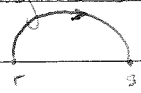
Robert Pollack — Families of overconvergent modular symbols

(with R. Harron, E. Dummitt, M. Hablicsek, L. Jean, D Rose)

1. Modular symbols

$\Delta_0 = Div^0(\mathbb{P}^1(\mathbb{Q}))$ degree 0 on $\mathbb{P}^1(\mathbb{Q})$. Γ = congruence subgp $\subset SL_2(\mathbb{Z})$

$\{s\} - \{r\}$ 

$V = \mathbb{Z}[\Gamma]$-module

$$MS_\Gamma(V) = \left\{ \varphi : \Delta_0 \to V \;\middle|\; \varphi \text{ additive}, \ D \in \Delta_0, \ \varphi(\gamma D)|\gamma = \varphi(D) \right\}$$

"V-valued modular symbols of level Γ"

Example: $V = Sym^k(\mathbb{C}^2)$ ← hom poly in X, Y deg k

$$S_{k+2}(\Gamma) \hookrightarrow MS_\Gamma(Sym^k(\mathbb{C}^2))$$

$$f \longmapsto \left(\{s\} - \{r\} \longmapsto \int_r^s f(z)(zX+Y)^k dz \right)$$

2. Overconvergent modular symbols

Define $\mathbb{A} = \mathbb{Q}_p\langle\langle z \rangle\rangle = \left\{ \sum_{n \geq 0} a_n z^n \;\middle|\; a_n \in \mathbb{Q}_p, \ a_n \to 0 \right\}$

$$\mathbb{D} = Hom_{cont}(\mathbb{A}, \mathbb{Q}_p)$$

Now let $k \in \mathbb{Z}$, $\gamma \in \Gamma_0(p)$, $f \in \mathbb{A}$ then $(\gamma \cdot_k f)(z) = (a+cz)^k f\left(\frac{b+dz}{a+cz}\right)$

$\mu \in \mathbb{D}$ then $(\mu|_k \gamma)(f) = \mu(\gamma \cdot_k f)$

This gives rise to $\mathbb{A}_k$, $\mathbb{D}_k$;

$\Gamma_0 \leq \Gamma_0(p)$; $MS_{\Gamma_0}(\mathbb{D}_k)$ ems of wt k

Specialization: $\mathbb{D}_k \longrightarrow Sym^k(\mathbb{Q}_p^2)$ $\quad$ $\Gamma_0(p)$ equivariant.

$$\mu \longmapsto \int (Y + zX)^k d\mu(z)$$

Comparison Theorem: (Stevens)

1) $MS_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \xrightarrow{\sim} MS_{\Gamma_0}(Sym^k(\mathbb{Q}_p^2))^{(<k+1)}$ $\quad$ (slope $< k+1$)

$\mathbb{D}_f$ $\qquad$ φ_f

2) $\mathbb{D}_f(\{e\} - \{\infty\}) =$ p-adic L-fn of f.

3. Making things explicit

$\mathbb{A} \geq$ (dense) $span\{z^j\}_{j=1}^\infty \implies \mu \in \mathbb{D}$ is uniquely det by $\{\mu(z^j)\}_{j=0}^\infty$ (sequence of moments)

$\mathbb{D}_k \longleftrightarrow$ bounded sequences in $\mathbb{Q}_p$.

unit ball

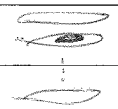
naive approximations: $\mu \in \mathbb{D}^\circ \subseteq \mathbb{D}$ look at first M moments mod p^M

NOT preserved by $\Gamma_0(p)$.

Alternatively: $\mathrm{Fil}^M(\mathbb{D}^\circ) = \{\mu \in \mathbb{D}^\circ \mid \mu(z^j) \in p^{M-j}\mathbb{Z}_p\} \circlearrowleft \Gamma_0$-stable!

$$\Gamma_0 \circlearrowright \underset{\substack{\uparrow \\ \text{finite}}}{\mathbb{D}^\circ / \mathrm{Fil}^M(\mathbb{D}^\circ)} \qquad (\text{consider the } j^{th} \text{ moment mod } p^{M-j})$$

4. Families of overconv. modular symbols

Weight space $W = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p) \longleftrightarrow$  $p-1$ open unit disks about 0

$\uparrow$ $\mathbb{Z}$ $\qquad (x \mapsto x^k)$ $\updownarrow$ k $\qquad \gamma$ top gen of $1+p\mathbb{Z}_p$

D = closed disk of radius $1/p$ in one of above disks.

