Lemma 3.8

Suppose that $M$ is a 3-manifold and that $\text{Int } M$ contains a 2-sphere $S$.

If $M \setminus S$ is connected, then $M \equiv M_1 \# M_2$, where $M_1$ is a 2-sphere bundle over $S'$.

Proof

There exists a 1-sphere $I$ crossing $S$ once transversely, $I \subseteq \text{Int } M$.

Let $N$ be a regular neighborhood of $S$ in $\text{Int } M$. 
\( \mathbb{V} \cong \mathbb{B} \cup S \times (-1,1) \) when \( \mathbb{B} = S^3 \) and

\( \partial \mathbb{V} \cap S \times (-1,1) \) consists of two discs, one on \( S \times f^{-1} \) and one on \( S \times f^1 \).

No \( \partial \mathbb{V} \cong S^2 \times S^1 \cong S^3 \).

\( \mathbb{V} \) cut along \( S \) is homeomorphic to

\( S^2 \times \mathbb{D}^1 \) with a ball removed.

\( S \times f^1 \) are boundary components of
Next, along \( S \), each homeomorphism to \( S^2 \) and filling these in results in \( S^3 \). So filling the third only gives \( S^3 \setminus (\beta^3 \times \beta^3) \cong S^1 \times \mathbb{R} \).

Thus \( M, = \hat{N} \) is an \( S^2 \)-bundle on \( S' \), and \( \partial \) a factor of \( M \). \( \Box \)

Note that \( M, \) is orientable if \( \hat{I} \) is an orientation preserving curve.

Let \( P \) be the non-orientable \( S^2 \) bundle over \( S' \).
Example 3.9

\[ P \# P = P \# (S^2 \times S') \]

Take \( S = S^2 \) in \( P \), precise \( P \), point in \( S' \).

Now in \( P \# P \) we easily find an orientation preserving curve cutting \( S \) transversely at a single point.

So \( N(SU2) = M' \) an \( S^2 \times S' \) pair (after fixing the \( S' \)-boundary).

Now \( P \# P \setminus M \cong P \) with two-three-bags removed, together with a funnel connecting them. So \( \overline{P \# P \setminus M} \cong P \).
Theorem 3.40

If $M$ is a compact 3-manifold, then $M = R \# M_1 \# \ldots \# M_k$, where each $M_i$ is an $S^2$-bundle over $S^1$, $0 \leq k \leq \nu h(\pi_1(M))$, and each 2-sphere in $R$ separates $R$.

Proof

The proof is an induction on $\nu h(M)$. If $\text{Int} M$ does not contain non-separating two-spheres, then $R = M$ and $k = 0$. 

Otherwise, \( M = M'_1 \# M, \) where
\( M_i \) is an \( S^- \)-bundle over \( S' \).

By Lemma 3.2 and 3.3 we have
\[
\rho_\pi_i(M) < \rho_\pi_i(M), \quad \text{and} \quad 20
\]
\[ M' = R \# M_2 \# \ldots \# M_n \] and hence
\[ M = R \# M_2 \# \ldots \# M_n \] as required.

\[ \text{Lemma 3.11} \]

Suppose that \( M \) is a compact 3-manifold,
\[ M = \hat{M}, \] \( \pi_1(M) \) is not a non-trivial free product. Then \( M \) is prime.
Proof.

Suppose not. Then $M$ admits a family $\mathcal{M}$ which is simply connected by Lemma 3.2.

Hence,

$$H_n(M, \Omega M; \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2) = 0$$

by Poincaré-Hopf duality and the universal coefficient theorem over fields.

The long exact sequence for pairs now gives:
\[ 0 = H_2(M, \mathbb{Z}; \mathbb{Z}) \rightarrow H_1(\partial M, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z}) \]

So \( H_1(\partial M, \mathbb{Z}) = 0 \).

But \( \partial M \) is a disjoint union of closed surfaces which are not spheres; each such contributes non-trivially to \( H_1(\partial M, \mathbb{Z}) \). Hence \( \partial M = \emptyset \).

Hence \( M \) is a homotopy \( S^3 \), and therefore \( M \cong S^3 \) by the Poincare' conjecture.
Corollary 3.12

Lens spaces and $S^2$-bundles over $S^1$ are prime.

Proof

It suffices to observe that $B$ and $B'$ do not admit non-trivial free splittings.

Def A 3-manifold is irreducible if $\forall$ every 2-sphere in $\mathcal{M}$ bounds a 3-cell.

Clearly, irreducible 3-manifolds are prime.
The converse is almost true:

**Lemma 3.13**

If $M$ is a prime 3-manifold which is not irreducible, then $M$ is an $S^2$-bundle over $S^1$.

**Proof**

Since $M$ is prime, every separating 2-sphere in $\text{Int} M$ bounds a 3-cell. Since $M$ is not irreducible, it must then contain a non-separating 2-sphere.
The result follows from Lemma 3.8 and Corollary 3.12. \[\square\]