Chain rule in Goodwillie calculus

<u>Setting</u>: We will be working with ∞-categories that are

- pointed ($\exists$ object that's initial & terminal)
- presentable
- admit filtered colims & finite lims, & filtered colims commute with finite lim

Call these <u>differentiable</u> (N.B. In Lurie's HA diff'ble is just the last bullet pt)

For functors $F: \mathcal{C} \to \mathcal{D}$ between diff'ble ∞-cats, recall that we've defined:

diff'ble ∞-cats	$\mathbb{R}$
n-excisive functors	deg $\leq n$ polynomials
n-reduced functors	$f^{(i)}(0) = 0 \ \forall\ i < n$
n-homogeneous functors	$f(x) = \text{const} \cdot x^n$
Taylor tower	(truncations of) Taylor series
$\partial_n F$, $cr_n F$	Taylor coeffs; nth sym diff: e.g. $(a,b) \mapsto f(a+b) - f(a) - f(b) + f(0)$ etc.

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_nF \arrow[d, dotted] \\ & P_1F \arrow[d] \\ F \arrow[r] \arrow[ur] \arrow[uur] & P_0F \end{tikzcd}$$

∂_* will be important today, so let's recall how we defined it:

<u>Thm</u>: n-homogeneous functors $F: \mathcal{C} \to \mathcal{D}$ $\xrightarrow[\sim]{cr_n}$ sym multilinear $L: \mathcal{C}^n \to \mathcal{D}$.

where cr_n is the <u>nth cross effect</u>:

$$cr_n F(X_1, \ldots, X_n) = \operatorname{hofib}\left(F\left(\coprod_i X_i\right) \longrightarrow \operatorname*{holim}_{\emptyset \neq S \subsetneq [n]} \left(\coprod_{i \in S} X_i\right)\right)$$

Apply this to D_n (which is n-homog), and we have $cr_n D_n : \mathcal{C}^n \to \mathcal{D}$ sym multilinear.

If $\mathcal{C}, \mathcal{D}$ are stable (e.g. $\mathcal{C} = \mathcal{D} = Sp$), set $\partial_n F := cr_n D_n F$

(In general, technical result [Lurie HA Cor 6.2.3.22] says

multilinear functors $Sp(\mathcal{C})^n \to Sp(\mathcal{D})$ $\xrightarrow{\sim}$ multilinear functors $\mathcal{C}^n \to \mathcal{D}$

so define $\partial_n F : Sp(\mathcal{C})^n \to Sp(\mathcal{D})$ s.t. $cr_n D_n(F) = \Omega^\infty \partial_n F \, \Sigma^\infty$)

In any case: $\partial_n F$ is a sym multilinear functor between stable ∞-cats or $F: Sp \to Sp$, finitary (preserves filtered hocolims) we in fact have

$$D_n F(X) = (\underbrace{cr_n(D_n F)(S, \ldots, S)} \wedge X^{\wedge n})_{h\Sigma_n}$$

so we abuse notation & write this as $\partial_n F$

These correspond to the nth derivatives at 0. [End of recap]

For general derivatives, define

$$F_X : Y \longmapsto \text{hofib}(F(Y \vee X) \to F(X))$$

F_X plays the role of nth derivative at X. — when $X = *$ this corresponds to setting $F_*(*) \simeq *$ while keeping all other derivatives the same.

The main goal of today is to prove the chain rule for $F: Sp \to Sp$

Thm: $F, G : Sp \to Sp$ s.t. F is finitary

Then $\partial_*((FG)_X) \simeq \partial_* F_{GX} \circ \partial_* G_X$

(If time permits we'll sketch how to generalise this statement to arbitrary diff'ble ∞-cats)

First, we have to understand "$\circ$". This is an operation on symmetric sequences

Def: A symmetric sequence (of spectra) is a sequence $A = (A_1, A_2, \ldots)$ of spectra with action of Σ_n on A_n $\forall n$.

