Our goal is a uniqueness statement

for a prime decomposition (via #).

This is powerfull, especially combined with a converse of Lemma 3.2 (which we will not prove).

<u>Def</u> The <u>rank</u> $rk(G)$ of a group is the smallest number of generators of G.

[this is sometimes known as the <u>Grushko rank</u>.].

We have

Lemma 3.3

If $G = G_1 * G_2$ is finitely generated, then so are G_1, G_2 and

$$rk(G) = rk(G_1) + rk(G_2).$$

This follows directly from

Theorem 3.4 [Grushko's theorem]

Let F_n be the n-generated free group. If $\varphi : F_n \longrightarrow G_1 * G_2$ is an epimorphism, then $\varphi : F_n \cong F_{d_1} * F_{d_2}$, $d_1 + d_2 = n$, and

$\psi \circ \varphi^{-1}\big|_{F_{\alpha_i}}$ is onto G_i.

[We will not prove it.]

Hence, $\pi_1(M)$ cannot have infinitely many connected sum components with non-trivial fundamental group.

But, a priori, it could have infinitely many simply connected such component.

Which brings us to:

Poincaré conjecture [Perelman's thm]

Every closed, connected, simply connected 3-manifold is homeomorphic to S^3.

Def

A homotopy n-sphere is an n-manifold homotopy equivalent to S^n.

A fake 3-cell is a compact contractible 3-manifold which is not homeomorphic to B^3.

Theorem 3.6

A 3-manifold M is a homotopy 3-sphere iff M is closed, connected and simply connected.

Proof

Clearly, if $M \simeq S^3$ then M is connected and simply connected. Also,

$$H_3(M;\mathbb{Z}) \cong H_3(S^3;\mathbb{Z}) \cong \mathbb{Z}$$

and so M is closed.

Now suppose M is closed, connected

and simply connected.

$\pi_1(M) = \mathbb{1} \Rightarrow H_1(M;\mathbb{Z}) = 0$.

By Poincaré duality, since M is closed:

$H_2(M;\mathbb{Z}) = 0$.

Let B be a 3-cell in M, and let

$C = \overline{M \setminus B}$.

$$\mathbb{1} = \pi_1(M) \cong \pi_1(C) \underset{\pi_1(S^2)}{*} \pi_1(B)$$

by van Kampen's theorem, and so

$\pi_1(C) = \mathbb{1}$.

Now we use a Mayer–Vietoris seq-uence:

$$\begin{array}{ccccccc} H_3(M) & \longrightarrow & H_2(C\cap B) & \longrightarrow & H_2(C)\oplus H_2(B) & \longrightarrow & H_2(M) \\ \| & & \| & & \| & & \| \\ \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & & H_2(C) & & 0 \end{array}$$

by inspection

$\therefore$ $H_2(C) = 0$.

Since C is a 3-manifold with boundary, we have $H_n(C) = 0$ $\forall n \geq 3$.

So $H_n(C) = 0$ $\forall n \geq 1$.

By Hurewicz's theorem, $\pi_i(M) = 1$ $\forall i$.

Since C is a simplicial complex, we conclude that $C \simeq *$ using Whitehead's theorem.

[Hence $C \cong B^3$ or C is a fake cell.]

We have a homeomorphism $f_+ : \underset{\substack{\cup \\ B_+}}{B^3} \longrightarrow B$.

Since $C \simeq *$, $f_+|_{\partial B^3} : S^2 \longrightarrow M$

extends to $f_- : \underset{\substack{\cup \\ B_-}}{B^3} \longrightarrow C$.

Hence we have $f : S^3 \longrightarrow M$

extending $f_+ : B^3 \to B$.

The inverse $f_+^{-1} : B \longrightarrow \underset{\substack{\cup \\ B_+}}{B^3}$ restricts to $\underset{\partial C}{\overset{\shortparallel}{f_+^{-1} : \partial B}} \longrightarrow S^2$, which extends to

$\underset{B_-}{\overset{\shortparallel}{f_-^{-1} : C \longrightarrow B^3}}$, Indeed:

∂C has a regular n'hood $N \cong S^2 \times [0,1]$.

This neighbourhood can be mapped onto B^3 with $S^2 \times \{1\}$ mapping to the cone point in B^3. Now we map $C \setminus N$ to this cone point as well.

Hence, f_+^{-1} and f_-^{-1} combine to a map

$$g : M \longrightarrow S^3.$$

We now look at the compositions:

$$g \circ f : S^3 \to S^3$$

on B_+ we have $g \circ f = id$.

on B_- we have $g \circ f \simeq id$ rel ∂B_-.

Also $f \circ g : C \to C$ is homotopic to id relative to ∂C, since $N \simeq C \simeq *$

[uses $\pi_4(C) = 1$]. $\square$

The proof also shows that Poincaré conjecture is equivalent to the non-existence of fake 3-cells, since C

is B^3 iff $M \cong S^3$ and otherwise C is a fake cell.

Primes

Def A 3-manifold M is *prime* iff if $M \cong M_1 \# M_2$ implies that $M_i \cong S^3$ for some i.

Note if $M = S^2 \times I$ then $M \cong B^3 \# B^3$ is a prime decomposition, but $\hat{M} \cong S^3$ is prime.

So weird things happen when M has

boundary components $\cong S^2$.

More generally we have:

Lemma 3.7

Suppose M is a compact 3-manifold with exactly k 2-spheres in ∂M. If $\hat{M} = M_1 \# M_2 \# \dots \# M_q$ is a prime factorisation of $\hat{M}$, then

$$M = M_1 \# M_2 \# \dots \# M_q \# \underbrace{B^3 \# \dots \# B^3}_{k \text{ factors}} .$$

Proof Doing $\# B_3$ adds a 2-sphere to the boundary.