Ore Localisation

R ring with unity, not necessarily abelian.

Def $x \in R$ is a zero-divisor iff $\exists y \in R \setminus \{0\}$:

$$xy = 0 \quad \text{or} \quad yx = 0.$$

R satisfies the (right) Ore condition iff $\forall p, q \in R$, q not a zero-divisor, $\exists r, s \in R$, s not a zero-div. s.t. $ps = qr$

[existence of common multiples].

R is a domain iff the only zero-divisor is 0.

Def R satisfies the Ore condition.

Let $Ore(R) = \{(p, q) \mid p, q \in R,\ q \text{ not a zero-divisor}\} / \sim$

where $(p, q) \sim (p', q') \iff \exists r, r'$ not zero-div. s.t. $pr = p'r'$ and $qr = q'r'$.

Because of $\sim$, we write p/q for (p, q).

We endow $Ore(R)$ with ring structure:

Take $p/q,\ p'/q' \in Ore(R)$. $\exists r, r'$ not ZD st.

$$qr = q'r'$$

Now $p/q + p'/q' = \dfrac{pr}{qr} + \dfrac{p'r'}{q'r'} = \dfrac{pr}{qr} + \dfrac{p's'}{qr} =$

$= \dfrac{pr + p's'}{qr} \in \mathrm{Ore}(R)$

Multiplication: Take p/q, $p'/q' \in \mathrm{Ore}(R)$.

$\exists\ r, s \in R$, s not ZD : $sp' = rq'$

Now: $p/q \cdot p'/q' = pq^{-1}p'q'^{-1} = pq^{-1}s^{-1}r$.

$\exists\ x, y \in R$; y not ZD : $q^{-1}s^{-1}r = xy^{-1}$.

$\therefore\ p/q \cdot p'/q' = pxy^{-1} = px/y$.

<u>Thm</u> Let R be a domain satisfying the Ore condition. Then $\mathrm{Ore}(R)$ is a skew-field, i.e., every non-zero element admits a two-sided inverse.

$\left[\ (p/q)^{-1} = q/p\ \right]$

<u>Thm</u> $W(G)$ satisfies Ore condition.

Note: $W(G)$ has <u>loads</u> of ZDs.

So $\mathrm{Ore}(W(G))$ exists, and contains $W(G)$.

Def $R \subseteq S$ rings. The <u>division closure</u> of R in S is the smallest subring D of S containing R and such that: $\forall x \in D$, if x is invertible in S, then it is invertible in D.

Def The division closure of $\mathbb{Q}G \hookrightarrow \mathcal{N}(G) \hookrightarrow \mathrm{Ore}(\mathcal{N}(G))$ is called the <u>Linnell ring</u>, and denoted $\mathcal{D}(G)$.

Recall [Atiyah Conjecture] G is torsion-free. G satisfies AC iff $\forall$ finite matrix A over $\mathbb{Q}G$ we have $\dim_{\mathcal{N}(G)} \ker A \in \mathbb{Z}$.

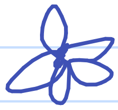
Thm [Linnell]
G torsion-free. G satisfies the AC $\iff$ $\mathcal{D}(G)$ is a skew-field.

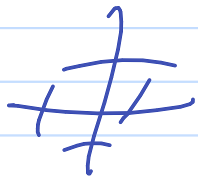
Fact $\beta_i^{(2)}(G) = \dim_{\mathcal{D}(G)} H_i(G; \mathcal{D}(G))$.

[$X = K(G,1)$. $H_i(G; \mathcal{D}(G)) = H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathcal{D}(G))$]

Thm [Linnell] F_n satisfies AC.

Example $F_n = \langle a_1, \dots, a_n \rangle$

$X =$  R_n

$\tilde{X} =$  $\mathrm{Cay}(F_n, \{a_1, \dots, a_n\})$.

$$C_*(\tilde{X}) = \quad 0 \to \mathbb{Z}F_n^{\,n} \xrightarrow{\begin{pmatrix}1-a_1\\ \vdots \\ 1-a_n\end{pmatrix}} \mathbb{Z}F_n \to 0 .$$

$$C_*(\tilde{X}) \otimes_{\mathbb{Z}G} D(G) \qquad 0 \to D(G)^n \xrightarrow[\partial]{\begin{pmatrix}1-a_1\\ \vdots \end{pmatrix}} D(G) \to 0$$

$1 - a_1 \in \mathbb{Z}F_n \setminus \{0\}$.

