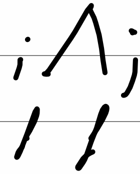
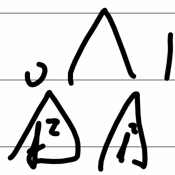
$Z' = \{$  $: J \in Z' \}$.

Let $h \in Z'$.

$h^2 =$  $=$ id

So Z' is abelian.

$$\bar{x}\ \underset{f\quad f}{{}_0\bigwedge_1}\ x = \underset{\bar{y}/x\quad \bar{x}/y}{{}_0\bigwedge_1} \leq \underset{\substack{\bar{y}/x \\ \wedge \\ z'}\quad \substack{r\bar{y}/ux \\ \wedge \\ z'}}{{}_0\bigwedge_1} = \underset{1\ \ r}{{}_0\bigwedge_1}.$$

Similar for other cases [satisfies def of z' up to length n]

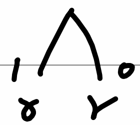
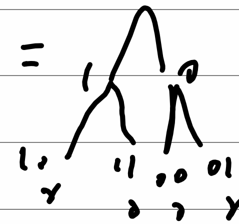
So $N \leq z'$. $\square$

$$\bar{x}\,\gamma\, x = \underset{\gamma\quad \gamma}{{}_1\bigwedge_0}, \quad \delta = \underset{\delta\quad \delta}{{}_1\bigwedge_0}$$

So $N = \{ \underset{f\quad f}{{}_i\bigwedge_j},\ f \in N \}$.

$N \cong N \times \mathbb{Z}/2$ so infinite.

Countable, $\mathbb{Z}/2$-mod so $\bigoplus_{\mathbb{N}} \mathbb{Z}/2$.

$x^{-1}\gamma x =$  , $x^{-2}\gamma x^{2} =$ 

in general: γ'_{y} in leaf i, trivial else.

So $\prod x^{-n_i}\gamma x^{n_i} \neq 1$ if n_i distinct.

$\therefore N \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$, x acts by shifting.

$\therefore$ G/N cyclic must be $\mathbb{Z}$.

$\therefore$ $G \cong \mathbb{Z}/_2 \wr \mathbb{Z}$.

4. L^2-Betti numbers. (algebraic def).

M is an $n \times m$ matrix over $\mathbb{Z}G$.

$$(\mathbb{Z}G)^n \xrightarrow{M} (\mathbb{Z}G)^m.$$

$$L^2(G)^n \xrightarrow{L^2(G)\otimes M} L^2(G)^m$$

$\dim \ker L^2(G) \otimes_G M$?

Note that this ker is G-invariant,

hence can find G-equivariant projection

$p : L^2(G)^n \longrightarrow L^2(G)^n$ with image $\ker L^2(G) \otimes M$.

p is represented by $(p_{ij})_{i,j=1..n}$,

with $p_{ij} \in N(G) = \{G$-equivariant bounded operator on $L^2(G) \} \subseteq L^2(G)$

$$x \longmapsto x(1).$$

$\dim_{N(G)} \ker = \operatorname{tr} p = \sum \operatorname{tr} p_{ii}$,

$\operatorname{tr} p_{ii}$ = coeff at 1.

<u>Atiyah conj</u> G is torsion free.

$\forall M \in M_{n \times m}(\mathbb{Z}G)$,

$$\dim_{N(G)} \ker L^2 \otimes_G M \in \mathbb{Z}.$$

<u>Strong Atiyah conj</u> (old version)

$\dim \ker \in \langle 1$, orders of finite groups in $G \rangle \leq \mathbb{Q}$.

This would mean that for M over $\mathbb{Z}L$, $L = \mathbb{Z}/2 \wr \mathbb{Z}$, we should have

$$\dim \ker \in \mathbb{Z}[\tfrac{1}{2}].$$

[Grigorchuk – Żuk]: $\exists\ m \in \mathbb{Z}L$ s.t.

$$\dim_{N(L)} \ell^2(L) \otimes_L m = \frac{1}{3}.$$

[Now Atiyah conjecture is formulated for groups with bounded torsion].

Key tool: Linnell's approximation thm.

Thm [Lück approximation]

Let $M \in M_{n \times m}(\mathbb{Z}G)$,

$G = G_0 > G_1 > \dots$, $G_i \trianglelefteq G$, $\bigcap G_i = 1$.

G/G_i finite.

$$\dim_{\mathcal{N}(G)} \ker L^2(G) \otimes M =$$

$$= \lim_{i \to \infty} \frac{1}{|G:G_i|} \dim_{\mathbb{Q}} \ker \mathbb{Z}G/G_i \otimes_G M.$$

Grigorchuk - Żuk look at the Markov operator $M = (x + \bar{x} + y + \bar{y}) \cdot \frac{1}{4} \in \mathbb{Q}L$.

They show that the spectrum of M is the same as the spectrum of

$M_{\partial T}$

M thought of as an operator on

$L^2(\partial T$, natural measure).

(this induces the measure).

So if $\lambda \in Sp(M)$ with multiplicity μ

then $\dim_{W(L)} L^2(L) \otimes_L (M - \lambda I) = \mu$.

We want $\mu = \frac{1}{3}$ for some λ.

Lück's thm gives

$$Sp(M_{\partial T}) = \lim_{i \to \infty} \frac{1}{2^i} Sp(M|_{\text{levels of } T}).$$

G Ż compute that $\lambda = 0$ has $\mu \to \frac{1}{3}$

as $i \to \infty$.