Thm [Lück, answering Gromov]

H a group of type F_n, $G = H \rtimes_f \mathbb{Z}$, $f \in \mathrm{Aut}(H)$.

Then $\beta_n^{(2)}(G) = 0$.

Recall:

Def [Semi-direct product]

A, B groups, $\varphi : B \longrightarrow \mathrm{Aut}(A)$ action.

$G = A \rtimes_\varphi B$ is a group given by:

- the underlying set is $A \times B$
- the multiplication is

$$(a, b) \cdot (a', b') = (a\, \varphi(b)(a'), bb')$$

$$\Big[\; a b a' b' = a \underbrace{b a' b^{-1}}_{\substack{b \cdot a' \\ \shortparallel \\ \varphi(b)(a')}} b b' \;\Big]$$

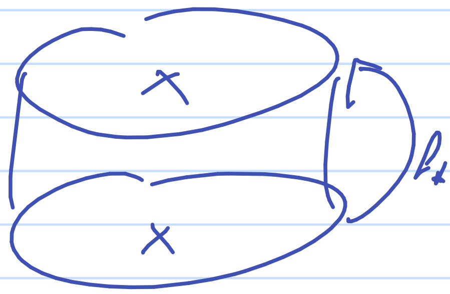
Proof X a $K(H,1)$ with finite n-skeleton.

$f : H \xrightarrow{\cong} H$ gives a htpy equivalence

$f_* : X \longrightarrow X$.

Now let $Y = X *_{f_*}$ be the mapping torus:

$$Y = X \times [0,1] \big/ (x,0) = (f_*(x), 1)$$



We endow $X \times [0,1]$ with the obvious CW-structure.

Y is a CW-complex.

Let $\kappa_i(Z)$ be the # of i-cells in Z.

We have $\kappa_i(Y) = \kappa_i(X) + \kappa_{i-1}(X)$.

Let $G_i = M \rtimes_{f^i} \mathbb{Z} \leq M \rtimes_f \mathbb{Z} = G$.

Note that $|G : G_i| = i$. Let Y_i be the mapping torus of f^i.

We have $\kappa_n(Y_i) = \kappa_n(X) + \kappa_{n-1}(X) \quad \forall i$

different group actions!

We have $\beta_n^{(2)}(G) = \beta_n^{(2)}(\tilde{Y}) = \frac{1}{i} \beta_n^{(2)}(\tilde{Y}_i)$

$$= \frac{1}{i} \beta_n^{(2)}(G_i)$$

since $Y_i \to Y$ is a covering.

Also, $\beta_n^{(2)}(\tilde{Y}_i) = \dim_{N(G_i)} H_n^{(2)}(\tilde{Y}_i) \le \dim_{N(G)} L^2(G)^{k_n(Y_i)}$

$$\le k_n(X) + k_{n-1}(X) \quad \forall i$$

$$\therefore \quad \beta_n^{(2)}(G) \le \frac{k_n(X) + k_{n-1}(X)}{i} \quad \forall i$$

$$\therefore \beta_n^{(2)}(G) = 0 \; .$$

<u>Def</u> Let P be a property of groups.
We say that G is <u>residually P</u> iff
$\forall g \in G \setminus \{1\} \; \exists \; \varphi : G \twoheadrightarrow Q$, Q with P :

$$\varphi(g) \neq 1 .$$

Very interesting case: P is "being finite".
Since being finite passes to subgroups, we may as well take $\varphi : G \to Q$ not necessarily surjective in the definition.

<u>Prop</u> G countable. G is residually finite iff
$\exists \; G = G_0 \geq G_1 \geq G_2 \geq \ldots$ s.t. $\{G_i\}$ is a chain
$|G : G_i|$ is finite and $\bigcap G_i = 1$.
$\{G_i\}$ is a <u>residual chain</u>

Proof $G = \{g_0, g_1, \ldots\}$, $g_0 = 1$

$\forall i > 0 \ \exists q_i : G \to Q_i$ finite : $q_i(g_i) \neq 1$.

Let $K_i := \ker q_i$, $G_i := \bigcap_{j \leq i} K_j \trianglelefteq G$ of finite index.

Clearly, $G_i \geq G_{i+1}$.

