Lecture 2

L^2-homology

Let X be a CW-cplx. We define the <u>L^2-chain complex</u> $C_*^{(2)}(X)$ by

$$C_n^{(2)}(X) = \left\{ \sum_{c \in n\text{-cells}} \lambda_c c \;\middle|\; \begin{array}{l} \lambda_c \in \mathbb{C} \\ \sum |\lambda_c|^2 < \infty \end{array} \right\}$$

Clearly the boundary maps give us linear maps $\partial_n : C_n^{(2)}(X) \longrightarrow C_{n-1}^{(2)}(X)$ between the Hilbert spaces $C_n^{(2)}(X)$.

The <u>L^2-homology</u> of X is

$$H_n^{(2)}(X) = \ker \partial_n \big/ \overline{\operatorname{im} \partial_{n+1}} .$$

At this point, we can say that $H_n^{(2)}(X) = 0$, but not much more.

Assume now that every ∂_n is <u>uniformly bounded</u>, i.e., $\exists k \; \forall n\text{-cell } c$, $\partial_n(c)$ is supported on at most k $(n-1)$-cells.

Under the extra assumption, ∂_n is bounded ($\Leftrightarrow$ continuous), and hence $\ker \partial_n$ is closed. $\therefore$ $\ker \partial_n$ is a Hilbert space.

$\therefore$ $\ker \partial_n / \overline{\operatorname{im} \partial_{n+1}} = H_n^{(2)}(X)$ is also a Hilbert space.

<u>Example</u>. If X is of finite type ($\Leftrightarrow$ every n-skeleton X_n is finite) then $C_n^{(2)}(X) = C_n(X)$, and $\operatorname{im} \partial_{n+1} = \overline{\operatorname{im} \partial_{n+1}}$.

$\therefore$ $H_n^{(2)}(X) = H_n(X)$.

In general, $H_n^{(2)}(X)$ is infinite dimensional as a vector space. To gain more information we need a better notion of dimension.

<u>Von Neumann Algebras</u>

Let $\mathcal{H}$ be a Hilbert space.

$$B(\mathcal{H}) = \left\{ T : \mathcal{H} \to \mathcal{H} \text{ linear} \;\middle|\; \begin{array}{l} \exists k \, \forall v \in \mathcal{H}: \\ \|v\| \leq 1 \Rightarrow \|Tv\| \leq k \end{array} \right\}$$

bounded linear operators on $\mathcal{H}$.

[smallest k = operator norm = $\|T\|$]

Fact

$\forall\, T \in B(\mathcal{H})\ \exists\, T^* \in B(\mathcal{H})\ \forall\, v, w \in \mathcal{H}$:

$\langle Tv, w\rangle = \langle v, T^* w\rangle$ [T^* is the adjoint]

Let G be a (discrete, countable) group.

$L^2(G) = \{ \sum_{g\in G} A_g\, g \mid A_g \in \mathbb{C}, \sum |A_g|^2 < \infty \}$

is a Hilbert space.

G acts on $L^2(G)$ by right multiplication, yielding $\rho : G \hookrightarrow B(L^2(G))$.

Def [Group von Neumann algebra]

$$W(G) = \{ T \in B(L^2(G)) \mid T\rho(g) = \rho(g)T \ \forall g \in G \}$$

= commutant of $\rho(\mathbb{C}G)$.

We also have the left action $G \curvearrowright L^2(G)$ yielding $\mathbb{C}G \hookrightarrow B(L^2(G))$.

Clearly, left action commutes with right action

$\therefore\ \mathbb{C}G \hookrightarrow W(G)$.

Lemma $*: B(L^2(G)) \to B(L^2(G))$ restricted to $\mathbb{C}G$

is given by

$$*: \sum \lambda_g g \longmapsto \sum \overline{\lambda_g}\, g^{-1}.$$

Proof Take $x = \sum \lambda_g g \in \mathbb{C}G$. Define $\bar{x} = \sum \overline{\lambda_g}\, g^{-1}$.

$\forall a, b \in G$: $\langle x a, b\rangle = \langle \sum \lambda_g g a, b\rangle =$

$= \lambda_{ba^{-1}}$.

OTOH: $\langle a, \bar{x} b\rangle = \lambda_{ba^{-1}}$

$\therefore \langle u, (\bar{x} - x^*) v\rangle = 0 \quad \forall u, v \in \mathbb{C}G$.

