Classifying spaces

Thm [Eilenberg – MacLane]
$\forall$ group G $\exists$ connected CW-cplx X s.t.:
1) $\pi_1(X) \cong G$
2) $\tilde{X}$ is contractible.
[Such an X is called a classifying space
or $K(G,1)$, or BG]
Moreover, if Y has the same properties, then
$X \simeq Y$ htpy equivalent.

In fact more is true: if X is a connected CW cplx with $\pi_1(X) = G$, then we can add cells in dimension 3 and above and construct a $K(G,1)$.

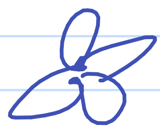
Def A group G is of type F_i iff it admits a $K(G,1)$ with finite i-skeleton.
G is of type F_∞ iff it is F_i $\forall i$.
[this is the same as having $K(G,1)$ of finite type]

G is of type F iff it admits a finite $K(G,1)$.

Prop G is of type F_1 iff it is finitely generated.

Proof

If G is of type F_1, we have $G = \pi_1(X)$ with X connected and with finite 1-skeleton. Pick a maximal tree T in X_1; then $\pi_1(X)$ is generated by loops lying in T except for one edge. There are finitely many such.

If G is n-generated, then G is the π_1 of a connected CW-cplx X such that X_1 is the n-rose  .

We can complete this X to a $K(G,1)$ without adding 1 and 0-cells. $\square$

Def $\beta_i^{(2)}(G) = \beta_i^{(2)}(\tilde{X})$ where $X = K(G,1)$.

Thm G is amenable of type F.

Then $\chi(G) = 0$ $\left[= \chi(X),\ X \text{ cpt } K(G,1) \right]$

Thm G is amenable, type F_∞.

$\beta_i^{(2)}(G) = 0 \quad \forall i \geq 1.$

If G is infinite, then also

$\beta_0^{(2)}(G) = 0.$

this is true in general

Proof We will prove that $\forall n, m \in \mathbb{N}$, $A \in M_{n,m}(\mathbb{C}G)$

we have $\ker A^{(2)} = \overline{\ker A}$.

$$\begin{tikzcd}
\mathbb{C}G^n \arrow[r, "A"] \arrow[d, hook] & \mathbb{C}G^m \arrow[d, hook] \\
L^2(G)^n \arrow[r, "A^{(2)} \approx A"] & L^2(G)^m
\end{tikzcd}$$

Once we have this:

$X = K(G,1)$ of finite type.

The cellular chain cplx of $\tilde{X}$ is

$$\longrightarrow \cdots \mathbb{C}G^k \xrightarrow{\partial = B} \mathbb{C}G^n \xrightarrow{\partial = A} \mathbb{C}G^m \longrightarrow \cdots$$

Since $\tilde{X}$ is contractible, $\mathrm{im}\, B = \ker A$.

Now
$$\begin{array}{c} \overline{\mathrm{im}\, B} = \overline{\ker A} = \ker A^{(2)} \\ \cap\!| \\ \overline{\mathrm{im}\, B^{(2)}} \end{array}.$$

So $C_*^{(2)}(X)$ is acyclic everywhere, except perhaps at 0.

By collapsing a maximal tree in X, we may assume that $C_0(X) = \mathbb{C}G$.

We have $\to \mathbb{C}G^n \xrightarrow{\partial} \mathbb{C}G \to 0$

with $\mathrm{im}\, \partial = I(G)$, augmentation ideal.

The proof we had for $\mathbb{Z}$ works here as well: $\overline{\mathrm{im}\, \partial} = \mathbb{C}G$.

Back to what we have to prove:

Let $K = \ker A$, $A : \mathbb{C}G^n \to \mathbb{C}G^m$.

We claim that $\overline{K} = \ker A^{(2)}$.

Let $p : L^2(G)^n \to \overline{K}^\perp \cap \ker A^{(2)}$ be the projection.

We need $\overline{K}^\perp \cap \ker A^{(2)} = 0 \iff \operatorname{tr}_{W(G)} p = 0$.

Let $A = (a_{ij})$; let $S = \bigcup_{i,j} \operatorname{supp} a_{ij} \subseteq G$

finite subset. Fix $\varepsilon > 0$.

