Basic theorems

Def A PL n-cell is a PL space PL homeomorphic to n-simplex

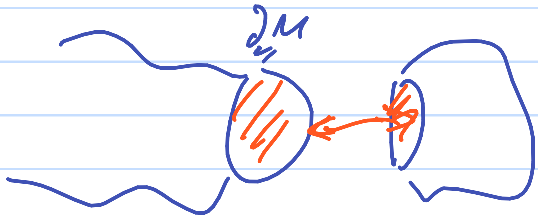
PL n-1-sphere — a — ∂ n-simplex.

Theorem 1.1 M a PL n-sphere, C a PL-subspace which is a PL n-cell, then $\overline{M \setminus C}$ is a PL-subspace which is a PL n-cell.

M  $M \setminus C \cong$  .

<u>Theorem 2.2</u> C PL n-cell. Every PL homeo of ∂C to ∂C can be extended to a PL homeo $C \to C$. [For homeo, take a cone]

<u>Thm 2.3</u> M is a PL n-mfld, C is a PL n-cell s.t. $M \cap C = \partial M \cap \partial C$ is a PL $(n-1)$-cell as a PL submanifold of both M and C, then M is PL homeo to $M \cup C$.



<u>Thm 2.4</u> M is a PL n-cell or PL-sphere, then every orientation-preserving PL homeo from M onto M is PL-isotopic to the identity.

<u>Thm 2.5</u> M is a PL n-mfld, C_1, C_2 are PL n-cells (as PL submanifolds) in $\operatorname{Int} M$ and X is a closed subset of M, such that $C_1 \cup C_2$ lies in a component of $M \setminus X$, then $\exists$ PL isotopy $\varphi : M \times I \to M$ s.t. $\varphi_0 = 1$, $\varphi_t|_X = 1$ $\forall t \in I$, $\varphi_1(C_1) = C_2$.

Regular n'hoods

Def K simplicial cplx, $\sigma \in K$, $\tau < \sigma$,

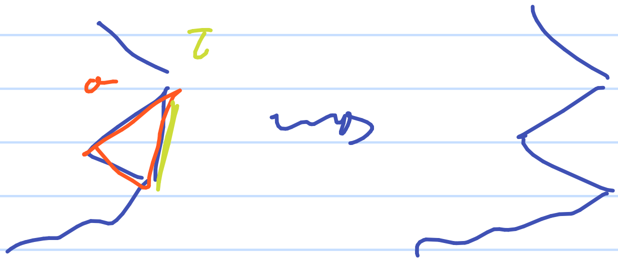
$\dim \tau = \dim \sigma - 1$, and

$\tau < \sigma' \Rightarrow \sigma' = \sigma$.

The cplx $K \setminus \{\sigma, \tau\}$ is obtained from

K by an <u>elementary collapse</u>.

We write $K \searrow K \setminus \{\sigma, \tau\}$

K  $K \setminus \{\sigma, \tau\}$.

Note: $K \searrow L$ then $|L|$ is a strong deformation

retract of $|K|$. (i.e. deformation is id on $|L|$).

$P \subseteq M$ is a <u>cpt polyhedron</u> iff it is the image of

a finite subcomplex of some triangulation in the

given PL structure of M.

A PL submfld N of M is a <u>regular n'hood</u> of P

iff $\exists$ triangulation (T, h) in the PL-structure of M

and finite subcomplexes K, L of T with $K \searrow L$,

$h(|K|) = N$, and $h(|L|) = P$.

Warning N is **not** necessarily a n'hood of P in the usual sense.

Thm 1.6 M a PL-mfd, (T,h) triangulation in the PL structure, $L \leq T$ finite.

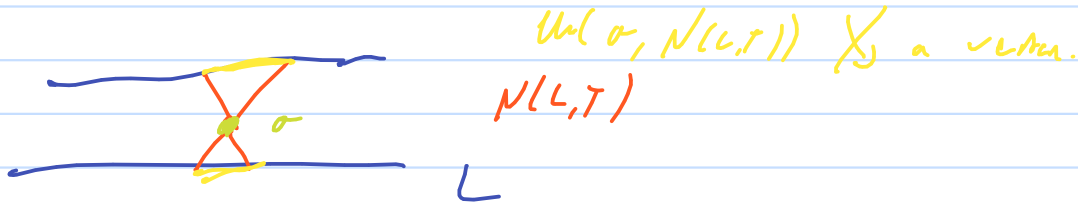
Let $N(L,T) = \bigcup_{\sigma \in L} St(\sigma)$. Then

$h(|N(L,T)|)$ is a regular n'hood of $|L|$

provided that:

(i) $\forall$ simplex of T with all vertices in L lies in L (L is full in T).

