Concluding remarks on general position.

**Def 1.13**

For $k$ a finite complex of dimension $k < n \leq 3$ and $M$ a PL manifold, a map $f : \text{Int } k \to \text{Int } M$

is in general position w.r.t. $k$ if $\mathcal{M}$ is a finite collection $B_1, \ldots, B_p$ of PL $n$-cells.
in $M$ with $f(141) \subseteq U \cap B_j$ and affine embeddings

$$h_i : B_i \to \mathbb{R}^n \text{ with } f^{-1}(B_i)$$

a subcomplex of $K$ and

$$h_i : f^{-1}(B_i) \to \mathbb{R}^n$$

a general position map for $i \in \{1, \ldots, p\}$.

**Theorem 1.4**

Suppose $K$ is a finite complex.
of dimension \( k \leq n \leq 3 \),

\( A, B, C \) are subcomplexes of \( \kappa \)

\( \text{w.r.t. } A \cap B \cap C = \emptyset \text{ and } A \cap C = \emptyset \).

Then given an \( n \)-manifold \( M \),

a PL map \( g: \mathbb{I}^n \to M \) with

\( g|_{\mathbb{I}^1} \) in general position w.r.t. \( \partial M \) (assuming \( g(1\mathbb{I}^1) \subseteq \text{Int}M \)),

and \( \exists \varepsilon > 0 \), \( \exists \) a PL map

\( f: \mathbb{I}^n \to M \) satisfying:

a) \( d(f(x), g(x)) < \varepsilon \) \( \forall x \in \mathbb{I}^n \)
b) \( \| A \|_{\text{LU}} = \| A \|_{\text{LU}} \)

c) \( \| A \|_{\text{LU}} \) is in general position with respect to some subdivision of \( BUC \), and

d) \( A \leq k, \| A \|_{\text{LU}} \) is injective

\( \Rightarrow \| A \|_{\text{LU}} \) is injective.
Heegaard splittings

Let $A$ a spot, not necessarily connected $(n-1)$-mfd $F$ is 2-sided in $M = M$

Then an embedding $h : F \times C(-1,1) \to M$

with $h(x,0) = x \forall x \in F$

and $h(F \times C(-1,1)) \cap \partial M = h(\partial F \times C(-1,1))$

Lemma 2.1

If $F$ is a compact $(n-1)$-mfd properly embedded in an $n$-mfd
and the image of

\[ i_* : H_\ast (F; \mathbb{Z}_2) \to H_\ast (M; \mathbb{Z}_2) \]

is 0 then F is two sided in M.

Recall Theorem 1.8: If a compact polyhedron has two regular n-skeleta
(which are n-submanifolds)
then they are homeomorphic.

let us record two corollaries:
Corollary 1.9

\( \mathcal{Y} (T,h) \) is a combinatorial triangulation of a PL n-manifold \( M \), and \( L \leq T \leq n \) are integers, then every regular neighborhood of \( h(l(L)) \) in \( M \) is a PL n-cell.

Corollary 2.10

If \( M \) is a PL n-manifold, then every regular neighborhood of \( \partial M \) in \( M \) is PL-homeomorphic to \( \partial M \times [0,1] \).
Proof of Lemma 2.2 (for n = 3)

Take $C$, a connected component of $F$.

Since $C$ is a submanifold of dimension 2, there is a triangulation $T$ (combinatorial)
in the PL structure of $M$.

Vertex $v$ in $C$

2. there is a subcomplex of $T$

containing $v$ in its interior,

which is a PL 3-cell, and
\$1B1NF \text{ is the image of a } 2\text{-cell in } B.\$

Passing to the second barycentric subdivison of $T$, we may assume that the star of 1B1NF is a regular neighborhood of 1B1NF.

Clearly, the star $\sim (1B12E) \times C.\$

The union of these stars is a regular neighborhood of $C$.\$
It might not however be homeomorphic to $C \times [-1, 1]$

Example: Mobius band

The problem is that there is a loop $l$ in $C$, if moving the edge going to the position division along the loop changes it into a vector going in the op-
giving direction.

\[ \ldots \]

Here, there were a loop disjoint
from \( F \) except at one point,
where the intersection is transverse,
while projects onto \( K \).

5. \( c' \) gives a cycle in \( H_1(\mathbb{M}_g; \mathbb{Z}) \)
in the image of \( H_1(F; \mathbb{Z}_l) \),
which is 0 by assumption.
But the number of framewise interaction mod 2 is a homo-
logical invariant.

Def. With notation as above, if

$F$ is transversally orientable, then we have an
embedding $F \times [-1,1] \to M$,

the n-manifold

$R = M \setminus \text{Int}(F \times (-1,1))$

is the result of cutting $M$ along

$F$.

Since regular slices are unique
up to homeomorphism, $R$ is well-defined.

Moreover, using regular neighborhoods of $F \times \{1\}$ and $F \times \{\frac{1}{3}\}$, we see that $\exists$ a map $g : R \to M$ taking $R \setminus F \times \{1, \frac{1}{3}\}$ onto $M \setminus F$, and $g(h(x, -1)) = g(h(x, 1)) = x$ for $x \in F$. 
Cubes with handles.

Def. Let \( M \) be a 3-manifold containing a collection \( \{ D_i \}_{i=1}^{n} \)
of pairwise disjoint, properly embedded 2-cells (which
limit to the two-sided by Lemma 2!)

such that cutting \( M \) along \( \partial D_i \);

results in a 3-cell is called

a cube with a handle.

By van Kampen's Theorem, \( \pi_1(M) \cong \mathbb{Z} \).
Theorem 2.2

Suppose $M_i$ is a cycle with $n_i$ handles, where $i < 1/2$. Then $M_1$ is homeomorphic to $M_2$ iff $n_1 = n_2$ and either both are orientable or both are non-orientable.