S. Dasgupta

Factorization of p-adic Rankin L-series

$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$

$\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$

Let $f \in S_{k_0}(N, \chi)$ $\quad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \bar{\mathbb{Q}}^*$

$$L(f \otimes f, s) = L(\mathrm{Sym}^2 f, s) L(\chi, s - k + 1)$$

this follows from:

- $\rho \otimes \rho = \mathrm{Sym}^2 \rho \oplus \det \rho = \mathrm{Sym}^2 \rho \oplus \chi \cdot \chi_{cyc}^{k_0 - 1}$
- Artin formalism for L-series

Thm

Assume that f is ordinary at p, $p \nmid N$

$$L_p(f \otimes f, s) = L_p(\mathrm{Sym}^2 f, s) L_p(\chi^{-1}, k_0 - s) \quad \text{for } s \in \mathbb{Z}_p$$

Rmks

(1) These p-adic L-fcns are functions on weight space

$$W = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{Q}_p^\times)$$

The formula above holds when k_0 even, s represents the weight $x \longmapsto \omega(x)^{k-1} \langle x \rangle^s$

2. LHS has no critical values, so defn of $L_p(f \otimes f, s)$ is subtle

3. In view of 2, thm cannot be proved by interpolating (*)

4. Thm was a conjecture of Citro. (thesis, student Hida)

How to define $L_p(f \otimes g, s)$ (esp if $wt(f) = wt(g)$, so no critical values)

Let F, G be Hida families s.t. $F_{k_0} = f_p$, $G_{k_0} = g_p$ (ordinary p-stabilisations)

Thm (Hida)

$\exists$ unique cont. function $L_p(F, G, K, \lambda, \sigma) : W^3 \to \mathbb{C}_p$

s.t. $\forall k, l, s \quad 2 \le l \le s \le k-1$,

$$L_p(F, G, k, l, s) = (*) \quad \frac{L(F_k \otimes G_l, s)}{(2\pi i)^{l+2s-1} \Omega_{F_k}}$$

(Shimura period)

note assymmetry for F & G!

$$L_p^f(f \otimes g, s) = L_p(F, G, k_0, k_0, s)$$

Our thm is: $L_p(F, F, K, K, \sigma) = L_p(\mathrm{Sym}^2 F, K, \sigma)\, L_p(\chi^{-1}, K\sigma^{-1})$

Specializing to $k(x) = x^{k_0}$, $\sigma(x) = \omega^{k_0}(x) \cdot \langle x \rangle^s$ gives the previous formulation

Motivation / Application of Thm

Notational assumption: $\gamma = 1$

$L_p(\mathrm{Sym}^2 f, s)$ has a "trivial zero" at $s = k_0 - 1$

Define $$\mathcal{L}_{an}(\mathrm{Sym}^2 f) = \frac{L_p'(\mathrm{Sym}^2 f, k_0 - 1)}{(*)\ L(\mathrm{Sym}^2 f, k_0 - 1)/\Omega_f}$$

(*) Euler like factors.

Building on his work with Glenn Stevens, R. Greenberg defined:

$\mathcal{L}_{dg}(f)$ = an ~~invariant~~ invariant of a class in $H^1(G_{\mathbb{Q}}, (\mathrm{Sym}^2 V_f)(1 - k_0))$

subject to local conditions

Conj $\mathcal{L}_{an}(\mathrm{Sym}^2 f) = \mathcal{L}_{anbg}(\mathrm{Sym}^2 f)$

Thm (Hida / Harron)

Let $\mathcal{L}_{GS}(\mathrm{Sym}^2 f) = -2 a_p'(k_0)/a_p(k_0)$

where $a_p(R) = U_p$ eigenvalue of F_k

Then $L_{alg}(Sym^2 f) = L_{GS}(Sym^2 f)$

Cor of fact. thm:

$L_{an}(Sym^2 f) = L_{GS}(Sym^2 f)$, so by H.H. we deduce Grenberg's Conj.

Remark Fact that fact thm $\Rightarrow$ cor was proved by Citro.

Pf of Cor:

$$L_p(f \otimes f, s) = \underbrace{L_p(Sym^2 f, s)}_{\text{trivial zero}} \underbrace{\zeta_p(k_0 - s)}_{\text{has a pole with res } 1 - \frac{1}{p}} \quad \text{at } s = k_0 - 1$$

$$L_p(f \otimes f, k_0 - 1) = L_p'(Sym^2 f, k_0 - 1) \cdot \left(1 - \frac{1}{p}\right)$$

Thm (Hida)

Let $P(F, \kappa, \lambda) = \dfrac{L_p(F, F, \kappa, \lambda, \kappa - 1)}{\left(1 - a_p(\lambda)/a_p(\kappa)\right)}$ still p-adic meromorphic.

then $(\kappa - \lambda) P(F, \kappa, \lambda)\Big|_{\kappa = \lambda} = -2\left(1 - \frac{1}{p}\right) L\left(Sym^2 F_k^{\#}, k - 1\right)$

$\Omega_{F_k^{\#}}$

Hida's thm $\Rightarrow$ $L_p(f\otimes f, k_0-1) = \mathcal{L}_{GS}(\mathrm{Sym}^2 f)(1-\frac{1}{p})\,\dfrac{L(\mathrm{Sym}^2 f, k_0-1)}{\Omega_f}$

<u>Pf of fact thm</u> (Gross)

Simplicity : $z=1$

WTS $\quad L_p(F,F,K,K,\sigma) = L_p(\mathrm{Sym}^2 F, K, \sigma)\cdot \zeta_p(K\sigma^{-1})$

