Valentina Di Proietto On p-adic invariant cycles theorem (+ Chiarellotto - Colemann - Iovita)

I Classical setting /ℂ

Δ unit disk, $\mathcal{X}$ smooth complex variety, Kähler manifold

$\pi : \mathcal{X} \longrightarrow \Delta$ semistable degeneration

π proper flat holomorphic, $\mathcal{X}_t = \pi^{-1}(t)$ is smooth for $t \neq 0$

$\mathcal{X}_0 = \pi^{-1}(0)$ is NCD $\mathcal{X}_t$ } { $\mathcal{X}_0$

t 0 Δ

Set $r : \mathcal{X} \longrightarrow \mathcal{X}_0$ s.t. $r^* : H^m(\mathcal{X}_0) \xrightarrow{\sim} H^m(\mathcal{X})$

$i : \mathcal{X}_t \longrightarrow \mathcal{X}$ for $t \neq 0$

$H^m(\mathcal{X}) \xrightarrow{i^*} H^m(\mathcal{X}_t)$ is not a morphism of MHS

One can define $H^m_{lim} \cong H^m(\mathcal{X}_t)$ for $t \neq 0$ as vector spaces but has limit Hodge structure:

$H^m_{lim} \supset N$ (monodromy operator) nilpotent

$N = \log T$.

INVARIANT CYCLES THEOREM

$H^m \xrightarrow{i^*} H^m_{lim} \xrightarrow{N} H^m_{lim}$ is exact, morphs compatible with MHS

Clemens - Schmid exact sequence

$$H_{2n+2-m} \longrightarrow H^m \xrightarrow{i^*} H^m_{lim} \xrightarrow{N} H^m_{lim} \longrightarrow H_{2n-m}$$

For $m = 1$, we extract

$$0 \longrightarrow H^1 \longrightarrow H^1_{lim} \xrightarrow{N} H^1_{lim}$$

is there a p-adic analogue?

Let k be perfect, char p, W Witt vectors, $K = \mathrm{Frac}\, W$

$X \longrightarrow W$ proper semistable curve

X_k is NCD, all the components are smooth

X_K is smooth.

For $m = 1$: the substitute for H^1 is $H^1_{rig}(X_k)$

the substitute for H^1_{lim} is $H^1_{log\text{-}crys}(X_k^\times / W^\times)$ $X_k^\times$ is a log scheme, $W^\times$

Theorem 1: The following sequence is exact

$$0 \to H^1_{rig}(X_k) \to H^1_{log\text{-}crys}(X_k^\times/W^\times) \xrightarrow{N} H^1_{log\text{-}crys}(X_k^\times/W^\times)$$

Remark: Theorem 1 was proven by Chiarellotto in '99, $|k|$ finite (Nakkajima)
We give a new proof, based on technique of Coleman-Iovita.

Let E be convergent F-isocrystal. Coleman and Iovita define N_E ∋ $H^1_{log\text{-}crys}(X_k^\times/W^\times, E)$
We ~~have the sequence~~ have

$$0 \to H^1_{rig}(X_k, E) \to H^1_{log\text{-}crys}(X_k^\times/W^\times, E) \xrightarrow{N_E} H^1_{log\text{-}crys}(X_k^\times/W^\times, E) \qquad (*)$$

Let E be a unipotent F-isocrystal (iterated extension of the trivial one)

Theorem 2: $(*)$ is not necessarily exact

$\exists \; 0 \to G \to E \to \mathcal{O}_X \to 0$ $\mathcal{O}_X$ trivial F-isocrystal, G unipotent.

$E \rightsquigarrow x \in H^1_{rig}(X_{\bar{k}}, G)$. If $\exists y \in H^1_{log\text{-}crys}(X_k^\times/W^\times, G)$ s.t. $N_G(y) = x$

$$\Rightarrow \quad H^1_{rig}(X_k, E) \Big/ \ker N_E \cong \text{1-dim vector space}$$

We do explicit calculation in the case of a Tate curve and E $(0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0)$

"Proof" (of Theorem 1)

$$\begin{array}{ccccc} X_k^{rig} & & \xrightarrow{sp_2} & & P_k^{rig} \\ {\scriptstyle sp_X}\downarrow & & & & \\ X_k & & \hookrightarrow & & P_k \end{array} \qquad P_k^{rig} \to P$$

We have $H^1_{rig}(X_k) = H^1_{dR}(sp_1^{-1}(X_k))$

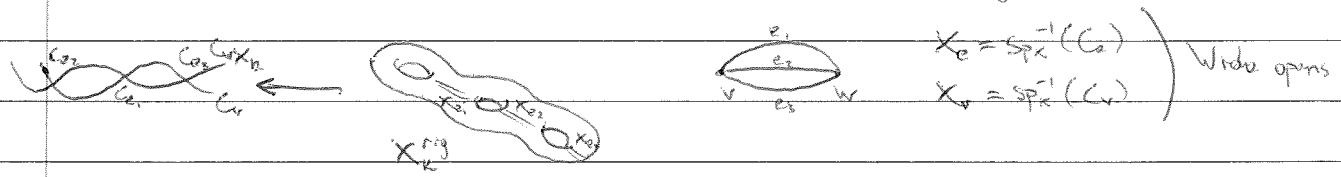
$H^1_{log\text{-}crys}(X_k^\times/W^\times) \cong H^1_{dR}(X_k^{rig})$

So this gives us: $0 \to H^1_{dR}(sp_P^{-1}(X_k)) \to H^1_{dR}(X_k^{rig}) \xrightarrow{N} H^1_{dR}(X_k^{rig})$

If X_k is NCD $\rightsquigarrow$ $Gr(X_k)$

C_v is an irred comp of X_k $\longleftrightarrow$ v is a vertex of $Gr(X_k)$

C_e is an intersection between C_v and C_w $\longleftrightarrow$ e an edge between v and w.



$\{X_v\}$ is an admissible covering of X_k^{rig}, so Mayer-Vietoris

$$\oplus_v H^0_{dR}(X_v) \xrightarrow{\beta} \oplus_e H^0_{dR}(X_e) \to H^1_{dR}(X_k^{rig}) \to \oplus_v H^1_{dR}(X_v) \to \oplus_e H^1_{dR}(X_e)$$

Colear $\beta = H^1_{Betti}(Gr(X_k))$. To define monodromy:

$$\begin{array}{ccccc} 0 \to & H^1_{Betti}(Gr(X_k)) & \to & H^1_{dR}(X_k^{rig}) & \\ & \curvearrowleft N & & \downarrow Res & \\ & & & \oplus H^0_{dR}(X_e) & \end{array}$$