Lemma 3.14

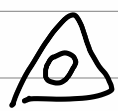
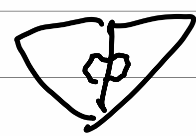
Suppose that M is a compact 3-manifold such that every 2-sphere in $\mathrm{Int}\, M$ separates. There exists an integer $k(M)$

such that if $\{S_1, \ldots, S_n\}$ is a collection of n pairwise disjoint 2-spheres in $\operatorname{Int} M$ with $n \geq k(M)$, then the closure of some component of $M \setminus \bigcup S_i$ is a punctured 3-cell.

Proof

(iii) components of $\bigcup S_i \cap \sigma$, σ 2-simplex are not spheres

(iv) they are also not arcs starting and ending at the same edge

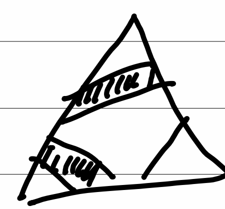
(v) Let τ be a 3-simplex. Every component of $\partial\tau \setminus J$ contains a vertex, where J is a component of $\partial\tau \cap \bigcup S_i$, as otherwise we would contradict (iii) or (iv).  or  .

(vi) $\forall$ 3-simplex τ in T, $\tau \cap \bigcup S_i$ is a disjoint union of 2-cells.

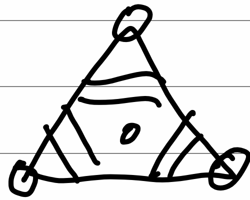
Now, let τ be a 3-simplex in $\mathcal{T}$

and let X be the closure of a component of $\tau \setminus \bigcup S_i$. Then X is a 3-cell whose boundary is a union of discs in $\tau \cap \bigcup S_i$ and a connected part of $\partial\tau$.

We say that X is <u>good</u> iff $X \cap \partial\tau$ is an annulus which contains no vertex of τ.

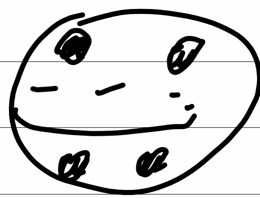


Note that each τ contains at most 6 bad (= not good) components; 4 containing the 4 vertices and one "central" component, which can be divided into two.



4 bad in dim 2

central component in dim 3:



Let R be the closure of a component of $M \setminus \cup S_i$. We say that R is <u>good</u> iff $R \cap \tau$ is good for every 3-simplex τ. Now it is easy to see that R is a fibre bundle over a 2-manifold.

with fibre I : this is clear on $R\cap\tau$ $\forall\tau$, where $R\cap\tau\cong D^2\times I$, and the structure is easily made compatible when passing to neighbouring simplices.

So $I\to R\to\Sigma$ is a fibring, and Σ has to be a closed surface.

Now either ∂R has two components homeomorphic to Σ, or one, a double cover of Σ.

In the first case $R\cong\Sigma\times I$, in the

Second it is a "twisted I-bundle over S^2".

Now, $\partial R \cong S^2$, and so $R \cong S^2 \times I$ or

R is the twisted I-bundle over P^2,

the projective space.

Put $k(M) = \dim H_1(M; \mathbb{Z}_2) + 6t$,

where t is the number of 3-simplices in T.

Suppose that we are given a collection

of n 2-spheres in $\mathrm{Int}\, M$ as in the statement, with $n \geq k(M)$.

Then some other collection satisfies (i) and (ii), and for this collection $\{S_i\}$ we have $M \setminus \cup S_i$ having $n+1$ components.

Since each 3-simplex intersects at most 6 components in a bad way, we have at most $6t$ bad components, and so at least $\dim H_1(M; \mathbb{Z}/2) + 1$ good ones.

Every good component contributes a free factor to $\pi_1(M)$: 1 in the case of $S^2 \times I$ and $\pi_1(P^3) = \mathbb{Z}/2$ in the other case.

Hence every twisted component contributes

1 dimension to $H_1(M;\mathbb{Z}_2)$, and so at least one component R is homeomorphic to $S^2 \times I$, a punctured 3-cell, contradicting (i). $\square$

<u>Theorem 3.15</u>

Every compact 3-manifold M can be written as a connected sum of finitely many prime factors.

<u>Proof</u> Assume first that $M \cong \hat{M}$.

Using Theorem 3.20, we can write

$M = R \# M_1 \# \ldots \# M_k$, where

every $M_i \cong S^2 \times S^1$ (is prime)

and R has no separating spheres.

We now factor $R = R_1 \# \ldots \# R_{n+1}$.

Then there are n disjoint spheres in Int R

such that the closures of the components

obtained from R by removing the spheres,

say $Q_1, \ldots, Q_n$, satisfy $\hat{Q}_i = R_i$

If $n \geq h(R)$ then, by Lemma 3.14,

some Q_i is a punctured 3-cell, hence $R_i \cong S^3$

$\cancel{\ }$

Now for general M we have

$$M \cong \hat{M} \# \underbrace{B^3 \# \dots \# B^3}_{\text{finitely many}}$$

by Lemma 3.7 $\square$

It can be shown that such a decomposition is unique, up to the ambiguity we discussed for non-orientable manifolds.