Recall:
Given \( f : X \to Y \), we define the image \( \text{Im}(f) \) to be the closure of \( \{ f(x) \mid f(x) \in Y \} \).

We write \( \text{Im}(f) = \overline{f[X]} \), \( \overline{f[X]} = \{ y \in Y \mid x \in X, f(x) \to y \} \).

We set \( e_i = f^*(s_i, f_1) \).

\( e_i \) are branch points
\( e_2 \) are lean points
\( e_3 \) - triple points etc.

For \( x \in \mathbb{C} \), the local dimension
\[ \text{loc dim}(x, \mathbb{C}) = \max \{ \text{dim}(x) \mid x \in \partial \mathbb{C}, x \in \mathbb{C} \} \]

\( x \in \mathbb{C} \) is regular if \( \mathbb{C} \) is an open Euclidean \( x \)-neighbourhood of \( x \) in \( \mathbb{C} \).

Homomorphic to \( \mathbb{C}^q \times [0, 2]^q \) where \( q \leq \text{loc dim}(x) \).
(The latter type is called boundary point of \( x \)).

**Def** 3.22 For \( k \leq n \leq 3 \) and a finite \( k \)-complex \( X \),

a map \( f : X \to \mathbb{R}^{n-k} \) is in general position with \( k \) if
(i) \( f \) is an affine embedding on each simplex of \( X \).
(ii) \( d_{N} S, f_1 \leq n-3 \), \( \forall x \in S, f_1 : d_{N}(f_1(x)) = n-k \).
(ii) For \( i > 2 \), \( \Delta_1 \cap \Delta_i \cap \Delta_{i+1} \subseteq \Delta_i - (i-1) \Delta_i \) (and hence \( \mathfrak{S}_i \cap \Delta_i = \emptyset \) for \( i > n \)). Furthermore, if \( y \in \mathfrak{S}_i \) and

\[
\mathfrak{g}^i(y) = \left\{ x \leq i \right\}
\]

then

\[
\exists j \in \Delta_i : (x, x_j) \geq \Delta_{i-1} \Delta_i
\]

(iii) For \( i > 2 \), \( \mathfrak{S}_i \cap \Delta_i \) contains a nonregular point only if on the cone \( n = 3, i > 2 \), in this case \( \mathfrak{S}_i \cap \Delta_i \)

contains only finitely many nonregular points and for each such point \( z \) the other point is \( \mathfrak{f}^i(y) \)

\( z \) a regular non-boundary point and

\[
\mathfrak{h}^i(x, x_j) = \mathfrak{h}^i(y, x_j) = 0.
\]

(v) For \( i > 2 \) and \( y \in \mathfrak{S}_i \), \( \mathfrak{f}^i(y) \) contains at least one boundary point; this occurs only when \( n = 3 \) and \( \mathfrak{h}^i(x, x_j) \) is not constant at every point in \( \mathfrak{f}^i(y) \).

(vi) For \( i > 2 \) and \( y \in \mathfrak{S}_i \), i.e., \( \mathfrak{f}^i(y) \) is regular at every \( x_j \in \mathfrak{g}^i(y) \), \( \mathfrak{f}^i(y) \) is transversal at \( y \), i.e., \( \mathfrak{f}^i(y) \) is maximally independent hypersurface \( \mathfrak{h}^i \) through \( y \) with \( \mathfrak{h}^i \cap \Delta_i = \mathfrak{h}(x, x_j) \), a neighborhood \( N \) of \( y \) in \( \mathbb{R}^n \) and a PL embedding.
h : \mathbb{N} \to \mathbb{R}^n \text{ with } h(y) = 0, \text{ and with } h' \text{ taking a lift } \tilde{x} \text{ of } x \in X \text{ into an } \hat{h}' \text{ which } \hat{h}' \mid_{h'(y) = 0} \text{ in } h_j \mid_{h'(y) = 0} \leq (0, \infty)^n, \text{ where } x \text{ is not } y \text{ is a boundary point.}

Lemma 1.22 \quad k \text{ is a } k \text{-complex, } k < k < 3,

A, B, C \subseteq X, \quad k = A \cup B \cup C, \quad A \cap B \cap C = \emptyset.

Therefore, given an affine map

g : X \to \mathbb{R}^n \text{ such that } g_1 \text{ is in general position w.r.t. } A \cup B \cup C \text{ and given } \delta > 0 \text{ define map } f : X \to \mathbb{R}^n

(a) \quad d_1 f_1 \leq \delta \quad \forall x \in X

(b) \quad f_1 \circ v_1 = g_1 \circ v_1

(c) \quad f_1 |_{B \cup C} \text{ is in general position w.r.t. } B \cup C

(d) \quad \forall L \subseteq X \text{ s.t. } g_1 \mid_L \text{ is an embedding, } \quad f_1 \mid_L \text{ is also an embedding.}

Unfortunately, the lemma is false.
Counterexample

\[ A = \pi, \quad B = \frac{A}{2} \text{ trim, and } \mathbb{S}^2. \]

\[ \mathbb{S} \text{ with both equators cut off.} \]

Now gluing identifies the equators transversally.

\[ \Gamma \text{ intersect with } \mathbb{S}^2. \]

Now this determines \( g \) uniquely, and \( g \) \( \frac{A}{2} - \gamma \).

Then \( g = f \). But \( \dim S_c(g) = 2 > 4 - 1 = 3 \).

So \( k = 3, \quad k = 3, \quad i = 2. \)