Cadoret: "ℓ-adic rep of étale fund groups of curves"

(j.w. A. Tamagawa)

Not k field char $k = p \geq 0$

$k \hookrightarrow \bar{k}$ alg closure

$\Gamma_k = \pi_1(\mathrm{Spec}(k))$

S normal scheme separated of f.t. geo. connected $/k$

$\pi_1(S)$ étale fundamental group

S^{cl} set of closed points $S^{\leq d} = \{ s \in S^{cl} \mid [k(s):k] \leq d \}$ $d \geq 1$

$s \in S$

$$\begin{array}{ccccc} \mathrm{Spec}\, k(s) & \xrightarrow{s} & S & \longleftarrow & S_{\bar{k}} \\ & \searrow & \downarrow & & \\ & & \mathrm{Spec}\, k & & \end{array}$$

funct $\Longrightarrow$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_{\bar{k}}) & \longrightarrow & \pi_1(S) & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ & & & & \uparrow & \nearrow & \uparrow & & \\ & & & & & \Gamma_{k(s)} & & & \end{array}$$

$\ell \neq p$ $\quad \rho : \pi_1(S) \longrightarrow Gl_2(\mathbb{Z}_\ell)$

$$\left.\begin{array}{lll} G = \mathrm{Im}\,\rho & \subset & Gl_2(\mathbb{Z}_\ell) \\ G^{geo} = \rho(\pi_1(S_{\bar{\ell}})) & \subset & G \\ G_s = \rho \circ \sigma_s(T_{k(s)}) & \subset & G \end{array}\right) \ell\text{-adic Lie groups} \qquad \left.\begin{array}{l} g \\ g^{geo} \\ g_s \end{array}\right) \text{Lie algebras}$$

$$\left.\begin{array}{l} G^2 \\ G^{geo\,2} \\ G_s^{\,1} \end{array}\right) \text{Zariski closures in } Gl_2$$

§. Motivation

$X \xrightarrow{f} S$ smooth proper with geo connected fibres

$$H_{\ell^n} = H^*(X_{\bar{\eta}}, \mathbb{Z}/\ell^n) = (\underset{①}{R^\bullet f_*} \mathbb{Z}/\ell^n)_{\bar{\eta}} \xrightarrow{\sim}$$

$$(\underset{②}{R^\bullet(f_* \mathbb{Z}/\ell^n)})_{\bar{s}} = \underset{③}{H^\bullet(X_{\bar{s}}, \mathbb{Z}/p^n)}$$

$$H_{p^\infty} = \varprojlim H_{p^n} \qquad \pi_1(S) = \overset{\text{acts on ①}}{\pi_1(S, \bar{\eta})} \xrightarrow{\sim} \overset{\text{acts on ②}}{\pi_1(S, \bar{s})} \xleftarrow[\sigma_s]{} \overset{\text{acts on ③}}{T_{k(s)}}$$

$$\begin{tikzcd} \rho : \pi_1(S) \arrow[r] & Gl(H_{\ell^\infty}) \\ T_{k(s)} \arrow[u, "\sigma_s"] \arrow[ur, "\rho_\sigma"'] & \end{tikzcd}$$

Compare this with $s \in S^{cl}$ $\hookrightarrow$ understanding how the image of $\rho: \pi_1(S) \to GL_2(H(X_s, \mathbb{Q}_\ell))$ varies with s

$\xi_s \to S/S^{geo}$ is open

$\Rightarrow \dim \xi \geq \dim \xi_S \geq \dim \xi - \dim \xi^{geo}$

Ex: $\begin{matrix} \mathcal{E} \\ \| \\ X \end{matrix} \to \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ $\rho: \pi_1(S) \to GL(T_\ell(\mathcal{E}_\eta))$

Serre's open image theorem.

§2 statements

Def An ℓ-adic rep $\rho: \pi_1(S) \to GL_2(\mathbb{Z}_\ell)$ is geometrically Lie perfect

$\xi \; L \; \rho$

if (1) $\forall U \subset_{or} \xi^{geo}$ $|U^{ab}| < +\infty$

(2) $(\xi^{geo})^{ab} = 0$

Thm Motivic rep are SLP

(:) if $p = 0$ - reduce to $k = \bar{k} \subset \mathbb{C}$

- comparison ℓ-adic $\hookrightarrow$ Betti
- Deligne semi-simplicity thm.

