $2(b^0 - b^1 + b^2 - b^3) + b^4 = 144$ and $b^4_+ - b^4_- = 64$, so $b^4_+ = 103 - b^2 + b^3$ and $b^4_- = 39 - b^2 + b^3$. In fact, given $\hat{A} = 1$, χ and σ determine each other, namely $24\hat{A} = 24 = 3\sigma - \chi$.

6.2 Topological Data for $\mathbb{R}^8/G$

In chapter 4, we found all the finite subgroups G of Spin(7) which have fixed point free representation in $SO(8)$. The spaces $\mathbb{R}^8/G$ are interesting, because they are candidates for singularities which might be resolved within Spin(7) holonomy. For this reason, we want to study the Betti numbers and signature associated to $\mathbb{R}^8/G$ and conjecture a topological criterion for the isolated singularity 0 in $\mathbb{R}^8/G$ to be resolved within holonomy Spin(7).

First we observe that when G acts on R^8 in a fixed point free fashion except at the origin, we can apply the work in the previous section to the only singular point 0. When $dim(M^g) = 0$ for $g \in G$, we see that $F_i(g)$ could be 1, 2, or 3 if $g \neq id$ and 0 if $g = id$. Further, we note that in the Spin(7) setting, weights 1 and 3 occur in pairs, the duality corresponds to the two $SU(4)$ structures on on the normal bundle of the fixed point.

Given a resolution $\pi : X \to \mathbb{R}^8/G$, where $F = \pi^{-1}(0)$, the conjecture of strong McKay correspondence [3] gives the cohomology of F with coefficient $\mathbb{C}$. Since X retracts to F, so the homology of X is the same as the homology of F. Regarding X as a manifold with boundary $Y = S^7/G$. $H_j(X) \simeq H_{8-j}(X)^*$ for $j \neq 0, 8$, and $H^j(X) \simeq (H_j(X))^*$ for $j < 8$. Since $dim(F) < 8$, $H_8(X) = 0$, and $H^8_c(X) = \mathbb{R}$, where H^*_c is cohomology with compact support.

From the Poincaré-Lefschetz duality theorem for manifolds with boundary, $H^j(X) \simeq H^{8-j}(X;Y)$ and the long exact homology sequence for (X,Y), we see that $H_0(X;Y) = 0$, $H_0(X) = \mathbb{R}$, $H_8(X;Y) = \mathbb{R}$, and $H_8(X) = 0$. Furthermore, $H_j(X) = H_j(X;Y)$ for $j \neq 0, 8$. In particular, even though X is non-compact,

$b^i(X) = b^{8-i}(X)$ for $i \neq 0, 8$. The strong McKay correspondence conjecture thus gives

$$\begin{aligned} b^4(X) &= \#\{\text{conjugacy classes } \{g\} \text{ in G with } wt(g) = 2\} \\ b^2(X) = b^6(X) &= \frac{1}{2}\#\{\text{conjugacy classes } \{g\} \text{ in G with } wt(g) = 1 \text{ or } 3\} \\ b^1(X) = b^3(X) &= b^5(X) = b^7(X) = 0 \end{aligned} \quad (6.7)$$

We can also find $b^4_+(X) - b^4_-(X)$ using the result from last section. As 0 is the only singular point, $b^4_+(X) - b^4_-(X)$ is just the number of conjugacy classes $\{g\}$ with $wt(g) = 2$ minus the number of conjugacy classes $\{g\}$ with $wt(g) = 1$ or 3. So b^4_- = half the number of conjugacy classes $\{g\}$ with $wt(g) = 1$.

