Digraph girth via chromatic number

Peter Keevash * Zhentao Li † Bojan Mohar ‡ Bruce Reed §

Abstract

Let D be a digraph. The chromatic number $\chi(D)$ of D is the smallest number of colours needed to colour the vertices of D such that every colour class induces an acyclic subdigraph. The girth of D is the length of a shortest directed cycle, or ∞ if D is acyclic. Let $G(k,n)$ be the maximum possible girth of a digraph on n vertices with $\chi(D) > k$. It is shown that $G(k,n) \geq \lfloor n^{1/k} \rfloor$ and $G(k,n) \leq (3\log_2 n \log_2 \log_2 n)^{1-1/k} n^{1/k}$ for $n \geq 3$ and $k \geq 2$.

1 Introduction

The *chromatic number* $\chi(D)$ of a digraph D is the minimum number k such that $V(D)$ can be partitioned into k parts, none of which contains a cycle of D (see [2, 10]). By a *cycle* we always mean a directed cycle, and we define the *girth* of D as the length of a shortest cycle in D (∞ if D is acyclic).

Given a digraph D with n vertices and chromatic number more than k, how large can the girth of D be? This question was posed by one of the authors [9], who conjectured a bound of $O(\sqrt{n})$ in the case $k = 2$. The analogous question for the usual chromatic number in graphs has a long history. A celebrated result of Erdős [5] shows that there are graphs with both girth and chromatic number larger than any specified constant. An example of a quantitative answer to the question is for graphs with girth at least 3 (i.e. triangle-free): the maximum chromatic number of a triangle-free graph on n vertices is $\Theta(\sqrt{n/\log n})$ by results of Ajtai, Komlós and Szemerédi [1] and of Kim [8]. At the other extreme, a graph on n vertices with chromatic number 3 could consist of single odd cycle of length n or $n-1$. On the other hand, any graph with chromatic number at least 4 contains a subgraph with minimum degree at least 3, and so a cycle of length $O(\log n)$; probabilistic constructions (see [4]) show that this is the correct order of magnitude. Similar bounds apply for the acyclic chromatic number of a graph G, which is the minimum number k such that $V(G)$ can be partitioned into k parts, none of which contains a cycle of G. In one direction this is because the

*School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK. Email: p.keevash@qmul.ac.uk. Research supported in part by ERC grant 239696 and EPSRC grant EP/G056730/1.

†Laboratoire de l'Informatique du Parallélisme, ENS de Lyon, France Email: zhentao.li@ens-lyon.fr

‡Department of Mathematics, Simon Fraser University, Burnaby, BC V5A 1S6. Email: mohar@sfu.ca. On leave from Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

§McGill University, Montreal, Canada. Email: breed@cs.mcgill.ca. On leave from Project Mascotte, INRIA, Laboratoire I3S, CNRS, Sophia-Antipolis, France.

chromatic number is at least the acyclic chromatic number; for the other, it is not hard to adapt the probabilistic construction to obtain graphs of girth $\Omega(\log n)$ and acyclic chromatic number larger than any fixed constant.

Another motivation to study the aforementioned question became apparent in a recent work of Harutyunyan and Mohar [7]. They generalized to digraphs an old result of Bollobás [3] that for every $k \geq 4$ there is $\alpha > 0$ and infinitely many graphs G of chromatic number k such that every 4-chromatic subgraph of G contains at least $\alpha|V(G)|$ vertices. The extension to digraphs obtained in [7] proves the same for all 3-chromatic subdigraphs, but the conclusion does not hold for 2-chromatic subdigraphs, as every digraph D with $\chi(D) \geq 3$ contains a cycle of length $o(|V(D)|)$, which gives a small 2-chromatic subdigraph. The last conclusion is a consequence of our Theorem 2 (the case $k = 2$).

2 Short cycles in digraphs

Let $G(k, n)$ be the maximum possible girth of a digraph on n vertices with $\chi(D) > k$. Note that the n-cycle C_n has $\chi(C_n) = 2$, so $G(1, n) = n$. Thus we may suppose that $k \geq 2$. We start with a lower bound for $G(k, n)$. Note that the order of magnitude is very different than that for graphs.

Theorem 1. *For every $k \geq 2$ we have $G(k, n) \geq \lfloor n^{1/k} \rfloor$.*

Proof. Consider the following construction. Let $C_r^1 = C_r$ denote the directed cycle of length r. For $i \geq 1$ let C_r^{i+1} denote the digraph on r^{i+1} vertices, divided into r parts V_j, $j \in \mathbb{Z}_r$ of size r^i, so that each part V_j induces a copy of C_r^i, and for each $j \in \mathbb{Z}_r$ we have all edges from V_j to V_{j+1}. Observe that the girth of C_r^i is equal to r. We claim that $\chi(C_r^i) \geq i + 1$ for $1 \leq i \leq r - 1$. This is clear for $i = 1$. Now we argue by induction for $i \geq 2$. Consider any colour class X in any colouring of C_r^i. Since $C_r^i[X]$ is acyclic there must be some part V_j disjoint from X. Then $C_r^i[V_j] = C_r^{i-1}$ is coloured using one fewer colour, so $\chi(C_r^i) \geq \chi(C_r^{i-1}) + 1 \geq i + 1$.

