## 6.2. The ordinals

Definition 6.2.1. *A set $X$ is* transitive *iff whenever $a \in b \in X$, $a \in X$.*

Definition 6.2.2. *A set $\alpha$ is an* ordinal *iff it is transitive and well-ordered by $\underline{\in}$.*

Theorem 6.2.3. *$\omega$ is an ordinal. So is $n$, for every $n \in \omega$.*

Proof: $\omega$ is transitive because if $m \in n \in \omega$, then $m \in \omega$. The statement that natural numbers are transitive is proved on the problem sheets.

We have also proved that $\omega$ and its elements are well-ordered by $\underline{\in}$. $\square$

Theorem 6.2.4. *If $\alpha$ is an ordinal, then so is $\alpha^+$.*

Proof: First we show that $\alpha^+$ is transitive.

Suppose $\gamma \in \beta \in \alpha^+$. $\alpha^+ = \alpha \cup \{\alpha\}$, so there are two cases.

If $\beta \in \{\alpha\}$, then $\beta = \alpha$, so $\gamma \in \alpha$, so $\gamma \in \alpha^+$.

If $\beta \in \alpha$, then $\gamma \in \alpha$ because $\alpha$ is transitive, so $\gamma \in \alpha^+$.

Now we show well-ordering. Suppose $S$ is a non-empty subset of $\alpha^+$.

There are two cases. If $S \cap \alpha \neq \varnothing$, then let $\beta$ be its least element. Since $\beta \in \alpha$, $\beta$ is the least element of $S$.

If $S \cap \alpha = \varnothing$, then $S = \{\alpha\}$, and obviously $\alpha$ itself is the least element of $S$. $\square$

Theorem 6.2.5. *If $\alpha$ and $\beta$ are ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof: Suppose not. Suppose $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$.

Then $\alpha \cap \beta$ is transitive, for if $\xi \in \eta \in \alpha \cap \beta$, then $\xi \in \eta \in \alpha$, so $\xi \in \alpha$ since $\alpha$ is transitive, and $\xi \in \eta \in \beta$, so $\xi \in \beta$ since $\beta$ is transitive. So $\xi \in \alpha \cap \beta$.

Let $\gamma = \min(\alpha \setminus \beta)$, and $\delta = \min(\beta \setminus \alpha)$.

We argue that $\gamma = \alpha \cap \beta$. If $\xi \in \gamma$, then $\xi \notin \alpha \setminus \beta$. On the other hand $\xi \in \gamma \in \alpha$, so $\xi \in \alpha$. So $\xi \in \alpha \cap \beta$. Now if $\xi \notin \gamma$, then $\xi \notin \alpha \cap \beta$, for otherwise, $\xi \in \alpha$, so since $\alpha$ is totally ordered, $\xi = \gamma$ or $\xi \ni \gamma$; and since $\alpha \cap \beta$ is transitive, $\gamma \in \alpha \cap \beta$, $\divideontimes$.

Similarly $\delta = \alpha \cap \beta$. Hence $\gamma = \delta \in \alpha \cap \beta$, $\divideontimes$. $\square$

Theorem 6.2.6. *If $\alpha$ and $\beta$ are ordinals, then $\alpha \subseteq \beta$ iff $\alpha \underline{\in} \beta$.*

Proof: $\Leftarrow$) If $\alpha = \beta$, then obviously $\alpha \subseteq \beta$. If $\alpha \in \beta$, then for all $\gamma \in \alpha$, $\gamma \in \beta$ by transitivity; so $\alpha \subseteq \beta$.

$\Rightarrow$) If $\alpha \subseteq \beta$, but $\alpha \neq \beta$, let $\gamma = \min(\beta \setminus \alpha)$.

As above, $\gamma = \alpha$, so $\alpha \in \beta$. $\square$

Definition 6.2.7. *If $\alpha$ and $\beta$ are ordinals, then $\alpha \leq \beta$ iff $\alpha \underline{\in} \beta$, iff $\alpha \subseteq \beta$.*

Theorem 6.2.8. *Suppose $A$ is a set of ordinals. Then $\bigcup A$ is an ordinal.*

Proof: We show first that $\bigcup A$ is transitive. Suppose $\gamma \in \beta \in \bigcup A$. Since $\beta \in \bigcup A$, there exists $\alpha \in A$ such that $\beta \in \alpha$. Since $\alpha$ is an ordinal, it is transitive, so $\gamma \in \alpha$. Then $\gamma \in \bigcup A$.