$\partial_n F = cr_n(D_n F)(S, \ldots, S)$ has Σ_n-action that comes from permuting the Ss

Def: The composition product of symmetric sequences A, B is given by

$$(A \circ B)_n = \bigvee_{\substack{\text{unordered} \\ \text{partitions of} \\ \{1, \ldots, n\}}} \underbrace{A_k}_{\substack{\text{no. of} \\ \text{sets in} \\ \text{partition}}} \wedge \underbrace{B_{n_1} \wedge \cdots \wedge B_{n_k}}_{\substack{\text{sizes of} \\ \text{sets in} \\ \text{the partition.}}}$$

Σ_n acts on $\{1, \ldots, n\}$, so acts on partitions, which induces $A_k \to A_k$ & $B_{n_i} \to B_{n_{\sigma(i)}}$ for some permutation $\sigma \in \Sigma_k$.

Together this defines an action on $(A \circ B)_n$

Note: If we let $P(n) = \{\text{partition types}\} = \left\{ ((k_1, \ldots, k_r), (l_1, \ldots, l_r)) \;\middle|\; \begin{array}{l} k_i \geq 1, \; 1 \leq l_1 < \cdots < l_r, \\ \sum k_i l_i = n \end{array} \right\}$

then this can be written as

$$(A \circ B)_n = \bigvee_{\lambda \in P(n)} (\Sigma_n)_+ \bigwedge_{H(\lambda)} (A_k \wedge B_{l_1}^{\wedge k_1} \wedge \cdots \wedge B_{l_r}^{\wedge k_r})$$

where $H(\lambda)$ = stabiliser of Σ_n acting on partition of type $P(n)$.

Rk: This agrees with the usual calculus on $\mathbb{R}\to\mathbb{R}$:

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{\lambda\in P(n)} \underbrace{\frac{n!\, k_1!\cdots k_r!}{k!\,(l_1!)^{k_1}\cdots(l_r!)^{k_r}}}_{= |\Sigma_n : H(\lambda)|} f^{(k)}(g(x))\, g^{(l_1)}(x)^{k_1}\cdots g^{(l_r)}(x)^{k_r}.$$

Outline
0. reduce.
1. simplify RHS
2. redefine Taylor tower for LHS
3. profit.

We proceed to prove the chain rule.

Step 0: reduction

We want to get rid of the dependence on X.

Claim: $(FG)_X \simeq F_{GX} G_X$

Proof: $F_{GX}(G_X(Z)) = F_{GX}(\mathrm{hofib}(G(Z\vee X)\to GX))$

$$= \mathrm{hofib}\left(\begin{array}{c} F(\mathrm{hofib}(G(Z\vee X)\to GX)\vee GX) \\ \downarrow \\ FGX \end{array} \right)$$

section $GX \to G(Z\vee X)$ of $G(Z\vee X)\to GX$ induces

$$\mathrm{hofib}(G(X\vee Z)\to GX)\vee GX \xrightarrow{\sim} G(X\vee Z)$$

so this is $\simeq \mathrm{hofib}(FG(X\vee Z)\to FGX) = (FG)_X$. $\square$

Now note that we may rewrite our chain rule as

$$\partial_*(F_{GX}\, G_X) \simeq \partial_* F_{GX} \circ \partial_* G_X$$

and F_{GX}, G_X are reduced, so it suffices to show for reduced F, G,

$$\partial_*(FG) \simeq \partial_* F \circ \partial_* G.$$

Step 1: simplify expression for composition.
(RHS)

Def: rth co-cross-effect of $F: Sp\to Sp$ is $cr^r F: Sp^r \to Sp$

$$cr^r F : (X_1,\ldots,X_r) \mapsto \mathrm{hocofib}\Big(\underset{J\subsetneq [r]}{\mathrm{hocolim}}\, F(\textstyle\prod_{j\in J} X_j) \to F(X_1\times\cdots\times X_r)\Big)$$

Lemma: $cr^r F(X_1,\ldots,X_r) \simeq cr_r F(X_1,\ldots,X_r)$.

Proof: Induct on r.

