Singularities of special Lagrangian submanifolds

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recommended reading:

math.DG/0111111
math.DG/0310460

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Almost Calabi-Yau $m$-folds

An *almost Calabi-Yau $m$-fold* $(M, J, g, \Omega)$ is a compact complex $m$-fold $(M, J)$ with a Kähler metric $g$ with Kähler form $\omega$, and a nonvanishing holomorphic $(m, 0)$-form $\Omega$, the holomorphic volume form. It is a *Calabi-Yau $m$-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla \Omega = 0$, the holonomy group $\text{Hol}(g) \subseteq \text{SU}(m)$, and $g$ is Ricci-flat.
Special Lagrangian $m$-folds

Let $(M, J, g, \Omega)$ be an almost Calabi-Yau $m$-fold. Let $N$ be a real $m$-submanifold of $M$. We call $N$ special Lagrangian (SL) if $\omega|_N \equiv \text{Im}\, \Omega|_N \equiv 0$, and SL with phase $e^{i\theta}$ if $\omega|_N \equiv (\cos \theta \text{Im}\, \Omega - \sin \theta \text{Re}\, \Omega)|_N \equiv 0$. If $(M, J, g, \Omega)$ is a Calabi-Yau $m$-fold then $\text{Re}\, \Omega$ is a calibration on $(M, g)$, and $N$ is an SL $m$-fold iff it is calibrated with respect to $\text{Re}\, \Omega$. 
Let \((M, J, g, \Omega)\) be an almost Calabi–Yau \(m\)-fold and \(N\) a compact \(\text{SL}_m\)-fold in \(M\). Let \(\mathcal{M}_N\) be the moduli space of \(\text{SL}_m\) deformations of \(N\). We ask:

1. Is \(\mathcal{M}_N\) a manifold, and of what dimension?

2. Does \(N\) persist under deformations of \((J, g, \Omega)\)?

3. Can we compactify \(\mathcal{M}_N\) by adding a ‘boundary’ of singular \(\text{SL}_m\)-folds? If so, what are the singularities like?
These questions concern the **deformations** of SL $m$-folds, **obstructions** to their existence, and their **singularities**. Questions 1 and 2 are fairly well understood, and we shall discuss them in the first half of this lecture. Question 3 is an active area of research, and will be discussed in the second half, and next lecture.
The answer to Question 1, on deformations of SL $m$-folds, was given by McLean in 1990 (in the Calabi-Yau case).

**Theorem.** Let $(M, J, g, \Omega)$ be an almost Calabi–Yau $m$-fold, and $N$ a compact SL $m$-fold in $M$. Then the moduli space $\mathcal{M}_N$ of SL deformations of $N$ is a smooth manifold of dimension $b^1(N)$, the first Betti number of $N$. 
Here is a sketch of the proof. Let $\nu \to N$ be the normal bundle of $N$ in $M$. Then $J$ identifies $\nu \cong TN$ and $g$ identifies $TN \cong T^*N$. So $\nu \cong T^*N$. We can identify a small tubular neighbourhood $T$ of $N$ in $M$ with a neighbourhood of the zero section in $\nu$, identifying $\omega$ on $M$ with the symplectic structure on $T^*N$. Let $\pi : T \to N$ be the obvious projection.
Then graphs of small 1-forms $\alpha$ on $N$ are identified with submanifolds $N'$ in $T \subset M$ close to $N$. Which $\alpha$ correspond to $SL_m$-folds $N'$?

Well, $N'$ is special Lagrangian iff $\omega|_{N'} \equiv \text{Im } \Omega|_{N'} \equiv 0$.

Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so this holds iff $\pi_*(\omega|_{N'}) = \pi_*(\text{Im } \Omega|_{N'}) = 0$.

We regard $\pi_*(\omega|_{N'})$ and $\pi_*(\text{Im } \Omega|_{N'})$ as functions of $\alpha$. 
Calculation shows that
\[ \pi_*(\omega|_{N'}) = d\alpha \] and
\[ \pi_*(\text{Im } \Omega|_{N'}) = F(\alpha, \nabla\alpha), \]
where \( F \) is nonlinear. Thus, \( M_N \) is locally the set of small 1-forms \( \alpha \) on \( N \) with \( d\alpha \equiv 0 \) and \( F(\alpha, \nabla\alpha) \equiv 0 \). Now
\[ F(\alpha, \nabla\alpha) \approx d(\ast\alpha) \] for small \( \alpha \). So \( M_N \) is locally approximately the set of 1-forms \( \alpha \) with \( d\alpha = d(\ast\alpha) = 0 \). But by Hodge theory this is the de Rham group \( H^1(N, \mathbb{R}) \), of dimension \( b^1(N) \).
Question 2, on obstructions to the existence of SL \( m \)-folds, can locally be answered using the same methods.

