Existence of factorizations

**Def** A compact 3-manifold $M$ with $\partial M \neq \emptyset$ and $\hat{M} \cong \mathbb{S}^3$ is called a punctured 3-cell.

**Lemma 3.19**

Suppose that $M$ is a compact 3-manifold such that every 2-sphere in $\text{Int} M$ separates. There exists an integer $k(M)$
such that if \( \{S_1, \ldots, S_n\} \) is a collection of \( n \) pairwise disjoint \( 2 \)-spheres in \( \text{Int} \ M \) with \( n \geq k(|M|) \), then the closure of some component of \( M \setminus V S_i \) is a punctured \( 3 \)-cell.

**Proof**

We fix a triangulation \( T \) of \( M \).

Take a collection \( \{S_1, \ldots, S_n\} \) of pairwise disjoint \( 2 \)-spheres in \( M \) in general position with respect to \( T \), meaning:
\[ \forall i, \ S_i \cap T_i^o = \emptyset \text{, and } S_i \text{ intersects edges transversely, and } S_i \notin 3\text{-simple.} \]

We easily homotope any given collection of spheres to satisfy this requirement.

We define a complexity \((i, \beta)\) of \(S_i\) by
\[ d = |T^{(2)} \cap \Omega| \quad \text{and} \quad \beta = \frac{\mathcal{E}}{|\mathcal{E} \cap \Omega|} \]

We order the pairs \((x, \beta)\) lexicographically.

Now suppose that the collection \(\mathcal{E}\) satisfies

(i) the closure of every component of \(M \setminus \mathcal{E}\) is not a punctured 3-cell.

(ii) Among all collections satisfying (i), \(\mathcal{E}\) has minimal complexity.
Let $D$ be a disc (2-cell) in $\text{Int} M$ with $D \cap V S_i = \emptyset$. Since the spheres are disjoint, we must have $\partial D \subseteq S_i$ for some $i$. Let $E'$ and $E''$ be the two 2-cells in $S_i$ bounded by $\partial D$.

Let $S_i' = D \cup E'$ and $S_i'' = D \cup E''$.

Claim: at least one of the collections $\{S_i, \ldots, S_i', \ldots, S_i'' \}$ satisfies (i).

Suppose otherwise. Since $M \setminus V S_i$ has no
punctured 3-cell component, if \( C' \)

is a connected component of \( M_1 \cup \bigcup_{j} U_{j}; s_i \)

which is a punctured 3-cell, then

\( S_i^{''} \subseteq C' \) or \( D \subseteq D C' \).

In the first case, \( S_i^{''} \) cuts \( C' \) into

two punctured 3-cells,

and so one of them

is a component of \( M_1 \cup U_{s_i} \).

Hence \( S_i^{''} \cap \text{Int} \ C' = \emptyset \).

Now, we also have a component \( C'' \)
of $M_1(\{\upsilon;\upsilon;\upsilon;\upsilon;\})$ which is a punctured 3-cell. By the same argument, $D \subseteq D'$. So $M \setminus \upsilon \upsilon \upsilon \upsilon$ contains $C' \cup C''$, which also is a punctured 3-cell. #

This proves the claim.

We say that a collection satisfying (i) obtained from $\{\upsilon;\upsilon;\upsilon;\upsilon;\}$ in the manner just described is a D-modification of $\{\upsilon;\upsilon;\upsilon;\upsilon;\}$.

We now establish further properties of the collection $\{\upsilon;\upsilon;\upsilon;\upsilon;\}$ satisfying (i)
(ii) $A$ is simple in $T$, on $V S i$ is to be a disc in $S$. Otherwise, take $D$ bounded by $o u s i$ and modify $S$ by $D$. By homotopying in a neighborhood of $D$, we arrange for the modifica-

cation to be in general position with respect to $T$. We see that $f$ did not increase and $\beta$ decreased.
(iv) let $o$ be a 2-simplex. Then $\gamma_i \sim I$ on $\partial \gamma_i$ cannot be an arc with both endpoints on the same edge of $o$. Otherwise, the arc and a part of the edge bound a disc $D$.

Let $N$ be a small regular n'hood of $D$ such that $N \cap \partial \gamma_i$ is a disc $E$, $\quad \nu^\gamma \circ \nu^* \circ \nu^{\partial D} = 0^2$. and such that $DE$ bounds a disc $E'$ on
2V, with \( E \cap T'' = \emptyset \).

We isolate \( S_i \) so that \( E \) is replaced by \( E' \).

This decreases \( t \).

(v) Let \( T \) be a 3-simplex. Every component of \( \partial T \setminus \{b\} \) contains a vertex, as otherwise we would contradict (iii).

\( \sim (iv) \).

(vi) A 3-simplex \( T \) in \( T \), \( \cap U_i \) is a disjoint union of 2-cells.
Otherwise, take a connected component \( C \) of \( T \cap V_S \); which is not homeomorphic to a disc.

Take the "innermost" such \( C \): 

\( \exists \) a component of \( 2C \) bounding a disc \( E \) in \( 2T \) such that if a component of \( 2 \cap V_S \) meets \( \text{Int } E \), then it is a disc. We "push \( E \)" to the inside of \( T \); \( \exists \) a disc \( D \) with \( \partial D = \partial \) and
First $D \subseteq \text{Int} \bar{c}$. We look at the $D$-modification of $\{S_i\}$ which satisfies

(i). If $C$ is contained in the new collection (as a subset), then we push $J$ to the inside of $\bar{T}$ as well, and reduce $d$. Otherwise, $I$ is already reduced, since we get rid of the other boundary component of $C$, which intersects $\bar{T}^{(0)}$. 