Thm If G is f.g. and non-amenable, then $\gamma_G \succeq e^n$.

Proof

$G = \langle S \rangle$, $|S| < \infty$.

G is non-amenable, so $\exists \varepsilon > 0$:

$$|\partial B_S(1,n)| \geq \varepsilon |B_S(1,n)|$$

Clearly $\varepsilon < 1$, and

$$(1-\varepsilon)|B_S(1,n)| \geq |B_S(1,n-1)|$$

So $\gamma_{G,S}(n) \geq \left(\frac{1}{(1-\varepsilon)}\right)^n$ exponential. $\square$

"Groups of subexponential growth are amenable"

Von Neumann - Day Problem

Is the converse true? If you have no F_2, are you amenable?.

No. [Olshanskii 1980] constructed non-amenable Tarski monster, i.e. all proper subgroups are 1 or $\mathbb{Z}$.

<u>Def</u> A group is a <u>Tarski monster</u> iff it is non-cyclic, f.g., and all proper non-tr subgroups are cyclic and isomorphic.

[A bit like $\mathbb{Q}$, but f.g.]

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<u>Thm</u> [Juschenko–Monod]

$\exists$ infinite simple f.g. amenable groups.

In particular, not every amenable group admits a virtual map to $\mathbb{Z}$.

1.3 Wolf-Milnor Theorem

<u>Thm</u> [Milnor-Wolf 1968]

Let G be a f.g. group.

1) If G is virt. nilpotent then

γ_G is polynomial

2) If G is virt. solvable but not v.

nilpotent then γ_G is exponential.

We need some preliminaries.

<u>Lemma</u>

G f.g., $N \trianglelefteq G$ of finite index.

Then $\gamma_N \asymp \gamma_G$.

Proof.

Let S be a finite gen set of N,

$T = S \sqcup S'$ — ″ — G

with $G \to G/N$ taking S' to G/N bijectively.

Let $k \in \mathbb{N}$ be such that $S^{S'} =$

$\{ s_i^t : t \in S', s \in S \} \subseteq B_S(1, k)$.

$t^{-1} s t$ and $S' \cdot S' \subseteq B_S(1, k) S'$.

$\gamma_{N,S}(n) = |B_S(1, n)| \leq |B_T(1, kn)|$

$\|$

$\gamma_{G,T}(kn)$.

so $\gamma_N \preceq \gamma_G$.

Consider a word in T. Going from the left, we replace

$t s_1 \dots s_\ell t'$, $t, t' \in S'$, $s_i \in S$

by $t s_1 \dots s_\ell t^{-1} s t''$ where

$s t'' = t t'$, $t'' \in S'$, $s \in S$.

Hence every element in $B_T(1, m)$ can be written as a word of length at most mk in S times a single element from S'.

There are at most $\gamma_S(mk) \cdot |G/N|$ such elements in G, and so

$$\gamma_G \preccurlyeq \gamma_N .$$

__Cor__ $H \underset{f.i.}{\leq} G$, G f.g., then

$$\gamma_H \simeq \gamma_G .$$

__Proof__ $\exists N \underset{f.i.}{\trianglelefteq} G$, $N \trianglelefteq H$. $\square$

and $\simeq$ is transitive.

Lemma

If G f.g. contains the free monoid on two generators $M(a,b)$ then

$$\gamma_G \simeq e^n.$$

Proof.

$\gamma_G \preceq e^n$ so ETS $e^n \preceq \gamma_G$.

$\exists k:$ $a, b \in B_G(1, k)$.

So $\gamma_G(mk) \geq B_M(1, m) \approx e^m$.

$$\gamma_G(m) \geq e^{\lfloor \frac{m}{k} \rfloor} \geq e^{\frac{m}{k} - 1} \simeq e^m. \qquad \square$$

Preparations for nilpotent groups.

Conventions: $g,h \in G$.

$$g^h = h^{-1} g h = \bar{h} g h$$

$$[g,h] = g h \bar{g} \bar{h}.$$

Very useful commutator relation:

$$[gh, x] = [g,x][h,x]^{x\bar{g}\bar{x}}.$$

In particular, if $[h,x]$ is central, then

$$[gh,x] = [g,x][h,x], \quad x \in G.$$

Key Lemma

G nilpotent of step k.

(G_i) lower central series, $G_0 = G$

G/G_1 f.g. $\quad q_i : G_i \to G_i/G_{i+1}$ $\quad G_{k+1} = \mathbb{1}$.

$\exists$ finite sets $T_i \subseteq G_i$ s.t.:

1) $q_i(T_i)$ is a minimal gen set of the abelian group G_i/G_{i+1}

2) $\forall i$, T_i consists of commutators from

$$[T_0, T_{i-1}] = \{ [t_0, t_{i-1}] : t_0 \in T_0,\ t_{i-1} \in T_{i-1} \}.$$

Proof Induction on i.

Base case : $i=0$. just pick T_0

Induction step : we have picked $T_0, \dots, T_{i-1}$.

Consider $\rho : G \longrightarrow H = G/G_{i+1}$

Note $H_j = \rho(G_j)$.

Let $S_j = \rho(T_j)$; note S_j has the same properties.

Key: H_i is central in H.

Every element of H_i is a product of commutators $[\alpha, \beta]$ with $\alpha \in H, \beta \in H_{i-1}$.

Since H_i is central, we may modify every α, β by multiplying it by elements of H_i. Since S_0 generates H/H_i (or T_0 generates G/G_i), we may assume that α is a word in S_0.

Similarly, β is a word in S_{i-1}.

$\alpha = x_1 \ldots x_a$, $x_i \in S_0$

$\beta = y_1 \ldots y_b$, $y_i \in S_{i-1}$

$[\alpha, \beta] = \prod_{i,j} [x_i, y_j]$ by the commutator identity.

So, $\langle S_0, S_{i-1} \rangle$ generates H_i.

Pick a minimal subset, call S_i, pull back by ρ to form T_i. $\square$