**Theorem.** Let \( M_t : t \in (-\epsilon, \epsilon) \) be a family of almost Calabi–Yau \( m \)-folds, and \( N_0 \) a compact SL \( m \)-fold of \( M_0 \). If \([\omega_t|_{N_0}] = [\text{Im } \Omega_t|_{N_0}] = 0 \) in \( H^*(N_0, \mathbb{R}) \) for all \( t \), then \( N_0 \) extends to a family \( N_t : t \in (-\delta, \delta) \) of SL \( m \)-folds in \( M_t \), for \( 0 < \delta \leq \epsilon \).
Singular SL $m$-folds

General singularities of SL $m$-folds may be very bad, and difficult to study. Would like a class of singular SL $m$-folds with nice, well-behaved singularities to study in depth. Would like these to occur often in real life, i.e. of finite codimension in the space of all SL $m$-folds. SL $m$-folds with isolated conical singularities (ICS) are such a class.
Let $N$ be an SL $m$-fold in $M$ whose only singular points are $x_1, \ldots, x_n$. Near $x_i$ we can identify $M$ with $\mathbb{C}^m \cong T_{x_i}M$, and $N$ near $x_i$ approximates an SL $m$-fold in $\mathbb{C}^m$ with singularity at $0$. We say $N$ has isolated conical singularities if near $x_i$ it converges with order $O(r^{\mu_i})$ for $\mu_i > 1$ to an SL cone $C_i$ in $\mathbb{C}^m$ nonsingular except at $0$. 
SL $m$-folds with ICS have a rich theory.

- **Examples.** Many examples of SL cones $C_i$ in $\mathbb{C}^m$ have been constructed. Rudiments of classification for $m = 3$.

- **Regularity near $x_1, \ldots, x_n$.** Let $\iota : N \to M$ be the inclusion. If $\nabla^k \iota$ converges to $C_i$ near $x_i$ with order $O(r^{\mu_i-k})$ for $k = 0, 1$ then it does so for all $k \geq 0.$
• **Deformation theory.** The moduli space $\mathcal{M}_N$ of deformations of $N$ is locally homeomorphic to $\Phi^{-1}(0)$, for smooth $\Phi : I \to O$ and fin. dim. vector spaces $I, O$ with $I$ the image of $H^1_{cs}(N', \mathbb{R})$ in $H^1(N', \mathbb{R})$, $N' = N \setminus \{x_1, \ldots, x_n\}$, and $\dim O = \sum_{i=1}^n s\text{-ind}(C_i)$. Here $s\text{-ind}(C_i) \in \mathbb{N}$ is the stability index, the obstructions from $C_i$. If $s\text{-ind}(C_i) = 0$ for all $i$ then $\mathcal{M}_N$ is smooth.
• **Desingularization.** Let $C$ be an SL cone in $\mathbb{C}^m$, non-singular except at 0. A non-singular SL $m$-fold $L$ in $\mathbb{C}^m$ is **Asymptotically Conical (AC)** $C$ if $L$ converges to $C$ at infinity with order $O(r^\nu)$ for $\nu < 1$. Then $tL$ converges to $C$ as $t \to 0_+$. Thus, AC SL $m$-folds model how families of nonsingular SL $m$-folds develop singularities modelled on $C$. 

