L^2-homology for G-CW complexes.

Setting: G countable, discrete group.

X a CW-complex with skeleta X_i.

Def X is a <u>G-CW complex</u> iff $G \curvearrowright X$ by sending n-cells to n-cells $\forall n$, and in such a way that if g stabilizes an n-cell, it fixes it pointwise.

The complex is <u>free</u> iff the action is free, i.e., the stabilizer of every cell is trivial.

A <u>G-cell</u> is a G-orbit of a cell.

X is <u>finite</u> iff it consists of finitely many G-cells.

X is of <u>finite type</u> iff every X_n is a finite G-complex.

Today, X is a free G-complex of finite type.

Def [L^2-chain cplx.]

$$C_n^{(2)}(X) = \left\{ \sum_{e \text{ n-cell}} \lambda_e e \;\middle|\; \begin{array}{l} \lambda_e \in \mathbb{C} \\ \sum |\lambda_e|^2 < \infty \end{array} \right\}.$$

But: $G \curvearrowright X_n / X_{n-1}$, and so $C_n(X)$ is a $\mathbb{Z}G$-module.

In fact, it is a free $\mathbb{Z}G$-module:

Pick $\{e_1, \ldots, e_m\}$ a set of n-cells, one in each G-n-cell.

Now $$\sum_{e \text{ n-cell}} \lambda_e e = \sum_{\substack{g \in G \\ i \in \{1,\ldots,m\}}} (\lambda_{ge_i} \, g \cdot e_i) \longleftrightarrow \begin{pmatrix} \sum \lambda_{ge_1} g \\ \sum \lambda_{ge_2} g \\ \vdots \end{pmatrix}$$

If the support is finite, $\begin{pmatrix} \sum \lambda_{ge_i} g \\ \vdots \end{pmatrix} \in (\mathbb{Z}G)^n$.

Clearly, we obtain an isomorphism $C_n(X) \cong (\mathbb{Z}G)^n$ of $\mathbb{Z}G$-modules. (depends on the choice of basis!).

The same trick gives $C_n^{(2)}(X) \cong L^2(G)^n$ as $\mathbb{C}G$-modules. So, $C_n^{(2)}(X)$ is a Hilbert module.

If we choose a different basis, we obtain

$$L^2(G)^n \cong C_n^{(2)}(X) \cong L^2(G)^n$$

$$\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} \qquad g_i \in G.$$

This is a unitary transformation, and so the Hilbert structure on $C_n^{(2)}(X)$ does <u>not</u> depend on the choice of basis.

Now, since X is of finite type, the differentials ∂ are bounded operators.

$\therefore$ $\ker \partial_n$ is closed in $C_n^{(2)}(X)$.

$\therefore$ it is a Hilbert module.

$$H_n^{(2)}(X) = \ker \partial_n \big/ \overline{\operatorname{im} \partial_{n+1}}.$$

$\overline{\operatorname{im} \partial_{n+1}}$ is closed in $\ker \partial_n$.

$\therefore$ it admits an orthogonal subspace, and

$$\ker \partial_n \big/ \overline{\operatorname{im} \partial_{n+1}} \cong \overline{\operatorname{im} \partial_{n+1}}^{\perp}, \text{ a Hilbert module.}$$

$\therefore$ $H_n^{(2)}(X)$ is a Hilbert module.

Def [L^2-Betti number]

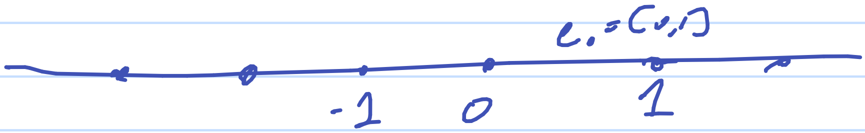
$$\beta_n^{(2)}(X) = \dim_{N(G)} H_n^{(2)}(X) \in [0,\infty)$$

Conjecture (Strong Atiyah conjecture)

If G is torsion-free, then $\beta_n^{(2)}(X) \in \mathbb{N}$

$\forall n$ $\forall$ free G-CW-cplx X of finite type.

