Existence of factorisations

Def A compact 3-manifold M with $\partial M \neq \emptyset$ and $\hat{M} \cong S^3$ is called a punctured 3-cell.

Lemma 3.14

Suppose that M is a compact 3-manifold such that every 2-sphere in $\operatorname{Int} M$ separates. There exists an integer $k(M)$

such that if $\{S_1, \dots, S_n\}$ is a collection of n pairwise disjoint 2-spheres in $\mathrm{Int}\, M$ with $n \geq k(M)$, then the closure of some component of $M \setminus \cup S_i$ is a punctured 3-cell.

Proof

We fix a triangulation T of M.

Take a collection $\{S_1, \dots, S_n\}$ of pairwise disjoint 2-spheres in M in general position with respect to T, meaning:

$\forall i$, $S_i \cap T^{(0)} = \phi$, and S_i intersects edges transversely, and $S_i \not\subseteq$ 3-simplex.

We easily homotope any given collection of spheres to satisfy this requirement.

We define a complexity (λ, β) of $\{S_i\}$ by

$\alpha = |T^{(1)} \cap \cup S_i|$ and $\beta = \sum_{\substack{2\text{-simplex} \\ \sigma}} |\pi_0(\sigma \cap \cup S_i)|$.

We order the pairs (α, β) lexicographically.

Now suppose that the collection $\{S_i\}$ satisfies

(i) the closure of every component of $M \setminus \cup S_i$ is <u>not</u> a punctured 3-cell.

(ii) Among all collections satisfying (i), $\{S_i\}$ has minimal complexity.

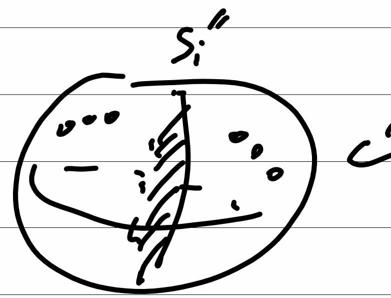
Let D be a disc (2-cell) in $\operatorname{Int} M$ with $D \cap \bigcup S_i = \partial D$. Since the spheres are disjoint, we must have $\partial D \subseteq S_i$ for some i. Let E' and E'' be the two 2-cells in S_i bounded by ∂D.

Let $S_i' = D \cup E'$ and $S_i'' = D \cup E''$.

Claim: at least one of the collections $\{S_1, \dots, S_i', \dots, S_n\}$, $\{S_1, \dots, S_i'', \dots, S_n\}$ satisfies (i).

Suppose otherwise. Since $M \setminus \bigcup S_i$ has no punctured 3-cell components, if C' is a connected component of $M \setminus (\bigcup_{j \neq i} S_j \cup S_i')$ which is a punctured 3-cell, then $S_i'' \subseteq C'$ or $D \subseteq \partial C'$.

In the first case, S_i'' cuts C' into two punctured 3-cells, and so one of these is a component of $M \setminus \bigcup S_i$ $\#$.



Hence $S_i'' \cap \operatorname{Int} C' = \emptyset$.

Now, we also have a component C''

of $M \setminus (\bigcup_{j \neq i} S_j \cup S_i'')$ which is a punctured 3-cell. By the same argument, $D \subseteq \partial C''$.

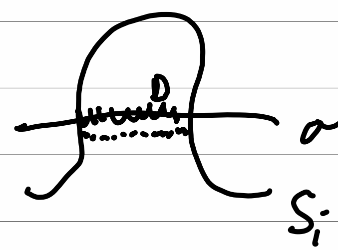
So $M \setminus \bigcup S_i$ contains $C' \cup_D C''$, which also is a punctured 3-cell. #

This proves the claim.

We say that a collection satisfying (i) obtained from $\{S_i\}$ in the manner just described is a <u>D-modification</u> of $\{S_i\}$.

We now establish further properties of the collection $\{S_i\}$ satisfying (i) and (ii):

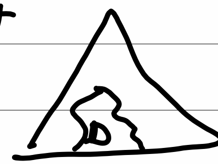
(iii) $\forall$ 2-simplex σ in T, $\sigma \cap \cup S_i$ is not an S^1. Otherwise, take D to be a disc in σ bounded by $\sigma \cap \cup S_i$ and modify $\{S_i\}$ by D. By homotoping in a n'hood of D



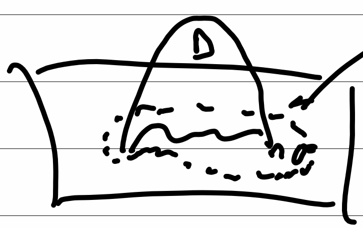
we arrange for the modification to be in general position with respect to T. We see that ξ did not increase and β decreased.

(iv) Let σ be a 2-simplex. Then $\sigma \cap \cup S_i$ cannot be an arc $\cong I$ with both endpoints on the same edge of σ.

Otherwise, the arc and a part of the edge bound a disc D.

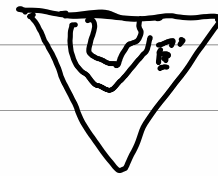


Let N be a small regular n'hood of D such that $N \cap \cup S_i$ is a disc E,



and such that ∂E bounds a disc E' on

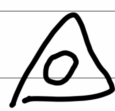
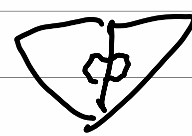
∂N, with $E' \cap T^{(1)} = \emptyset$.



We isotope S_i so that E is replaced by E'.

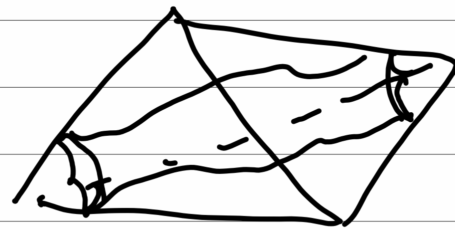
This decreases d.

(v) Let τ be a 3-simplex. Every component of $\partial\tau \setminus \cup S_i$ contains a vertex, as otherwise we would contradict (iii) or (iv).

 or .

(vi) $\forall$ 3-simplex τ in T, $\tau \cap \cup S_i$ is a disjoint union of 2-cells.

Otherwise, take a connected component C of $\tau \cap US_i$ which is not homeomorphic to a disc.



Take "the innermost" such C:

$\exists\, \partial$ a component of ∂C bounding a disc E in $\partial\tau$ such that if a component of $\tau \cap US_i$ meets $\mathrm{Int}\, E$, then it is a disc. We "push E" to the inside of τ; $\exists$ a disc D with $\partial D = \partial$ and

First $D \subseteq \operatorname{Int} \Sigma$. We look at the D-
-modification of $\{S_i\}$ which satisfies
(i). If C is contained in the new
collection (as a subset), then we push
∂ to the inside of Σ as well, and reduce
d. Otherwise, d is already reduced, since
we got rid of the other boundary com-
ponent of C, which intersected $T^{(1)}$.