In summary, using the results on orbifold Hodge number formulae and strong McKay conjecture for $SU(4)$-manifolds as is done in [3], we conjecture that for X, a resolution of $\mathbb{R}^8/G$, where G acts fixed point freely except at the origin, the topological data are:

$$b^2 = b^6 = b^4_- = \frac{1}{2}(\text{the number of conjugacy classes in G of weight 1 or 3})$$

$$\begin{aligned} b^4_+ &= \text{the number of conjugacy classes in G of weight 2)} \\ &-\frac{1}{2}(\text{the number of conjugacy classes in G of weight 1 or 3}) \end{aligned}$$

We also have a conjecture similar to Theorem D in [13] as follows:

Conjecture 6.2.1 *Suppose X admits ALE torsion-free* Spin(7)*-structures. The moduli space of ALE torsion-free* Spin(7)*-structures is a smooth manifold of dimension $b^4_-(X)$.*

In particular, if $b^4_- < 0$, then $\mathbb{R}^8/G$ admits no resolution with holonomy Spin(7). If $b^4_- = 0$, then the only Spin(7) structure is the flat one on $\mathbb{R}^8/G$. Since given a resolution X of $\mathbb{R}^8/G$, X has ALE Spin(7) structure Ω, then $t^4\Omega$ is a 1-parameter family of ALE Spin(7) structures on X for $t \in (0, \infty)$. Either they are all isomorphic, in which case $\Omega = \mathbb{R}^8/G$, as nothing else can be both ALE and homothetic to itself, or they are non-isomorphic, thus the moduli space of such structures is at least 1 dimensional, i.e. $b^4_- > 1$.

Now, we can calculate $b^4(X)$ and $b^4_+(X) - b^4_-(X)$ for the resolution X of $\mathbb{R}^8/G$ for the group G found in chapter 4. We have the following result.

(i) G is a solvable Frobenius complement. G is cyclic of order n. If $n = 2, 3$ or 6, then in all possible fixed point free $SU(4)$ representations of G, the conjugacy classes have either weight 0 or 2. Hence, $b^4_- = 0$. If $n \neq 1, 2, 3$ and 6, there exist fixed point free $SU(4)$ representations of G such that an even number of conjugacy classes have either 1 or 3, so b^4_- is a positive integer.

(ii) G is a solvable Frobenius complement. $G = D_{n_1} \times C_{n_2}$where $(n_1, n_2) \neq 1$ and $(n_2, 2) = 1$. There are no fixed point free irreducible representation of Dihedral groups. So this case is excluded.

(iii) G is a solvable Frobenius complement. $G = Q_{k+1} \times C_{m'}$, where k is the largest integer such that $2^k | m$, $k \geq 2$, and $m' = \frac{m}{2^k}$. The fixed point free irreducible representations of generalized quaternion groups are all 2-complex dimensional sitting in $SU(2)$. From the characters of these representations, we see that the weight of each non-trivial conjugacy class of G is 2. Hence, $b^4_- = 0$.

(iv) G is a solvable Frobenius complement. $G = \langle x, y | x^4 = y^m = 1, x^{-1}yx = y^r \rangle$, where $(4, m) = (r-1, m) = 1$. $r^2 = 1(m)$, so $r = m - 1$.

$$y = \begin{bmatrix} \xi & 0 \\ 0 & \xi^r \end{bmatrix}, x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where ξ is a primitive m-th root of 1.
m has to be odd, say $m = 2l + 1$, then G has $4 + 2l$ conjugacy classes. Its representation in $SU(4) = SO(8)$ must be the direct sum of two copies of the given $SU(2)$ representation and has the following weights: class $\{1\}$ has weight 0; class $\{xy^k\}_{k=0}^{2l}$ has weight 2; class $\{x^3y^k\}_{k=0}^{2l}$ has weight 2; class $\{x^2\}$ has weight 2; classes $\{y^k, y^{-k}\}$ have weight 2 and $\{x^2y^k, x^2y^{-k}\}$ have weights 2 for $1 \leq k \leq 2l$. So $b^4 = 2l + 3 = \sigma$, so $b^4_- = 0$.