R = conv p.s $/\mathbb{Q}_p$ on D

$$\mathbb{Z}_p^\times \simeq \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p) \longmapsto \chi(\gamma) - 1$$
$$\downarrow \chi$$
$$\mathbb{C}_p$$

$\mathbb{D}(R) = \mathrm{Hom}_{cont}(\mathbb{A}, R)$ "R-valued distributions"

Specialisation: $k \in \mathbb{Z} \cap D \rightsquigarrow R \to \mathbb{Q}_p \rightsquigarrow \mathbb{D}(R) \xrightarrow{\mathrm{sp}_k} \mathbb{D}_k$

$w \longmapsto \gamma^k - 1$ $\qquad \circlearrowleft \Gamma_0(p) \qquad \circlearrowleft \Gamma_0(p)$

Fact: $\mathbb{D}(R)$ has a natural R-linear $\Gamma_0(p)$-action making specialisation $\Gamma_0(p)$-equivar.

$\rightsquigarrow MS_{\Gamma_0}(\mathbb{D}(R)) \longrightarrow MS_{\Gamma_0}(\mathbb{D}_k)$

$\Phi \qquad \text{———} \qquad \Phi_k$

5. Making this explicit.

R° = unit ball of R, $\mathbb{D}(R^\circ) = R^\circ$-valued dist.

$$\mathrm{Fil}^M(\mathbb{D}(R^\circ)) = \{\mu \in \mathbb{D}(R^\circ) \mid \mu(z^j) \in p^{\lceil (M-j)\frac{p-2}{p-1} \rceil} R_0\} \circlearrowleft \Gamma_0\text{-stable}$$

$\mathbb{D}(R^\circ) / \mathrm{Fil}^M(\mathbb{D}(R^\circ)) + w^M \mathbb{D}(R^\circ)$ finite stable under Γ_0

6. Examples

① $\Gamma_0(11)$ $p=11$ $(k \equiv 0 \pmod{p-1})$ $S_2(\Gamma_0(11))$ 1-dim $\langle f \rangle \leftarrow \chi_0(11)$ Hida family rk 1

$\to$ Produce a "random" $\Phi \in MS_{\Gamma_0(11)}(\mathcal{F}_M)$

$\to$ Compute $\Phi^{ord} = \Phi \mid U_p^{M'}$ $M' \gg 0$ (proj to ordinary space)

Observations:

1) Φ^{ord} Hecke-eigensymbol

2) Hecke-eigenvalues extend to $\mathbb{Z}_p[[w]]$

Data:

$M = 6 \pmod{11^6}$

$a_2(w) = -2 - 7236w - 6775w^2 + \cdots + 2w^5$

specialise $k=0 \longrightarrow w = \gamma^0 - 1 = 0 \qquad a_2(X_0(11)) = -2$

specialise $k=10 \longrightarrow a_2(\gamma^0 - 1) = -24 \quad (11^6)$

Compute $-2a_{11}'(0) = L\text{-invariant} \quad (11^6)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad \log q / \operatorname{ord} q$

② $N = 11,\ p = 3$

$S_2(\Gamma_0(33))^{ord}$ 2-dim Hida family rk 2

$\longrightarrow$ produce 2 random symbols Φ_1, Φ_2

$\downarrow$

$\Phi_1^{ord}, \Phi_2^{ord}$ {basis}

$\longrightarrow$ compute char poly of U_3:

$y^2 + (8w + 503)y + (50w^2 + 502)$

$\text{disc} = 225 + 29w + 26w^2 + \cdots = (w - \alpha) \times \text{Unit}$