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If $N$ is an SL $m$-fold with ICS at $x_1, \ldots, x_n$ and cones $C_i$, and $L_1, \ldots, L_n$ are AC SL $m$-folds in $\mathbb{C}^m$ with cones $C_i$, then under cohomological conditions we can construct a family of compact nonsingular SL $m$-folds $\tilde{N}^t$ for small $t > 0$ converging to $N$ as $t \to 0$, by gluing $tL_i$ into $N$ at $x_i$, all $i$. 
Here is how this works. Let \( B_\varepsilon(0) \) be an open ball of small radius \( \varepsilon > 0 \) in \( \mathbb{C}^m \), and choose a local diffeomorphism \( \Upsilon_i : B_\varepsilon(0) \to M \) with \( \Upsilon_i(0) = x_i \), that identifies \( C_i \) in \( \mathbb{C}^m \) with the tangent cone to \( N \) at \( x_i \), and \( \Upsilon_i^*(\omega) = \omega_0 \), for \( \omega \) the Kähler form on \( M \) and \( \omega_0 \) the Hermitian form on \( \mathbb{C}^m \). Write \( \Sigma_i = C_i \cap S^{2m-1} \). Then \( \iota_i : (\sigma, r) \mapsto r\sigma \) is a diffeomorphism \( \iota_i : \Sigma_i \times (0, \infty) \to C_i \setminus \{0\} \).
For $0 < \epsilon' < \epsilon$ small there is a unique $\phi_i : \Sigma_i \times (0, \epsilon') \to \mathbb{C}^m$ such that $\text{Im}(\gamma_i \circ \phi_i)$ coincides with $N \setminus \{x_i\}$ near $x_i$, and $(\phi_i - \iota_i)(\sigma, r)$ is perpendicular to $T_{r\sigma}C_i$ in $\mathbb{C}^m$ for all $(\sigma, r) \in \Sigma_i \times (0, \epsilon')$. These are distinguished coordinates on $N$ near $x_i$. Regard $\phi_i - \iota_i$ as a small closed 1-form on $C_i$. Regularity theory gives

$$\nabla^k(\phi_i - \iota_i) = O(r^{\mu_i - k})$$

as $r \to 0$ for some $\mu_i > 1$ and all $k \geq 0$. 

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Similarly, for $R \gg 0$ there is a unique $\psi_i : \Sigma_i \times (R, \infty) \to \mathbb{C}^m$ such that $\text{Im} \psi_i$ coincides with $L_i$ near $\infty$, and $(\phi_i - \nu_i)(\sigma, r)$ is perpendicular to $T_{r,\sigma}C_i$ in $\mathbb{C}^m$ for all $(\sigma, r) \in \Sigma_i \times (R, \infty)$. These are distinguished coordinates on $L_i$ near $\infty$. Regularity gives $\nabla^k(\psi_i - \nu_i) = O(r^{\nu_i-k})$ as $r \to \infty$ for some $\nu_i < 1$ and all $k \geq 0$. We assume $\nu_i < -1$ for no obstructions, or $\nu_i = -1$ and $m < 6$. 

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Fix $\tau \in (0, 1)$. Let $t > 0$ with $2t^\tau < \epsilon'$ and $t^\tau > tR$. Define a compact, nonsingular Lagrangian $N^t$ in $M$ to be $N$ outside $\gamma_i \circ \phi_i(\Sigma_i \times (0, 2t^\tau))$ for all $i$, to be $\gamma_i(tL_i)$ outside $\psi_i(\Sigma_i \times (t^{\tau-1}, \infty))$ in $L_i$, and to interpolate smoothly between these on $\Sigma_i \times [t^\tau, 2t^\tau]$. On $\Sigma_i \times [t^\tau, 2t^\tau]$ we have 
\[
\phi_i(\sigma, r) \equiv \nu_i(\sigma, \tau) + O(t^{\mu_i \tau}) \quad \text{and} \quad t\psi_i(\sigma, t^{-1}r) \equiv \nu_i(\sigma, r) + O(t^{\nu_i(\tau-1)+1}),
\]
so $|\phi_i(\sigma, r) - t\psi_i(\sigma, t^{-1}r)|$ is small.
This \( N^t \) is approximately special Lagrangian, as \( \omega|_{N^t} \equiv 0 \) and \( \text{Im } \Omega|_{N^t} \) is small. Banach norms of \( \text{Im } \Omega|_{N^t} \) measure the ‘error’, e.g. 
\[
\| \text{Im } \Omega|_{N^t} \|_{C^0} = O(t^{(\mu_i-1)\tau}) + O(t^{(\nu_i-1)(\tau-1)})
\]
for small \( t \). But also, \( N^t \) is nearly singular for small \( t \), with second fundamental form \( \| B \|_{C^0} = O(t^{-1}) \), Riemann curvature \( \| R(g|_{N^t}) \|_{C^0} = O(t^{-2}) \) and injectivity radius \( \delta(g|_{N^t}) = O(t) \).
We show using analysis that we can deform $N^t$ to a nearby SL $m$-fold $\tilde{N}^t$. We must solve the nonlinear elliptic p.d.e. $Q(\tilde{N}^t) = \text{Im } \Omega|_{\tilde{N}^t} \equiv 0$. We make the solution as the limit of a series of Lagrangians $(N^t_k)_{k=0}^\infty$ with $N^t_0 = N^t$, which roughly inductively satisfy
\[
\text{d}Q|_{N^t_k}(N^t_{k+1} - N^t_k) = -\text{Im } \Omega|_{N^t_k}.
\]
The series converges if the initial ‘error’ is small enough, in terms of $\|B\|_{C^0}, \|R(g|_{N^t})\|_{C^0}, \delta(g|_{N^t}), \ldots$. 

