John Voight — Semi-arithmetic points

A Belyi map is a map $X \longrightarrow \mathbb{P}^1$ of curves/$\mathbb{C}$ that is ramified over at most 3 points. Belyi proved that $X/\mathbb{C}$ has a Belyi map $\Leftrightarrow$ X can be defined over a number field.

Example: $j : X_0(N) \longrightarrow \mathbb{P}^1$ is ramified over $0, 1728, \infty$ for $N \geq 1$,

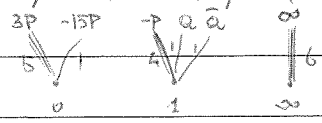
as is $j : X(N) \longrightarrow \mathbb{P}^1$, a Galois map with group $\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$

Question: Let $\varphi : X \longrightarrow \mathbb{P}^1$ be a Galois Belyi map defined over a # field F. Let $H \leq \mathrm{Jac}(X)$ be the subgroup of degree zero divisors over F supported over ramification points of φ. How big is H?

Ex: For $j : X_0(N) \longrightarrow \mathbb{P}^1$ Manin-Drinfeld says any deg 0 divisor supported at cusps $(j^{-1}(\infty))$ is torsion in $J(N)$. The divisors supported in $j^{-1}(0)$ and $j^{-1}(1728)$ are CM divisors ($\mathbb{Z}[\omega], \mathbb{Z}[i]$) and generate a subgroup of rank ≤ 1 on each new isogeny factor A_f.

Ex: $C : y^2 + (\sqrt{6}+1)xy + (\sqrt{6}+1)y = x^3 + x^2 + (18\sqrt{6}+43)x + (130\sqrt{6}+323)$

φ deg 6 $\to \mathbb{P}^1$



$$\varphi(x,y) = \frac{(342\sqrt{6} - 847)x^3 + \cdots + (3036\sqrt{6} + 13674)}{50\,000}$$

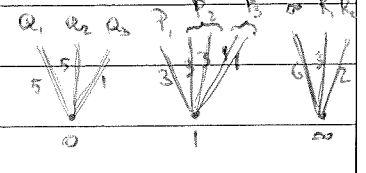
C has conductor $\sqrt{6}\,\mathfrak{p}_5^2\,\mathfrak{p}_3' = \mathfrak{N}$ with $\mathrm{Norm}(\mathfrak{N}) = 750$

$C(F) = \mathbb{Z}P$, $P = (20\sqrt{6} + 51, 174\sqrt{6} + 424)$, $C(F(i)) \cong \mathbb{Z} \oplus \mathbb{Z}$

P, Q linearly independent, $Q + \bar{Q} = 4P$.

Ex: E/K $\quad \mathbb{Q}(j(E)) = K = F(\beta)$ $\quad \beta^2 - \frac{1+\sqrt{5}}{2}\beta - (\sqrt{3}-1) = 0$

φ deg 11 $\to \mathbb{P}^1$ $\quad F = \mathbb{Q}(\sqrt{5}, \sqrt{3})$



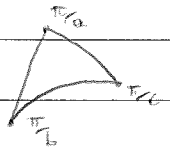
$\langle P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2 \rangle \leq E(K)$ has rank 4

$(\mathrm{div}\,\varphi - 1)$ $\quad 3P_1 + 3P_2 + P_3 = 3R_1 + 2R_2 = 5Q_1 + 5Q_2 + Q_3$ $\quad (\mathrm{div}\,\varphi)$

$(\mathrm{div}\,d\varphi)$ $\quad 2P_1 + 2P_2 + 4Q_1 + 4Q_2 = 4R_1 + 3R_2$

Any point can occur like this, so we seek to restrict the φ's.

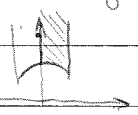
Let $a,b,c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ satisfy $a \leq b \leq c$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 1$

Let T be  in $\mathcal{H}$

The reflections in sides of T form a discrete group; the subgroup of orientation-preserving isometries

$$\bar{\Delta} = \bar{\Delta}(a,b,c) = \langle \gamma_a, \gamma_b, \gamma_c \mid \gamma_a^a = \gamma_b^b = \gamma_c^c = \gamma_a\gamma_b\gamma_c = 1 \rangle$$

$\bar{\Delta}\backslash\mathcal{H}$ is a (punctured) Riemann-surface of genus 0.



Ex: $\bar{\Delta}(2,3,\infty) \cong PSL_2(\mathbb{Z})$

Let $\Delta = \Delta(a,b,c) = \langle \gamma_a, \gamma_b, \gamma_c \mid \gamma_a^a = \gamma_b^b = \gamma_c^c = \gamma_a\gamma_b\gamma_c = -1 \rangle$, then $\Delta \hookrightarrow SL_2(\mathbb{R})$ and $\bar{\Delta} = \Delta/\{\pm 1\}$. Let $B = \mathbb{Q}\langle\Delta\rangle \subseteq M_2(\mathbb{R})$ the $\mathbb{Q}$-algebra generated by Δ.

Theorem (Takeuchi) B is a quaternion algebra over $F = \mathbb{Q}(\operatorname{trd}\Delta) = \mathbb{Q}(\lambda_{2a}, \lambda_{2b}, \lambda_{2c})$ where $\lambda_s = 2\cos 2\pi/s$, $\lambda_\infty = 2$. $\mathcal{O} = \mathbb{Z}_F\langle\Delta\rangle$ is an order in B, $\bar{\Delta} \hookrightarrow \mathcal{O}^\times/\{\pm 1\}$

Theorem (Takeuchi) Δ is arithmetic $\iff$ (a,b,c) is in a list of 85 triples.

Let $\mathfrak{N}$ be an ideal of F unramified in B. Let $\mathcal{O}(\mathfrak{N})^\times_1 = \{\gamma \in \mathcal{O}^\times : \gamma \equiv 1 \pmod{\mathfrak{N}}\}$ and $\Delta(\mathfrak{N}) = \Delta \cap \mathcal{O}(\mathfrak{N})^\times_1$, $X(\mathfrak{N}) = \bar{\Delta}(\mathfrak{N})\backslash\mathcal{H}$. $\Delta(\mathfrak{N})$ is a congruence subgp. of Δ

Conjecture: (Voight) Suppose $\Phi : X(\mathfrak{N}) \to E$ is a nonarithmetic uniformizer of an elliptic curve E over F. Let $H = \mathbb{Z}^r$ the gp of degree 0 divisors on $X(\mathfrak{N})$ defined over F supported over ramification points of $j : X(\mathfrak{N}) \to X(1)$.
Then $\Phi(H)$ has rank $\min(E(F), r-3)$.