(v) G is a solvable Frobenius complement with a normal subgroup G_0 of index 2, where $G_0 = \langle x, y | x^4 = y^m = 1, x^{-1}yx = y^r \rangle$, where $(4, m) = (r-1, m) = 1$. $r^2 = 1(m)$, so $r = m-1$. $G = \langle G_0, \tau \rangle$, where τ is an involution. The $SU(4)$ representation of G is

$$\tilde{\rho}(x) = \begin{bmatrix} \rho(x) & 0 \\ 0 & \rho(x^3) \end{bmatrix} \tilde{\rho}(y) = \begin{bmatrix} \rho(y) & 0 \\ 0 & \rho(y^\gamma) \end{bmatrix}$$

$$\tilde{\rho}(\tau) = \begin{bmatrix} 0 & \rho(x) \\ 1 & 0 \end{bmatrix}$$

where $\gamma^2 = -1(m)$, and $\rho(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\rho(y) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^r \end{bmatrix}$, and ξ is a primitive m-th root of 1. So $G = \langle x, y, \tau | x^4 = y^m = \tau^2 = 1, x^{-1}yx = y^r, \tau x \tau^{-1} = x^{-1}, \tau y \tau^{-1} = y^{-\gamma} \rangle$.
G has $5 + \frac{m-1}{2}$ many conjugacy classes. $m = 2l + 1$. Its representation in $SU(4)$ has the following weights: $\{1\}$ has weight 0; conjugacy classes $\{xy^k, x^3y^k\}_{k=0}^{2l}$, $\{\tau, \tau x^2, \tau y^{k\frac{\gamma+1}{\gamma}}, \tau x^2 y^{-\frac{k}{\gamma-k+1}}\}_{k=0}^{2l}$, $\{\tau x, \tau x^3, \tau x y^{\frac{k(\gamma-1)}{\gamma}}, \tau x^3 y^{\frac{k}{\gamma-k+1}}\}_{k=0}^{2l}$, the class $\{x^2\}$ all have weight 2. And for each $1 \leq k \leq \frac{m-1}{4}$, the $\frac{m-1}{2}$ classes $\{y^k, y^{-k}, y^\gamma, y^{-k\gamma}\}$ have weight 2, classes $\{x^2y^k, x^2y^{-k}, x^2y^\gamma, x^2y^{-k\gamma}\}$ have weight 2. So $b^4 = 4 + \frac{m-1}{4} = \sigma$, and $b^4_- = 0$.

(vi) G is a solvable Frobenius complement. $G = \langle x, y | x^8 = y^m = 1, x^{-1}yx = y^r \rangle$, $r^4 = 1(m)$, $(8, m) = (r-1, m) = 1$. $m = 2l + 1$.

$$y = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^r & 0 & 0 \\ 0 & 0 & \xi^{r^2} & 0 \\ 0 & 0 & 0 & \xi^{r^3} \end{bmatrix}, x = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where ξ is a primitive m-th root of 1.

G has $8 + \frac{m-1}{2}$ conjugacy classes, its representation in $SU(4)$ has the following weights: $\{1\}$ has weight 0; conjugacy classes $\{x^4\}$, $\{xy^{k(r-1)}\}_{k=0}^{2l}$, $\{x^2y^{k(r-1)}\}_{k=0}^{2l}$, $\{x^3y^{k(r-1)}\}_{k=0}^{2l}$, $\{x^5y^{k(r-1)}\}_{k=0}^{2l}$, and $\{x^7y^{k(r-1)}\}_{k=0}^{2l}$ all have weight 2. The other $\frac{m-1}{2}$ conjugacy classes are of the form $\{y^k, y^{kr}, y^{kr^2}, y^{kr^3}\}$ and $\{x^4y^k, x^4y^{kr}, x^4y^{kr^2}, x^4y^{kr^3}\}$, these can have weight $1, 2$ or 3, depending on m. So depending on m, b^4_- may be greater than 0. In fact, for $m \leq 100$, the only m's for which in some representation b^4_- is a positive integer are listed as follows

m	15	35	39	45	51	55	65	75	85	95
b^4_-	1	3	3	4	3	4	5	4	7	8

In fact, one can show that in order for b^4_- to possibly be a positive integer, m must be odd and have at least two distinct prime factors at least one of which, say p, is such that $4|(p-1)$.