But $\mathbb{C}G$ is dense in $L^2(G)$, and

$$L^2(G) \times L^2(G) \to \mathbb{C}$$
$$(u, v) \longmapsto \langle u, (\bar{x} - x^*) v\rangle$$

is continuous.

$\therefore \bar{x} - x^* = 0$ $\square$

Lemma $W(G)$ is a $*$-algebra (i.e. is

$*$-invariant).

Proof Take $T \in W(G)$. Then $T^* \in B(L^2(G))$.

Take $g \in G$.

$T \in W(G) \implies T\rho(g) - \rho(g) T = 0$

$\therefore \rho(g)^* T^* - T^* \rho(g)^* = \rho(g^{-1}) T^* - T^* \rho(g^{-1}) = 0$

$\forall g \in G$. So T^* commutes with $\rho(G)$. $\square$

Hence $W(G)$ is a C^*-algebra.

(we are now doing noncommutative geometry!)

Open problem: are $W(F_n)$ and $W(F_m)$ $(n \neq m > 1)$ isomorphic as (C^*-)algebras?

Theorem (von Neumann)

- $W(G)$ is its own bicommutant in $B(L^2(G))$.

- $W(G)$ is weakly closed in $B(L^2(G))$
- $W(G)$ is strongly closed in $B(L^2(G))$.

Def $\mathcal{H}$ a Hilbert space.

The weak topology on $B(\mathcal{H})$ is the coarsest topology in which all maps $E_{x,y} : B(\mathcal{H}) \to \mathbb{C}$
$E_{x,y}(T) = \langle Tx, y \rangle$ are continuous.

The strong topology on $B(\mathcal{H})$ is the coarsest topology in which all maps $E_x : B(\mathcal{H}) \to \mathcal{H}$
$E_x(T) = Tx$ are continuous.

Example Let Q be a finite group.

Then $W(Q) = \mathbb{C}Q$

The map $W(Q) \longrightarrow \mathbb{C}Q$ is obtained by

$$T \longmapsto \sum_{g \in Q} \langle T\mathbb{1}, g\rangle g$$

Def [von Neumann trace]

$$tr_{W(G)} : W(G) \longrightarrow \mathbb{C}$$
$$T \longmapsto \langle T\mathbb{1}, \mathbb{1}\rangle$$

[Note: $x = \sum \lambda_g g \in \mathbb{C}G$, $tr_{W(G)}(x) = \lambda_1$]

Lemma $tr_{W(G)}$ i) is linear

ii) satisfies $tr_{W(G)}(ST) = tr_{W(G)}(TS)$.

iii) is weakly continuous.

Proof i) and iii) are immediate.

ii) For $S = \sum \lambda_g g$, $T = \sum \mu_g g \in \mathbb{C}G$ we have

$$tr_{W(G)}(ST) = \sum_g \lambda_g \mu_{g^{-1}} = \sum_{g^{-1}} \lambda_{g^{-1}} \mu_g = \sum_{g^{-1}} \mu_g \lambda_{g^{-1}}$$

$$= tr_{W(G)}(TS).$$

Now $\forall S \in \mathbb{C}G$, the map

$W(G) \to \mathbb{C}$, $T \mapsto \mathrm{tr}(ST) - \mathrm{tr}(TS)$

is weakly cont. and trivial on $\mathbb{C}G$.

$\therefore$ it is trivial on $W(G) = \overline{\mathbb{C}G}^{\text{weak}}$.

Now $\forall S \in W(G)$, the map $T \mapsto \mathrm{tr}(ST) - \mathrm{tr}(TS)$

is again trivial. $\square$

<u>Fact</u> Let $\mathcal{H}$ be ∞-dim Hilbert space.

If $\varphi : B(\mathcal{H}) \to \mathbb{C}$ is linear and

$\varphi(ST) = \varphi(TS) \quad \forall S, T \in B(\mathcal{H})$, then

$\varphi = 0$.

There is <u>no trace</u> on $B(\mathcal{H})$!

<u>Dimensions of Hilbert modules.</u>

Let $\mathcal{H}$ be a Hilbert space with an isometric linear right G-action.

Let $\iota : \mathcal{H} \hookrightarrow L^2(G)^n$, $n \in \mathbb{N}$, be an isometric linear G-equivariant embedding.

