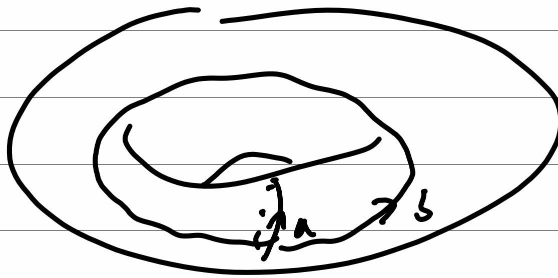
Recall: $\pi_1(T^2) \cong \mathbb{Z}^2$

generated by a, b



Lemma 2.9

A pair $a^p b^q$, $a^r b^s$ of nontrivial elements of $\pi_1(T^2)$ can be represented by a pair of simple loops meeting transversely in a single point iff $\det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \pm 1$.

<u>Proof</u>

Suppose that $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \pm 1$.

Let $\omega_1 : t \mapsto (e^{2\pi i p t}, e^{2\pi i q t})$

$\omega_2 : t \longmapsto (e^{2\pi i r t}, e^{2\pi i s t})$

be two simple closed loops in T^2.

Suppose that $\omega_1(t_1) = \omega_2(t_2)$ for some t_1, t_2.

Then $e^{2\pi i p t_1} = e^{2\pi i r t_2}$, etc. so

$pt_1 - rt_2 \in \mathbb{Z}$.

Similarly $qt_1 - st_2 \in \mathbb{Z}$.

So $(t_1, -t_2)\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathbb{Z}^2$.

Since $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \pm 1$, the matrix

$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is invertible over $\mathbb{Z}$.

Thus $(t_1, -t_2) \in \mathbb{Z}^2$.

So ω_1, ω_2 intersect only at $(0,0)$, where the intersection is clearly transverse.

Now, for the harder direction:

Suppose we have ω_1, ω_2 intersecting transversely in a single point, both being simple closed curves.

We have $T^2 \setminus \omega_1, \omega_2 \cong D^2$, and

so $\exists$ homeomorphism $h: T^2 \to T^2$ s.t.

$\operatorname{im} \omega_1 \cup \operatorname{im} \omega_2 = h(\operatorname{im} a \cup \operatorname{im} b)$

[classification of surfaces].

Since ω_1 intersects ω_2 transversely, we have $\tilde{h} \circ \omega_1 = a$ and $\tilde{h} \circ \omega_2 = b$

(up to reparametrizing).

Now $h_* \in Out(\pi_1(T^2)) = Aut(\pi_1(T^2)) / Inn$.

where Inn is the normal subgroup of

inner automorphisms, i.e. of conjugations.

Since $\pi_1(T^2)$ is abelian, $Inn = 1$.

So $h_* \in Aut(\mathbb{Z}^2) \cong GL_2 \mathbb{Z}$.

So h_* is represented by a matrix,

which is invertible over $\mathbb{Z}$.

But this matrix is $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and its determinant must also be invertible over $\mathbb{Z}$, i.e.

$$\left|\begin{matrix} p & q \\ r & s \end{matrix}\right| = \pm 1 \qquad \square$$

Let $V_1 = S^1 \times B^2$. We have generators a, b for $\pi_1(\partial V_1) = \pi_1(T^2)$.

Let b be a generator of

$$\ker\left(\pi_1(\partial V_1) \to \pi(V_1)\right) \cong \mathbb{Z}.$$

Given coprime integers p, q there is a unique 3-manifold $L_{p,q}$

with Heegaard diagram

(V_1, J) where J represents

the element $a^p b^q \in \pi_1(T^2)$.

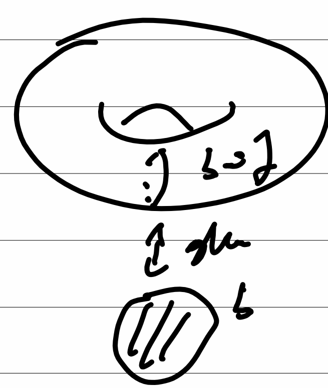
$L_{p,q}$ is the <u>lens space</u> of type (p,q).

