Surface subgroups in dimension 3

Lecture 2

Recall:

Main Theorem 1.2: [L] Any finitely generated Kleinian group Γ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

Lemma 1.4: If O is a compact orientable 3-orbifold, and each arc and circle of $\mathrm{sing}(O)$ has order a prime p, then

$$\dim\, H_1(O; \mathbb{F}_p) \geq b_1(\mathrm{sing}(O)).$$

Endgame of proof of 1.2

Let Γ be a Kleinian group with a finite non-cyclic subgroup.

Simplifying assumptions:

1. $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.
2. Γ is cocompact

Then $O = \Gamma \backslash \mathbb{H}^3$ is a closed hyperbolic 3-orbifold.

Goal: find an infinite covering space O_i of O, containing a compact 3-dimensional suborbifold N_i such that:

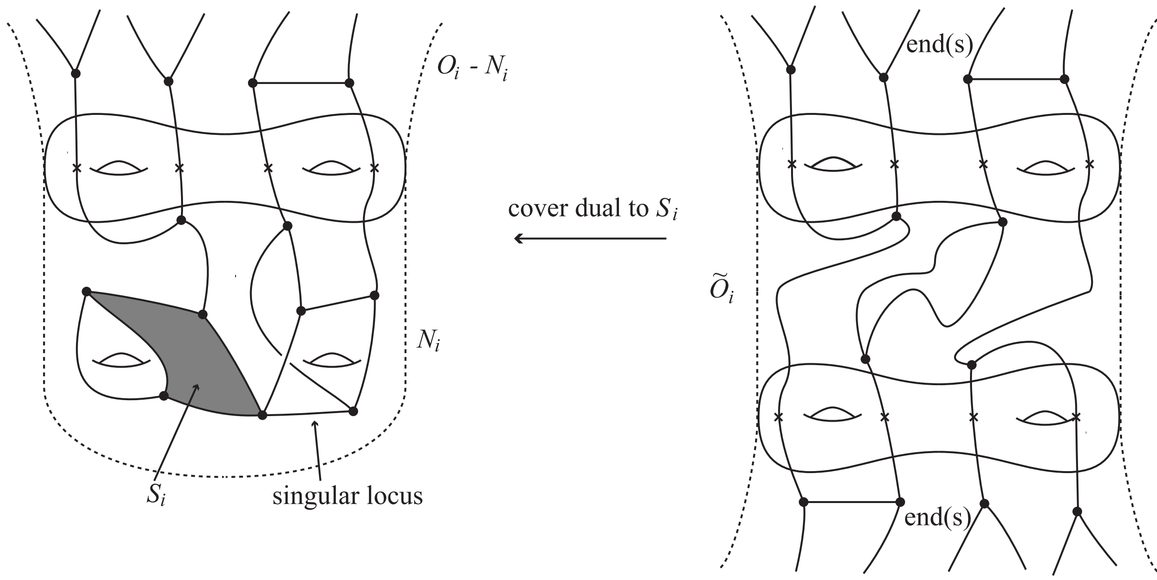
1. every arc and circle of $\text{sing}(N_i)$ has order 2;
2. $b_1(\text{sing}(N_i)) > d_2(\partial N_i)$

By Lemma 1.4, $d_2(N_i) > d_2(\partial N_i)$.

So, ker $H^1(N_i;\mathbb{F}_2) \to H_1(\partial N_i;\mathbb{F}_2)$ is non-trivial.

Let S_i be a surface properly embedded in $N_i - \mathrm{sing}(N_i)$ dual to a non-trivial element of this kernel, and that is disjoint from ∂N_i.

Let $\tilde{O}_i$ be the 2-fold cover of O dual to S_i.



This has at least two ends.

We may find a finite manifold cover M_i of $\tilde{O}_i$.

This also has at least two ends.

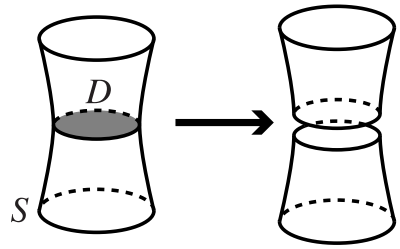
Now apply ...

