6-functor formalism, L5.

Recall our setup:

$D : \mathrm{Corr}(C, E) \longrightarrow \mathrm{Cat}_\infty$ lax symmetric monoidal

$f_! = D\left(\begin{smallmatrix} & X & \\ {\scriptstyle id}\swarrow & & \searrow {\scriptstyle f} \\ X & & Y \end{smallmatrix}\right)$ $g^* = D\left(\begin{smallmatrix} & X & \\ {\scriptstyle g}\swarrow & & \searrow {\scriptstyle id} \\ Y & & X \end{smallmatrix}\right)$, 1_X = object specified by $D\left(\begin{smallmatrix} & X & \\ \swarrow & & \searrow {\scriptstyle id} \\ * & & X \end{smallmatrix}\right)$.

We defined a new category (Liu-Zheng) $LZ = LZ^C_D$:

- objects: objects of C
- morphisms: $\mathrm{Hom}_{LZ}(X, Y) = D(X \times Y)$

composition:

$$\begin{tikzcd} & X_1 \times X_2 \times X_3 \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\ X_1 \times X_2 & X_2 \times X_3 & X_1 \times X_3 \end{tikzcd}$$

$F \in D(X_1 \times X_2)$, $G \in D(X_2 \times X_3)$

$$G \circ F = p_{1,3!}(p_{1,2}^* F \otimes p_{2,3}^* G)$$

identity: $\Delta_!(1_X) \in D(X \times X)$

This is naturally a 2-category (2-morphisms induced by 1-morphisms of $D(-$

We also defined Fourier-Mukai functor

$FM : LZ^D_C \longrightarrow \mathrm{Cat}_\infty$

$$\begin{tikzcd} X_1 \arrow[r, mapsto] \arrow[d, "F"'] & D(X_1) \arrow[d, "FM(F) = p_{2!}(p_1^*(-) \otimes F)"] \\ X_2 \arrow[r, mapsto] & D(X_2) \end{tikzcd}$$

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_2 \\ X_1 & \end{tikzcd} \qquad F \in D(X_1 \times X_2)$$

Fact: this is a functor of 2-categories.

Aim for today: Want to know when $f^*, f^!$ are isomorphic.

Assume E = all morphisms

(actually just need $f : X \to Y \in E \Rightarrow \Delta_f : X \to X \times_Y X \in E$

e.g. can have property "any 2 of $f, g, f \circ g \in E \Rightarrow$ all $\in E$")

x $f: X \to Y$.

By considering the slice $C_{/Y}$ (defined in last lecture) we may assume Y is the terminal object $\rightsquigarrow X \times Y = Y \times X = X$

Def: $A \in D(X)$ is <u>f-smooth</u> if $A \in \mathrm{Hom}_{LZ}(X, Y)$ is a left-adjoint in LZ

" <u>f-proper</u> if $A \in \mathrm{Hom}_{LZ}(Y, X)$ "

Rk (on names) In LCHS proper $\Rightarrow f_! = f_*$ (lecture 1)

smooth $\Rightarrow f^! = f^*$ (?) (but there are nonstandard terms)

We will see more about this next week.

e.g. · X LCHS that's a retract of $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$. Then $\mathbb{Z}$ f-smooth for $X \xrightarrow{f} *$

· LCHS: f-proper $\Leftrightarrow$ A locally isomorphic to constant sheaf on a perfect complex & supported on compact subset of X.

· no nice characterisation of f-smooth objects.

Rk (on duality) $D^{op}: \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty$ is also a 3-functor formalism and f-smooth $\leftrightarrow$ f-proper

but also f_*, $f^!$ are all right-adjoints

so results below appear asymmetric between f-smooth, f-proper.

<u>What does f-smooth mean?</u> A f-smooth iff

$$\begin{tikzcd} X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\exists$ unit $\alpha: \underbrace{\Delta_!(1_X)}_{\mathrm{id} \in \mathrm{Hom}_{LZ}(X,X)} \longrightarrow \underbrace{p_1^* A \otimes p_2^* B}_{B \circ A}$

$\exists$ counit $\beta: \underbrace{f_!(A \otimes B)}_{A \circ B} \longrightarrow \underbrace{1_Y}_{\mathrm{id} \in \mathrm{Hom}_{LZ}(Y,Y)}$

s.t. $A \xrightarrow{\cong} A \circ \mathrm{id} \xrightarrow{\alpha} A \circ (B \circ A) \xrightarrow{\cong} (A \circ B) \circ A \xrightarrow{\beta} \mathrm{id} \circ A \xrightarrow{\cong} A$

with $A \circ \mathrm{id} = p_{1!}(p_2^* A \otimes \Delta_! 1_X)$, $A \circ (B \circ A) = p_{1!}(p_2^! A \otimes p_1^* A \otimes p_2^* B)$, $(A \circ B) \circ A = f^* f_!(A \otimes B) \otimes A$, $\mathrm{id} \circ A = f^* 1_Y \otimes A$

is the identity & likewise for the analogous expression for B.

