Grigorchuk group

Plan

1. Growth of groups
   - definition, examples
   - amenability
   - Gromov's thm
   - Grigorchuk's group
2. Atiyah conjecture
   - von Neumann algebras,
   - von Neumann dimension
   - Atiyah conj.
   - Grigorchuk's group.

1. Growth of groups 1.2. Defns and examples.

Def [Asymptotic equivalence]

$f, g : \mathbb{N} \to \mathbb{R}$ non-decreasing.

$f \precsim g$ iff $\exists A \in \mathbb{N}$:

$$f(n) \leq A g(An + A) + An + A$$

$\forall n$.

$f \simeq g$ iff $f \precsim g$ and $g \precsim f$.

[f and g are asymptotically equivalent]

Examples

- $p, q \in \mathbb{Z}[t]$. $p \simeq q$ iff they are of the same deg or both are linear or constant.

- $p \not\simeq e^t$.

<u>Def</u> [Growth]

G a group generated by a finite set S.

$\gamma_{G,S}(n) = |B(1,n)|$ where

B is the ball in $\mathrm{Cay}(G,S)$.
vertex set of

[the growth function]

<u>Lemma</u> G generated by S and T,

then $\gamma_{G,S} \simeq \gamma_{G,T}$.

Proof: Let $A \in \mathbb{N}_*$ be such that

$$S \subseteq B_T(1, A).$$

Then $B_S(1,n) \subseteq B_T(1, An)$

so $\gamma_{G,S}(n) \leq \gamma_{G,T}(An)$

$$\gamma_{G,S} \preceq \gamma_{G,T}.$$

By symmetry $\gamma_{G,T} \preceq \gamma_{G,S}$. $\square$

So we can talk about the growth of G.

Example $G = F_n$ freely generated by $S = \{a_1, \dots, a_n\}$.

$$|\partial B(1,m)| = \begin{cases} 1 & m=1 \\ 2n & \text{if } m=2 \\ 2n(2n-1)^{m-2} & m>2 \end{cases}$$

$$\gamma_{F_n,S}(m) = 1 + 2n \sum_{k=0}^{m-2} (2n-1)^k$$

$$= 1 + 2n \frac{(2n-1)^{m-1}-1}{2n-2} \asymp c^m .$$

1.2. Amenability.

One of the key concepts in group theory.

"Amenable groups are small".

<u>Def</u> (Følner) discrete group G.

Given finite $S \subseteq G$ and $\varepsilon > 0$, a

non-empty
finite subset $F \subseteq G$ is an <u>(S, ε)-Følner set</u> iff $\frac{|\partial_S F|}{|F|} < \varepsilon$,

where $\partial_S F = \{ f \in F : fs \notin F$ for some $s \in S \}$.

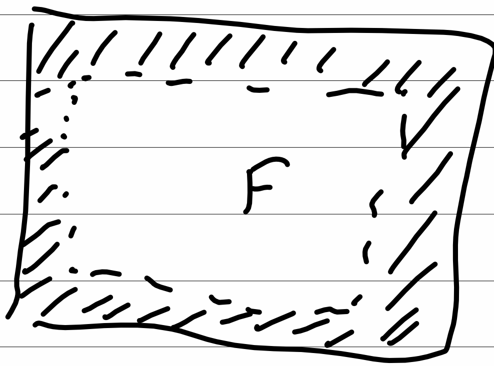
G is <u>amenable</u> iff for every S, ε there is a Følner set.

There are <u>many</u> equivalent definitions.

<u>Examples:</u>

- finite group G. Take $F = G$. Works for every S and every $\varepsilon > 0$.

• $\mathbb{Z}^n$. Take $F = [-N, N]^n$



∂F, depends on S

$|\partial F| \simeq (2N)^{n-1}$

$|F| = (2N)^n$

$\forall \varepsilon > 0$ can take n large enough.

$$\left[\ \gamma_{\mathbb{Z}^n, \text{standard}}(m) \asymp m^n \ \right]$$

• One can easily show that all abelian groups are amenable.

• <u>Fact / Exercise</u>

$K \to G \to Q$ extension, K, Q amenable. Then G is amenable.

<u>Corollary</u>
All solvable groups are amenable.
Hence all virt. solvable groups are amenable.

<u>However</u> : amenable groups can be complicated!

<u>Prop</u> If G is amenable, then $F_2 \not\leq G$.

<u>Proof</u> $F_2 = F(a,b)$, $S = \{a,b\}$.

Suppose that A is a (S,ε) Følner set, $\varepsilon \approx 0$.

$A = \bigsqcup A_i$, $A_i = A \cap$ coset of F_2.

For at least one i we must have

A_i a Følner set.

So we have found a Følner set

in F_1. But: in $\mathrm{Cay}(F_2, S)$,

full subgraphs are forests, and a tree

has $\chi = 1 = V_1 + V_2 + V_3 + V_4$

$$- \tfrac{1}{2}(V_1 + 2V_2 + 3V_3 + 4V_4)$$

$$= \tfrac{1}{2}V_1 - \tfrac{1}{2}V_3 - V_4$$

where $V_i = |\text{vertices of valence } i|$.

$$V_4 + 1 = \frac{1}{2}(V_1 + V_3)$$

$$|\text{tree}| + 1 = \frac{3}{2}(V_1 + V_3) + V_2 \le \frac{3}{2}|\partial\,\text{tree}|$$

$$\vee$$

$$|\text{tree}|$$

So $\dfrac{|\partial\,\text{forest}|}{|\text{forest}|} \ge \dfrac{2}{3}$ $\#$.