Lemma 1.5: Let M be an orientable hyperbolic 3-manifold with at least 2 ends. Then $\pi_1(M)$ contains a surface subgroup.

Proof:

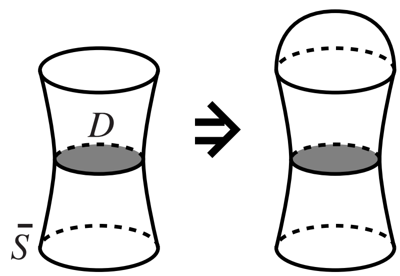
Let S be a closed orientable surface separating two ends of M.

Compress S as much as possible to $\overline{S}$:



compress

$\overline{S}$ is incompressible:



Incompressible

Old theorem: Any properly embedded orientable incompressible surface is π_1-injective.

Some component of $\overline{S}$ still separates two ends of M.

It's π_1-injective.

It's not a sphere, because M is irreducible (as M is hyperbolic).

Hence, $\pi_1(M)$ contains a surface subgroup. □ (1.5)

So, surface subgroup $\leq \pi_1(M_i) \leq \pi_1(O)$. □ (1.2)

Three main theorems

Let Γ be a cocompact Kleinian group with a finite non-cyclic subgroup.

Let $O = \Gamma \backslash \mathbb{H}^3$.

Simplifying assumption: $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.

Theorem 2.1: O has a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\mathrm{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

Let $M = |\tilde{O}|$.

Theorem 2.2: If a closed orientable 3-manifold M has infinite π_1, then either

1. M is hyperbolic; or
2. M has a finite cover $\tilde{M}$ with $b_1 > 0$.

In case 2, there is an induced finite cover of $\tilde{O}$ with underlying space $\tilde{M}$. So, its $\pi_1 \twoheadrightarrow \pi_1(\tilde{M}) \twoheadrightarrow \mathbb{Z}$.

So, wlog M is hyperbolic.

Theorem 2.3: [L-Long-Reid] Any closed hyperbolic 3-manifold M has a sequence of infinite covers M_i s.t. $h(M_i) \to 0$.

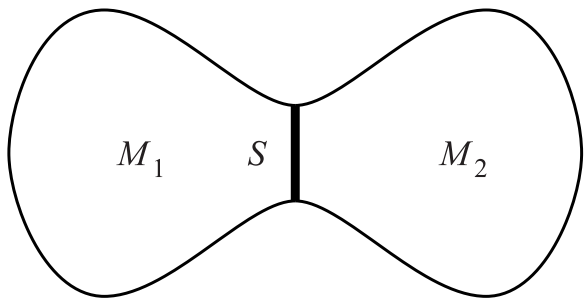
Here $h(M_i)$ is the 'Cheeger constant' of M_i

Let M be a complete Riemannian manifold.

If M has finite volume, then its Cheeger constant $h(M)$ is

$$\inf_S \left\{ \frac{\text{Area}(S)}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}} \right\},$$

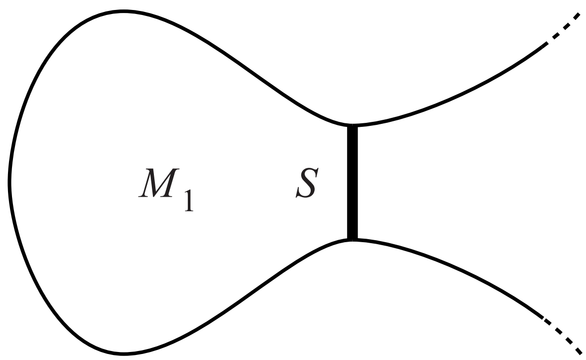
as S ranges over all embedded codimension 1 submanifolds that separate M into M_1 and M_2.



If M has infinite volume, then its Cheeger constant $h(M)$ is

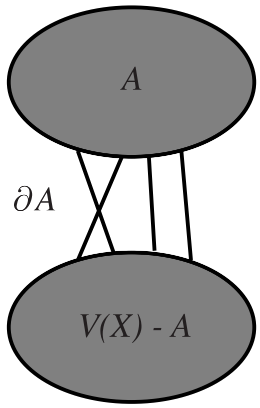
$$\inf_S \left\{ \frac{\text{Area}(S)}{\text{Vol}(M_1)} \right\},$$

as S ranges over all embedded codimension 1 submanifolds that bound a finite volume submanifold M_1.