What are the $\cong$ maps?

$$\begin{tikzcd} X \arrow[dr, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & X\times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

- $A = p_{1!}\Delta_! A = p_{1!}\Delta_!(A \otimes 1_X) = p_{1!}\Delta_!(\Delta^* p_2^* A \otimes 1_X) \overset{\text{proj formula}}{\cong} p_{1!}(p_2^* A \otimes \Delta_! 1_X)$.
- $p_{1!}(p_2^* A \otimes p_1^* A \otimes p_2^* B) \cong p_{1!}(p_1^* A \otimes p_2^*(A \otimes B)) \overset{\text{proj formula}}{\cong} p_{1!}p_2^*(A\otimes B) \otimes A \overset{\text{base change}}{\cong} f^* f_!(A \otimes B) \otimes A$.
- $f^* 1_Y \otimes A = 1_X \otimes A = A$.

Observations: Let A f-smooth, $A \dashv B$ $(B \in D(X))$.

- B f-smooth, $B \dashv A$. (by symmetry)
- $\exists$ natural isomorphism of functors $B \otimes f^*(-) \cong \underline{hom}(A, f^!(-)) : D(Y) \to D$

Proof: FM functor of 2-category preserves adjunctions.

$A \dashv B \xrightarrow{FM} f_!(A \otimes -) \dashv B \otimes f^*(-)$.

But also $f_!(A \otimes -) \dashv \underline{hom}(A, f^!(-))$ $\square$.

- in particular, $B = B \otimes f^* 1_Y \cong \underline{hom}(A, f^! 1_Y)$

can also view this as arising from $\beta : f_!(A \otimes B) \to 1_Y$

Def: Verdier dual $\mathbb{D}_f A = \underline{hom}(A, f^! 1_Y)$.

- $A \mapsto \mathbb{D}_f \mathbb{D}_f A$ isomorphism
- $\mathbb{D}_f$ commutes with base change (base change functor of 2-categories)

So in general, Verdier dual $\mathbb{D}_f A$ is our candidate for right-adjoint B.

Prop: $A \in D(X)$. A f-smooth iff the natural map

$$p_1^* A \otimes p_2^* \mathbb{D}_f A \xrightarrow{\cong} \underline{hom}(p_2^* A, p_1^! A).$$

This is the map adjoint to

$$p_1^* A \otimes p_2^* \mathbb{D}_f A \otimes p_2^* A \longrightarrow p_1^* A \otimes p_2^* f^! 1_Y \xrightarrow{\text{base change}} p_1^* A \otimes p_1^! f^* 1_Y \xrightarrow{(*)} p_1^! A.$$

counit of $- \otimes A \dashv \underline{hom}(A, -)$

$f^* f_! \cong p_{1!} p_2^*$

$p_1^! f^* \longrightarrow p_1^! f^* f_! f^! \cong p_1^! p_{1!} p_2^* f^! \longrightarrow p_2^* f^!$

adjoint to $p_{1!}(p_1^* A \otimes p_1^! 1_X) \xrightarrow{\text{proj formula}} A$

Pf: "only if" Given pullback square, any $B' \in D(Y')$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$$\underbrace{g'^* B}_{\mathbb{D}_f A} \otimes f'^* B' \xrightarrow{\cong} \underline{\hom}(g'^* A, f'^! B')$$

(2nd observation under pullback)

Set $g = f : X \to Y$ $\quad B = A$.

"if": counit β is natural:

$\beta : f_!(A \otimes \mathbb{D}_f A) \to 1_Y$ adjoint to $A \otimes \mathbb{D}_f A \to f^! 1_Y$ adjoint to $\mathbb{D}_f A \xrightarrow{id} \underline{\hom}(A, f^! 1_Y)$

unit: Define $\alpha : \underbrace{\Delta_! 1_X \to \underline{\hom}(p_2^* A, p_1^! A)}_{\text{what is this map?}} \xrightarrow{\cong} p_1^* A \otimes p_2^* \mathbb{D}_f A$.

take adjoints: $\Delta_! 1_X \otimes p_2^* A \to p_1^! A$.

we know $p_{1!}(\Delta_! 1_X \otimes p_2^* A) \xrightarrow{\cong} A$.

check $\underbrace{A \to \cdots \to A}_{\text{refer to big scary diagram}}$ & $\underbrace{B \to \cdots \to B}_{\text{similar.}}$ are id

□

R: We needed a weaker condition:

Wanted $\Delta_! 1_X \to \underline{\hom}(p_2^* A, p_1^! A)$ to induce $\Delta_! 1_X \to \underline{\hom}(p_2^* A, p_1^! A)$

so sufficed to have $\mathrm{Hom}_{D(X)}(1_X, \Delta^!(p_1^* A \otimes p_2^* \mathbb{D}_f A)) \xrightarrow{\cong} \mathrm{Hom}_{D(X)}(1_X, \Delta^! \underline{\hom}(p_2^* A, p_1^! A))$

"isomorphism on global sections"

Why is it called that? In sheaves: global sections are $\mathrm{Hom}(\underset{\text{monoidal unit.}}{\mathbb{Z}}, \mathcal{F})$

Reading analogously, we obtain

Df: $A \in D(X)$ f-proper, $A \dashv B$ in LZ.

