Recall: $\pi_1(T^2) \cong \mathbb{Z}^2$

generated by $a, b$

**Lemma 2.9**

A pair $a^m b^n, a^n b^m$ of nontrivial elements of $\pi_1(T^2)$ can be represented by a pair of simple loops meeting transversely in a single point.
If \[ \det \left( \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) = \pm 1. \]

\text{Proof}

Suppose that \[ \left| \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right| = \pm 1. \]

Let \[ \omega_1 : t \mapsto (e^{2\pi i p t}, e^{2\pi i q t}) \]

\[ \omega_2 : t \mapsto (e^{2\pi i r t}, e^{2\pi i s t}) \]

be two simple closed loops in \( \mathbb{T}^2 \).

Suppose that \( \omega_1(t) = \omega_2(t) \) for some \( t \in \mathbb{R} \).

Then \( e^{2\pi i p t} = e^{2\pi i r t} \), and so

\[ pt, -rt, \in \mathbb{Z}. \]

Similarly, \( qt_1 - 2t_2 \in \mathbb{Z}. \)
So \( \langle t, v \rangle \neq 0 \) implies \( \langle p, q \rangle \in \mathbb{Z}^2 \).

Since \( |v_x| = \pm 1 \), the vector \( \langle p, q \rangle \) is invertible on \( \mathbb{Z} \).

Thus \( \langle e_1, e_2 \rangle \in \mathbb{Z}^2 \).

So \( w \) is an integer only at \( (0,0) \), where the intersection is clearly transverse.

Now, for the harder direction:
Suppose we have two interesting

functions in a high point, both

being simple closed curves.

We have \( T^2 \setminus \omega, \omega \cong \mathbb{D} \), and

a \( \mathbb{Z} \)-homomorphism \( h : T^2 \to T^2 \) satisfies

\[
\lim_{a \to \omega} v_1 \omega = h(\lim_{a \to \omega} v_1) \]

\[
\text{classification of surfaces.}
\]

Since \( \omega \), interest is transcendental,
we have $\phi(w_1 \cdot a \cdot w_2 \cdot b) = b$ (up to reparametrization).

No $h_+ \in \text{Out}(\mathbb{T}^2) = \text{Aut}(\mathbb{T}^2)/\text{Inn}.$

where $\text{Inn}$ is the normal subgroup of
inner automorphisms, i.e., of conjugations.

Since $\mathbb{T}^2$ is abelian, $\text{Inn} = 1.$

So $h_+ \in \text{Aut}(\mathbb{T}^2) \cong \text{GL}_2 \mathbb{Z}.$

So $h_+$ is represented by a matrix, which is invertible over $\mathbb{Z}.$

But the matrix is $\left( \begin{smallmatrix} p & q \\ r & s \end{smallmatrix} \right),$ and
its determinant must also be
invariant on $Z$, i.e.

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 1$$

Let $V_1 = S'^2 \times B^2$. We have generators $a, b$ for $\pi_1(\mathbb{R}P^3) = \mathbb{Z}, (\mathbb{T}^2)$.

Let $b$ a generator of

$$\ker (\pi_1(\partial V) \to \pi_1(V)) \cong \mathbb{Z}.$$ 

Given coprime integer $p, q$ there

is a unique $3$-manifold $\mathbb{L}_{p,q}$
with Haagerup diagram

\((U, J)\) where \(J\) represents the element \(a^p b^q \in \pi(\Gamma^2)\).

\(L_{p,q}\) is the lens space of type \((p,q)\).

\[\text{Note: sometimes we do not count } L_{0,0} \text{ and } L_{1,0} \text{ as lens spaces, i.e., }\]

\[L_{0,0} \cong S^2 \times S^1\]

\[L_{1,0} \cong S^3 \text{ (Hopf fibration)}\]
We compute:
\[
\bar{n}_1(L_{0,9}) = \langle a, b \mid [a, b], a^9 b^7, b^2 \rangle = \langle a \mid a^9 \rangle \cong \mathbb{Z}_9^2.
\]

In particular, \( \bar{n}_1(L_{0,1}) = \mathbb{Z} \) and
\[
\bar{n}_1(L_{1,0}) = 1.
\]

We will need the following classical result:

**Lemma 2.10**

If \( F \) is any \( \mathbb{Z} \)-manifold and
\[
f_0, f_1 : S^1 \to F \text{ are homotopic}
\]
Embeddings which are not null-homotopic, then there exists an isotopy $g : F \times [0, 1] \to F$ such that

$g_0 = \text{id}, \quad g_1 \circ f_0 = f_1$.

Consider a homeomorphism $h : T^2 \to T^2$ that extends to $\mathcal{V}$, i.e., $h_* (\mathcal{V}) = \mathcal{V}$.

Proof: If $h$ extends to $\mathcal{V}$, then letting $N = \ker \left( \pi_1 (\mathcal{V}) \to \pi_1 (\mathcal{V}_1) \right)$, we have $h_* (N) = N$. Thus, $h_* (\mathcal{V})$ generates $\mathcal{V}$, and so $h_* (\mathcal{V}) = \mathcal{V}$.
Now suppose that $h_4(b) = b$.

Hence, $h_0 b = b$ a $h_1 b = b$

where $\overline{b} : S^1 \to T^2$ is given by

$$\overline{b}(z) = b(z) \frac{6}{z}.$$

By Lemma 2.10, there exists an isotopy $g : T^2 \times I \to T^2$ taking $h_0 b$ to $b'$, where $b' \in S^3b, b^S$.

So $h$ is isotopic to a homeomorphism $h'$ fixing $im b$. Hence $h'$ induces
a homeomorphism of $T \setminus b \cong \mathbb{R}^2$.

\[ \begin{array}{c}
0 \\
\hline
i
\end{array} \]

But each node can be extended to $B^2 \times I$.

Example 2.11

i) $L_{1,9} \cong S^3 \cup q$.

$L_{1,9}$ has Heegaard diagram
Perforation of Dehn twist around $b$

(with correct orientation):

\[ \sim \]

which is a Heegaard diagram

of $L_{1,0} \cong S^3$.

ii) $L_{p,q} \cong L_{p,q'}$ provided that

a) $q = \pm q'$ mod $p$ or

b) $qq' = \pm 2 \text{ mod } p$. 