(vii) G is a non-solvable Frobenius complement. $G = SL(2,5) \times C_m$, $(m, 30) = 1$, G has $SU(4)$ representation ρ in the form $\phi_{i_1} \otimes \chi \oplus \phi_{i_2} \otimes \chi^{-1}$ where ϕ_i are the 2 2-dimensional fixed point free $SU(2)$ representation of $SL(2,5)$ and χ is a nontrivial character of C_m. From the character table of $SL(2,5)$, we see that G has 9m conjugacy classes. If $i_1 = i_2$, then these conjugacy classes except the identity all have weight 2, so $b^4 = \sigma = 8 + 9(m-2)$ and $b^4_- = 0$.

Now consider the case $i_1 \neq i_2$, $\rho = \phi_1 \otimes \chi \oplus \phi_2 \otimes \chi^{-1}$. From the character table of $SL(2,5)$, we see that ϕ_1 and ϕ_2 have the same characters on all conjugacy classes except the ones with order 5 or 10. The conjugacy classes on which ϕ_i's have the same character can all be seen to have weight 2 in the representation ρ. Some careful computation of the weight for the conjugacy classes of elements of order 5 or 10 yields the following result:

$$b^4_- = 2\lceil \frac{m-5}{10} \rceil + 4\lceil \frac{m-4}{10} \rceil + 2 \begin{cases} \lceil \frac{m}{10} \rceil & \text{if } \lceil \frac{m}{10} \rceil \equiv 0(3) \\ \lceil \frac{m}{10} \rceil - 1 & \text{if } \lceil \frac{m}{10} \rceil \equiv 1(3) \\ \lceil \frac{m}{10} \rceil - 1 & \text{if } \lceil \frac{m}{10} \rceil \equiv 2(3), m \equiv 1, 2(5) \\ \lceil \frac{m}{10} \rceil & \text{if } \lceil \frac{m}{10} \rceil \equiv 2(3), m \equiv 3, 4(5) \end{cases}$$

$(viii)$ G is a non-solvable Frobenius complement, $G = \{(h, \alpha) \in \tilde{S_5} \times D_m | sgn(h) = sgn(\alpha)\}$, $(m, 30) = 1$. $G_0 = SL(2,5) \times C_m$. Since the representation ρ for

G is $Ind_{G_0}^G(\phi_1 \otimes \chi)$, and ρ on G_0 is $\phi_1 \otimes \chi \oplus \phi_2 \otimes \chi^{-1}$, previous computations for ρ on G_0 show that b^4_- is positive. Hence, a fortiori, b^4_- must be positive for ρ on G. In fact, it's easy to see from the character table and calculations for the G_0 case that

$$b^4_- = \lceil \frac{m-5}{10} \rceil + 2\lceil \frac{m-4}{10} \rceil + \begin{cases} \lceil \frac{m}{10} \rceil & \text{if } \lceil \frac{m}{10} \rceil \equiv 0(3) \\ \lceil \frac{m}{10} \rceil - 1 & \text{if } \lceil \frac{m}{10} \rceil \equiv 1(3) \\ \lceil \frac{m}{10} \rceil - 1 & \text{if } \lceil \frac{m}{10} \rceil \equiv 2(3), m \equiv 1, 2(5) \\ \lceil \frac{m}{10} \rceil & \text{if } \lceil \frac{m}{10} \rceil \equiv 2(3), m \equiv 3, 4(5) \end{cases}$$

In summary, for the resolution X of $\mathbb{R}^8/G$ to have $b^4_- > 0$, i.e. for X to admit torsion-free Spin(7)-structure, then G must either be cyclic of order n, where $n \neq 1, 2, 3$ and 6; or $G = \langle x, y | x^8 = y^m = 1, x^{-1}yx = y^r \rangle$, where $r^4 = 1(m)$, $(2, m) = (r-1, m) = 1$, and m is odd with at least two distinct prime factors, one of which, say p, is such that $4|(p-1)$; or G is a non-solvable Frobenius complement such that $G = SL(2,5) \times C_m$ where $(m, 30) = 1$; or G is a non-solvable Frobenius complement such that $G = \{(h, \alpha) \in \tilde{S_5} \times D_m | sgn(h) = sgn(\alpha)\}$, $(m, 30) = 1$.