Let $g \in \bigcap G_i$. Then $g \in K_i \ \forall i$

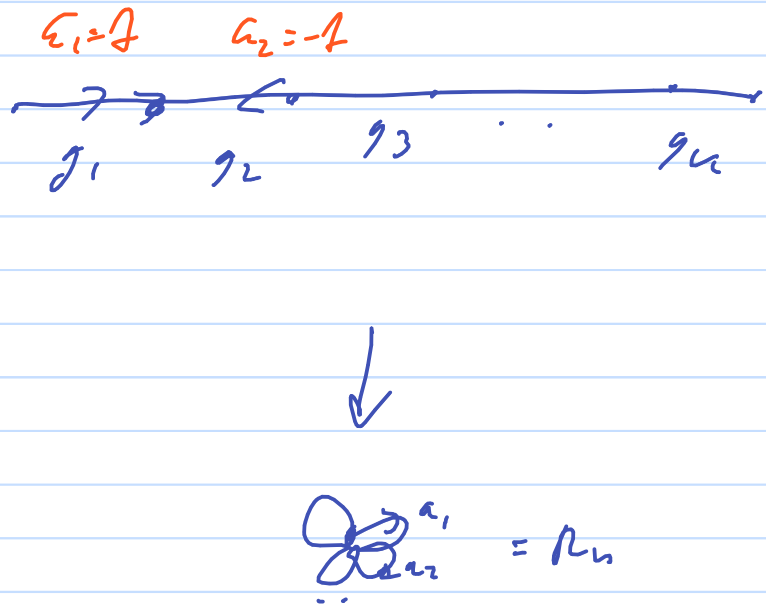
$\therefore q_i(g) = 1 \ \forall i$. But $g = g_k$, some k

$\therefore k = 0 \ \therefore g = 1$. □

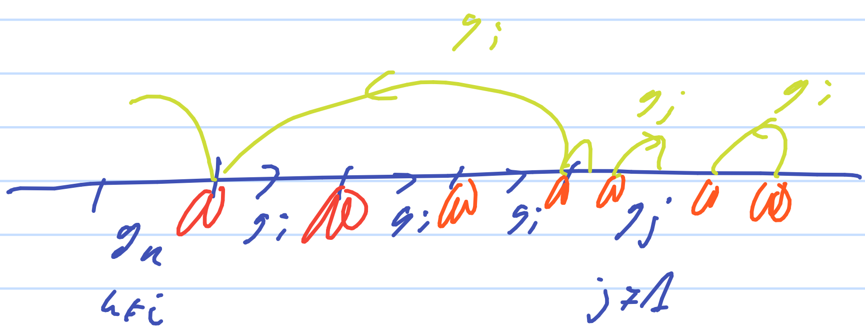
Example • $G = \mathbb{Z}$. $G_i := i\mathbb{Z}$.

• $G = F_n = \langle a_1, \ldots, a_n \rangle$ free group.

$g \in G \setminus \{1\}$, $g = \prod g_i^{\varepsilon_i}$ (reduced), $g_i \in \{a_1, \ldots, a_n\}$, $\varepsilon_i \in \{\pm 1\}$.



We now complete g_i-cycles:



Add loops:

The resulting covering yields a finite index subgroup $H \leq G$ s.t. $g \notin H$. (it is not a loop).

Let $K = \bigcap_{x \in F_n} H^x \trianglelefteq G$ finite index, $g \notin K$.

Take $q: G \twoheadrightarrow G/K$.

- linear groups

Thm [Malcev]

Every f.g. linear group is residually finite.

Def G is linear iff $G \hookrightarrow GL_n(k)$, k a field.

What's the idea: $GL_n \mathbb{Z}$ is residually finite,

since:

Take $g = (a_{ij}) \in GL_n\mathbb{Z} \setminus \{1\}$.

$\exists\, k \in \mathbb{N} : k > |a_{ij}|$.

$$GL_n \mathbb{Z} \to GL_n(\mathbb{Z}/k)$$
$$(a_{ij}) \mapsto (a_{ij}) \neq \mathrm{id}.$$

Clearly $GL_n(\mathbb{Z}/k)$ is finite.

<u>Thm</u> [Lück's approximation thm]

G residually finite of type F_{n+1}.

(G_i) a residual chain of finite index subgroups.

$$\beta_n^{(2)}(G) = \lim_{i \to \infty} \frac{1}{|G:G_i|} \beta_n(G_i; \mathbb{C}).$$

We can use this to compute:

<u>Lemma</u> $G = F_n$ free group.

$$\beta_i^{(2)}(G) = \begin{cases} 0 & i \neq 1 \\ n-1 & i = 1 \end{cases}.$$

<u>Proof</u> G is of type F.

(G_i) is a residual chain.

Abstractly, $G_i \cong F_{n_i}$, a free group.

Let $k_i := |G : G_i|$. What is n_i?

$G_i = \pi_1(R_{n_i})$, $R_{n_i} \longrightarrow R_n$ covering of index k_i

$\therefore$ $\underbrace{\chi(R_{n_i})}_{=-(n_i-1)} = k_i \cdot \chi(R_n) = -(n-1)$

$\therefore$ $n_i = k_i(n-1)+1$.

$$\beta_j(G_i) = \beta_j(R_{n_i}) = \begin{cases} 1 & \text{if } j=0 \\ n_i & j=1 \\ 0 & \text{o/w} \end{cases}$$

So

$$\beta_0^{(2)}(G) = \lim_{i\to\infty} \frac{1}{k_i}\cdot 1 = 0$$

$$\beta_j^{(2)}(G) = \lim_{i\to\infty} 0 = 0 \quad \text{if } j \geq 2.$$

$$\beta_1^{(2)}(G) = \lim \frac{1}{k_i}\cdot n_i = \lim \frac{k_i(n-1)+1}{k_i} = n-1.$$

$\square$