Example 2.11

ii) $L_{p,q} \cong L_{p,q'}$ provided that

a) $q = \pm q' \mod p$ <u>or</u>

b) $qq' = \pm 1 \mod p$.

In a), we again use Dehn twists around b: the loop $a^p b^q$ becomes $a^p b^{q+p}$ or $a^p b^{q-p}$, depending on whether we do right or left twist. Reversing the orientation of T^2 via $(z,w) \mapsto (z,\bar{w})$ takes $a^p b^q$ to $a^p b^{-q}$.

Hence $L_{p,q} \cong L_{p,q'}$ if $q' \equiv \pm q \mod p$.

In b), we change the roles of V_1 and V_2:

the loop γ in ∂V_1 is the second generator of $\pi_1(T^2)$ seen from the perspective of V_2.

The first generator is $a^r b^s$, and by Lemma 2.9 we must have

$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \pm 1$, and we may chose the sign.

Now the Heegaard diagram of $L_{p,q}$ after switching the roles of V_1 and V_2 is $(V_2; \gamma')$, where γ' represents b.

Letting a', b' be the generators of $\pi_1(T^2)$ from V_2-perspective, we have $b' = a^p b^q$ and $a' = a^r b^s$.

Then J' is given by $\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1}$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\det} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} = (\pm r, p)$$

and so J' is represented by $b'^{\pm r} a'^{p}$.

Hence $L_{p,q} \cong L_{p,\pm r}$.

By assumption, $qq' \equiv \pm 1 \mod p$,

and so $\begin{vmatrix} p & q \\ q' & s \end{vmatrix} = ps - qq' = \pm 1$

for some $s \in \mathbb{Z}$.

Hence $L_{p,q} \cong L_{p,q'}$.

[It can actually be shown that (a) and (b) are also necessary conditions.]

<u>Non-orientable case:</u>

Given an n-manifold M and a path $\alpha : I \to M$, we may subdivide the path : take $0 = t_0 < t_1 < \dots < t_n = 1$

s.t. $\gamma|_{[t_i, t_{i+1}]}$ has image lying in

a Euclidean n-ball.

We take N_i, a n'hood of $\gamma([t_i, t_{i+1}])$

in the ball. In particular, N_0 is a

n'hood of $\gamma(0)$. We can translate N_0

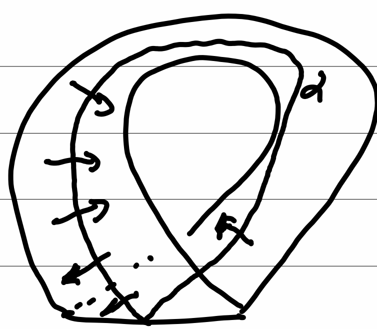
by adding $\gamma(t_1) - \gamma(0)$ and intersect

the image with N_1. We proceed this

way recursively, and obtain a n'hood

U of $\gamma(0)$ which by a sequence of

Translations can be sent to a n'hood U of $\gamma(1)$. If γ is a loop, then we say that γ is <u>orientation preserving</u> iff given an orientation of V, U is oriented in the same way.



<u>not</u> orientation preserving.

Check: the notion of being orientation preserving is invariant under homotopy py. Also, if α, β are loops, then the concatenation $\alpha \cdot \beta$ is orientation reversing iff exactly one of α, β is.

Hence $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ given by

$$\alpha \mapsto \begin{cases} 0 & \text{if } \alpha \text{ is orientation preserving} \\ 1 & \text{o/w} \end{cases}$$

is a well-defined homomorphism.

Let $w(M)$ denote the kernel

of this morphism.

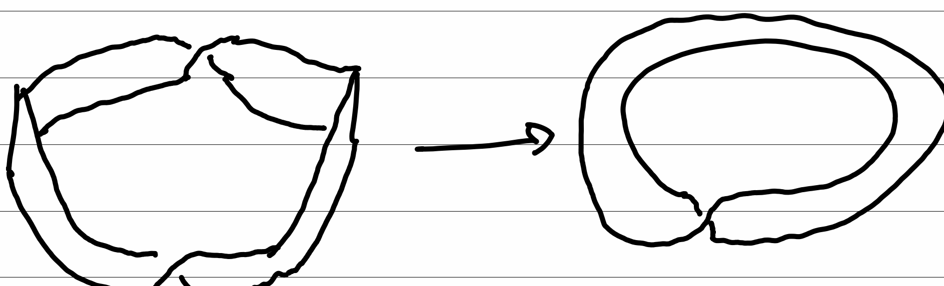
It is clear that $w(M) = \pi_1(M)$ iff M is orientable.

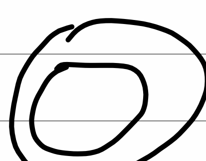
Theorem 2.12

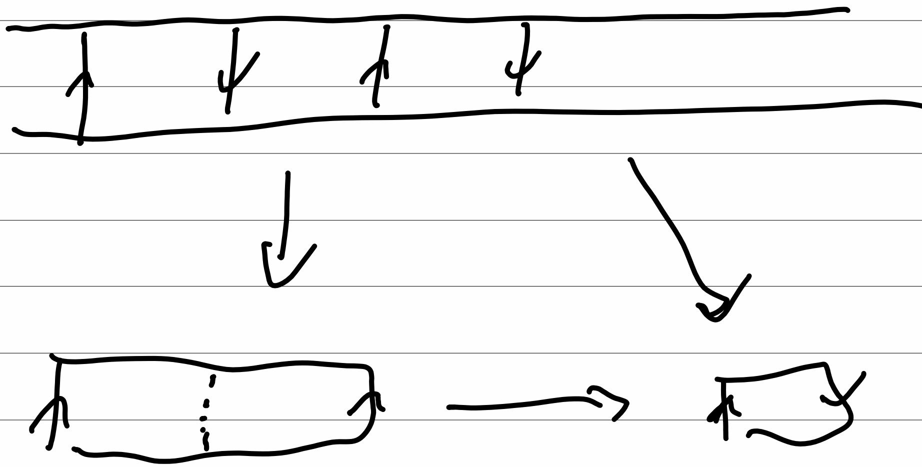
Let M be a non-orientable manifold. Then the covering $p: M' \to M$ corresponding to $w(M) \triangleleft \pi_1(M)$ is 2-sheeted, and regular. Also, M' is orientable, and universal in the sense that if $q: M'' \to M$ is a covering and M'' is orientable, then we have the commutative diagram

