Recall: $G = \mathbb{Z}^n \rtimes_A \mathbb{Z}$.

If all e-values of A lie on S^1,

then G is v.f. nilpotent.

2° $\exists$ e-value λ of A with $|\lambda| \neq 1$.

WLOG $|\lambda| > 1$ is maximal.

$v \in \mathbb{Z}^n \otimes \mathbb{C} = \mathbb{C}^n$ s.t. $|v| = 1$, $Av = \lambda v$.

Replace A by a power s.t. $|\lambda| \geq 3/_1$.

Find $w \in (\mathbb{Z} \oplus i\mathbb{Z})^n$: $|w - \langle v,w\rangle v| < \frac{1}{5}|w|$.

Consider free monoid on $\{t, tw\}$.

If it does not embed, then some

$t^j w t^{-j}$ a difference of $\sum_{i=1}^{\lambda} t^{k_i} w t^{-k_i}$

and $\sum_{\ell=1}^{\beta} t^{k_\ell} w t^{-k_\ell}$, with $k_i, k_\ell < j$,

$\{k_i\}, \{k_\ell\}$ without repetitions.

Now: $t^j w t^{-j} = A^j w =$

$= A^j v/\langle v,w\rangle + A^j(w - v\langle v,w\rangle)$

has ~~norm~~ length at least $\lambda^j \langle v,w\rangle - \frac{1}{5}\lambda^j |w|$

$\geq \frac{3}{5}\lambda^j |w|$.

In fact, $\langle v, A^j w \rangle \geq \frac{3}{5}|\lambda|^j |w|$.

The projection of $A^{k_i} w$ onto v is positive, so we ignore it.

We get $\displaystyle\sum_{i=1}^{j} \frac{6}{5}|\lambda|^{k_i} |w| \geq \frac{3}{5}|\lambda|^j |w|$.

$$\sum_{i=1}^{j} \frac{6}{5}|\lambda|^{k_i} |w| \leq \frac{6}{5}|w| \frac{1}{|\lambda|-1} |\lambda|^j \leq \frac{2}{5}|w|\,|\lambda|^j .$$

Hence $(\mathbb{Z}+i\mathbb{Z})^4 \rtimes_A \mathbb{Z}$ has exp. growth.

$$(\mathbb{Z}^4 \rtimes_A \mathbb{Z}) \times (\mathbb{Z}^4 \rtimes_A \mathbb{Z}) \twoheadrightarrow (\mathbb{Z}+i\mathbb{Z})^4 \rtimes_A \mathbb{Z} .$$

So $\mathbb{Z}^4 \rtimes \mathbb{Z}$ has exponential growth.

Now we are ready for:

Lemma $\underset{\text{nil}}{N} \hookrightarrow \underset{\text{f.g.}}{G} \twoheadrightarrow \mathbb{Z}^n$

is v. nilpotent or has exponential growth.

Proof

From the key example: $\mathbb{Z}^n \curvearrowright N_i/N_{i+1} \otimes \mathbb{Q}$

has all eigenvalues on S^1, and

hence we may pass to a finite index

subgroup of $\mathbb{Z}^n$ and get $\mathbb{Z}^n \curvearrowright N_i/N_{i+1}$

trivial for all i. So G has center,

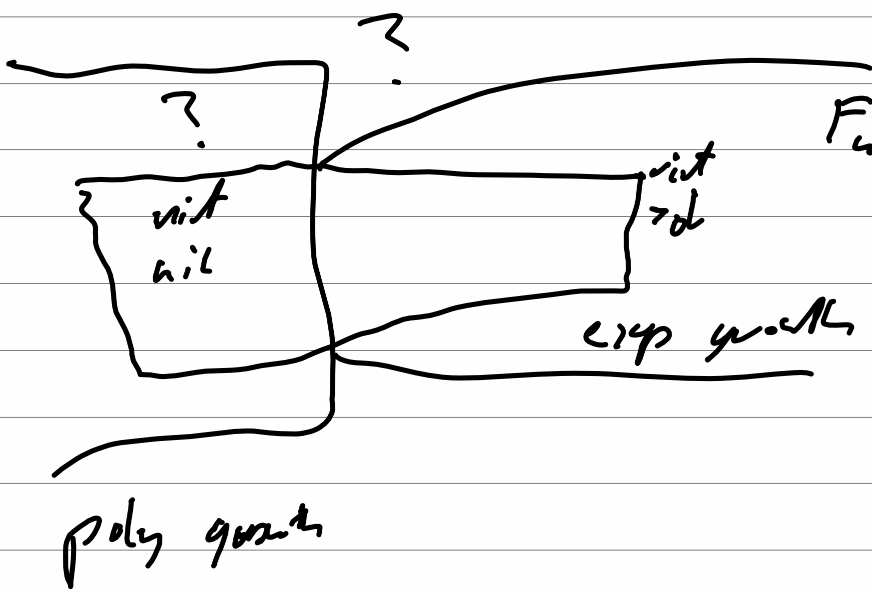
and in fact a finite central series. $\square$