To deduce the theorem, let $r = \lfloor n^{1/k} \rfloor$, and let D be the digraph obtained from C_r^k by adding $n - r^k$ isolated vertices. (We can assume $r \geq 3$, as the theorem is obviously true when $r \leq 2$.) Then $\chi(D) > k$ and the girth of D is r. □

We remark that $\chi(C_r^i) = i + 1$ for $1 \leq i \leq r - 1$; this is easy to prove by induction.

Our main result is an upper bound that matches the lower bound up to a polylogarithmic factor. Define $g(k, n) = (3 \log_2 n \log_2 \log_2 n)^{1-1/k} n^{1/k}$.

Theorem 2. *For $n \geq 3$ and $k \geq 2$ we have $G(k, n) \leq g(k, n)$.*

To prove Theorem 2 we introduce an additional digraph parameter. We say that $S \subseteq V(D)$ is a *hitting set* if every cycle in D contains at least one vertex of S. Let $H(r, n)$ be the smallest number h such that any digraph D on n vertices with girth more than r has a hitting set of size h. Note that if $r \geq n$ such a digraph is acyclic, so $H(r, n) = 0$. Let $h(r, n) = 3(n/r) \log_2 n \log_2 \log_2 n$.

Theorem 3. *For $n \geq r \geq 3$ we have $H(r, n) \leq h(r, n)$.*

Proof. For every fixed $r \geq 3$, we argue by induction on n. The base case is $n = r$, when $H(r,n) = 0$. Now suppose that $n > r$. Note that we can assume that $h(r,n) < n$, since the entire vertex set is trivially a hitting set, so we have $r > 3\log_2 n \log_2\log_2 n$. Since $3\log_2 n\log_2\log_2 n > n$ for $3 \leq n \leq 37$ we can assume that $r \geq 38$. The idea for the induction step is as follows. Suppose D is a digraph on n vertices with no cycle of length at most r. We find a small set S of vertices and a partition of $V(D) \setminus S$ as $A \cup B$ so that there are no edges of D from A to B. Then we apply the induction hypothesis to find hitting sets in $D[A]$ and $D[B]$, to which we add S to obtain a hitting set in D.

To find S we fix any vertex v and consider its iterated neighbourhoods, defined as follows. Given a vertex u, the *out-distance* of u from v is the length of a shortest path in D from v to u (or ∞ if there is no such path). The *in-distance* of u from v is the out-distance of v from u. Let $N_i^+(v)$ be the set of vertices at out-distance i from v and $N_i^-(v)$ be the set of vertices at in-distance i from v. Let $N_{\leq i}^+(v) = \cup_{j=1}^i N_j^+(v)$ and $N_{\leq i}^-(v) = \cup_{j=1}^i N_j^-(v)$. Let $t = \lfloor r/2 \rfloor$. Note that $N_{\leq t}^+(v) \cap N_{\leq t}^-(v) = \emptyset$, since there is no cycle of length at most r.

Now we suppose for a contradiction that D does not have a hitting set of size $h(r,n)$. We will see that this forces the iterated neighbourhoods of v to grow rapidly (see [6] for a similar argument based on edge expansion). To see this, fix $i < t$, let $S_i^+ = N_{i+1}^+(v)$, $A_i^+ = N_{\leq i}^+(v)$, $B_i^+ = V(D) \setminus (A_i^+ \cup S_i^+)$, and note that there are no edges of D from A_i^+ to B_i^+. Write $m = |A_i^+|$. By induction hypothesis, $D[A_i^+]$ has a hitting set of size $h(r,m)$ and $D[B_i^+]$ has a hitting set of size $h(r,|B_i^+|) \leq h(r,n-m)$. Adding S_i^+ gives a hitting set of D, which by assumption has size more than $h(r,n)$, so $|S_i^+| > h(r,n) - h(r,m) - h(r,n-m)$. We estimate $h(r,n) - h(r,m) - h(r,n-m) \geq 3r^{-1}\log_2\log_2 n(n\log_2 n - m\log_2 m - (n-m)\log_2 n) \geq c_i^+|A_i^+|$, where $c_i^+ = 3r^{-1}\log_2\frac{n}{|A_i^+|}\log_2\log_2 n$. Therefore $|A_{i+1}^+| = |A_i^+| + |S_i^+| > (1 + c_i^+)|A_i^+|$.

