Unknot recognition in quasi-polynomial time

Marc Lackenby

March 2021

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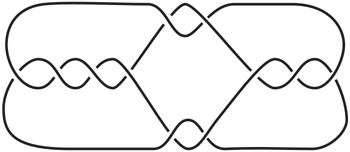

Goeritz's unknot

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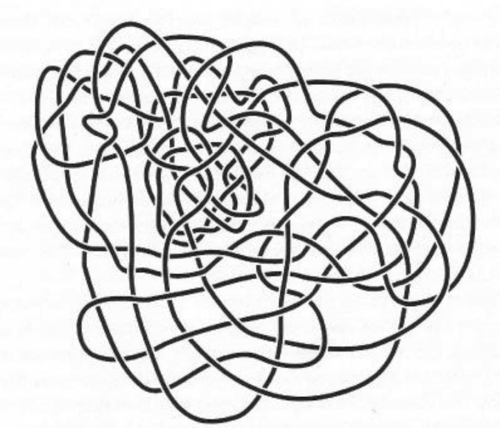

Haken's unknot

Early history

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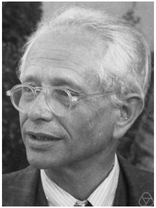

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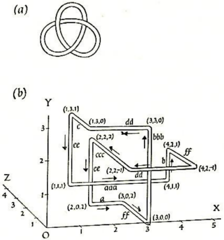
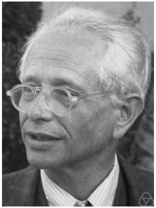
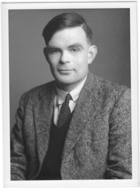


Fig. 1. (*a*) The trefoil knot (*b*) a possible representation of this knot as a number of segments joining points.





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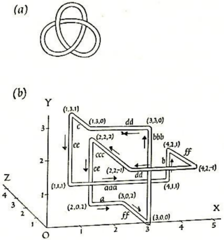

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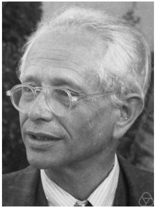

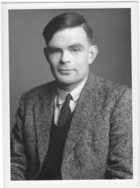

Theorem: [Haken, 1961] There is an algorithm to determine whether a given knot is the unknot.

Many other approaches

- Normal surfaces [Haken, Hass-Lagarias-Pippenger]
- Geometric structures [Thurston]
- Representations of π_1 [Kuperberg]
- Hierarchies [Agol, L]
- Khovanov homology [Kronheimer-Mrowka]
- Heegaard Floer homology [Ozsváth-Szabó, Sarkar-Wang, Manolescu-Ozsváth-Sarkar]
- Arc presentations [Dynnikov, L]
- Reidemeister moves [Hass-Lagarias, L]
- Pachner moves [Mijatovic]

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[Thurston 2011] ‘A lot of people have thought about this question.’ ‘I think it’s entirely possible that there’s a polynomial-time combinatorial algorithm to unknot an unknottable curve, but this has been a very hard question to resolve.’

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Main Theorem: [L 2021] There is an algorithm to determine whether a diagram with n crossings is the unknot that completes in time $2^{O(\log n)^3}$. 'quasi-polynomial time'

Hierarchies

Let M be a compact orientable 3-manifold, for example $S^3 - \mathrm{int}(N(K))$ for K a knot.

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A hierarchy is a sequence of 3-manifolds $M = M_1, \ldots, M_{\ell+1}$ and orientable surfaces $S_1, \ldots, S_\ell$ such that each S_i is properly embedded in M_i and $M_{i+1} = M_i \backslash\backslash\, S_i$.

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We do not require the surfaces to be incompressible or for the final manifold $M_{\ell+1}$ to be balls (although we will be aiming to produce such hierarchies).

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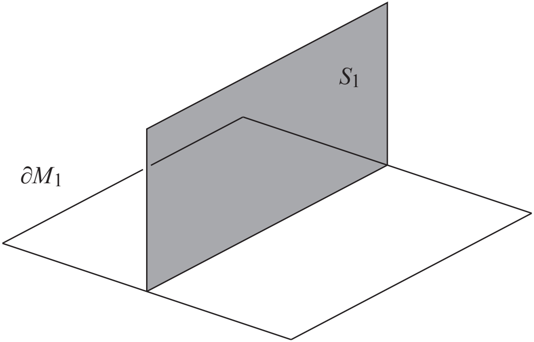