Now we show that $\bigcup A$ is well-ordered. Suppose $S$ is a non-empty subset of $\bigcup A$. Let $\alpha \in S$. If $\alpha$ is the least element of $S$, then we are done, so suppose not. Then there exists $\beta \in \alpha$ such that $\beta \in S$; that is, $S \cap \alpha$ is a non-empty subset of $\alpha$. Since $\alpha$ is an ordinal, it is well-ordered. Let $\beta$ be the least element of $S \cap \alpha$. If $\gamma$ now is any element of $S$, then either $\gamma \in \alpha$, when $\gamma \geq \beta$ as shown, or $\gamma \notin \alpha$. Then $\gamma \overset{\ni}{=} \alpha$, so $\gamma \overset{\ni}{=} \beta$, as required. □

DEFINITION 6.2.9. *An ordinal $\alpha$ is said to be a* successor ordinal *iff there exists and ordinal $\beta$ such that $\alpha = \beta^+$.*

*An ordinal $\alpha$ is said to be a* limit ordinal *iff it is not* 0 *and is not a successor ordinal.*

EXAMPLE 6.2.10. *$\omega$ is a limit ordinal. All non-zero natural numbers are successor ordinals.*

THEOREM 6.2.11. *$\lambda$ is a limit ordinal iff for all $\alpha \in \lambda$, $\alpha^+ \in \lambda$.*

PROOF: If $\alpha \in \lambda$, then $\lambda \neq \alpha^+$ or $\lambda$ would be a limit. Also $\alpha \subseteq \lambda$ but $\lambda \not\subseteq \alpha$. Let $\beta \in \lambda \setminus \alpha$. Then $\beta \notin \alpha$. So $\alpha \overset{\in}{=} \beta$. So $\alpha \in \lambda$, so $\alpha^+ \subseteq \lambda$. Hence $\alpha^+ \leq \lambda$, so $\alpha^+ \in \lambda$. □

THEOREM 6.2.12. *Suppose $A$ is a non-empty set of ordinals.*

*If $A$ has a greatest element $\alpha$, then $\bigcup A = \alpha$.*

*Otherwise, $\bigcup A$ is a limit ordinal, and is the least ordinal greater than all elements of $\alpha$.*

PROOF: If $\alpha$ is the greatest element of $A$, then for all $\beta \in A$, $\beta \subseteq \alpha$. Thus $\bigcup A \subseteq \alpha$. The reverse is clear.

Now suppose $A$ has no greatest member.

If $\beta \in A$, then there exists $\gamma > \beta$ such that $\gamma \in A$. So $\gamma \subseteq \bigcup A$, so $\gamma \leq \bigcup A$, so $\beta < \bigcup A$.

Now suppose $\gamma > \beta$ for all $\beta \in A$. Then $\gamma \supseteq \beta$ for all $\beta \in A$, so $\gamma \supseteq \bigcup A$. So $\gamma > \bigcup A$. □

PROPOSITION 6.2.13. *Let $\alpha$ be an ordinal. Then $\alpha^+$ is the least ordinal greater than $\alpha$.*

PROOF: Certainly $\alpha^+ > \alpha$. Suppose $\gamma > \alpha$. Then $\gamma \ni \alpha$, so $\{\alpha\} \subseteq \gamma$. Also $\gamma \supseteq \alpha$. So $\gamma \supseteq \alpha \cup \{\alpha\} = \alpha^+$. So $\gamma \geq \alpha^+$ as required. □

PROPOSITION 6.2.14. *Let $\alpha$ be an ordinal, and let $f : \alpha \to \alpha$ be an order-preserving function (ie. $\gamma \in \delta$ implies $f(\gamma) \in f(\delta)$).*

*Then for all $\beta \in \alpha$, $f(\beta) \geq \beta$.*

PROOF: Suppose not. Let $S = \{\beta \in \alpha : f(\beta) < \beta\}$.

By assumption, $S \neq \varnothing$, so $S$ has a least element $\gamma$. Since $\gamma \in S$, $f(\gamma) < \gamma$. Since $f(\gamma) < \gamma$ and $\gamma$ is the least element of $S$, $f(f(\gamma)) \geq f(\gamma)$.

Now, $f(\gamma) < \gamma$ while $f(f(\gamma)) \geq f(\gamma)$, ※ to the assumption that $f$ is order-preserving. □

So $\bigcup$ is the supremum operator on sets of ordinals.

COROLLARY 6.2.15. *$\lambda$ is a limit ordinal iff $\lambda$ is non-empty and*

$$\lambda = \bigcup\{\alpha : \alpha < \lambda\}.$$

*(More snappily, $\lambda = \bigcup \lambda$.)*

*$\bigcup \alpha^+ = \alpha$.*