$$
\begin{tikzcd}
M'' \arrow[r] \arrow[dr, "q"'] & M' \arrow[d, "p"] \\
 & M
\end{tikzcd}
$$

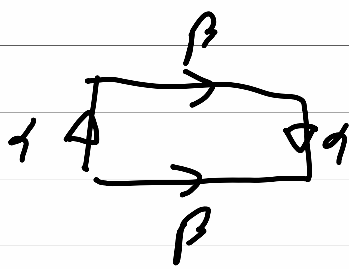
M' is called the <u>orientable double cover of M</u>.



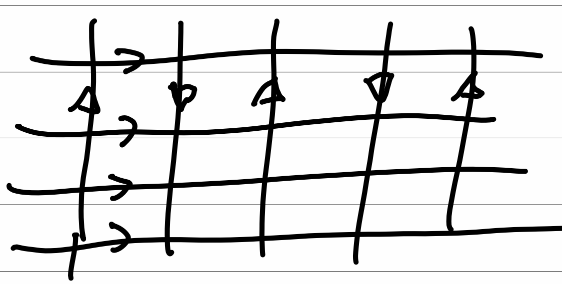
is  annulus.



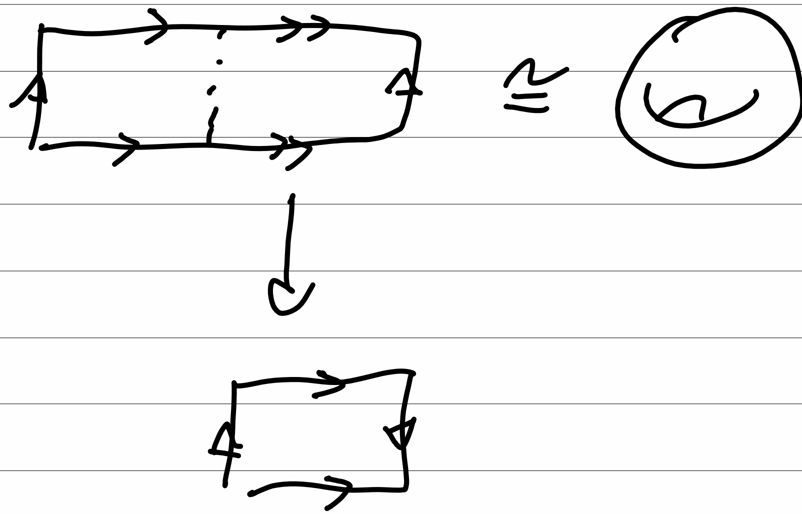
Now consider the Klein bottle:

$K =$ 

Universal cover $\cong \mathbb{R}^2$



Orientable double cover:



α, β are simple loops intersecting at a single point transversely.

$$\pi_1(K) = \langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} \rangle$$

$\alpha \in w(K)$, $\beta \notin w(K)$.

Now, $\alpha, \beta^2 \in \omega(K)$. We have

$$\pi_1(K)/\langle\langle \alpha, \beta^2 \rangle\rangle = \langle \alpha, \beta \mid \alpha, \beta^2, \alpha\beta\alpha\beta^{-1} \rangle$$

$$\cong \langle \beta \mid \beta^2 \rangle \cong \mathbb{Z}/2.$$

So $\langle\langle \alpha, \beta^2 \rangle\rangle = \omega(K)$.

Also, α normalizes $\langle \alpha, \beta^2 \rangle$ since it lies inside, β normalizes β^2, and $\beta\alpha\beta^{-1} = \alpha^{-1}$ and

$$\beta^{-1}\alpha\beta = \beta^{-1}\beta\alpha^{-1} = \alpha^{-1}.$$

$\therefore \langle \beta^2, \alpha \rangle$ is normal in $\pi_1(K)$,

and hence $\langle \beta^2, \alpha \rangle \cong \langle\langle \beta^2, \alpha \rangle\rangle \cong$

$\cong \omega(K)$.

Also $\alpha^{-1}\beta^2\alpha\beta^{-2} = \alpha^{-1}\beta\beta\alpha\bar{\beta}\bar{\beta} =$

$= \alpha^{-1}\beta\bar{\alpha}\bar{\beta} = \bar{\alpha}(\alpha\beta\alpha\bar{\beta})^{-1}\alpha = \bar{\alpha}\alpha = 1$.

So $\omega(K)$ is abelian.

Since it acts on $\mathbb{R}^2$ cocompactly,

$\omega(K) \cong \mathbb{Z}^2$.

Hence, $\forall$ element in $\pi_1(K)$ can be written uniquely as $\alpha^n \beta^m$, where $\alpha\beta = \beta\bar{\alpha}$.

<u>Lemma 2.13</u>

A pair $\alpha^p\beta^q, \alpha^r\beta^s \in \pi_1(K)$ of non-trivial elements is represented up to conjugacy by a pair ω_1, ω_2 of simple loops meeting transversely in a single point, with ω_1 orientation preserving and

ω_2 orientation reversing iff $p = \pm 1$,

$q = 0$ and $r = \pm 1$.