Base case: Split fibre sequences $F* \rightleftarrows FX \to cr^1F(X)$, $cr_1F(X)\to FX \rightleftarrows F*$

$\leadsto F(X)\simeq F(*)\vee cr^1 F \simeq F(*)\times cr_1 F$. $\Rightarrow$ $cr_1 F(X)\to FX \to cr^1 F(X)$ is an equiv.

reduction step:

$$cr_r(F)(X_1,\dots,X_r) = \mathrm{hofib}\Big(F(\textstyle\coprod_i X_i) \to \underset{\emptyset\neq S\subseteq[r]}{\mathrm{holim}}(\coprod_{i\in S} X_i)\Big)$$

hofib between vertex of cube & rest of cube

Goodwillie see L2

$$\simeq \mathrm{hofib}\begin{pmatrix} cr_{r-1}(F(-\vee X_r))(X_1,\dots,X_{r-1}) \\ \downarrow \\ cr_{r-1}(F)(X_1,\dots,X_{r-1})\end{pmatrix}$$

hofib between hofibs of layers

& similarly $cr^r(F)(X_1,\dots,X_r) = \mathrm{hocofib}\begin{pmatrix} cr^{r-1}(F(-\times X_r))(X_1,\dots,X_{r-1}) \\ \downarrow \\ cr^{r-1}F(X_1,\dots,X_{r-1})\end{pmatrix}$

Use the fact that $-\vee- \simeq -\times-$ in spectra, then we have

$\mathrm{hofib}\begin{pmatrix}A\\ \downarrow\\ B\end{pmatrix}$ & $\mathrm{hocofib}\begin{pmatrix}B\\ \downarrow\\ A\end{pmatrix}$ so argument as before works $\square$

$\lambda \vdash P(n)$, define

$$(F,G)_\lambda = \Big(\underbrace{[P_{k_1},\dots,P_{k_r}]}\, cr^r(F)\Big)(P_{l_1}G,\dots,P_{l_r}G)$$

polynomial truncation in each variable of

$cr^r(F): Sp \to Sp$.

Prop: $\partial_n((F,G)_\lambda) \simeq (\Sigma_n)_+ \underset{H(\lambda)}{\wedge} \big(\partial_k F \wedge \partial_{l_1} G^{\wedge k_1} \wedge \dots \wedge \partial_{l_r} G^{\wedge k_r}\big)$ $(n = \sum k_i l_i)$

Lemma: $H: Sp^r \to Sp$ $(k_1,\dots,k_r)$-excisive, $G_1,\dots,G_r: Sp \to Sp$, G_i l_i-excisive $\forall r$.
Then $H\circ(G_1,\dots,G_r): Sp \to Sp$ is $\sum k_i l_i$-excisive.
Technical proof but result is intuitive when compared to degs of polys!

Proof of Prop: By Lemma, $(F,G)_\lambda$ is n-excisive.

Moreover $[D_{k_1},P_{k_2},\dots,P_{k_r}]cr^r F(P_{l_1}G,\dots,P_{l_r}G) \longrightarrow [P_{k_1},\dots,P_{k_r}]cr^r F(P_{l_1}G,\dots,P_{l_r}G)$

has hofib $[P_{k_1-1},P_{k_2},\dots,P_{k_r}]\, cr^r(F)(P_{l_1}G,\dots,P_{l_r}G)$

which by Lemma is $n-l_1 < n$ -excisive so has same derivatives.

Repeat inductively; can replace $(F,G)_\lambda$ with

$[D_{k_1},\dots,D_{k_r}]\,cr^r(F)(P_{l_1}G,\dots,P_{l_r}G)$

want to replace the P_{l_i} inside with D_{l_i} as well

Lemma: $[D_{k_1},\dots,D_{k_r}]\,cr^r(F)(X_1,\dots,X_r) \simeq \underbrace{[D_1,\dots,D_1]}_{D_1^{(k)}}\, cr^k(F)(\underbrace{X_1,\dots,X_1}_{k_1},\dots,\underbrace{X_r,\dots,X_r}_{k_r})_{h(\Sigma_{k_1}\times\dots\times\Sigma_{k_r})}$