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Three things can go wrong in this proof:
(A) For the ‘error’ to be small and the series to converge, we need $\tau \approx 1$ and $\nu_i < -1$ for all $i$, or $\nu_i = -1$ and $m < 6$.
(B) To make the Lagrangian $N^t$ we join $N \backslash \{x_1, \ldots, x_n\}$ and $\Upsilon(tL_1), \ldots, \Upsilon(tL_n)$. Effectively we must find a closed 1-form on $\Sigma_i \times [t^\tau, 2t^\tau]$ interpolating between small closed 1-forms $\phi_i(\sigma, r) - \nu_i(\sigma, \tau)$ and $t\psi_i(\sigma, t^{-1}r) - \nu_i(\sigma, r)$. 
Now $\phi_i(\sigma, r) - \nu_i(\sigma, \tau)$ is exact, and $t\psi_i(\sigma, t^{-1}r) - \nu_i(\sigma, r)$ is exact if $\nu_i < -1$, but if $\nu_i \geq -1$ then we can have $[t\psi_i(\sigma, t^{-1}r) - \nu_i(\sigma, r)] \neq 0$ in $H^1(\Sigma_i, \mathbb{R})$. This is a global topological obstruction to making $N^t$ Lagrangian. To overcome it, we modify $N' = N \backslash \{x_1, \ldots, x_n\}$ by a small closed 1-form $\alpha^t$ whose cohomology class $[\alpha^t] \in H^1(N', \mathbb{R})$ satisfies $[\alpha^t]|_{\Sigma_i} = [t\psi_i(\sigma, t^{-1}r) - \nu_i(\sigma, r)]$ in $H^1(\Sigma_i, \mathbb{R})$ for all $i$. Such $\alpha^t$ need not exist.
(C) Suppose $N$ is connected, but $N' = N \setminus \{x_1, \ldots, x_n\}$ has $l > 1$ connected components, which meet at $x_1, \ldots, x_n$. Then the Laplacian $\Delta^t$ on functions on $N^t$ has $l - 1$ small eigenvalues of size $O(t^{m-2})$. The corresponding eigenfunctions are approximately constant on each component of $N'$, and change on the ‘necks’ $\Upsilon(tL_i)$. The linearization $dQ|_{N^t}$ of $Q$ at $N^t$ is basically $\Delta^t$. So small eigenvalues of $\Delta^t$ can cause the series $(N^t_k)_{k=0}^{\infty}$ to diverge.
To overcome this, the components of $N^t_k - N^t$ in the directions of the $l - 1$ eigenfunctions with small eigenvalues must remain small for all $k \geq 0$. There is a *global cohomological obstruction* to doing this, that there should be a small closed $(m - 1)$-form $\beta^t$ on $N'$ whose cohomology class $[\beta^t] \in H^{m-1}(N', \mathbb{R})$ satisfies $[\beta^t]|_{\Sigma_i} = [\ast (t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r))]$ in $H^{m-1}(\Sigma_i, \mathbb{R})$ for all $i$. Such $\beta^t$ need not exist.
We understand obstructions (B),(C) using relative cohomology. As \( \omega|_{\tilde{N}^t} \equiv \text{Im } \Omega|_{\tilde{N}^t} \equiv 0 \), we have classes \([\omega], [\text{Im } \Omega]\) in \( H^k(M, N^t; \mathbb{R}) \) for \( k = 2, m \). Also we have \([\omega_0], [\text{Im } \Omega_0]\) in \( H^k(\mathbb{C}^m, L_i; \mathbb{R}) \). An exact sequence gives \( H^k(\mathbb{C}^m, L_i; \mathbb{R}) \cong H^{k-1}(L_i; \mathbb{R}) \), and as \( \Sigma_i \) is the ‘boundary’ of \( L_i \) we restrict to \( H^{k-1}(\Sigma_i; \mathbb{R}) \). So \([\omega_0], [\text{Im } \Omega_0]\) induce classes in \( H^{k-1}(L_i; \mathbb{R}) \) for all \( i \), which must lie in the image of \( H^{k-1}(N'; \mathbb{R}) \).