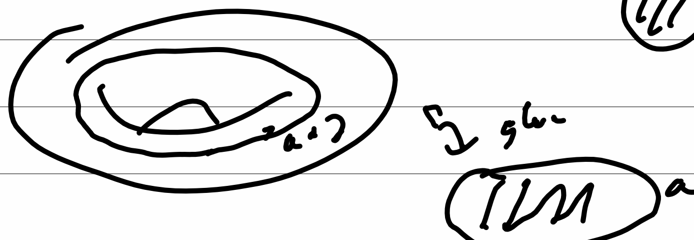
[Note: sometimes we do not count

$L_{0,1}$ and $L_{1,0}$ as lens spaces, since

$L_{0,1} \cong S^2 \times S^1$

$L_{1,0} \cong S^3$ (Hopf fibration)





]

We compute:

$$\pi_1(L_{p,q}) = \langle a,b \mid [a,b], a^p b^q, b \rangle$$
$$= \langle a \mid a^p \rangle \cong \mathbb{Z}/p\mathbb{Z}$$

In particular, $\pi_1(L_{0,1}) = \mathbb{Z}$ and

$$\pi_1(L_{1,0}) = \mathbb{1}.$$

We will need the following classical result:

<u>Lemma 2.10</u>

If F is any 2-manifold and $f_0, f_1 : S^1 \to F$ are homotopic embeddings which are not null-homotopic, then $\exists$ an isotopy $g : F \times \underline{I} \to F$ such that $g_0 = \mathrm{id}$, $g_1 \circ f_0 = f_1$.

<u>Cor</u> A homeomorphism h of T^2 extends to V_1 iff $h_*(b) = \pm b$.

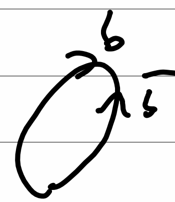
<u>Proof</u> If h extends then, letting $N = \ker(\pi_1(\partial V_1) \to \pi_1(V_1))$, we have $h_*(N) = N$. Thus $h_*(b)$ generates N, and so $h_*(b) = \pm b$.

Now suppose that $h_*(b) = \pm b$.

Hence, $h \circ b \simeq b$ or $h \circ b \simeq \bar{b}$

where $\bar{b} : S^1 \to T^2$ is given by

$\bar{b}(z) = b(\bar{z})$



By Lemma 2.10, there exists an

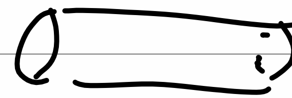
isotopy $g_t : T^2 \times I \to T^2$ taking $h \circ b$

to b', where $b' \in \{b, \bar{b}\}$.

So h is isotopic to a homeomorphism h'

fixing im b. Hence h' induces

, a homeomorphism of $T^2 \setminus b \cong$ annulus .

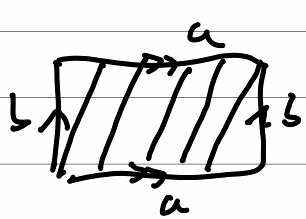


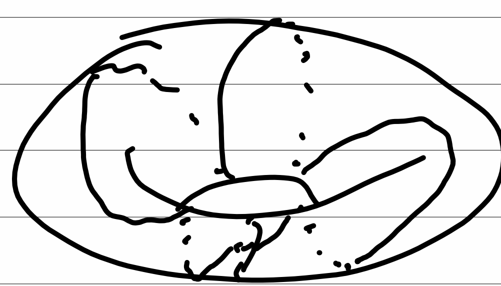
But each such can be extended to $B^2 \times I$. $\square$

<u>Example</u> 2.11

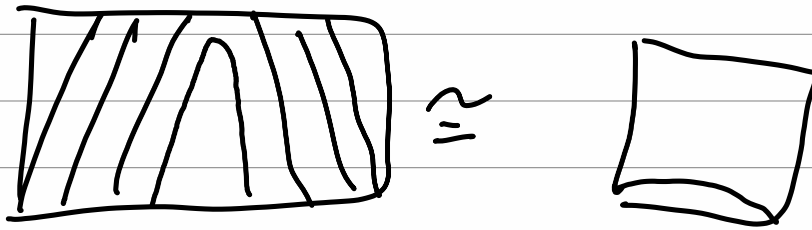
i) $L_{1,q} \cong S^3 \ \forall q$.

$L_{1,q}$ has Heegaard diagram





Perform q Dehn twists around b (with correct orientation):



which is a Heegaard diagram of $L_{1,0} \cong S^3$.

ii) $L_{p,q} \cong L_{p,q'}$ provided that

a) $q = \pm q' \mod p$ <u>or</u>

b) $qq' = \pm 1 \mod p$.