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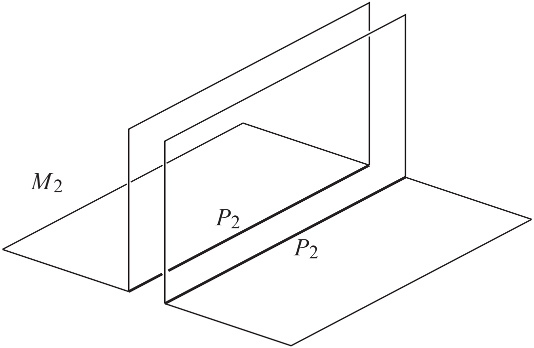

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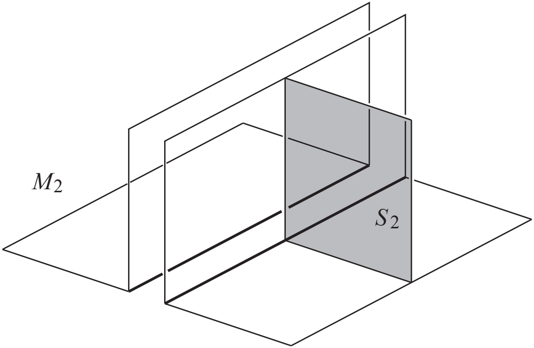

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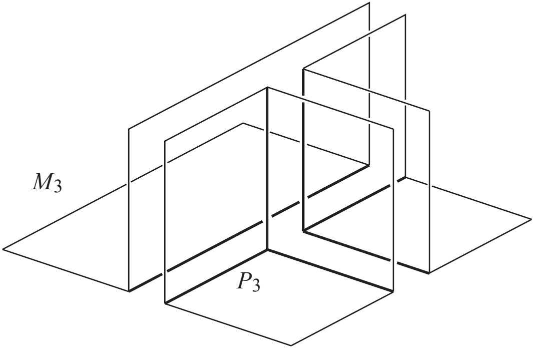
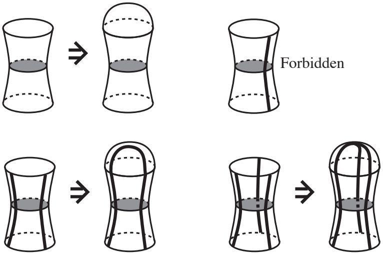


Essential boundary patterns

A boundary pattern P is essential if, for any properly embedded disc D that intersects P at most three times, ∂D bounds a disc D' in ∂M that intersects P in one of the following:

- the empty set,
- an arc,
- a tripod.



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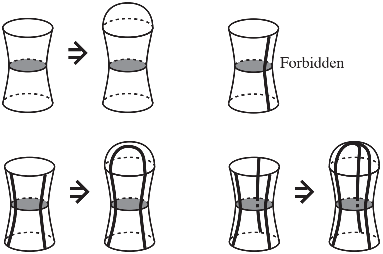

A disc D properly embedded in M that intersects P at most three times for which ∂D does not bound a disc D' in ∂M as above is a violating disc.

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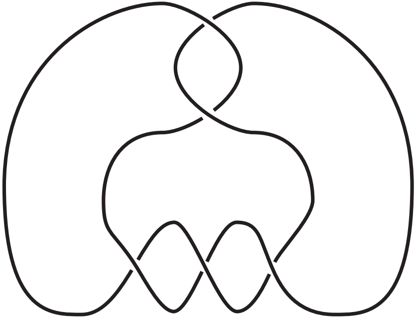
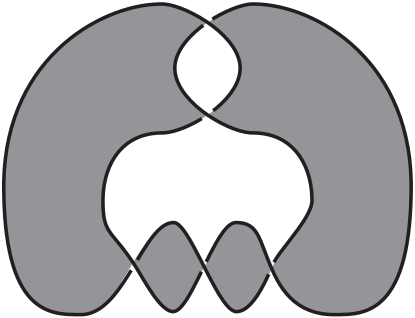
So, we will use essential hierarchies as a way of proving that a knot is non-trivial.

