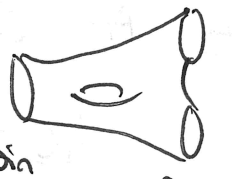
Central extensions of bordism bicategories

(Functorial) def^n of TQFT

[Atiyah] Sym. mon. functor $Z : \mathrm{Bord}_{d-1, d} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$



∂in ∂out

(d-1)-dim closed mfds — f.d. $\mathbb{C}$-vector spaces

d-dim bordisms / diffeo rel ∂ — linear maps

$\sqcup$ — $\otimes$

allow mfds / bordisms to have extra structure

e.g.

- orientation
- spin structure

?

monoidal: $Z(- \sqcup -) \cong Z(-) \otimes Z(-)$

functoriality:

M M'

$$Z(M \cup M') = Z(M') \circ Z(M).$$

allow for <u>projective functors</u>

$$Z(M \cup M') = k\, Z(M') \circ Z(M)$$

$k(M', M) \in \mathbb{C}^\times$. c.f. [Turaev].

<u>anomaly</u>

can recover from stricter def by adding "extra structure", i.e. <u>central exts</u>

consider proj functor

$F : \mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$.

$$F(gf) = k(g, f)\, F(g) \circ F(f).$$

$\xrightarrow{f} \xrightarrow{g} \xrightarrow{h}$

$$F(hgf) = k(hg, f)\, k(h, g)\, F(h) \circ F(g) \circ F(f)$$

$= k(h, gf)\, k(g, f)$

$k(hg,f)\,k(h,g) = k(h,gf)\,k(g,f)$ cocycle condition.

also appears in the context of central exts of gps / categories.

central ext of gps: central ext of G by A: $\tilde{G}$

$$0 \to A \to \tilde{G} \to G \to 0. \qquad A \subseteq Z(\tilde{G}).$$

each preimage of $g \in G$ is an A-torsor

A action compatible with multiplication.

to describe central ext: pick set theoretic $G \to \tilde{G}$
$g \mapsto \tilde{g}$

$g,h \in G$. $\quad \tilde{g}\tilde{h} = a\,\widetilde{gh}$

$a(g,h) \in A$.

a satisfies cocycle condition: $a(hg,f)\,a(h,g) = a(h,gf)\,a(g,f)$

$\rightsquigarrow [a] \in H^2(G;A)$.

diff choice differ by cob

$a(g,h) = \varphi(g) + \varphi(h) - \varphi(gh) \qquad \varphi : G \to A.$

central ext of G by A $\leftrightsquigarrow$ $H^2(G;A)$.

central ext of categories: central ext of $\mathcal{C}$ by A: $\tilde{\mathcal{C}}$.

same objs as $\mathcal{C}$

quotient functor $q: \tilde{\mathcal{C}} \to \mathcal{C}$

preimage of each morphism is an A-torsor.
i.e. mors of $\tilde{\mathcal{C}}$ have A-action.

$\xrightarrow{\tilde{f}} \xrightarrow{\tilde{g}}$ $\quad (a\cdot\tilde{g})\circ\tilde{f} = a\cdot(\tilde{g}\tilde{f}) = \tilde{g}(a\cdot\tilde{f}).$

cocycle condition $\forall \xrightarrow{f} \xrightarrow{g} \xrightarrow{h}$

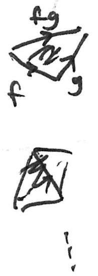
central ext of $\mathcal{C}_e$ by A $\leftrightsquigarrow$ $H^2(\mathcal{C}_e; A)$.

$$\begin{array}{c} H^2(\mathcal{C}_e; A) \\ \text{``}\| \\ H^2(N(\mathcal{C}_e); A) \\ | \\ \underline{\text{nerve}}. \end{array}$$

simplicial cx

• obj

— morphisms.



resolving projectivity:

$F: \mathcal{C}_e \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ projective $\rightsquigarrow$ anomaly k

$[h] \in H^2(\mathcal{C}_e; \mathbb{C}^\times)$.

central ext $\tilde{\mathcal{C}}_e \to \mathcal{C}_e$ by $\mathbb{C}^\times$.

$\tilde{\mathcal{C}}_e \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ genuine functor.

mor f in $\mathcal{C}_e$ $\rightsquigarrow$ lift $\tilde{f}$ in $\tilde{\mathcal{C}}_e$

$\hookrightarrow F(\tilde{f})$.

~~refer~~ if image of $k \in \{\alpha^n \mid n \in \mathbb{Z}\}$ $\alpha \in \mathbb{C}^\times$.

$$\begin{array}{c} \{\alpha^n \mid n \in \mathbb{Z}\} \\ \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} \end{array}$$

think of it as a central ext by $\mathbb{Z}$.

turning to $\mathrm{Bord}_{d-1,d}$: think of central ext. as "adding extra structure":

$\widetilde{\mathrm{Bord}}_{d-1,d}$: mor:  + elt in $\mathbb{Z}$

Example

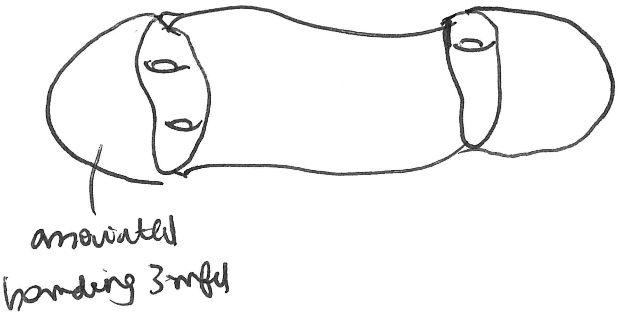
signature central extensions. [Walker]

oriented surfaces & 3D bordisms.