$p > 0$ - Bertini argument — S curve

- Specialization $\to k$ is finite

- Frobenius weight argument

Thm 1 (Main thm)

k f.g. field of char 0

S curve$/k$

$\rho : \pi_1(S) \to GL_2(\mathbb{Z}_\ell)$ ℓ-adic rep

1/ uniform open image thm for GLP rep) If ρ is GLP

$\forall d \in \mathbb{Z}_{\geq 1}$

- $E_\rho^{\leq d} = \{ s \in S^{\leq d} \mid \operatorname{codim}_S G_s \geq 1 \}$ is finite

- $\exists B_{\rho,d} \in \mathbb{Z}_{\geq 1}$ s.t $\forall s \in S^{\leq d} \mid E_\rho^{\leq d}[G : G_s] \leq B_{\rho,d}$

2/ If ρ is arbitrary $\forall d \in \mathbb{Z}_{\geq 1}$

a/ the set $\{ s \in S^{\leq d} \mid \operatorname{codim}_G G_s \geq 3 \}$ is finite

b/ for all but finitely many $s \in S^{\leq d}$ G_s is controlled by the first terms of derived series of G

e.g. the set $\{ s \in S^{\leq d} \mid D^2 g^{geo} \not\subset g_s \}$ is finite arithmetic invariant.

Rem $\ell \neq p > 0$
$\ell = p$?

Thm 2 (ℓ indep of the excepcional locus for motivic rep) ~~If~~

k f.g of char 0

S of arbitrary dim

$\rho_p : \pi_1(S) \to Gl_r(\mathbb{Z}_\ell)$ family of ℓ-adic motivic rep
ℓ?

E_{ρ_ℓ} is independent of ℓ.

Rem Thm 2 is predicted by motivic conj. $X \xrightarrow{f} S$

$G_{\bar{E}}$ motivated motivic Galois group of $X_{\bar{E}}$

$\hookrightarrow Sl_{HB}$ alg reductive group / $\mathbb{Q}_\ell$

$G_{\bar{s}} \hookrightarrow G_{\bar{\eta}}$ and $G_{\bar{s}} \xrightarrow{\sim} G_{\bar{\eta}} \iff U \subset_{op} G^{geo}$
$\cap$
$G_{\bar{s}}(\mathbb{Q}_\ell)$

$E_{\rho_\ell} \supset \{ s \in S \mid G_s \underset{\neq}{\hookrightarrow} G_{\bar{\eta}} \} = E^{mot}$

Tate's semisimplicity and motivated fullness $\Rightarrow$

$Gt_{\mathbb{Q}_\ell} = Gt^3$
$= E_{\rho_\ell} = E^{mot}$

$$F_{\mathcal{G}_\ell} = F^{\text{mot}}$$

$$\| \quad \{ s \in S \mid \operatorname{codim}_{S^2} S^2_s \geq 1 \}$$

Hint of proof of theorem 2

char 0 → - ℓ-adic independency results for family of ℓ-adic Galois rep
- Borel de Serre-Thal thm

char 0 → - Deligne semisimplicity.

$s \in S$ Assume $s \notin F_{\mathcal{G}_\ell}$ for some prime ℓ

↳ Serre's ℓ-independence of $\left(\ker \left(\pi_1(S) \to \mathcal{G}^2_\ell / (\mathcal{G}^2_\ell)^0 \right) \text{ vs } \ker \left(\pi_1(s) \to \mathcal{G}^2_{\ell,s} / (\mathcal{G}^2_{\ell,s})^0 \right) \right)$ is independent of ℓ.