Example



(i) The knot 5_2

Example



(ii) The first surface in the hierarchy

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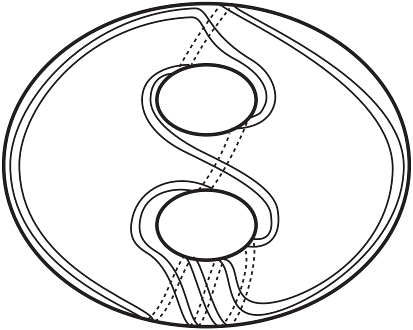
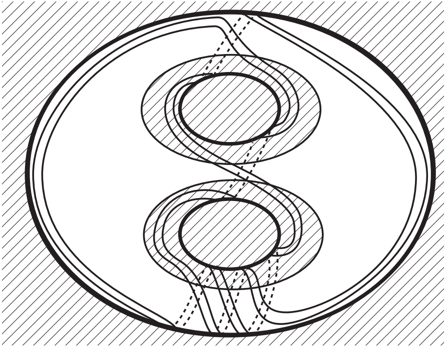


(iii) The exterior of this surface

Example



(iv) The second surface in the hierarchy

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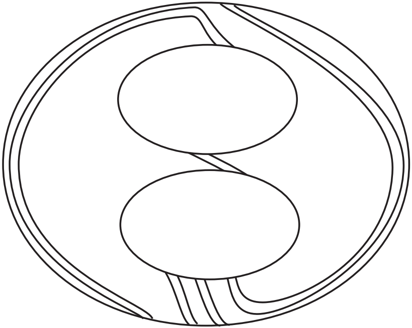

(v) The pattern of one of the balls

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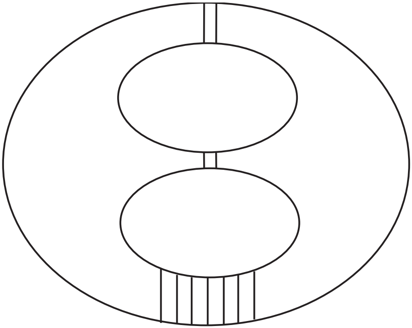

(vi) A simplified copy of the pattern

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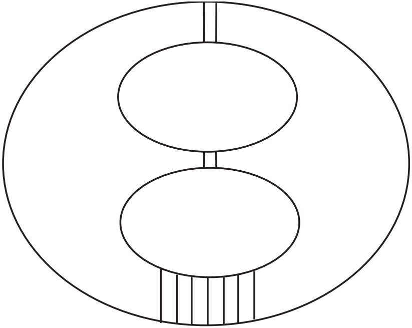

(vi) A simplified copy of the pattern

This is an essential boundary pattern, and hence the knot 5_2 is not the unknot

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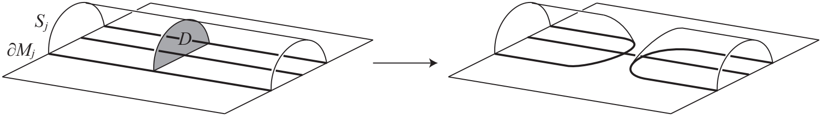

A new unknot recognition algorithm

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Build a triangulation of $M_1 = S^3 - \mathrm{int}(N(K))$ and set $\ell = 1$.

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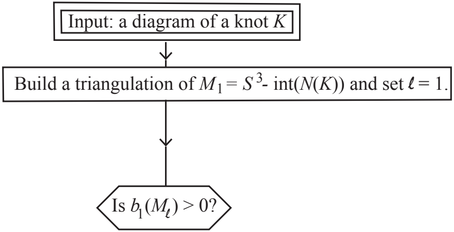

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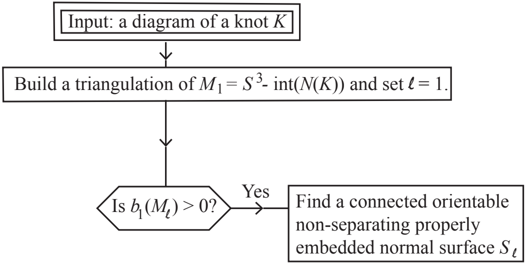

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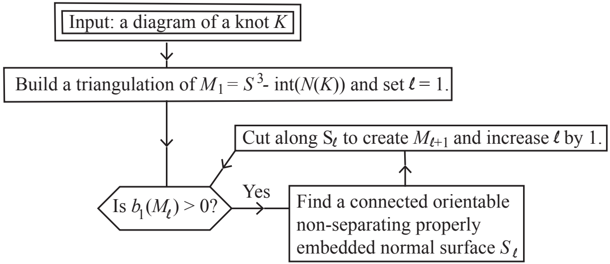

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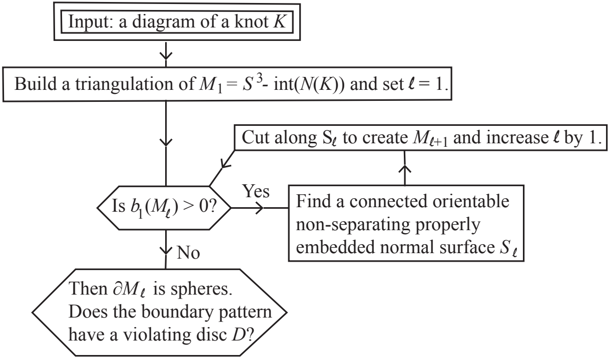

