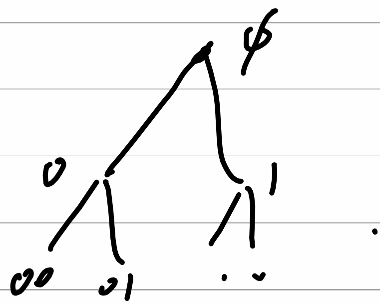
2. Grigorchuk's group.

Let T be a rooted binary tree.

$V(T)$ = free monoid on $\{0,1\} \cong \mathbb{Z}/2\mathbb{Z}$.



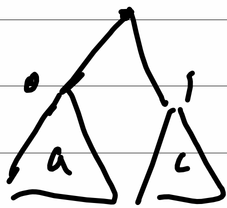
Define $a \in \mathrm{Aut}(T)$ by

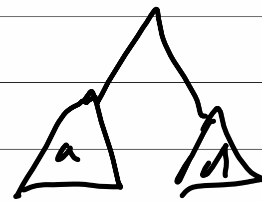
$$a \cdot v_1 v_2 \cdots v_n = (v_1+1) v_2 v_3 \cdots$$

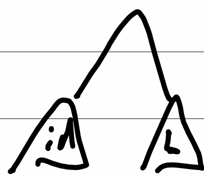
with $v_1 + 1$ mod 2.

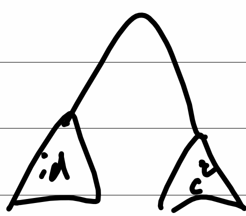
Note $a^2 = id$.

Define $b, c, d \in \mathrm{Aut}(T)$ by

b : 

c : 

$d =$  .

$b^2 =$  $= id$

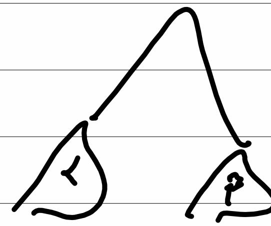
$c^2 = id$, $d^2 = id$. Also, b, c, d commute.
and $bcd = 1$.

<u>Def</u> $G = \langle a, b, c, d \rangle \leq \mathrm{Aut}(T)$
the Grigorchuk group.

Lemma

$$\varphi : \left(\mathrm{Aut}(T) \times \mathrm{Aut}(T)\right) \rtimes \mathbb{Z}/2 \longrightarrow \mathrm{Aut}(T)$$

$$((\alpha, \beta), i) \longmapsto$$



$\cdot\, a^i$

is an isomorphism.

Proof. Immediate.

Let $H = \{ g \in G : g \cdot 0 = 0 \}$.

Lemma $H < G$ of index 2,

$$H = \langle b, c, d, b^a, c^a, d^a \rangle .$$

$\varphi^{-1}(H) \leq G \times G$ is a subdirect product.

Proof $a \notin H$ and clearly $G = H \sqcup Ha$.

$$H = \ker(G \underset{\substack{\text{kills} \\ b,c,d}}{\twoheadrightarrow} \mathbb{Z}/2).$$

So $H = \langle b, c, d, b^a, c^a, d^a \rangle$

by Schreier formula.

$$\varphi^{-1}(H) = \langle (a,c), (a,d), (1,b), (c,a), (d,a), (b,1) \rangle$$

is a subdirect product of $G \times G$. $\square$

Cor G is infinite.

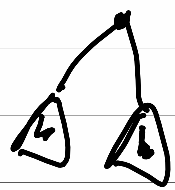
Proof $H \twoheadrightarrow G$, $|G:H| = 2$. $\square$

Let $B = \langle\langle b \rangle\rangle \trianglelefteq G$.

Lemma $B \leq_{f.i.} G$.

Proof G/B is generated by a and d

since $bcd = cbd = 1$.

Now $(ad)^2 =$  so $(ad)^4 = id$.

So G/B is a quotient of

$\langle a, d \mid a^2, d^2, (ad)^4 \rangle = D_4$ $\square$.

We now write $\approx$ for commensurability.

Lemma $G \approx G^2$.

Proof. $G \approx H$,

$$\varphi^{-1}(H) = \langle (1,b), (1,c), (1,d), (b,1), (c,1), (d,1) \rangle$$

$$\geq \langle (1,b), (b,1) \rangle.$$

Take $g \in G$.

$$\varphi((b^g, 1)) = b^g \in H.$$

So $B \times B \leq \varphi^{-1}(H) \cong H \approx G$

$\cong$

$G \times G$ $\square$

Cor $\gamma_G \sim \gamma_{G^2} \sim (\gamma_G)^2$,

Since $\gamma_G(n) \geq n$ as $|G| = \infty$, we get that γ_6 is not a polynomial.

Let H_3 denote the stabilizer of the first 3 levels.

We will rewrite every $g \in G$ as

Can compute g_i using the rewriting procedures Φ_0, Φ_1, $i \in \{0,1\}$. Both like a.

$$\Phi_0(b) = \begin{cases} c & \text{if } \#a\text{'s before } b \text{ is even} \\ \epsilon & \text{odd} \end{cases}$$

b	Φ_0	Φ_1
#a even	c	a
odd	a	c

c		
	d	a
	a	d

d		
	b	1
	1	b

<u>Lemm</u> $\sum \ell(g_{i|x}) \le \frac{5}{6}\ell(g) + 8$, $g \in H_3$.

<u>Proof</u> Write $g = a x_1 a x_2 \dots$, $x_i \in \{b, c, d\}$

or $g = x_1 a x_2 a \dots$.

Call the word w. $|w| = \ell(g)$.

We rewrite w into w_0, w_1 as above.

Get $|w_0| + |w_1| \le |w| + 1 - |w|_d$

where $|w|_x$ is the # of x's in w.

[+1 comes from $\underbrace{\quad}_{4}\underbrace{\quad}_{4} x_i = w$].

$\sum |w_{ij}| \leq |w| + 3 - |w|_c$

$\sum |w_{ijk}| \leq |w| + 7 - |w|_b$.

Now $|w|_b + |w|_c + |w_d| \geq \frac{|w|-1}{2}$.

So for some x, $|w|_x \geq \frac{|w|}{6} - 1$.

Now $\sum l(g_{ijk}) \leq \min\{ \sum |w_{ijk}|, \sum |w_{ij}| + 4,$

$\sum |w_i| + 2 + 4 \}$

$\leq |w| + 7 - \frac{|w|}{6} + 1 =$

$= \frac{5}{6}|w| + 8$ $\square$

Note $|G : H_3| \leq 2^6$.

We can find coset reps of length at most 2^6.

So $\gamma_6(n) \le \left| B(1, n+2^{6}) \cap H_3 \right|$

$$\le \sum_{\substack{n_{ijk} \\ a_{n_{ijk}} \le (n+2^6)\cdot\frac{5}{6}+8}} \prod \gamma_6(n_{ijk})$$

If $\gamma_6(n) = e^{n}$ fln

$$e^{n} \le \sum e^{a_{n_{ijk}}} \le \text{polys in } n \cdot e^{5/6 n + \text{const.}}$$

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