The last missing ingredient is

Th (Malcev)

Every polycyclic group is virtually f.g. nilpotent-by-f.g.-by-f.g. free abelian.

This concludes our study of solvable groups.



1.4 Gromov's theorem

<u>Thm</u> [Gromov]

G f.g. has polynomial growth :⟺ G is virt. nilpotent.

We will follow Ozawa's proof.

First we need some property T.

<u>Def</u> [Kazhdan]

G loc. cpct group, $S \subseteq G$ finite, $\varepsilon > 0$, $\pi : G \to \mathcal{U}(\mathcal{H})$ unitary rep on a Hilbert space $\mathcal{H}$.

A unit vector $v \in \mathcal{H}$ is (S, ε)-almost invariant iff $|\langle s \cdot v, v \rangle| > 1 - \varepsilon$ $\forall s \in S$.

G has T iff $\exists S$ $\exists \varepsilon$ $\forall \pi$:

if there is an (S, ε)-almost inv. vector then G fixes a unit vector in π.

Example: • finite groups

• $SL_n \mathbb{Z}$, $n \geq 3$

• lattices in higher rank semi-simple Lie groups.

Non-examples: • free groups

• indicable groups,

i.e. $G \twoheadrightarrow \mathbb{Z}$.

<u>Prop</u> G infinite amenable.

G does not have T.

<u>Proof</u> Suppose it does for some

$S \subseteq G$ finite, $\varepsilon > 0$.

Let F be a (S, ε)-Følner set.

Let $\pi : G \to U(L^2(G)) = \{ \sum_{g \in G} \lambda_g g : \sum |\lambda_g|^2 < \infty \}$, $\lambda_g \in \mathbb{C}$

$$g . \sum \lambda_h h = \sum \lambda_h g h.$$

$\mathbb{1}_F \in L^2(G)$. Clearly, the G-invariant vectors in $L^2(G)$ are constant functions. $|G| = \infty$, so the only G-inv vector is 0.

Now $\forall \, \partial \in \mathcal{F}$:

$$\left\langle \partial \frac{1}{\sqrt{|F|}} \mathbb{1}_F , \frac{1}{\sqrt{|F|}} \mathbb{1}_F \right\rangle =$$

$$\frac{1}{|F|} \left| F \setminus \partial_s F \right| = 1 - \frac{|\partial_s F|}{|F|} \geq$$

$$\geq 1 - \frac{|\partial_S F|}{|F|} \geq 1 - \varepsilon. \qquad \square.$$

[Mok, Korevaar–Schoen]

Thm G has (T) iff $\exists\, \pi : G \to U(\mathcal{H})$

s.t. $H^1_{cont}(G; \mathcal{H}) \neq 0$.

We will not prove this.

$$H^1_{cont}(G; \mathcal{H}) := Z^1(G; \mathcal{H}) \big/ \overline{B^1(G; \mathcal{H})}.$$

$$Z^1(G; \mathcal{H}) = \left\{ f : G \to \mathcal{H} \;\middle|\; \begin{array}{l} \forall g, h \in G : \\ f(gh) = f(g) + g.f(h) \end{array} \right\}$$

cocycles.

$$B^1(G; \mathcal{H}) = \{ g \mapsto x - g.x \mid x \in \mathcal{H} \}.$$

Closure is taken in the product Topology i.e. pointwise. on G.