Let's compute $\beta_n^{(2)}$ of $\mathbb{Z} \curvearrowright \mathbb{R}$.



$$\mathbb{Z} = \langle t \rangle, \quad t \cdot x = x+1.$$

$$C_0 = \mathbb{Z}\mathbb{Z}$$

$$C_1 = \mathbb{Z}\langle e_0 \rangle \cong \mathbb{Z}\mathbb{Z}.$$

$$C_1 \xrightarrow{1-t} C_0$$

$$[n,n+1] \longmapsto (n+1)-(n)$$

$$t^n \cdot e_0 \qquad t^{n+1}-t^n$$

$$\partial$$

$$C_*^{(2)}(X) = \quad L^2(\mathbb{Z}) \xrightarrow{1-t} L^2(\mathbb{Z}).$$

1) ∂ is injective: $x \in L^2(\mathbb{Z})$, $x(1-t) = 0$.

$x = \sum x_n t^n$, $\quad x_n = x_{n+1}$ $\forall n$.

$\therefore \; x = \sum x_0 t^n \in L^2(\mathbb{Z})$. $\;\therefore\; x_0 = 0$.

2) $\overline{\operatorname{im} \partial} = \ell^2(\mathbb{Z})$.

since $\mathbb{C}G$ is dense in $\ell^2(G)$, suffices to show $\overline{\operatorname{im} \partial} \supseteq \mathbb{C}\mathbb{Z}$.

<u>Def</u> [the augmentation ideal]

$$\underset{R}{A}(G) = \{ x = \sum x_g g \in \underset{R}{\mathbb{C}G} \mid \sum x_g = 0 \}.$$

<u>Exercise</u> $A(G)$ is generated (as RG module.) by $\{1-g \mid g \in S\}$ for every generating set S of G.

<u>So</u> $A(\mathbb{Z}) \subseteq \overline{\operatorname{im} \partial} = \overline{\operatorname{im}(1-t)}$.

So, it suffices to show that $\overline{A(\mathbb{Z})} \supseteq \mathbb{C}\mathbb{Z}$.

Take $x = \sum x_g g \in \mathbb{C}\mathbb{Z}$. Let $\lambda = \sum x_g$.

Since $\mathbb{Z}$ is infinite, pick a sequence g_i of distinct elements in $\mathbb{Z} \setminus \operatorname{supp} x$.

Let $x_n = x - \sum_{i=1}^{n} \frac{\lambda}{n} g_i$.

augmentation of x_n is $\lambda - n \cdot \lambda/n = 0$

$\therefore x_n \in A(G) \quad \forall n$.

$$\| x - x_n \| = \sum_{i=1}^{n} \frac{\lambda^2}{n^2} = \frac{\lambda^2}{n} \longrightarrow 0.$$

Euler characteristic

Let Y be a finite connected CW-complex.

Let $G = \pi_1(Y)$.

$G \curvearrowright \tilde{Y}$, $\tilde{Y}$ is a finite G-CW complex.

$\chi(Y) = \sum (-1)^i c_i$, $c_i = |$ i-cells in $Y|$.

$= |$ i-G-cells in $\tilde{Y}|$.

$$\therefore\ C_i(\tilde{Y}) \cong (\mathbb{Z}G)^{c_i}.$$

$$C_i^{(2)}(\tilde{Y}) \cong \left(L^2(G)\right)^{c_i}.$$

$$\therefore\ \dim_{W(G)} C_i^{(2)}(\tilde{Y}) = c_i$$

$$\therefore\ \chi(Y) = \sum (-1)^i \dim_{W(G)} C_i^{(2)}(\tilde{Y}).$$

Prop $0 \to A \to B \to C \to 0$ weakly exact sequence of Hilbert modules, then

$$\dim_{W(G)} B = \dim_{W(G)} A + \dim_{W(G)} C.$$

$$\begin{array}{ccc} C_{i+1}^{(2)} & \xrightarrow{\partial_{i+1}} & C_i^{(2)} \\ \Vert & & \Vert \\ \ker \partial_{i+1} \oplus \ker \partial_{i+1}^{\perp} & \longrightarrow & \ker \partial_i \oplus \ker \partial_i^{\perp} \end{array}$$

$$0 \to \ker \partial_{i+1}^{\perp} \xrightarrow{\partial} \ker \partial_i \to \ker \partial_i / \overline{\operatorname{im} \partial_{i+1}} \to 0$$

weakly exact. $\qquad \qquad \ker \partial_i / \overline{\operatorname{im} \partial_{i+1}} = H_i^{(2)}(\tilde{Y})$.