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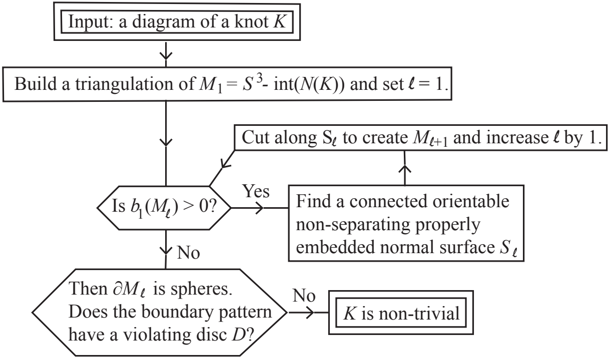

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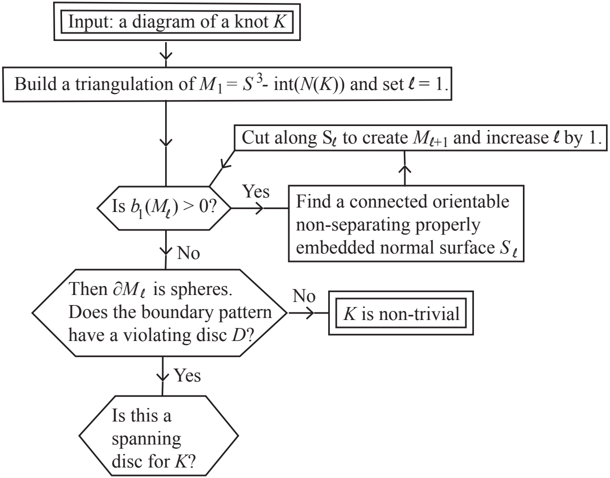

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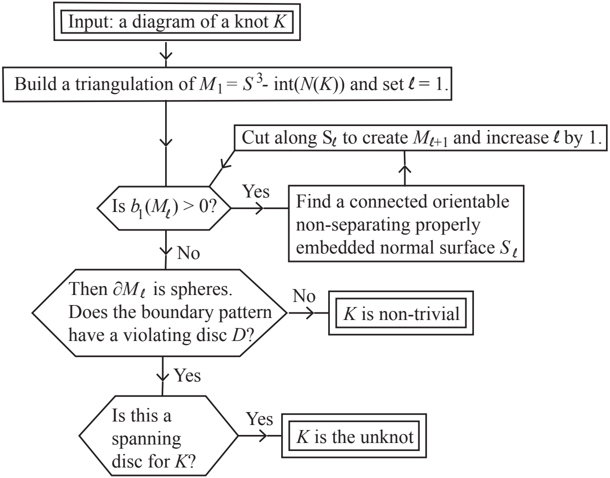

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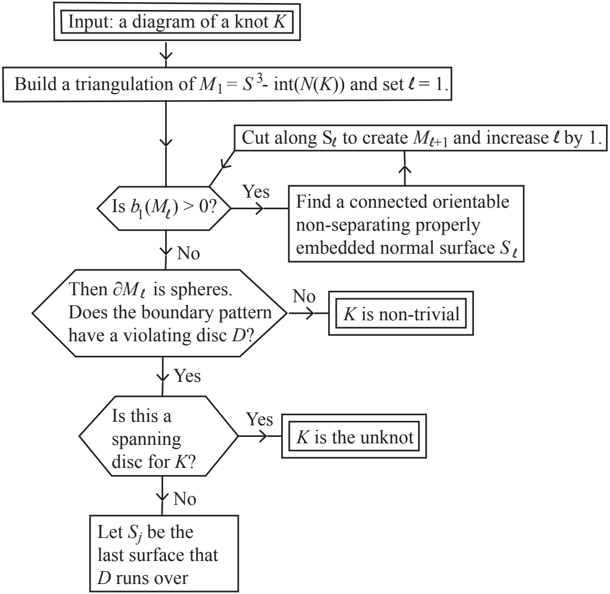