(ii) If $\sigma \in N(L,T)$ and $\sigma \cap L \neq \emptyset$ then $lk(\sigma, N(L,T)) \cap L$ is a vertex.



Cor 1.7 $h(|N(L'',T'')|)$ is a regular n'hood of $h(|L|)$, where T'' denotes the second barycentric subdivision of T.

Thm 1.8 M a PL-mfd, P compact polyhedron in M. N_1, N_2 reg. n'hoods of P in M. Then

(i) $\exists$ PL homeo $h: N_1 \longrightarrow N_2$

(ii) If $P \subseteq \operatorname{Int} N_i$, we can require that $h|_P = id$.

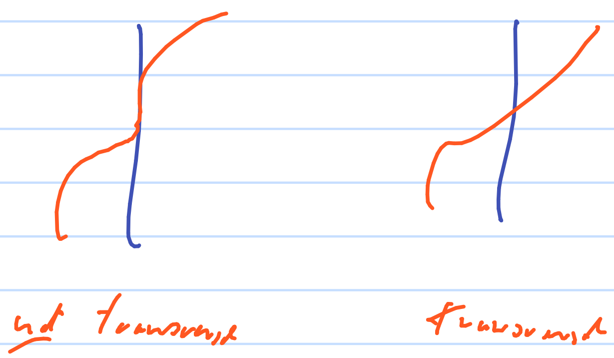
(iii) If $N_i \cap \partial M$ is a reg. n'hood of $P \cap \partial M$, (hence $N_i \cap \partial M = \phi$ if $P \cap M = \phi$), $\exists$ PL isotopy $f: M \times I \longrightarrow M$ s.t.

$$f_0 = id, \quad f_1(N_1) = N_2.$$

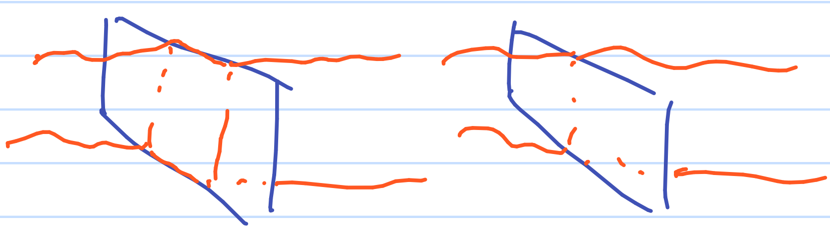
(iv) If, in (iii), $P \subseteq \operatorname{Int} N_i$, we can require that $f_t|_P = id \quad \forall t \in I$.

General position

Idea: to mimic transversality in diff geometry.



tangent space should intersect as little as possible.



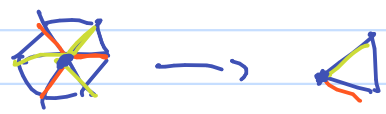
Def A map $f : |K| \longrightarrow \mathbb{R}^n$ is *affine* iff it maps simplices linearly, once the image of vertices is specified.

Given $f : X \to Y$, we define the *singular set* $S(f)$ to be the closure of $\{x \in X \mid |f^{-1}f(x)| > 1\}$.

We write $S = \bigcup_{i \in \mathbb{N}} S_i(f)$, $S_i(f) = \{x \in S(f) \mid |f^{-1}f(x)| = i\}$

We set $\Sigma_i = f(S_i(f))$.

Σ_1 are *branched points*



Σ_2 are *double points*

Σ_3 — " — *triple points* etc.

For $x \in |K|$, the local dimension

$\mathrm{loc\,dim}(K, x) = \max\{\dim(\sigma) \mid x \in |\sigma|, \sigma \in K\}$

$x \in |K|$ is *regular* iff $\exists$ an open nbhd of x in $|K|$ homeomorphic to $\mathbb{R}^q$ or $[0, \infty)^q$ where $q = \mathrm{loc\,dim}\, x$.

(the latter type is called *boundary point of K*).

Def 2.11 For $k < n \leq 3$ and a finite k-complex K, a map $f : |K| \longrightarrow \mathbb{R}^n$ is *in general position* w.r.t. K iff

(i) f is an affine embedding on each simplex of K.

(ii) $\dim S_1(f) \leq n-3$, $\forall x \in S_1(f)$: $\mathrm{loc\,dim}(K, x) = n-1$,