b	Φ_0	Φ_1
# a even	c	a
odd	a	c

c		
	d	a
	a	d

d		
	b	1
	1	b

Lemma $\sum \ell(g_{ijk}) \le \frac{5}{6}\ell(g) + 8$, $g \in H_3$.

Proof Write $g = a x_1 a x_2 \dots$, $x_i \in \{b, c, d\}$

or $g = x_1 a x_2 a \dots$.

Call the word w. $|w| = \ell(g)$.

We rewrite w into w_0, w_1 as above.

Get $|w_0| + |w_1| \le |w| + 1 - |w|_d$

where $|w|_x$ is the # of x's in w.

[$+1$ comes from $\underbrace{\quad}_{4}\underbrace{\quad}_{4} x_i = w$].

$\sum |w_{ij}| \leq |w| + 3 - |w|_c$

$\sum |w_{ijk}| \leq |w| + 7 - |w|_b$.

Now $|w|_b + |w|_c + |w_d| \geq \frac{|w|-1}{2}$.

So for some x, $|w|_x \geq \frac{|w|}{6} - 1$.

Now $\sum l(g_{ijk}) \leq \min\{ \sum |w_{ijk}|, \sum |w_{ij}| + 4,$

$\sum |w_i| + 2 + 4 \}$

$\leq |w| + 7 - \frac{|w|}{6} + 1 =$

$= \frac{5}{6}|w| + 8$ $\square$

Note $|G : H_3| \leq 2^6$.

We can find coset reps of length at most 2^6.

So $\gamma_6(n) \le \left| B(1, n+2^{6}) \cap H_3 \right|$

$$\le \sum_{\substack{n_{ijk} \\ a\, n_{ijk} \cdot (n+2^{6}) \cdot \frac{5}{6} + 8}} \prod \gamma_6(n_{ijk})$$

If $\gamma_6(n) = e^{n}$ fln

$$e^{n} \le \sum e^{a\, n_{ijk}} \le \text{polyin } n \cdot e^{5/6\, n + \text{const.}}$$

Lamplighter group

$$\mathbb{Z}/_2 \wr \mathbb{Z} = \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}/_2\Big) \rtimes \mathbb{Z}$$

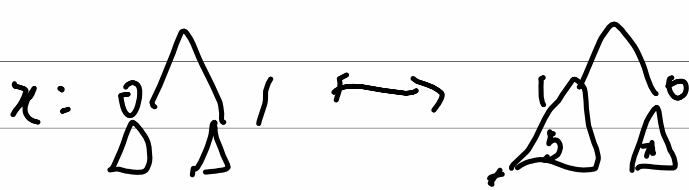
with $\mathbb{Z} \curvearrowright \bigoplus_{\mathbb{Z}} \mathbb{Z}/_2$ by addition on the indexing $\mathbb{Z}$.

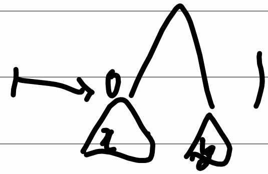
$$\mathbb{Z}/_2 \wr \mathbb{Z} = \langle a, b \mid [b^{a^i}, b] = 1,\ b^2 = 1 \rangle$$

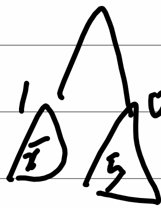
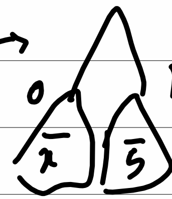
b^{a^i} generates $\mathbb{Z}/_2$ in position i.

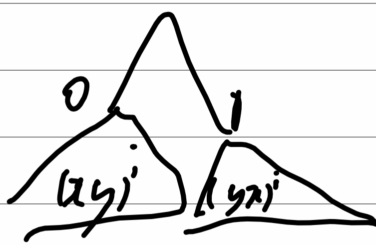
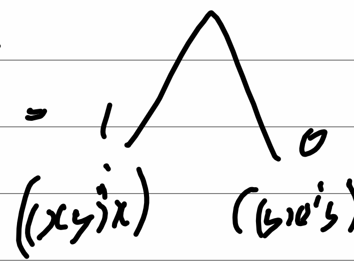
This group is <u>not</u> finitely presented.

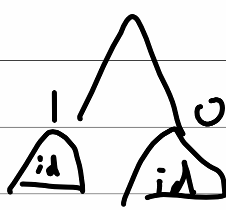
Consider $x, y \in \mathrm{Aut}(T)$ given by



and y: 

$\bar{x}$:  , $\bar{y}$: 

$x^{2i} =$  $x^{2i+1} =$ 

$r = \bar{x} y =$  so $r^2 = 1$.

Def Let $N = \langle\langle r \rangle\rangle$ in $G = \langle x, y \rangle$.

Lemma N coincides with the smallest subgroup Z of G s.t.

$\forall h \in Z$: $\eta h \xi^{-1} \in Z$ with $\eta, \xi \in \{x, y\}$

or $\eta, \xi \in \{\bar{x}, \bar{y}\}$.

Moreover, N is abelian, all elts of order 2.

Proof $r = \bar{x} y \in Z$, so $N \leq Z$.

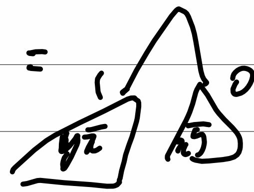
Take $h \in N$. $x h \bar{y} = x h \bar{x} x \bar{y} =$

$= x h \bar{x} \, x \bar{r} \bar{x} \in N$. Similarly for other.

So $N = Z$.

Let $Z' = \left\{ \underset{1 \quad 1}{i \bigwedge j}, \ 1 \in Z' \right\}$, fixed by sym of levels.

Exercise: Z' is a group. It contains γ and

$\delta := y\bar{x} = x\gamma\bar{x} =$ 

$(y\bar{x})^2 = \underset{id}{0} \bigwedge \underset{id}{1}$ so $y\bar{x} = x\bar{y}$ so $y\bar{x} \in Z'$.