↳ assume $\mathcal{G}^2_\ell$, $\mathcal{G}^2_{\ell,s}$ are connected

Replace reductive $\mathcal{G}_\ell$ by its semisimplification ↳ assume $\mathcal{G}^2_\ell$ is

$$\mathcal{G}^2_\ell \leftarrow \mathcal{G}^2_\ell / R_u(\mathcal{G}^2_\ell)$$

$$\mathcal{G}^2_{\ell,s} \leftarrow \mathcal{G}^2_{\ell,s} / R_u(\mathcal{G}^2_\ell) \cap \mathcal{G}^2_{\ell,s}$$

Choose a Levi functor $L_{\ell,s} \hookrightarrow \mathcal{G}_{\ell,s}^{\lambda}$

$s \notin \mathbb{F}_{p_\ell}$ $\quad \mathcal{G}_{\ell}^{\lambda}{}_{s} = L_{\rho,s} = \mathcal{G}_{\ell}^{\lambda}$

Serre's ℓ independence of reduction $\longrightarrow$

$$\begin{array}{ccccc} \text{rank } L_{\ell,s} & = & \text{rank } \mathcal{G}_{\ell}^{\lambda} \\ \| & & \| \\ \text{rank } L_{\rho',s} & = & \text{rank } \mathcal{G}_{\ell'}^{\lambda} \end{array} \quad ?$$

$$\begin{array}{ccc} L_{\ell',s} & \hookrightarrow & \mathcal{G}_{\ell'}^{\lambda} \\ \uparrow & & \\ T_{\ell'} & \hookleftarrow & Z(\mathcal{G}_{\ell}^{\lambda}) \end{array}$$

max torus

$Z(\mathcal{G}_{\ell'}^{\lambda}) \subset Z(L_{\ell',s})$

$Z(L_{\rho',s}) / Z(\mathcal{G}_{\ell}^{\lambda})$ is finite

$\Downarrow$

$$\mathcal{G}_{\ell'}^{\lambda} = \mathrm{Con}_{\mathcal{G}_{\ell'}^{\lambda}}(Z(\mathcal{G}_{\ell'}^{\lambda}))^{\circ} =$$

$$L_{\ell',s} = \mathrm{Con}_{\mathcal{G}_{\ell}^{\lambda}}(Z(L_{\rho',s}))^{\circ}$$

§7.2 Applications

1. $E^{\mathrm{mot}\,\leq d} \subset \mathbb{F}_{p_\ell}^{\leq d} \Rightarrow \{ s \in S^{\leq d} \mid \mathcal{G}_s \not\hookrightarrow \mathcal{G}_{\bar{\eta}}$ is finite $\}$

does not require uniformity in Thm 1

$\Rightarrow$ finiteness thm. of NS rank Endomorphism ring rank.

2) Strong ℓ-primary torsion conjecture $X_s \longrightarrow k(s)$,

$s \in S^{\leq d}$, $X \longrightarrow S$ Abelian scheme

$\forall d \in \mathbb{Z}_{\geq 1}$ $\exists B_{d,\ell} \geq 1$ s.t. $\forall s \in S^{\leq d}$

$$\left| A_s(k(s))[\ell^\infty] \right| \leq B_{d,\ell}.$$

§ 3 Hints of proof of thm 1

$$\mathcal{G}(n) = \ker\left(G \longrightarrow Gl_2(\mathbb{Z}_\ell) \longrightarrow Gl_2(\mathbb{Z}_{\ell^n}) \right) \quad n \geq 0$$

G top f.g. $\Rightarrow \forall B$ $S(G,B) = \{ U \subset_{op} G \mid [G : U] \leq B \}$

u finite

$$S(N_B) \subset \bigcap_{u \in S(G,B)} U \subset_{op} G$$

$$H_n(G) = \{ U \subset_{op} G \mid \mathcal{G}(u+1) \subset U \text{ but } \mathcal{G}(u) \not\subset U \}$$

$$H_n^{\Phi}(G) = \{ \text{---} \ \Phi \ \mathcal{G}(u) \subset U \ \text{---} \}$$

$\#H_n^{\Phi}$ finite.

+ Frattini property

$(\star)$ $G_S \subset_{op} G$ not open $\Leftrightarrow \forall n$ $\exists U \in H_n^{\Phi}(G)$ s.t.

$G_S \subset U$

Lemma: $G < d\, Gl_2(\mathbb{Z}_\ell)$ $\Phi(G(u)) = G(u+1)$, $u \geq 0$

$$\begin{tikzcd}
U \arrow[r, hook, "\text{cy}"] & \pi_1(S) \arrow[r, hook] & \mathcal{G}_U \arrow[r] & S \\
 & & \operatorname{Spec} k(S) \arrow[ul] \arrow[u] &
\end{tikzcd}$$

missed last part —