Then $\iota(\mathcal{H})$ is a closed subspace of $L^2(G)^n$, and so $\exists$ projection $p_\iota : L^2(G)^n \twoheadrightarrow \iota(\mathcal{H})$.

The projection is G-equivariant, and so

$$p_\iota \in M_n(W(G)).$$

[It is clear that $p \in M_n(B(L^2(G)))$.

G-equivariance $\Rightarrow$ p_ι commutes with $\begin{pmatrix} \rho(g) & & 0 \\ & \rho(g) & \\ 0 & & \ddots \ \rho(g) \end{pmatrix}$

$\forall g \in G$]

Take $T \in B(\mathcal{H})^G$. Then $\iota \circ T \circ p_\iota \in M_n(W(G))$,

and we can define $\operatorname{tr}(\iota \circ T \circ p_\iota) = \sum \operatorname{tr}(\text{diagonal entries})$.

<u>Proposition</u> $\operatorname{Tr}(\iota \circ T \circ p_\iota)$ is independent of ι.

<u>Proof</u> Take $\iota' : \mathcal{H} \to L^2(G)^{n'}$. By enlarging n or n' if necessary, we may assume that $n = n'$. Let $p = p_\iota$, $p' = p_{\iota'}$.

Non-trivial but true: we can invert ι, and find $\iota^{-1} : \iota(\mathcal{H}) \to \mathcal{H}$ an isometry, inverse to ι.

Now $L^2(G)^n = \iota(\mathcal{H}) \oplus \iota(\mathcal{H})^\perp = \iota'(\mathcal{H}) \oplus \iota'(\mathcal{H})^\perp$.

We have $\iota' \circ \iota^{-1} : \iota(\mathcal{H}) \to \iota'(\mathcal{H})$ an isometry

We extend it by 0 on $\iota(\mathcal{H})^\perp \to \iota'(\mathcal{H})^\perp$

and obtain a partial isometry $j : L^2(G)^n \to L^2(G)^n$.

We have $j \circ \iota = \iota'$, and taking adjoints:

$$p\, j^* = \iota^* j^* = \iota'^* = p'.$$

Now $tr(\iota' \circ T \circ p') = tr(j \circ \iota \circ T \circ p \circ j^*) =$

$$= tr(\iota \circ T \circ p \circ j^* j) = tr(\iota \circ T \circ p \circ p) =$$

$$= tr(\iota \circ T \circ p) \qquad \square$$

Def A Hilbert space $\mathcal{H}$ with a G-action is a Hilbert G-module iff it can be G-equivariantly isometrically embedded into some $L^2(G)^n$.

Let $\mathcal{H}$ be a Hilbert G-module. We define $tr_{W(G)} : B(\mathcal{H}) \to \mathbb{C}$ by taking $\iota : \mathcal{H} \hookrightarrow L^2(G)^n$ and

$$tr_{W(G)}(T) = tr_{W(G)}(\iota \circ T \circ p_{\iota(\mathcal{H})}).$$

Morphisms of Hilbert modules are bounded G-operators.

The <u>dimension of $\mathcal{H}$</u> is $\dim_{W(G)} \mathcal{H} = \mathrm{tr}_{W(G)}\, \mathrm{id}_{\mathcal{H}}$.

$\mathcal{H}$ is <u>free</u> iff $\exists\, \iota : \mathcal{H} \hookrightarrow L^2(G)^n$ which is an isomorphism.

<u>Thm</u> [Properties of the von Neumann trace]

- linearity
- weak continuity
- trace property ($\mathrm{tr}(ST) = \mathrm{tr}(TS)$).
- faithfulness : $\mathrm{tr}(T^*T) = 0 \iff T = 0$
- positivity : $T \le S$ (i.e. $S - T = X^*X$) then $\mathrm{tr}(S) \ge \mathrm{tr}(T)$

<u>Def</u> $K \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} Q$ of

Hilbert G-modules is <u>weakly exact</u> at $\mathcal{H}$

$\ker \beta = \overline{\mathrm{im}\, \alpha}$.

<u>Thm</u> [Properties of von Neumann dimension]

i) $\dim_{W(G)} L^2(G) = 1$

ii) $\dim \mathcal{H} = 0 \iff \mathcal{H} = 0$

iii) $0 \to K \to \mathcal{H} \to Q \to 0$ weakly exact $\Rightarrow \dim \mathcal{H} = \dim K + \dim Q$.