$\therefore \exists\, x \in D(G) : \quad x(1-a_1) = 1$.

$$(x \ \ 0 \dots 0)\begin{pmatrix}1-a_1\\ \vdots \\ 1-a_n\end{pmatrix} = 1$$

$\therefore$ ∂ is onto $\therefore$ $\ker \partial$ is $n-1$-dim as $D(G)$-vector space.

$$\therefore \quad \beta_i^{(2)}(F_n) = \begin{cases} n-1 & i = 1 \\ 0 & \text{o/w} \end{cases}.$$

Universal L^2-torsion

Let G be a group of type F, satisfying AC, and L^2-acyclic, i.e., $\beta_i^{(2)}(G)=0 \ \forall\, i$.

Examples: $\mathbb{Z}^n$, $F_n \rtimes \mathbb{Z}$, $\Sigma \rtimes \mathbb{Z}$, Σ surface group,

in general $G \rtimes \mathbb{Z}$ where G is of type F

and satisfies AC.

Take $X = K(G,1)$ finite. We have $C_*(\tilde{X}) \otimes D(G)$

$$0 \longrightarrow D(G)^{n_n} \xrightarrow{A_n} D(G)^{n_{n-1}} \longrightarrow \cdots \longrightarrow D(G)^{n_0} \longrightarrow 0$$

is exact.

Starting at $D(G)^{n_0}$, using right multiplication by elementary matrices $\begin{pmatrix} 1 & & x \\ & \ddots & \\ & & 1 \end{pmatrix}$ $x \in D(G)$,

we put A_1 into lower-diagonal form

$$\begin{pmatrix} * & & 0 \\ & * & \\ & & \ddots & \\ * & & & \ddots \end{pmatrix}$$

Now, using similar operation on $D(G)^{n_1}$, we put A into diagonal form

$$\left(\begin{array}{ccc|c} * & & 0 & \\ & * & & 0 \\ 0 & & \ddots & \\ \hline 0 & & & 0 \end{array}\right).$$

The kernel of A_1 is now blatantly visible, and so

$A_2 = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right)$. As before, we modify $D(G)^{n_1}$ so that $A_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right)$,

and $D(G)^{n_2}$ so that A_2 is diagonal

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{matrix} \end{array}\right).$$ We proceed this way.

We are left with

$$0 \to D(G)^{n_k} \xrightarrow{\left(\begin{array}{c|c} D_k & 0 \end{array}\right)} D(G)^{n_{k-1}} \xrightarrow{\left(\begin{array}{c|c} 0 & D_{k-1} \end{array}\right)} \cdots \to 0$$

Let $\rho_u^{(2)}(X) = \prod_k$ non-zero entries in $D_k \in D(G)\setminus\{0\} = D(G)^\times$

<u>Problem</u>: this depends on the choice of basis!

Solution: $\rho_u^{(2)}(X) \in D(G)^\times / [D(G)^\times, D(G)^\times]$.

This is the universal L^2-torsion of X.

It is <u>still unknown</u> if $\rho_u^{(2)}(X)$ depends on X!

We can turn $\rho_n^{(v)}(X)$ into a pair of polytopes:

G is f.g., so its abelianisation G^{ab} is also.

$\therefore\ G^{ab} = \mathbb{Z}^n \oplus T$, T is torsion.

Define $G^{fab} = \mathbb{Z}^n = G^{ab}/T$.

Let $K = \ker(G \twoheadrightarrow G^{fab})$.

We have $K \hookrightarrow G$ inducing

$\mathbb{Z}K \hookrightarrow \mathbb{Z}G$

We can form a <u>twisted group ring</u> $(\mathbb{Z}K)G^{fab}$, isomorphic to $\mathbb{Z}G$ as a ring:

pick a set-theoretic section

$s: G^{fab} \to G$.

Now $\sum_{a \in G^{fab}} \lambda_a a \mapsto \sum \lambda_a s(a) \in \mathbb{Z}G$

$\lambda_a \in \mathbb{Z}K \ \forall a$.

In the same way we have $D(K)G^{fab} \hookrightarrow D(G)$.

<u>Fact</u>: $D(G) = \mathrm{Ore}(D(K)G^{fab})$.

Pick representative $\rho_n^{(v)}(X) \in D(G) = pq^{-1}$,

$p, q \in D(K)G^{fab}$.

Take P_p = convex hull in $\mathbb{C}^{2n} \otimes \mathbb{R}$ of supp p

P_q = " " — in — supp q.

$P^{(2)}(X) = P_p - P_q$, this is independent of X!