Let X be a graph, with vertex set $V(X)$.

For $A \subseteq V(X)$, ∂A is the set of edges with one endpoint in A and one endpoint not in A.



If $V(X)$ is finite,

$$h(X) = \min\left\{\frac{|\partial A|}{|A|} : A \subset V(X), 0 < |A| \leq |V(X)|/2\right\}.$$

If $V(X)$ is infinite,

$$h(X) = \inf\left\{ \frac{|\partial A|}{|A|} : A \subset V(X), 0 < |A| < \infty \right\}.$$

Let

$$\begin{aligned} M &= \text{closed Riemannian manifold} \\ \Gamma &= \pi_1(M) \\ S &= \text{finite generating set for } \Gamma \\ M_i &= \text{covering space of } M \\ \Gamma_i &= \pi_1(M_i) \\ X_i &= \text{coset diagram of } \Gamma/\Gamma_i \text{ w.r.t. } S \end{aligned}$$

Theorem 2.4: There are constants $c, C > 0$ s.t. for all covers $M_i \to M$,

$$c\ h(X_i) \leq h(M_i) \leq C\ h(X_i).$$

2.1, 2.2, 2.3, 2.4 $\Rightarrow$ Goal

Let Γ be a cocompact Kleinian group with a finite non-cyclic subgroup.

Let $O = \Gamma \backslash \mathbb{H}^3$.

Simplifying assumption: $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.

2.1 $\Rightarrow$ we may pass to a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

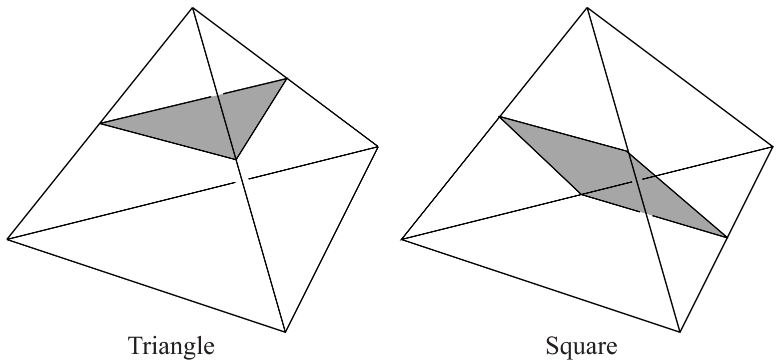
Let $M = |\tilde{O}|$.

2.2 $\Rightarrow$ wlog M is hyperbolic.

Let T be a triangulation of M with one vertex.
Wlog, this vertex is a singular vertex of $\tilde{O}$.

Its edges (when oriented) $\longrightarrow$ a generating set S for $\pi_1(M)$.

In the interior of each edge, pick a 'midpoint'.
In each face, pick three arcs running between the midpoints.
In each tetrahedron, pick triangles and squares with these arcs as edges:



Wlog each triangle and square intersects $\text{sing}(\tilde{O})$ transversely.

$2.3 \Rightarrow M$ has covers M_i with $h(M_i) \to 0$.

Let X_i = coset diagram of $\pi_1(M)/\pi_1(M_i)$ w.r.t. S.

X_i = 1-skeleton of M_i.

$2.4 \Rightarrow h(X_i) \to 0$.

Let A_i be a finite subset of $V(X_i)$ s.t.

$$\frac{|\partial A_i|}{|A_i|} \to 0 \text{ as } i \to \infty.$$

Construction of N_i:

Let $|\partial N_i| \cap X_i = \partial A_i$.
Join up these points using lifts of arcs, triangles and squares.
This bounds a compact 3-dimensional suborbifold N_i.

Must check: $b_1(\mathrm{sing}(N_i)) > d_2(\partial N_i)$.

$\mathrm{sing}(N_i)$ is a graph with:
$\sim |A_i|$ trivalent vertices;
$\precsim |\partial A_i|$ univalent vertices.

