David Loeffler — Euler systems for Rankin-Selberg convolutions

1) Iwasawa theory for one modular form

Let f be a new eigenform wt k, level N, $k \geq 2$, $p \nmid N$ prime.

Fix $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and assume $v_p(a_p(f)) > 0$

$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$

Then $\exists$ two p-adic L-functions for f: $L_{p,\alpha}(f)$ and $L_{p,\beta}(f)$ α,β roots of $X^2 - a_p X + p^{k-1}\varepsilon(p)$ living in $H(\mathbb{Z}_p^\times)$ = locally analytic distributions = $\mathcal{O}(\mathcal{W})$ $\mathcal{W}$ = weight space.

More intrinsically: V_f p-adic Galois rep^n

$\mathbb{D}_{\mathrm{cris}}(V_f^*)$ = 2-dim space $\circlearrowright \varphi$, φ eigenvalues α^{-1}, β^{-1}

Can define $L_p(f) \in H(\mathbb{Z}_p^\times) \otimes \mathbb{D}_{\mathrm{cris}}(V_f^*) \rightsquigarrow L_{p,\alpha}, L_{p,\beta}$ by projection.

We want to link these to cohomology of V_f^*.

Define $H^1_{\mathrm{Iw}}(\mathbb{Q}(\mu_{p^\infty}), V_f^*) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^1(\mathbb{Q}(\mu_{p^n}), T)$ $T \subset V_f^*$ lattice

Similarly local $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), V_f^*)$

Theorem (Perrin-Riou)

There is a map

$$\mathcal{L}_{V_f} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), V_f^*) \longrightarrow H(\Gamma) \otimes \mathbb{D}_{\mathrm{cris}}(V_f^*) \qquad \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times$$

called regulator, big logarithm, big dual exponential.

Theorem (Kato)

1) $\exists$ element $z^{(f)} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\mu_{p^\infty}), V_f^*)$ whose image under $\mathcal{L}_{V_f} \circ \log_p$ is $L_p(f)$

2) $z^{(f)} = z_1^{(f)}$ is the first step in a compatible system

$z_m^{(f)} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\mu_{mp^\infty}), V_f^*)$, $m \geq 1$ squarefree + coprime to pN

s.t. $\mathrm{cores}_m^{ml}(z_{ml}^{(f)}) = P_\ell(\sigma_\ell^{-1})\, z_m^{(f)}$ P_ℓ = Hecke poly at ℓ

(i.e. an Euler system) = Euler factor

2) Rankin-Selberg case

Now let f, g be two modular forms. We have Gal rep^n $V_f \otimes V_g$ $\longleftrightarrow$ Rankin-Selberg L-fn

Bertolini-Darmon-Rotger: class in $H^1(\mathbb{Q}, V_f^* \otimes V_g^*)$ related to Hida's p-adic RS L-fn.

Recall: S surface/k, $CH^2(S,1) = \left\{ \sum_i (Z_i, u_i) : Z_i \subset S \text{ curves}; u_i \in k(Z_i)^\times, \sum \text{div} = 0 \right\} / \sim$

Theorem: 1) Given integers $m \geq 1$, $N \geq 5$, $c > 1$ coprime to $6mN$

$\exists$ a class ${}_c\Xi_{m,N} \in CH^2(Y_1(N)^2/\mathbb{Q}(\mu_m), 1)$ reducing to BDR's Beilinson-Flach element for $m=1$

2) Let χ Dirichlet char mod m, $e_\chi \in \bar{\mathbb{Q}}[Gal(\mathbb{Q}(\mu_m)/\mathbb{Q})]$ idempotent. Then image of $e_\chi \cdot {}_c\Xi_{m,N}$ under Beilinson regulator calculates $L'(f,g,\chi,1)$.

3) Norm relations relating ${}_c\Xi_{m,N}$ with ${}_c\Xi_{m,pN}$ and ${}_c\Xi_{pm,N}$

Now fix a prime p

Huber: $\exists$ map $r_{\text{ét}} : CH^2(Y_1(N)^2/\mathbb{Q}(\mu_m), 1) \longrightarrow H^1(\mathbb{Q}(\mu_m), H^2_{\text{ét}}(Y_1(N)^2/\bar{\mathbb{Q}}, \mathbb{Z}_p(2)))$

For f,g wt 2 eigenforms of level N, define

$${}_cZ_m^{(f,g)} \in H^1(\mathbb{Q}(\mu_m), V_f^* \otimes V_g^*)$$

by projecting to $(\bar{f},\bar{g})$ component.

4) If $p \nmid mN$ and f,g ordinary at p, image of ${}_cZ_m^{(f,g)}$ under Bloch-Kato logarithm calculates p-adic Rankin-Selberg L-values of twists of f,g mod m

Dependence on c

$\exists\, \Xi_{m,N} \in CH^2(\cdots,1) \otimes \mathbb{Q}$ s.t. ${}_c\Xi_{m,N} = (c^2 - \sigma_c)\cdot \Xi_{m,N}$, $\forall c$ coprime to $6mN$

Funny thing

The elements $\left({}_cZ_{mp^k}^{(f,g)}\right)_{k\geq 1}$ do not define an Iwasawa coh class

$$\text{cores}_{mp^k}^{mp^{k+1}}\left({}_cZ_{mp^{k+1}}\right) = (\alpha_f \alpha_g)\cdot {}_cZ_{mp^k} \qquad (p \mid N) \quad (\alpha_f, \alpha_g \ U_p \text{ eigenval})$$

If $\alpha_f\alpha_g$ is a unit

$$\left((\alpha_f\alpha_g)^{-k}\, {}_cZ_{mp^k}^{(f,g)}\right)_{k\geq 1} \in H^1_{Iw}(\mathbb{Q}(\mu_{mp^\infty}), V_f^* \otimes V_g^*)$$

If $v_p(\alpha_f\alpha_g) < 1$ and f not a twist of g, get an elt of $H(\Gamma) \otimes_{\Lambda(\Gamma)} H^1_{Iw}$

5) If f,g ordinary, can put $Z_m^{(f,g)}$ in analytic families as f,g vary in Hida families

3) Conjectures on higher rank Euler systems

Philosophy: L-functions $\longleftrightarrow$ determinants of global cohomology

$H^1_{Iw}(\mathbb{Q}(\mu_{p^\infty}), V_f^* \otimes V_g^*)$ rank 2 $\rightsquigarrow$ expect elts in Λ^2 living in an Euler system w. L-factor as comput. relation.

Observation: A linear functional on H^1_{Iw} $\rightsquigarrow$ map $\Lambda^2 H^1_{Iw} \to H^1_{Iw}$

$\mathcal{L}$ gives such a functional after $\otimes\, \mathcal{H}(\Gamma)$

$\rightsquigarrow \Lambda^2 H^1_{Iw} \to H^1_{Iw} \otimes_\Lambda \mathcal{H}(\Gamma) \otimes \mathbb{D}_{cris}$

Using ideas of Otsuki: add in a $\mathbb{Q}(\mu_m)$

Fact: The properties of $z_m^{(f,g)}$ are consistent with being image of a rank 2 Euler system