- B f-proper, $B \dashv A$.

- $f_!(B \otimes -) \cong f_* \underline{\hom}(A,-) : D(X) \to D(Y)$ natural iso of functors
- $\alpha : p_1^* A \otimes p_2^* B \to \Delta_! 1_X$ induces isomorphisms

$$B \cong p_{2*} \underline{\hom}(p_1^* A, \Delta_! 1_X)), \quad A \cong \underline{\hom}(p_2^* B, \Delta_! 1_X)$$

- Above properties compatible with base change.

Can derive the 3rd from 2nd; apply base change

$$\begin{tikzcd} X \times X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

& push $\Delta_! 1_X$

$$B \cong p_{2!}(p_1^* B \otimes \Delta_! 1_X) \overset{\text{2nd point}}{\cong} p_{2*} \underline{\hom}(p_1^* A, \Delta_! 1_X)$$

Rk: If $\Delta_! 1_X = \Delta_* 1_X$ then

$$B \cong p_{2*} \underline{\hom}(p_1^* A, \Delta_* 1_X) \underset{\text{use that } *, \otimes \text{ compatible.}}{=} p_{2*} \Delta_* \underline{\hom}(\Delta^* p_1^* A, 1_X) = \underline{\hom}(A, 1_X)$$

Additionally, the same approach gives

Prop: $A \in D(X)$. A f-proper iff the natural map

$$f_!(A \otimes \underbrace{p_{2*} \underline{\hom}(p_1^* A, \Delta_! 1_X)}_{\text{"}B\text{"}}) \to f_* \underline{\hom}(A, A)$$

is an $\cong$ on global sections.

As before, this is a special case of a more general natural transformation

$$f_!(- \otimes p_{2*} \underline{\hom}(p_1^* A, \Delta_! 1_X)) \to f_* \underline{\hom}(A, -) \quad (\dagger)$$

(always defined, $\cong$ if f-proper)

⌈ how to derive this?

adjoint to $f^* f_!(\ldots) \overset{\text{base change}}{\cong} p_{1!}(p_2^*(-) \otimes \underbrace{p_2^* p_{2*}} \underline{\hom}(p_1^* A, \Delta_! 1_X))) \xrightarrow{p_2^* \dashv p_{2*}} p_{1!}(\ldots) \to \underline{\hom}(A, -)$

adjoint to $p_2^*(-) \otimes \underline{\hom}(p_1^* A, \Delta_! 1_X) \to \underline{\hom}(p_1^* A, p_1^!(-)) \underset{\substack{\text{proj formula} \\ \text{\& some adjunctions.}}}{\overset{\cong}{\leftarrow}} p_1^! \underline{\hom}(A, -)$

adjoint to $p_2^*(-) \otimes \underline{\hom}(\quad) \otimes p_1^* A \to p_2^*(-) \otimes \Delta_! 1_X \to p_1^!(-)$

$p_1^* A \otimes - \dashv \underline{\hom}(p_1^* A, -)$

adjoint to

$p_{1!}(p_2^*(-) \otimes \Delta_! 1_X) \xrightarrow{\cong} (-)$.

⌋

Def: $f : X \to Y$ in C cohomologically proper if it's n-truncated, some n ($\forall g : X \to Z$ space of lifts $f \overset{Y}{\underset{X \xrightarrow{g} Z}{}}$ has $\pi_i = 0$ $\forall i > n$), $1_X \in D(X)$ f-proper,

& $\Delta : X \to X \times X$ cohomologically proper

Rk: f n-truncated $\iff$ Δ $(n-1)$-truncated (Lurie p401) so this is well-defined if taken inductively.

Prop: $f: X \to Y$ s.t. Δ cohomologically proper.
- $\exists$ natural transformation of functors $f_! \to f_*$.