To estimate the growth of $|A_i^+|$ we divide the steps into groups G_j, $j \geq 1$, such that for $i \in G_j$ we have $n^{1-2^{-j+1}} \leq |A_i^+| < n^{1-2^{-j}}$. Then for each $i \in G_j$ we have $|A_{i+1}^+|/|A_i^+| > 1 + d_j$, where $d_j := 3r^{-1}2^{-j}\log_2 n\log_2\log_2 n$. Also, the total expansion factor over $i \in G_j$, excluding the last element of G_j, is at most $n^{2^{-j}}$. Therefore $(1+d_j)^{|G_j|-1} \leq n^{2^{-j}}$. Note that $d_j < 1$, as $r > 3\log_2 n\log_2\log_2 n$. Using the inequality $(1+1/x)^x \geq 2$ for $x \geq 1$ we obtain $n^{2^{-j}} \geq 2^{d_j(|G_j|-1)}$, so $|G_j| - 1 \leq d_j^{-1}2^{-j}\log_2 n = \frac{r}{3\log_2\log_2 n}$. Let $\ell \in \mathbb{N}$ be such that $n^{1-2^{-\ell+1}} \leq n/2 < n^{1-2^{-\ell}}$; then $\log_2\log_2 n < \ell \leq 1 + \log_2\log_2 n$. Thus we reach a set $|A_{i^+}^+| > n/2$ for some i^+, where

$$i^+ \leq \sum_{j=1}^{\ell} |G_j| \leq (1 + \lfloor \log_2\log_2 n \rfloor)\left(1 + \frac{r}{3\log_2\log_2 n}\right). \tag{1}$$

Since $n > r \geq 38$, we have $\log_2\log_2 n \geq 2$. If $\lfloor \log_2\log_2 n \rfloor = 2$, then (1) implies that

$$i^+ \leq 3 + \frac{r}{\log_2\log_2 n} \leq \frac{3r}{38} + \frac{r}{\log_2\log_2 39} \leq \frac{r}{2}.$$

On the other hand, if $\log_2\log_2 n \geq 3$, then $r > 3\log_2 n\log_2\log_2 n \geq 72$. In this case, (1) gives the same conclusion as above:

$$\begin{aligned} i^+ &\leq (1+\log_2\log_2 n)\left(1+\frac{r}{3\log_2\log_2 n}\right) \\ &\leq \frac{r}{3}+1+\frac{r}{3\log_2\log_2 n}+\frac{r}{3\log_2 n} \\ &\leq \frac{r}{3}+\frac{r}{72}+\frac{r}{9}+\frac{r}{24}=\frac{r}{2}. \end{aligned}$$

In the last calculation we used $r > 3\log_2 n \log_2\log_2 n$ and $r \geq 72$.

The same argument applies to $A_i^- = N_{\leq i}^-(v)$, so we reach a set $|A_{i^-}^-| > n/2$ for some $i^- \leq r/2$. But then $A_{i^+}^+$ and $A_{i^-}^-$ intersect, contradicting the assumption that there is no cycle of length at most r. Thus D does have a hitting set of size $h(r,n)$, which completes the proof by induction. □

Proof of Theorem 2. Suppose that D is a digraph on n vertices with girth more than $r = g(k,n)$. We claim that D has chromatic number at most k. To see this, we repeatedly apply Theorem 3 (we can assume $n \geq r \geq 3$). Let $S_1 = V(D)$. For $i \geq 2$ we apply Theorem 3 to find a hitting set S_i for $D[S_{i-1}]$. Then $|S_i| \leq h(r,|S_{i-1}|) \leq (3r^{-1}\log_2 n\log_2\log_2 n)^{i-1}n$ and $D[S_{i-1}\setminus S_i]$ is acyclic for $i \geq 2$. Since $|S_k| \leq (3r^{-1}\log_2 n \log_2\log_2 n)^{k-1}n \leq r$ and D has girth more than r, $D[S_k]$ is acyclic. Thus we have a k-colouring, whose colour classes are $S_{i-1}\setminus S_i$ $(2 \leq i \leq k)$ and S_k. □

Acknowledgements. We thank the Bellairs Research Institute for their hospitality during the meeting at which this research was conducted.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

[2] D. Bokal, G. Fijavž, M. Juvan, P. M. Kayll and B. Mohar, The circular chromatic number of a digraph, *J. Graph Theory* **46** (2004), 227–240.

[3] B. Bollobás, Chromatic number, girth and maximal degree, *Discrete Math.* **24** (1978), 311–314.

[4] B. Bollobás, *Random Graphs*, Academic Press, London, 1985.

[5] P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.

[6] J. Fox, P. Keevash and B. Sudakov, Directed graphs without short cycles, *Combin. Probab. Comput.* **19** (2010), 285–301.

[7] A. Harutyunyan and B. Mohar, Two results on the digraph chromatic number, *Discrete Math.* **312** (2012), 1823–1826.

[8] J.H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Struct. Algorithms* **7** (1995), 173–207.

[9] B. Mohar, Talk at Bellairs Research Institute meeting on Graph Theory, 2011.

[10] V. Neumann-Lara, The dichromatic number of a digraph, *J. Combin. Theory, Ser. B* **33** (1982), 265–270.