7. The Axiom of Replacement

The Axiom of Replacement is the following family of axioms (stated slightly more formally than in lectures):

Replacement Axiom Scheme *Suppose that X is a set, and $\psi(y, x, w_1, \ldots, w_n)$ is a formula of the language of set theory having the property that for all $x \in X$ and for all $w_1, \ldots, w_n$, there is a unique set y such that $\psi(y, x, w_1, \ldots, w_n)$ holds.*

Then for all $w_1, \ldots, w_n$, there exists a set z whose elements are all sets y such that there exist $x \in X$ and such that $\psi(y, x, w_1, \ldots, w_n)$ holds.

Intuitively, the Axiom of Replacement allows us to take a set X, and form another set by replacing the elements of X by other sets according to any definite rule. As an example, we may take the set ω:

$$\omega = \{0, 1, 2, \ldots, n, \ldots\}$$

and use the Replacement Axiom Scheme to replace each n by the ordinal $\omega + n$:

$$\{\omega, \omega + 1, \omega + 2, \ldots, \omega + n, \ldots\}$$

and then use the Axiom of Unions to take the union of this set, to form the ordinal $\omega + \omega$.

So why is this harmless-looking axiom controversial?

7.1. Is it consistent?

As I may have mentioned before, we do not know whether the Axioms of Set Theory are consistent or not*. And if there *is* a problem, it's quite possible that it's with the Axiom of Replacement. It has the property, which it shares with the Axiom of Infinity and the Powerset Axiom, of enormously increasing the universe of sets that we can prove to exist. And maybe that is "too much of a good thing".

I don't know how many set theorists wake in a cold sweat at four in the morning wondering, "Is the Axiom of Replacement inconsistent?" I suspect it's not many.

7.2. Is it true?

A colleague of mine, with whom I was discussing these issues the other week, said, "But surely the Axiom of Replacement is obviously true". As an example of the sort of thing that might happen if it is false, is it really credible that

$$\omega = \{0, 1, 2, \ldots, n, \ldots\}$$

exists as a set but

$$\{\omega, \omega + 1, \omega + 2, \ldots, \omega + n, \ldots\}$$

does not?

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* If they're inconsistent, we'll maybe find out some day; if not, we'll never know; if you take the section C course on the Gödel Incompleteness Theorems, you'll find out why.

The overwhelming majority of Set Theorists would agree with my colleague that yes, the Axiom of Replacement is true*.

To make the point a little more sharply, abandoning the Axiom of Replacement essentially means abandoning one of the following:

1. Hartog's Theorem, or
2. Transfinite Recursion.

And I think you would have trouble convincing a set theorist to part company with either of these.

7.3. Is it necessary?

This is really the point at which the objections to Replacement arise. Is the Axiom of Replacement necessary to do "real mathematics" (in the way that, for instance, the Axiom of Infinity plainly is)? Granted that Set Theorists would rather lose their eye-teeth than the Axiom of Replacement, need other mathematicians feel the same way?

The debate here is not, I have to say, of uniformly high quality. Mathematicians are human, and just as given to tribalism as everyone else; some people in this context seem to have a circular definition of "real mathematics".

However there is an argument to be made here. Much mathematics, for example, most of what occurs in the part I syllabus, can be carried out without the Axiom of Replacement. One's theory of the ordinals is hamstrung without Replacement; but then you don't often use the ordinals if you're doing, for example, Quantum Mechanics. To date there is one theorem that is reasonably well-known about subsets of $\mathbb{R}$ which relies on (a certain amount of) Replacement, and that is the theorem that every Borel set is determined†.

7.4. To summarise...

It should be fairly clear what my own view is. But perhaps it is possible to accommodate the objectors (who include some very eminent names) by saying that the Axiom of Replacement could be taken, roughly, to divide "low-tech" from "high-tech" applications of Set Theory. Without Replacement, our theory of sets would be greatly impoverished. But if your next research paper uses Replacement in an essential way, then the chances are good that either you have had some serious set-theoretic training, or one of your co-authors is a set theorist.

For an intelligent discussion of Replacement by a sceptic, see Appendix A of M. Potter's *Set Theory and its Philosophy.*

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* Of course, for this point of view to make sense, one needs to have a philosophy of set theory according to which such an axiom *could* meaningfully be said to be true. Maybe there's a psychological point at issue here: why would a mathematician devote their professional life to studying set theory unless the sets they were studying really existed, and they were finding out things about them that really were the case?

† The Borel subsets of $\mathbb{R}$ are the following: 1. All open sets are Borel; 2. If B is Borel, then so is $\mathbb{R} \setminus B$; 3. Any countable union of Borel sets is Borel. The property of being determined is a set-theoretic/topological property.