So $\quad \dim_{\mathcal{N}(G)} H_i^{(2)}(\tilde{Y}) = \dim \ker \partial_i - \dim \ker \partial_{i+1}^{\perp}$

$= \dim \ker \partial_i + \dim \ker \partial_{i+1} + \dim C_{i+1}$

$\therefore \sum (-1)^i \dim H_i^{(2)}(\tilde{Y}) = \sum (-1)^i \dim C_i = \chi(Y)$

$\sum (-1)^i \dim H_i^{(2)}(\tilde{Y}) = \chi^{(2)}(\tilde{Y})$.

<u>How to prove the proposition?</u>

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

weakly exact.

$p^* : C \to B$ satisfies $\ker p^{\perp} = \overline{\operatorname{im} p^*}$

$\ker p^{*\perp} = \overline{\operatorname{im} p}$.

Consider $i \oplus p^* : A \oplus C \to B$.

Since $\overline{\operatorname{im} p} = C$, we have $\ker p^* = 0$.

Also, $\ker i = 0$.

$\therefore \; i \oplus p^*$ is injective.

$$\overline{\mathrm{im}(i\oplus p^*)} = \overline{\mathrm{im}\, i} \oplus \overline{\mathrm{im}\, p^*} =$$

$$= \ker p \oplus \ker p^\perp = B$$

So $i\oplus p^* : A\oplus C \longrightarrow B$ is a weak isomorphism.

Polar decomposition:

$\forall\ x : X \longrightarrow Y$ bounded operator on Hilbert spaces,

$\exists\ q : X \to X$ positive and $u : X \to Y$ partial isometry

st. $\ker u = \overline{\mathrm{im}\, q}^{\perp}$ and

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr, "q"'] & & Y \\ & X \arrow[ur, "u"'] & \end{tikzcd}$$

commutes.

In our case, take $i\oplus p^* = uq$

$\mathrm{im}(i\oplus p^*) \le \mathrm{im}\, u$ ∴ $\overline{\mathrm{im}\, u} = Y$.

u is a partial isometry, so u^*u is a projection.

∴ $u^*u = (u^*u)^2 = u^*uu^*u$

$$(u-uu^*u)^*(u-uu^*u) = u^*u - u^*uu^*u - u^*uu^*u + u^*uu^*uu^*u = 0$$

∴ $u - uu^*u = 0$

$u = uu^*u$

$uu^* = uu^*uu^* = (uu^*)^2$

$\therefore$ uu^* is also a projection.

uu^* is a projection.

Now $\overline{\operatorname{im} u^*} = \ker u^\perp$

$\therefore$ $\overline{\operatorname{im} uu^*} = \overline{\operatorname{im} u} = B$.

But projection with dense image is an isometry!

$\therefore$ $u : A \oplus C \to B$ is an isometry.

So $A \oplus C \cong B$.

We still need to show that $\dim A \oplus C =$
$\dim A + \dim C$.

$A \le L^2(G)^n$, $C \le L^2(G)^m$,

closed subspaces.

Let p, q be the corresponding projections.

Then $p \oplus q : L^2(G)^{n+m} \twoheadrightarrow A \oplus C$, a projection.

Pick a basis $e_1, \ldots, e_{n+m}$ for $L^2(G)^{n+m}$,

respecting $L^2(G)^{n+m} = L^2(G)^n \oplus L^2(G)^m$.

Now $f_v(p+q) = \sum \langle p \oplus q(e_i), e_i \rangle =$

$$= \sum_{i=1}^{n} \langle p(e_i), e_i \rangle + \sum_{i=n+1}^{m} \langle q(e_i), e_i \rangle =$$

$$= f_v(p) + f_v(q) = \dim A + \dim C. \quad \square$$