Proof: Put $A = 1_X$ in (†)

$$f_!(- \otimes p_{2*} \underline{\mathrm{hom}}(\underbrace{p_1^* 1_X, \Delta_! 1_X}_{\mathrm{hom}(1_X, 1_X) = 1_X})) \longrightarrow f_* \underline{\mathrm{hom}}(1_X, -)$$

$$\Delta_! 1_X \;\; = \;\; \Delta_* 1_X \qquad\qquad f_* \underline{\mathrm{hom}}(1_X, -) = f_*(-) \quad \because 1_X \text{ monoidal unit.}$$

(Rk after observations)

□

- f cohomologically proper $\iff$ $f_!(1_X) \to f_*(1_X)$ is an $\cong$

$\iff$ it's an $\cong$ on global sections

Lemma:

Def: $f: X \to Y$ cohomologically étale if it's n-truncated, some n; $1_X \in D(X)$

f-smooth, Δ cohomologically étale

Prop $f: X \to Y$, Δ cohomologically étale
- $\exists$ natural transformation $f^! \to f^*$
- f cohomologically étale $\iff$ $f^! 1_Y \to f^* 1_Y (= 1_X)$ $\cong$ $\iff$ $\cong$ on global sections

$$
\begin{tikzcd}[column sep=small]
p_{1!}(p_2^*A \otimes \Delta_! 1_X) \arrow[r, leftrightarrow, "\text{proj formula}", "\cong"'] \arrow[d, "\text{by def this } \iota"] \arrow[dd, bend right, "\alpha"'] & p_{1!}\Delta_!(1_X \otimes \Delta^* p_2^* A) & p_{1!}\Delta_! A \arrow[l, "\cong"'] \arrow[r, "\text{unit of } p_{1!} \dashv p_1^!"] & p_{1!}\boxed{p_1^! p_{1!}}\Delta_! A \arrow[r, "\text{counit of } p_{1!}\dashv p_1^!"] \arrow[r, phantom, bend left, "\circlearrowright"] & p_{1!}\Delta_! A \arrow[r, "\cong"] & A \arrow[lll, bend right, "\cong"'] \\
p_{1!}(p_2^*A \otimes \underline{\mathrm{hom}}(p_2^*A, p_1^!A)) \arrow[rrrr, "\text{counit of } -\otimes p_2^*A \dashv \underline{\mathrm{hom}}(p_2^*A,-)"] & & & & p_{1!}p_1^!A \arrow[u, leftrightarrow, "\cong"] \arrow[r, "\text{counit of } p_{1!} \dashv p_1^!"] \arrow[r, phantom, "\circlearrowright"'] & A \\
p_{1!}(p_1^*A \otimes \boxed{p_2^* D_f A \otimes p_2^* A}) \arrow[u, "(*)", "\cong \text{ (assumed)}"'] \arrow[r, "\cong"] \arrow[r, phantom, bend left, "\circlearrowright \text{ by def of } (*)"] & p_{1!}(p_1^*A \otimes p_2^*(A \otimes D_f A)) \arrow[r, "\text{counit of } -\otimes A \dashv \underline{\mathrm{hom}}(A,-)"] \arrow[d, "\text{proj formula}"] \arrow[dr, phantom, "\circlearrowright"] & p_{1!}(p_1^*A \otimes p_2^* \overbrace{f^! 1_Y}^{1_X}) \arrow[r, "\text{base change (unit?)}"] \arrow[d, "\text{proj formula}"] & p_{1!}(p_1^*A \otimes p_1^! X) \arrow[u, "(*)"] \arrow[r, phantom, "\circlearrowright \text{ by def of } (*)"] & & \\
& p_{1!}p_2^*(A \otimes D_f A) \otimes A \arrow[r, "\text{base change}"] & f^* f_!(A \otimes D_f A) \otimes A \arrow[r, "\beta", "\text{counit of } -\otimes A \dashv \underline{\mathrm{hom}}(A,-)"'] & f^* 1_Y \otimes A \arrow[uurr, bend right, "\cong"'] & &
\end{tikzcd}
$$

$$p_{1!}p_2^* f^! 1_Y \otimes A \xrightarrow[f_! \dashv f^!]{\text{base change}} f^* 1_Y \otimes A \quad \text{proj formula}$$

$$
\begin{tikzcd}
p_{1!}(p_1^*A \otimes p_2^* f^! 1_Y) \arrow[r, "p_{1!} \dashv p_1^!"] \arrow[d, "\text{proj formula}"] & p_{1!}(p_1^* \otimes p_1^! p_{1!} p_2^* f^! 1_Y) \arrow[r, "\text{base change}"] & p_{1!}(p_1^* \otimes p_1^! f^* f_! f^! 1_Y) \arrow[r, "f_! \dashv f^!"] & p_{1!}(p_1^*A \otimes p_1^! 1_X) \arrow[d, "\text{proj formula}"] \\
p_{1!}p_2^* f^! 1_Y \otimes A \arrow[r, "\text{base change}"] & f^* f_! f^! 1_Y \otimes A \arrow[rr, "f_! \dashv f^!"] & & A \otimes 1_X
\end{tikzcd}
$$

ι