We are proving: G has poly growth $\Rightarrow$ G is virt. nilpotent.

Goal: produce a fin dim rep of G.

We have $\pi : G \to \mathcal{U}(\mathcal{H})$ with

$H^1(G, \mathcal{H}) \neq 0$. Fix $S = S^{-1}$ finite gen set.

<u>Lemma</u> Every coclass in $H^1(G; \mathcal{H})$ is represented by a unique S-harmonic cocycle $G \to \mathcal{H}$.

<u>Proof</u> Let $\mu : \mathcal{P}(G) \to [0, \infty)$

$$\mu(T) = \frac{|T \cap S|}{|S|}$$ be

a prob. measure associated to S.

$\xi : G \to \mathcal{H}$ is μ-*harmonic*

(or S-*harmonic*) iff $\sum_{g\in G} \mu(g)\xi(g) = 0$.

We endow $Z^1(G;\mathcal{H})$ with a Hilbert space structure via $\langle \xi, \eta \rangle = \sum_g \mu(g) \langle \xi(g), \eta(g) \rangle$ (it requires checking that this is a Hilbert space).

It is easy to see that the topology this induces coincides with the product top, since $\xi \in Z^1(G;\mathcal{H})$ is uniquely determined by $(\xi(s), s \in S)$. So $\overline{B^1(G;\mathcal{H})}$ is the Hilbert-space closure.

Now $\overline{B^1(G;\mathcal{H})} = B^1(G;\mathcal{H})^{\perp} =$

$$= \left\{ \xi \in Z^1(G;\mathcal{H}) \;\middle|\; \sum \mu(g) \langle \xi(g), x - gx \rangle = 0 \quad \forall x \in \mathcal{H} \right\}$$

Note: $\langle \xi(g), gx \rangle = \langle g^{-1}\xi(g), x \rangle$

and $0 = \xi(g^{-1}g) = \xi(g^{-1}) + g^{-1}\xi(g)$

so $g^{-1}\xi(g) = -\xi(g^{-1})$.

$\therefore \sum \mu(g) \langle \xi(g), x - gx \rangle =$

$= \langle \sum \mu(g) \xi(g), x \rangle + \sum \mu(g) \langle \xi(g^{-1}), x \rangle$

$= 2 \langle \sum \mu(g) \xi(g), x \rangle$

Taking $x = \sum \mu(g)\, \xi(g)$ yields

$$B^1(G;\mathcal{H})^\perp = \{\xi \in Z^1 : \sum \mu(g)\xi(g) = 0\}$$

i.e. the set of harmonic cocycles.

Hence $H^1(G;\mathcal{H})$ lifts to the set of harmonic cycles in Z^1. $\square$

Suppose $b : G \to \mathcal{H}$ is a non-trivial harmonic cocycle.

We may also assume that $\mathcal{H}$ has no trivial subrep.

Def The entropy of a measure ν is

$$H(\nu) = \sum_g \nu(g) \log \frac{1}{\nu(g)}$$

$\left[H(\mu) = \log |S|. \right]$

$$\delta(\nu,\nu') = H\left(\frac{\nu+\nu'}{2}\right) - \frac{1}{2}\left(H(\nu) + H(\nu')\right).$$

Fact: H is concave, i.e., $\delta(\nu,\nu') \geq 0$.

Def ν,ν' two measures on G.

The convolution $\nu * \nu'$ is

$$\nu * \nu'(g) = \sum_h \nu(h)\, h.\nu'(g) \text{ where}$$

$$h.\nu'(g) = \nu'(h^{-1}g).$$

$G = \langle S \rangle$, $S = S^{-1}$, $1 \in S$.

Lemma G has poly growth then it has *slow entropy growth* i.e.

$$\liminf n\left(H(\mu^{*n+1}) - H(\mu^{*n})\right) < \infty.$$

Proof Since log is concave,

$$H(\mu^{*n}) = \sum_g \mu^{*n}(g) \log \frac{1}{\mu^{*n}(g)} \leq$$

$$\leq \log \sum_{g \in \operatorname{supp} \mu^{*n}} \frac{\mu^{*n}(g)}{\mu^{*n}(g)} = \log |\operatorname{supp} \mu^{*n}|$$

$$= \log \gamma_{G,S}(n) \leq d \log n$$

when $\gamma(n) \leq n^d$.

Now if $d' < \liminf n\left(H(\mu^{*n+1}) - H(\mu^{*n})\right)$

then $H(\mu^{*n}) = \sum_{k=0}^{n-1} H(\mu^{*k+1}) - H(\mu^{*k}) \geq$

$$\geq \sum_{k=1}^{n-1} \frac{d'}{k} + H(\mu) \geq d' \log n + H(\mu)$$

So $d \log n \geq d' \log n + H(\mu)$ $\forall n$.

$\therefore d \geq d'$. $\square$

A series of (locally lossy) estimates gives:

for $b : G \to \mathcal{H}$ harmonic cocycle, $\xi \in \mathcal{H}$, $g \in S$:

$$|\langle b(g), \xi \rangle|^2 \leq \frac{4}{|S|}\left(H(\mu^{*n+1}) - H(\mu^{*n})\right)$$

$$\cdot \sum_{x \in G} |\langle b(x), \xi \rangle|^2 (g\mu^{*n} + \mu^{*n})(x).$$

$$\lesssim d \cdot \frac{4}{|S|} \cdot \frac{1}{n} \sum_{x} |\langle b(x), \xi \rangle|^2 (g\mu^{*n} + \mu^{*n})(x)$$

Involved analysis: if $\mathcal{H}$ has no non-trivial finite dim subrepresentation, then

$$\frac{1}{n}\sum_{x}\mu^{*n}(x)\langle b(x), \xi\rangle \longrightarrow 0 \text{ as } n\to\infty.$$

So if $\mathcal{H}$ is as above, then

$$\langle b(x), \xi\rangle = 0 \quad \forall\, x\in S,\ \xi\in\mathcal{H}.$$

$\therefore\ b(x) = 0 \quad \forall\, x\in S$

$\therefore\ b = 0 \quad \lightning .$

So $\exists$ fin dim subrep of $\mathcal{H}$.

One can quite easily check that

$$\mathcal{H}' = \mathcal{H}/\overline{\langle \text{fin dim subreps}^{\pi} \text{ with } H'(G,\pi) = 0\rangle}$$

is still a grp with $H^1(G;\lambda)\neq 0$

with no finite dim subgrps with $H^1=0$.

<u>So</u> we have produced $\rho: G \to GL_n\mathbb{C}$

s.t. there exists a non-trivial harmonic

cocycle.

Finite groups have T, so $|\rho(G)|=\infty$.

Tits alternative tells us that $\rho(G)$ is virt.

solvable, so $\exists\, G_1 \underset{\text{f.i.}}{\leq} G$:

$$G_2 \longrightarrow G_1 \twoheadrightarrow \mathbb{Z}$$

Poly growth, using the argument of

Milnor then G_2 is f.g.

Clearly G_2 has growth n^{d_2} with

$d_2 < d_1$, G_1 of growth n^{d_1}.

Now use induction.

virt.

G is poly-v. $\mathbb{Z}$ so soluble, so

v. nilpotent.