G is amenable, and so $\exists\ F \subseteq G$ finite s.t. (nonempty)

$\dfrac{|\partial F|}{|F|} < \varepsilon$, where $\partial F = \{ f \in F \mid fs^{\pm 1} \notin G$ for some $s \in S \}$

Let $\Delta = \partial F \cup \{ g \in G \setminus F \mid gs^{\pm 1} \in F$ for some $s \in S \}$.

Δ is finite, and so $p_\Delta : L^2(G)^n \to L^2(G)^n$

has image of finite dimension, where p_Δ is the projection onto $L^2(\Delta)^n$, $L^2(\Delta) = \{ \sum_{g \in \Delta} \lambda_g g,\ \lambda_g \in \mathbb{C} \}$.

We define $L^2(F)^n$ analogously; let p_F be the corresponding projection.

Now $\operatorname{tr}_{W(G)} p = \langle p(1), 1 \rangle =$

$= \dfrac{1}{|F|} \sum_{f \in F} \langle p(f), f \rangle =$

$$= \frac{1}{|F|} \sum \langle p \circ \rho_F(g), g \rangle = \frac{1}{|F|} \mathrm{tr}_{\mathbb{C}}(p \circ \rho_F)$$

when we view $p \circ \rho_F : L^2(F)^n \longrightarrow L^2(G)^m$.

$$\begin{tikzcd} L^2(F)^n \arrow[rr] \arrow[dr] & & L^2(G)^m \\ & \mathrm{im}\, p \circ \rho_F \arrow[ur] & \end{tikzcd}$$

the ker.

A bit of functional analysis:

$$\mathrm{tr}(p \circ \rho_F) \leq \| p \circ \rho_F \| \dim_{\mathbb{C}}\big((p \circ \rho_F)(\mathrm{im}(p \circ \rho_F)) \big).$$

We have $\| p \circ \rho_F \| \leq \|p\| \, \|\rho_F\| \leq 1$.

Also, $\mathrm{im}(p \circ \rho_F) \leq \mathrm{im}\, p \leq \ker A^{(2)}$ since

p is the projection onto $\overline{K}^{\perp} \cap \ker A^{(2)}$.

So $\mathrm{tr}_{\mathcal{N}(G)}\, p \leq \frac{1}{|F|} \dim_{\mathbb{C}}\big((p \circ \rho_F)(\ker A^{(2)}) \big)$.

What is $p \circ \rho_F(\ker A^{(2)})$?

Let $u \in \ker \rho_\Delta \leq L^2(G)^n$

We have $\rho_F(\gamma \cdot u) = \gamma \cdot \rho_F(u) \quad \forall \gamma \in S$.

$\therefore \rho_F A u = A_{\rho_F} u$

But $p_F(u) \in L^2(F) \subseteq \mathbb{C}G$.

Thus, if $u \in \ker A^{(2)} \cap \ker p_\Delta$ then

$A\, p_F(u) = p_F(Au) = 0$

and $p_F(u) \in \mathbb{C}G$

$\therefore$ $p_F(u) \in K$.

$\therefore$ $p\, p_F(u) = 0$ as $\operatorname{im} p \subseteq \overline{K}^{\perp}$

So $\dim_G \operatorname{im} p\, p_F(\ker A^{(2)}) \leq \dim_G L^2(G)^n / \ker p_\Delta = n \cdot |\Delta|$

So $\operatorname{tr}_{N(G)} p \leq \frac{1}{|F|} \dim \operatorname{im} p\, p_F(\ker A^{(2)}) \leq$

$\leq \frac{1}{|F|} \cdot n \cdot |\Delta| \leq \frac{(|S|+1)n \cdot |\partial F|}{|F|} \leq$

$\leq (|S|+1) \cdot n \cdot \varepsilon \quad \forall \varepsilon > 0$.

$\therefore$ $\operatorname{tr}_{N(G)} p = 0$ $\square$

Def G, H groups, $f: H \to \mathrm{Aut}(G)$ homom.
The *semi-direct product*
$G \rtimes_f H$ is a group with underlying set (G, H), and multiplication

$$(g,h)\cdot(g',h') = g\, f(h)(g')\, hh'.$$

Thm H of type F_∞, $G = H \rtimes_f \mathbb{Z}$.

$$\beta_i^{(2)}(G) = 0 \quad \forall i.$$

Proof $X = K(H,1)$ of finite type.
Up to subdividing, we realise f as a cellular map $X \to X$; build Y to be the corresponding mapping torus.
Now count cells. $\square$