As $|\partial A_i|/|A_i| \to 0$, $b_1(\mathrm{sing}(N_i)) \sim |A_i|$.

$d_2(|\partial N_i|) \precsim |\partial A_i|$
$|\mathrm{sing}(\partial N_i)| \precsim |\partial A_i|$
$\Rightarrow d_2(\partial N_i) \precsim |\partial A_i|$

So, for $i >> 0$, $b_1(\mathrm{sing}(N_i)) > d_2(\partial N_i)$, as required.

Theorem 2.3: [L-Long-Reid] Let M be a closed hyperbolic 3-manifold. Then M has infinite-sheeted covers M_i such that $h(M_i) \to 0$.

This is a consequence of:

Theorem 2.5: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated free subgroups Γ_i such that $\delta(\Gamma_i) \to 2$.

Here $\delta(\Gamma_i)$ = the 'critical exponent' of Γ_i

Theorem 2.6: [Sullivan]

$$\lambda_1(\Gamma_i \backslash \mathbb{H}^3) = \begin{cases} \delta(\Gamma_i)(2 - \delta(\Gamma_i)) & \text{if } \delta(\Gamma_i) \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Here $\lambda(\Gamma_i \backslash \mathbb{H}^3)$ = the first eigenvalue of the Laplacian of $\Gamma_i \backslash \mathbb{H}^3$.

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M_i, $\lambda_1(M_i) \geq h(M_i)^2/4$.

Preliminaries on Riemannian manifolds

Let M be a closed n-dimensional Riemannian manifold.

Let $C^\infty(M)$ be the smooth functions $M \to \mathbb{R}$.

There is an inner product on $C^\infty(M)$:

$$\langle f, g \rangle = \int_M fg \; d\text{vol}.$$

Let $*$ be the Hodge star operator on differential forms on M:

$$*\colon \Omega^k(M) \to \Omega^{n-k}(M)$$

If $dx_1, \ldots dx_n$ forms an orthonormal basis at a point of $T^*(M)$. Then at this point

$$*(dx_1 \wedge \ldots \wedge dx_k) = dx_{k+1} \wedge \ldots \wedge dx_n.$$

Then there is an inner product on differential k-forms:

$$\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge *\omega_2 \; d\mathrm{vol}.$$

Stokes theorem $\Rightarrow$ for $\omega_1 \in \Omega^{k-1}(M), \omega_2 \in \Omega^k(M)$,

$$\langle d\omega_1, \omega_2 \rangle = \langle \omega_1, (-1)^{k(n-k)} * d * \omega_2 \rangle$$

And so $(-1)^{k(n-k)} * d*$ is the formal adjoint of d.

We denote it by d^*.

$$\Omega^0(M) \;\underset{d^*}{\overset{d}{\rightleftarrows}}\; \Omega^1(M) \;\underset{d^*}{\overset{d}{\rightleftarrows}}\; \dots \;\underset{d^*}{\overset{d}{\rightleftarrows}}\; \Omega^n(M)$$

The Laplacian is

$$\Delta: C^\infty(M) \to C^\infty(M)$$
$$f \mapsto d^* df$$

This is self-adjoint:

$$\langle f, d^* dg \rangle = \langle df, dg \rangle = \langle d^* df, g \rangle$$

There is an orthonormal set of smooth eigenfunctions u_n such that any $f \in C^\infty(M)$ is

$$f = \sum_n \mu_n u_n.$$

Say that

$$\Delta u_n = \lambda_n u_n,$$

where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$$

Definition:

$$\lambda_1(M) = \lambda_1$$

Note: u_0 is the constant function $1/\sqrt{\mathrm{vol}(M)}$.

Note:

$$\langle f, f \rangle = \sum_n \mu_n^2.$$

$$\langle df, df \rangle = \langle f, \Delta f \rangle = \sum_n \mu_n^2 \lambda_n.$$

So:

$$\lambda_1(M) = \inf \left\{ \frac{||df||^2}{||f||^2} : f \in C^\infty(M) \text{ and } \int_M f = 0 \right\}.$$

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M,

$$\lambda_1(M) \geq h(M)^2/4.$$

Proof: (when M is closed)

Let f be an eigenfunction with eigenvalue λ_1. Let

$$M_+ = \{x \in M : f(x) \geq 0\}$$
$$M_- = \{x \in M : f(x) \leq 0\}$$

Wlog, $\mathrm{vol}(M_+) \leq \mathrm{vol}(M_-)$.