Fix $\quad \kappa(x) = x^2\cdot\alpha(x),\ \sigma(x) = x\cdot\beta(x)$

Simplicity in the talk: $\beta = 1$, α = even char

$$\begin{tikzcd} & \mathbb{Q}(\zeta_{p^2}) = K \\ \mathbb{Q} \arrow[ur, dash, "\alpha"] & \end{tikzcd}$$

$$\alpha : (\mathbb{Z}/p^2\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^*$$

$$f = F\kappa \in S_2(Np^2, \alpha)$$

$X = X_1(Np^2) \qquad S = X\times X$

~~$CH(S)\otimes\overline{\mathbb{Q}}$~~

$$Z_f = W_{Np^2}\ T_f \qquad CH^1(S)\otimes\overline{\mathbb{Q}}$$

(Atkin Lehner)

idempotent corresponding to f in $\mathbb{T}$ (Hecke algebra)

$\Delta_{\mathcal{Z}^2} \in CH^2(S,1) = \{ \sum(Z_i; u_i)$, $Z_i \subset S$ curves, u_i functions on Z_i s.t $\sum \operatorname{div} u_i = 0 \} / \sim$

there is a pairing:

$$CH^2(S,1) \times CH^1(S) \xrightarrow{\cap} \mathbb{S}_m$$

$$(\sum(Z_i; u_i)) \cap Y = \prod u_i(Z_i \cap Y)$$

Define Beilinson Flach unit $b_f = \Delta_{\mathcal{Z}^2} \cap \mathcal{Z}_f \in \bar{\mathcal{Q}}^* \otimes \bar{\mathcal{Q}}$

Prop

$b_f \in \mathcal{U}_\alpha = (\mathcal{O}_K^* \otimes \bar{\mathcal{Q}})^{\bar{\alpha}'}$ (has dim 1)

Special Values Formulae:

① $\dfrac{L'(f \otimes f, 1)}{\Omega_f} = \log_{\mathcal{O}} b_f$

$\mathcal{O}_K^* \hookrightarrow \mathbb{C}^\times \xrightarrow{\log|\cdot|} \mathbb{R}$ ($\log_{\mathcal{O}}$)

extend by $\bar{\mathcal{Q}}$ linearity to $\mathcal{O}_K^\times \otimes \bar{\mathcal{Q}} \longrightarrow \mathbb{C}$

Pf: ② $\dfrac{L'(f \otimes f, 1)}{\Omega_f} = \operatorname{reg}_{\mathbb{C}}(\Delta_{\mathcal{Z}^2}(\omega_f \otimes \bar{\omega}_f - \bar{\omega}_f \otimes \omega_f))$

Beilinson – Shimura – Rankin

(b)
$$\begin{tikzcd} CH^2(S,1) \times CH^1(S) \arrow[r, "\cap"] \arrow[d, "cl_{dR}"] & \mathbb{C}^* \arrow[d, "\log|\ |"] \\ CH^2(S,1) \times H^{1,1}(S,\mathbb{R}) \arrow[r, "\mathrm{reg}_{\mathbb{C}}"'] & \mathbb{R} \end{tikzcd}$$

commutes

(c) $cl_{dR}(Z_f) = \omega_f \otimes \overline{\omega_f} - \overline{\omega_f} \otimes \omega_f$

(2) (Dirichlet)

$$L'(\alpha,0) = (*)\log_\infty(c_\alpha) \qquad c_\alpha = \prod_{\substack{a=1 \\ (a,p)=1}}^{p^2} \left(\frac{1-\zeta^a}{1-\zeta}\right) \otimes \alpha(a) \in U_\alpha$$

"circular unit"

(3) recall $\kappa(x) = x^2\alpha(x)$, $f = F_2, \alpha, \sigma(x) = x \leftrightarrow 1 \in \mathbb{Z}$

$$L_p(F,F,\kappa,\kappa,\sigma) = \log_p(b_f)$$

Pf: (a) Thm BDR (Bertolini, Darmon, -)

$$L_p(F,F,\kappa,\kappa,\sigma) = \mathrm{reg}_p(\Delta_\alpha - 2)(\omega_f \otimes \eta_f - \eta_f \otimes \omega_f)$$

(b)
$$\begin{tikzcd} CH^2(S,1) \times CH^1(S) \arrow[r, "\cap"] \arrow[d, equal] & K_p^* \arrow[dr, "\log_p"] & \\ CH^2(S,1) \times \mathrm{Fil}^1 H^2_{dR}(S/K_p) \arrow[rr, "\mathrm{reg}"] & & K_p \end{tikzcd}$$

synthomic regulator

(uses Bessers thm relating reg_p to $\mathrm{reg}_{ét}$.)

(c) $d_p(\tilde{z}_f) = \omega_f \otimes \eta_f - \eta_f \otimes \omega_f$

(4) $L_p(\tilde{\alpha}^{-1}, 1) = * \log_p(c_\alpha)$ Leopoldt.

$$L(f \otimes f, s) = L(\mathrm{Sym}^2 f, s)\, L(\alpha, s-1)$$

take derivatives at $s=1$, apply $\log_\infty$

$$\log_\infty b_f = \frac{L(\mathrm{Sym}^2 f, 1)}{\Omega_f} \cdot \log_\infty(c_\alpha)$$

$$\Rightarrow b_f = \frac{L(\mathrm{Sym}^2 f, 1)}{\Omega_f} \, c_\alpha \qquad c_\alpha \in \bar{\alpha}$$

take $\log_p$

$$\log_p(b_f) = \frac{L(\mathrm{Sym}^2 f, 1)}{\Omega_f} \log_p c_\alpha$$

$$L_p(F, F, K, K, \sigma) = L_p(\mathrm{Sym}^2 F, K, \sigma) \cdot \zeta_p(K\sigma^{-1})$$