Bibliography

[1] M. Atiyah & G. Segal, 'On Equivariant Euler Characteristic', J. Geom. and Phys. 6 (1989), 671-677.

[2] D.V. Alekseevskii, 'Riemannian Spaces with Exceptional Holonomy', Functional Anal. Appl. 2 (1968) 97-105.

[3] V.V. Batyrev & D.I. Dais, 'Strong McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry', Preprint 1995.

[4] M. Berger, 'Sur les groupes d'holonomie homogène des variétés à connexion affines et des variétés riemanniennes', Bull. Soc. Math. France 83 (1955) 279-330.

[5] R.L. Bryant, 'Metrics with Exceptional Holonomy', Ann. Math. 126 (1987) 525-576.

[6] R.L. Bryant & S.M. Salamon, 'On the Construction of Some Complete Metrics with Exceptional Holonomy', Duke Math. J. 58 (1989) 829-850.

[7] M.J. Collins, private conversations, Hilary Term 1996.

[8] T. Eguchi & A.J. Hanson, 'Asymptotically Flat Solutions to Euclidean Gravity', Phys. Lett. 74B (1978) 249-251.

[9] W. Fulton, Introduction to Toric Varieties, P.U.P.

[10] R. Harvey, H.B. Lawson Jr., Calibrated Geometry, Acta Math., 148 (1982), 47-157.

[11] D.D. Joyce, 'Compact Riemannian 7-manifolds with Holonomy G_2. I', J. of Diff. Geom. 43 (1996), 291-328.

[12] D.D. Joyce, 'Compact Riemannian 7-manifolds with Holonomy G_2. II', J. of Diff. Geom. 43 (1996), 329-375.

[13] D.D. Joyce, 'Compact Riemannian 8-manifolds with Holonomy Spin(7)', Invent. Math. 123, 507-552 (1996).

[14] D.D. Joyce, 'Compact Manifolds with Exceptional Holonomy Spin(7) ', Math. Ann. 300 165-180(1994).

[15] P. B. Kronheimer, 'The Construction of ALE Spaces as HyperKähler Quotients', J. Diff. Geom. 29 (1989), 665-683.

[16] C. LeBrun, M. Singer, 'A Kummer-type Construction of Self-dual 4-manifolds', preprint (1993).

[17] D.G. Markushevich, M.A. Olshanetsky, A.M. Perelomov, 'Description of a Class of Superstring Compactifications Related to Semi-Simple Lie Algebras', Comm. Math. Phys. 111, 247-274 (1987).

[18] D.N. Page, 'A Physical Picture of the $K3$ Gravitational Instanton', Phys. Lett. 80B (1978), 55-57.

[19] D.S. Passman, 'Permutation Groups', W.A. Benjamin Inc., 1968.

[20] M. Reid, 'Young Person's Guide to Canonical Singularities', In: Algebraic Geometry, Bowdoin 1985, Proc. Symp. Pure Math. 46(1987), 345- 416.

[21] S.S. Roan, 'On the Generalization of Kummer Surfaces', J. Diff. Geom. 30(1989) 523-537.

[22] S.S. Roan, S.-T. Yau, 'On Ricci Flat 3-fold', Acta Math Sinica, New Series, 1987, Vol 3, No. 3, 256-288.

[23] S.M. Salamon, 'Quarternionic KählerManifolds Invent. Math. 67, 143-171 (1982).

[24] S.M. Salamon, 'Riemannian Geometry and Holonomy Groups', Pitman Res. Notes in Math. 201, Longman, Harlow, 1989.

[25] P. Topiwala, 'A New Proof of the Existence of Kähler- Einstein Metrics on $K3$', Inv. Math. 89 (1987), 425-448.

[26] M. Y. Wang, 'Parallel Spinors and Parallel Forms', Ann. Global Anal. Geom. 7, 59-68 (1989).

[27] J. A. Wolf, 'Spaces of Constant Curvature', McGraw-Hill, 1967.

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