Focus on M_+ and take all integrals over M_+:

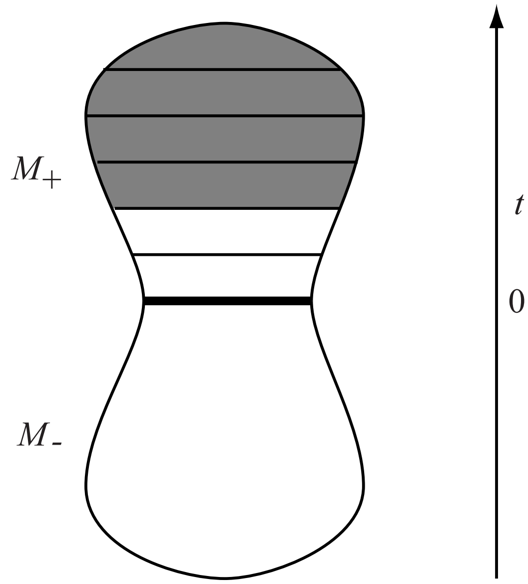
$$\begin{aligned}
\lambda_1 &= \frac{\int |df|^2}{\int |f|^2} \\
&= \frac{\int |df|^2 \int |f|^2}{\left(\int f^2\right)^2} \\
&\geq \frac{\left(\int |df|.|f|\right)^2}{\left(\int f^2\right)^2} \text{ by Cauchy-Schwarz} \\
&= \frac{1}{4}\frac{\left(\int |df^2|\right)^2}{\left(\int f^2\right)^2}
\end{aligned}$$

Claim: RHS $\geq h(M)^2/4$.

For any $t \geq 0$, let

$$A(t) = \text{Area}(\{x \in M_+ : f(x)^2 = t\})$$

$$V(t) = \text{Vol}(\{x \in M_+ : f(x)^2 \geq t\})$$



Co-area formula: $\int |df^2| = \int A(t)\ dt$

For each t, $A(t) \geq h(M)V(t)$, because at least half the volume lies in M_-.

So:

$$\begin{aligned}\int A(t)\ dt &\geq h(M) \int V(t)\ dt \\ &= h(M) \int_t \left(\int_{\{x : f(x)^2 \geq t\}} d\mathrm{vol} \right) dt \\ &= h(M) \int f^2\ d\mathrm{vol}\end{aligned}$$

So,

$$\lambda_1(M) \geq \frac{1}{4} \frac{\left(\int |df^2|\right)^2}{\left(\int f^2\right)^2} \geq \frac{h(M)^2}{4}. \qquad \square$$

Manifolds with infinite volume

If M is complete and has infinite volume,

$$h(M) = \inf_S \left\{ \frac{\text{Area}(S)}{\text{Vol}(M_1)} \right\},$$

as S varies over all codim 1 submanifolds bounding a finite volume submanifold M_1.

We now consider the Laplacian

$$\begin{aligned} \Delta\colon L^2(M) &\to L^2(M) \\ f &\mapsto d^* df \end{aligned}$$

This has spectrum in $[0, \infty)$.

0 is no longer an eigenvalue of Δ, because no non-zero constant function is in $L^2(M)$.

$\lambda_1(M)$ is the infimum of the spectrum.

$$\lambda_1(M) = \inf \left\{ \frac{||df||^2}{||f||^2} : f \in L^2(M) \cap C^\infty(M) \right\}.$$

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M,

$$\lambda_1(M) \geq h(M)^2/4.$$

Proof: (infinite volume case)

For any $t \geq 0$, let

$$A(t) = \text{Area}(\{x \in M : f(x)^2 = t\})$$
$$V(t) = \text{Vol}(\{x \in M : f(x)^2 \geq t\})$$

Then $V(t) < \infty$ because $f \in L^2$.

Same proof as before. $\square$