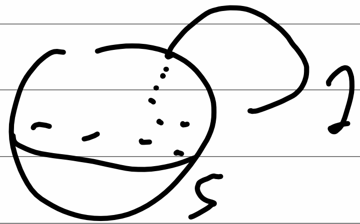
Lemma 3.8

Suppose that M is a 3-manifold and that $\operatorname{Int} M$ contains a 2-sphere S s.t. $M \setminus S$ is connected. Then $M \cong M_1 \# M_2$ where M_1 is a 2-sphere bundle over S^1.

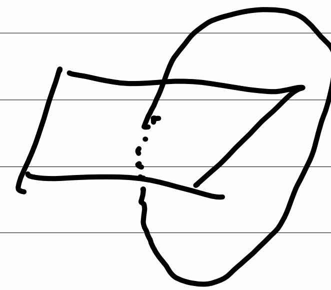
Proof

There exists a 1-sphere J crossing S once transversely, $J \subseteq \operatorname{Int} M$.

Let N be a regular n'hood of $S \cup J$ in $\operatorname{Int} M$.



Locally



$N \cong B \cup S \times [-1,1]$ where $B \cong B^3$ and $\partial B \cap S \times [-1,1]$ consists of two discs, one on $S \times \{-1\}$ and one on $S \times \{1\}$.

Now $\partial N \cong S^2 \# S^2 \cong S^2$.

N cut along S is homeomorphic to $S^2 \times I$ with a ball removed:

$S \times \{\pm 1\}$ are boundary components of

N cut along S, each homeomorphic to S^2, and filling these in results in B^3. So filling the third only gives $S^3 \setminus (B^3 \sqcup B^3) \cong S^2 \times I$.

Thus $M_1 = \hat{N}$ is an S^2-bundle over S^1, and is a factor of M. $\square$

Note that M_1 is orientable iff J is an orientation preserving curve.

Let P be the non-orientable S^2 bundle over S^1.

Example 3.9

$P \# P = P \# (S^2 \times S^1)$.

Take $S \cong S^2$ in P, preimage of a point on S^1.

Now in $P \# P$ we easily find an orientation preserving curve J cutting S transversely at a single point.

So $N(S \cup J) = M_1$ is an $S^2 \times S^1$ factor (after filling the S^2-boundary).

Now $P \# P \setminus M_1$ is P with 2 three-balls removed, together with a tunnel connecting them. So $\widehat{P \# P \setminus M_1} \cong P$.

Theorem 3.10

If M is a compact 3-manifold, then $M = R \# M_1 \# \cdots \# M_k$ where each M_i is an S^2-bundle over S^1, $0 \leq k \leq rk(\pi_1(M))$, and each 2-sphere in R separates R.

Proof

The proof is an induction on $rk\, \pi_1 M$.

If Int M does not contain non-separating two-spheres, then $R = M$ and $k = 0$.

Otherwise, $M = M' \# M_1$, where

M_1 is an S^2-bundle over S^1.

By Lemmata 3.2 and 3.3 we have

$\mathrm{rk}\, \pi_1(M') < \mathrm{rk}\, \pi_1(M)$, and so

$M' = R \# M_2 \# \dots \# M_n$ and thus

$M = R \# M_1 \# \dots \# M_n$ as required. $\square$

Lemma 3.11

Suppose that M is a compact 3-manifold, $M = \hat{M}$, $\pi_1(M)$ is not a non-trivial free product. Then M is prime.

Proof.

Suppose not. Then M admits a factor M_1 which is simply connected by Lemma 3.2.

Hence

$$H_2(M_1, \partial M_1; \mathbb{Z}/2\mathbb{Z}) \cong H^1(M_1; \mathbb{Z}/2) = 0$$

by Poincaré - Lefschetz duality and the universal coefficient theorem over fields.

The long exact sequence for pairs now gives:

$$0 = H_2(M_1, \partial M_1; \mathbb{Z}/2) \to H_1(\partial M_1; \mathbb{Z}/2) \to \underset{\substack{\| \\ 0}}{H_1(M_1; \mathbb{Z}_2)}$$

So $H_1(\partial M_i; \mathbb{Z}/2) = 0$.

But ∂M_i is a disjoint union of closed surfaces which are not spheres; each such contributes non-trivially to $H_1(\partial M_1; \mathbb{Z}/2)$. Hence $\partial M_1 = \emptyset$.

Hence M_1 is a homotopy S^3, and therefore $M_1 \cong S^3$ by the Poincaré conjecture. $\square$

Corollary 3.12

Lens spaces and S^2-bundles over S^1 are prime.

Proof

It suffices to observe that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ do not admit non-trivial free splittings.

$\square$

Def A 3-manifold is irreducible iff every 2-sphere in M bounds a 3-cell.

Clearly, irreducible 3-manifolds are prime.

The converse is almost true:

<u>Lemma 3.13</u>

If M is a prime 3-manifold which is not irreducible, then M is an S^2-bundle over S^1.

<u>Proof</u>

Since M is prime, every separating 2-sphere in $\mathrm{Int}\, M$ bounds a 3-cell. Since M is not irreducible, it must then contain a non-separating 2-sphere.

The result follows from Lemma 3.8 and Corollary 3.12. $\square$