We will look at piecewise linear (PL) 3-manifolds. This is equivalent to the other two perspectives:

- smooth, by Munkres and Whitehead.
- topological, by Moise.

Def $B^n = \{ x \in \mathbb{R}^n \mid \|x\| \le 1 \}$ unit ball,

$\|\cdot\|$ is the Euclidean / L^2-norm.

$S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$.

A space homeomorphic to S^{n-1} is an $(n-1)$-sphere

— " — B^n — " — n-cell.

A top. n-manifold M is a separable metric space s.t.
(countable dense subset)

every point has a n'hood homeomorphic to $\mathbb{R}^n$ or

$\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \ge 0 \}$.

The boundary of M, ∂M, is the set of points

whose n'hoods are homeom. to $\mathbb{R}^n_+$.

Exercise why is $\mathbb{R}^n \not\cong \mathbb{R}^n_+$?

The interior $\mathrm{Int}\, M$ is $M \setminus \partial M$.

∂M is empty or an $(n-1)$-mfld with $\partial\partial M = \emptyset$.

M is <u>closed</u> iff it is cpt & $\partial M = \emptyset$.

M is <u>open</u> iff every component is non-cpt and $\partial M = \emptyset$.

<u>Def</u> A <u>simplicial cplx</u> K is a set of simplices in some $\mathbb{R}^n$ which is locally finite, closed under taking faces, and $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is a face of both σ and τ.

$|K| = \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^n$.

L is a <u>subdivision</u> of K iff $|L| = |K|$ and every simplex in L lies in a simplex of K.

Given simplicial cplxs K_1, K_2, a map $f : |K_1| \to |K_2|$ is <u>piecewise linear</u> (PL) iff $\exists$ subdivision L_i of K_i s.t. $f : |L_1| \to |L_2|$ is <u>simplicial</u>, i.e. takes vertices to vertices and each simplex linearly onto a simplex.

[

$f(\alpha x_1 + \beta x_2 + \gamma x_3) = \alpha f(x_1) + \beta f(x_2) + \gamma f(x_3)$.

with $\alpha + \beta + \gamma = 1$, non-negative

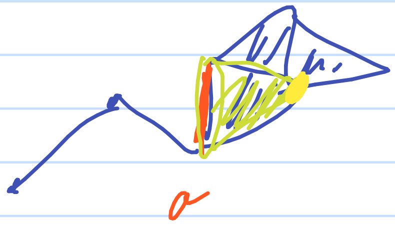
Exercise A composition of PL maps is PL.

Def A *triangulation* of a space X is a pair (T,h) where T is a simplicial complex and $h : |T| \to X$ is a homeomorphism.

Two triangulations (T,h), (T',h') are *compatible* iff $h'^{-1}h : |T| \to |T'|$ is PL.

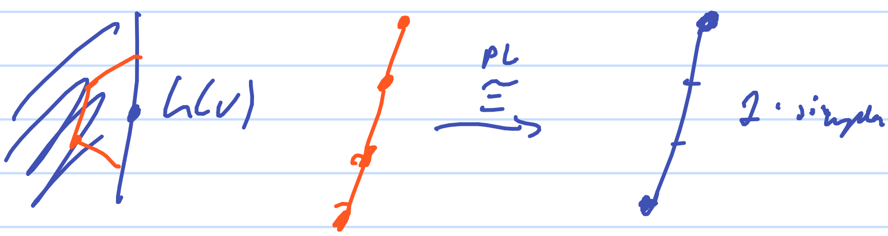
Def Given $\sigma \in K$, simplicial complex, the *star* of σ

$$st(\sigma, K) = \{\tau \in K \mid \exists \rho : \tau \leq \rho \text{ and } \rho \cap \sigma \neq \emptyset\}$$



The *link* $lk(\sigma, K) = \{\tau \in st(\sigma,K) \mid \tau \cap \sigma = \emptyset\}$

Def A triangulation (T,h) of an n-manifold M is *combinatorial* iff $\forall$ vertex v of T, $|lk(v,T)|$ is PL homeomorphic to an $(n-1)$-simplex or the boundary boundary of an n-simplex, if $h(v) \in \partial M$ or $h(v) \in \operatorname{Int} M$ respectively.



If (K, h) is a combinatorial triangulation of M and L is a subdivision of K, then (L, h) is also a combinatorial triangulation.

Def A PL-structure on a mfld M is a maximal, non-empty collection of compatible combinatorial triangulations of M.

A PL-mfld is a manifold with a PL-structure.

A map $f: M_1 \to M_2$ between PL-mflds is a PL-map provided that for some (hence any) triangulations $(T_1, h_1), (T_2, h_2)$ of M_1, M_2 resp. the map $h_2^{-1} f h_1$ is PL.

Warning $\exists$ top mfld with 2 non-compatible triangulations.

$\exists$ top mfld without any PL-structure.

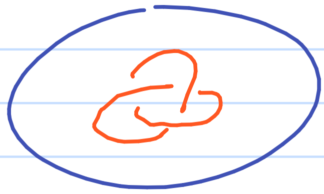
But in dim 3 everything is fine.

Def A submanifold N of a PL-mfld M is a PL-submanifold iff $\exists$ triangulation (T, h) in the PL structure on M and a subcplx $S \leq T$ s.t. $(S, h|_{|S|})$ is a combinatorial triangulation of N.

Note that this gives a PL-structure on N.

Warning we could have vertex $v \in S \subseteq T$ s.t.

in M: $lk(v,M)$ is a 3-sphere



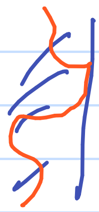
"local knotting"

$lk(v,N)$ — a 1-sphere

i.e. $(lk(v,T), lk(v,S))$ need not be PL homeomorphic to a $(\partial\text{simplex}, \partial\text{face})$ pair of appropriate dimensions.

Known in dim ≤ 3, there is no local knotting.

Def A subcomplex N is **proper** iff $N \cap \partial M = \partial N$.



not proper.

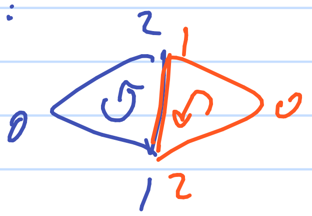
Orientation

Two orderings of the vertices of a simplex are **equivalent** iff they differ by the action of the alternating group on the set.

Thus every simplex of dim ≥ 1 has exactly two equivalence classes of orderings, which we call **orientations**.

An *orientation* of a PL-manifold M is a consistent orientation of every n-simplex in some triangulation T in the PL-structure.

Consistent means:



i.e. the ordering induced to $n-1$ face of two n-simplices is opposite.

[slightly off for ∂-manifolds]

Clearly, if M can be oriented (i.e. if M is *orientable*) $\iff$ $H_n(M, \partial M) = \mathbb{Z}$, and an orientation chooses a generator.

M is *unoriented* iff we didn't pick an orientation.

(whether possible or not).

M is *unorientable* iff it is not orientable.