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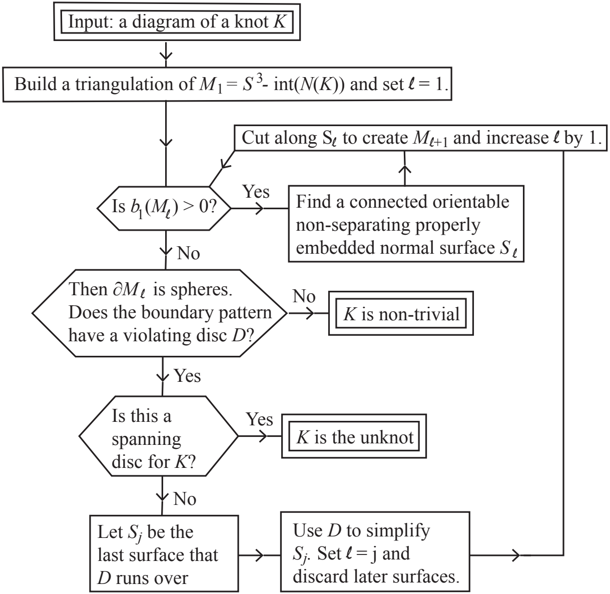

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(Throughout the talk, I'll refer to the 'genus' $g(S_i)$ but I may mean some related notion, for example χ_- or a version that also counts intersections with the pattern.)

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So the 'running time' would be at most Lg^L.

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Hence, the running time is $n^{O(\log n)^2}$.

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Seifert's algorithm creates a surface S_1 with $g(S_1) \leq O(n)$.

A quadratic bound on the genus of surfaces

The first surface in the hierachy is a Seifert surface.

Seifert's algorithm creates a surface S_1 with $g(S_1) \leq O(n)$.

For the later surfaces in the hierarchy, we use a generalisation of Seifert's algorthm: we do not forget that our manifolds M_i lie in S^3.

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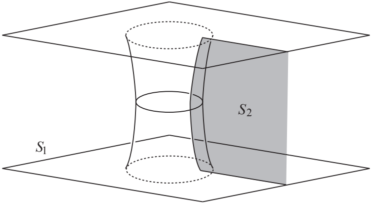
The problem is that if we compress a surface S_j, we have to discard the later surfaces.



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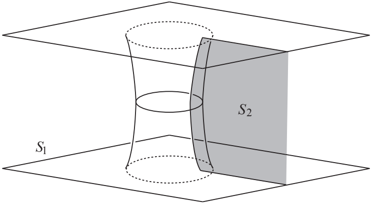
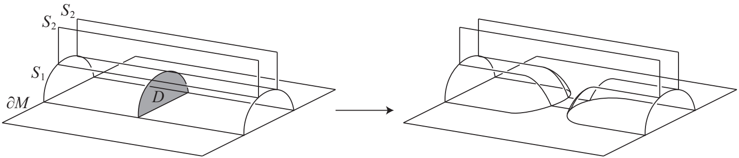


This is not the case with boundary-compressions:



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We can use a multi-surface $S_1, \dots, S_k$ to create k steps in the hierarchy using the surfaces

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Poincaré duality implies that a compact orientable 3-manifold M contains a multi-surface of rank at least $g(\partial M)$.

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So as far as our algorithm is concerned, a multi-surface behaves like a single surface.

Our aim is to find a hierarchy of multi-surfaces of length $O(\log n)^2$.

Logarithmic length

We hope to find a hierarchy

$$M_1 \xrightarrow{\mathcal{S}_1} M_2 \xrightarrow{\mathcal{S}_2} \dots \xrightarrow{\mathcal{S}_\ell} M_{\ell+1}$$

where

- $\mathcal{S}_i$ is a multi-surface of rank $g(\partial M_i)$;
- $g(\partial M_i)$ grows exponentially as a function of i;

and so the hierarchy terminates after $O(\log n)^2$ steps.

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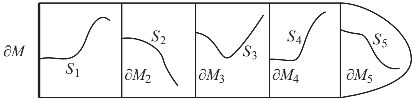

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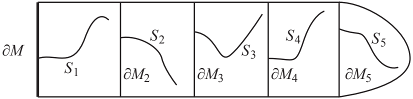

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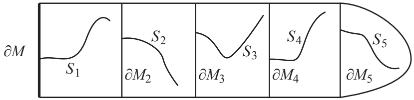

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Recall that the Cheeger constant of a Riemannian n-manifold M is

$$\inf\left\{\frac{\mathrm{Area}(\partial M')}{\min\{\mathrm{Vol}(M'), \mathrm{Vol}(M - M')\}}\right\}$$

as M' ranges over all n-dimensional submanifolds of M.

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The 'size' of M_i is $g(H \cap M_i)$.

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- If we come across a Cheeger region, we can simplify the initial generalised Heegaard splitting.
- If we never see a Cheeger region, the hierarchy completes in $O(\log n)^2$ steps.

The algorithm that runs in quasi-polynomial time