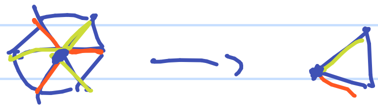
Recall:

Given $f: X \to Y$, we define the singular set $S(f)$ to be the closure of $\{x \in X \mid |f^{-1}f(x)| \geq 2\}$.

We write $S = \bigcup_{i \in \mathbb{N}} S_i(f)$, $S_i(f) := \{x \in S(f) \mid |f^{-1}f(x)| = i\}$.

We set $\Sigma_i := f(S_i(f))$.

Σ_1 are branched points



Σ_2 are double points

Σ_3 — " — triple points etc.

For $x \in |K|$, the local dimension

$\mathrm{locdim}(K,x) = \max\{\dim(\sigma) \mid x \in |\sigma|, \sigma \in K\}$

$x \in |K|$ is regular iff $\exists$ an open n'hood of x in $|K|$ homeomorphic to $\mathbb{R}^q$ or $[0,\infty)^q$ where $q = \mathrm{locdim}\, x$.

(the latter type is called boundary point of K).

Def 2.11 For $k < n \leq 3$ and a finite k-complex K, a map $f: |K| \to \mathbb{R}^n$ is in general position w.r.t. K iff

(i) f is an affine embedding on each simplex of K.

(ii) $\dim S_1(f) \leq n-3$, $\forall x \in S_1(f)$: $\mathrm{locdim}(K,x) = n-1$, and $f|_{|K| \setminus S_1(f)}$ is an immersion.

(iii) for $i \geq 2$, $\dim S_i(f) \leq ik - (i-1)n$ (and hence $S_i(f) = \emptyset$ for $i > n$). Furthermore, if $y \in S_i(f)$ and
$$f^{-1}(y) = \{x_1, \dots, x_i\} \quad \text{then}$$
$$\sum_{j=1}^{i} \operatorname{loc\,dim}(K, x_j) \geq (i-1)n$$

(iv) for $i \geq 2$, $S_i(f)$ contains a nonregular point only in the case $n = 3$, $i = 2$. In this case $S_2(f)$ contains only finitely many nonregular points and for each such point x the other point x_1 in $f^{-1}f(x)$ is a regular non-boundary point and
$\operatorname{loc\,dim}(K, x) = \operatorname{loc\,dim}(K, x_1) = 2$.

v) for $i \geq 2$ and $y \in S_i(f)$, $f^{-1}(y)$ contains at most one boundary point; this occurs only when $n = 3$ and K has loc dim 2 at every point in $f^{-1}(y)$.

(vi) for $i \geq 2$ and $y \in S_i(f)$ s.t. f is regular at every $x_j \in f^{-1}(y)$, f is transverse at y, i.e., $\exists$ maximally independent hyperplanes $H_1, \dots, H_i$ through 0 with $\dim H_j = \operatorname{loc\,dim}(K, x_j)$, a n'hood N of y in $\mathbb{R}^n$ and a PL embedding

$h : N \longrightarrow \mathbb{R}^n$ with $h(y) = 0$, and with hf taking a n'hood of x_j in $|K|$ onto an n'hood of 0 in M_j or $M_j^+ \subseteq [0,\infty)^m$, when x is not or is a boundary point.

<u>Lemma 1.22</u> K is a k-complex, $k < n \leq 3$, $A, B, C \leq K$, $K = A \cup B \cup C$, $A \cup B$ is a full subcomplex of K and $A \cap C = \phi$. Given an affine map $g : |K| \longrightarrow \mathbb{R}^n$ such that $g|_{|B|}$ is in general position w.r.t. to B, and given $\varepsilon > 0$ $\exists$ affine map $f : |K| \to \mathbb{R}^n$:

(a) $d(f(x), g(x)) < \varepsilon \quad \forall x \in |K|$

(b) $f|_{|A \cup B|} = g|_{|A \cup B|}$

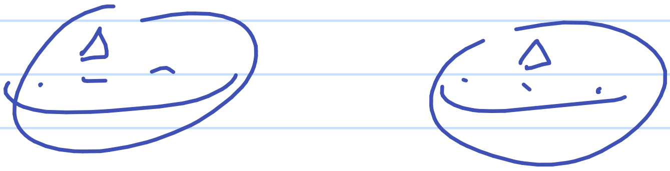
(c) $f|_{|B \cup C|}$ is in general position w.r.t. $B \cup C$,

(d) $\forall L \leq K$ s.t. $g|_{|L|}$ is an embedding, $f|_{|L|}$ is also an embedding.

Unfortunately, the lemma is <u>false</u>.

Counterexample

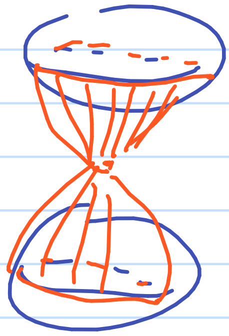
$A = \emptyset$. $B = \coprod_2$ triangulated S^2.



$K =$ B with both equators coned off

Now $g|_B$ identifies the equators

transversely



[intersect with $\mathbb{R}^2$:

sphere 1

equator

sphere 2]

Now this determines g uniquely, and so if $f|_B = g|_B$,

then $g = f$. But $\dim S_2(g) = 2 > ik - (i-1)n$

for $n=3$, $k=2$, $i=2$.