Bord^{or}_{23}

central ext: Bord^{sig}_{23}

construction: for each surface, pick a 3-mfd that bounds it (normally handlebody)



⇝ closed 3-mfd.

extra structure: bounding 4-mfd

equivalent if they have the same signature $\in \mathbb{Z}$.

Reshetikhin-Turaev construction: Input MTC

⇝ projective functor $\mathrm{Bord}^{or}_{23} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

resolved by signature ext $\mathrm{Bord}^{sig}_{23} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ genuine functor.

Bicategories

- obj
- 1-mor
- 2-mor

composition:

associative

interchange law

horizontal-comp

$\xrightarrow{f}\xrightarrow{g}\xrightarrow{h}$ don't have to be associative

$(hg)f \longrightarrow h(gf)$

associator (invertible 2-mor)

similarly, identity 1-mors have $f\circ id \longrightarrow f$, $id\circ f \longrightarrow f$ } unitors

+ other coherence conditions.

alternatively: it's a category but instead of sets of morphisms, we have categories of morphisms.

e.g. monoidal category $\mathcal{C}$ ⇝ bicategory

- one obj
- 1-mor: objs of $\mathcal{C}$
- 2-mor: mors of $\mathcal{C}$

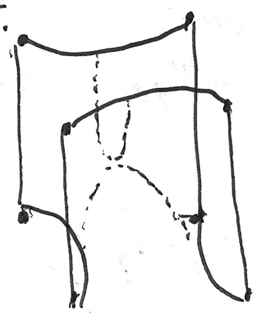
 comp. in $\mathcal{C}$

 $\otimes$

- bordism bicategory $\mathrm{Bord}_{d-2,d-1,d}$
  - obj: closed $(d-2)$-dim mfds
  - 1-mor: $(d-1)$-dim bordisms
  - 2-mor: bordisms between bordisms / diffeo.

  (subtlety with smooth structure & gluing — we'll ignore that)

$d=2$:



comp - gluing in relevant direction.

sym mon structure — $\sqcup$.

• $2\mathrm{Vect}_{\mathbb{C}}$ (bicategorical version of $\mathrm{Vect}_{\mathbb{C}}$)

- obj — linear categories
- 1-mor — linear functors
- 2-mor — linear transformations.

look at ~~specific~~ distinguished object — $\mathrm{Vect}_{\mathbb{C}}$

$$\mathrm{Hom}_{2\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Vect}_{\mathbb{C}}, \mathrm{Vect}_{\mathbb{C}}) \cong \mathrm{Vect}_{\mathbb{C}}$$

"delooping of $\mathrm{Vect}_{\mathbb{C}}$".

sym. mon structure — $\otimes$ (of linear cats).

<u>once-extended TQFT</u>: sym. mon functor $Z : \mathrm{Bord}_{d-2,d-1,d} \to 2\mathrm{Vect}_{\mathbb{C}}$

~~There~~ But there are examples that don't satisfy functoriality on the nose

[Bumura] extended R-T $(d=3)$ vertically compose 2-mors ⇝ anomaly.

<u>central extension of bicategories</u>: bicat $\mathcal{C}$, ab grp A

$\tilde{\mathcal{C}}$ — same objs & 1-mor as $\mathcal{C}$

— 2-mor: free A-action

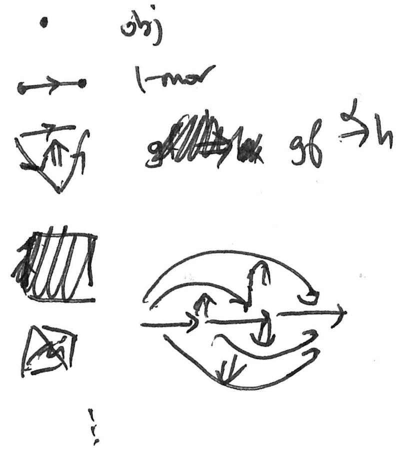
quotient ⇝ $\tilde{\mathcal{C}} \to \mathcal{C}$.

(associators ↦ associators
unitors ↦ unitors ?)

cat: associativity of comp ⇝ cocycle condition.

bicat: more coherence rules ⇝ system of cocycle conditions.

<u>Thm</u>: central ext of $\mathcal{C}$ by A ⟷ $H^3(\underbrace{N(\mathcal{C})}_{\text{Duskin nerve (simplicial cx)}}; A)$



for bordism hints:

[BDSV] extend construction of Bord^{sig}_{23} to $\mathrm{Bord}^{sig}_{123}$

centr. ext. of Bord^{or}_{123}.

[Wall's nonadditivity of signature]

cocycle.

$\in H^3(N(\mathrm{Bord}^{or}_{123}); \mathbb{Z})$

divisible by 2 $\rightsquigarrow$ cocycle of index 2 subextension $\mathrm{Bord}^{sig/2}_{123}$.

[S-P] not further divisible. (string theory)

Alternate proof directly from cong. computation

[BDSV] $\mathrm{Bord}^{sig}_{123} \longrightarrow 2\mathrm{Vect}_{\mathbb{C}} \longleftrightarrow \mathrm{MTC} + \sqrt{D}$

global dimension (invariant of MTC)

(depends on unpublished work regarding a presentation of Bord^{or}_{123})

(

assuming the same work, can show

$\mathrm{Bord}^{sig/2}_{123} \to 2\mathrm{Vect}_{\mathbb{C}} \longleftrightarrow \mathrm{MTC}$

Another thing that André & I are working on:

$\mathrm{Bord}^{sig/2}_{12\varepsilon} \longrightarrow 2\mathrm{Vect}^{ss}_{\mathbb{C}} \longleftrightarrow \mathrm{MTC}.$

restrict to only invertible 2-mor

restrict to only ss lin. cats.