Note: The case $r=1$ $D_n F(X) \simeq D_1^{(n)} cr^n F(X,\dots,X)_{h\Sigma_n}$ in L2

We can deduce the Lemma from this:

$$\mathrm{LHS} \overset{r=1\text{ case}}{\simeq} \underbrace{D_1^{(k)}([cr^{k_1}, \dots, cr^{k_r}]}_{\text{actually } [D_1^{(k_1)} cr^{k_1}, \dots, D_1^{(k_r)} cr^{k_r}]} cr^r(F))(\overbrace{X_1, \dots, X_1}^{k_1}, \dots, \overbrace{X_r, \dots, X_r}^{k_r})_{h(\Sigma_{k_1} \times \dots \times \Sigma_{k_r})}$$

Unwind def^n of cr^r: it's an iterated hocofib, which is the same as taking a large total hocofib.

So applying this, we get

$$D_1^{(k)} cr^k(F)(\underbrace{P_{l_1}G, \dots, P_{l_1}G}_{k_1}, \dots, \underbrace{P_{l_r}G, \dots, P_{l_r}G}_{k_r})_{h(\Sigma_{k_1} \times \dots \times \Sigma_{k_r})}$$

now we can replace the P_{l_i} with D_{l_i} as $D_1^{(k)} cr^k F$ is multilinear.

Reverse this & we get

$$D_n((F,G)_\lambda) \simeq D_n\left(D_1^{(k)} cr^k(F)(D_{l_1}G, \dots, D_{l_r}G)_{h(\Sigma_{k_1} \times \dots \times \Sigma_{k_r})}\right)$$

$$\overset{\text{finiteness}}{\simeq} D_n\left(\partial_k F \wedge \left((\partial_{l_1}G \wedge X^{\wedge l_1})_{h\Sigma_{l_1}}\right)^{\wedge k_1} \wedge \dots \wedge \left((\partial_{l_r}G \wedge X^{\wedge l_r})_{h\Sigma_{l_r}}\right)^{\wedge k_r}\right)_{h(\Sigma_{k_1} \times \dots \times \Sigma_{k_r})}$$

$$\overset{\wedge \text{ preserves } (-)_{h\Sigma}}{\simeq} D_n\left(\underbrace{(\partial_k F \wedge \partial_{l_1}G^{\wedge k_1} \wedge \dots \wedge \partial_{l_r}G^{\wedge k_r} \wedge X^{\wedge n})_{h(\Sigma_{l_1}^{k_1} \times \dots \times \Sigma_{l_r}^{k_r})})_{h(\Sigma_{k_1} \times \dots \times \Sigma_{k_r})}}_{H(\lambda)}\right)$$

$$\simeq D_n\left(\partial_k F \wedge \partial_{l_1}G^{\wedge k_1} \wedge \dots \wedge \partial_{l_r}G^{\wedge k_r} \wedge X^{\wedge n}\right)_{H(\lambda)}$$

$$\simeq D_n\left(\underbrace{\left((\Sigma_n)_+ \wedge_{H(\lambda)} (\partial_k F \wedge \partial_{l_1}G^{\wedge k_1} \dots \wedge \partial_{l_r}G^{\wedge k_r}) \wedge X^{\wedge n}\right)_{\Sigma_n}}_{n\text{-homog.}}\right)$$

so we may read off $\partial_n((F,G)_\lambda) \simeq (\Sigma_n)_+ \wedge_{H(\lambda)} (\partial_k F \wedge \partial_{l_1}G^{\wedge k_1} \wedge \dots \wedge \partial_{l_r}G^{\wedge k_r})$ $\square$

Step 2: ~~Redefine~~ Taylor tower

Idea: construct $p_n(FG)$ which also form a Taylor tower

Assume $F(X) \simeq L(X, \dots, X)$.

Let $\Pi_k(n)$ be the full subcat of $(\overbrace{\mathbb{N} \times \dots \times \mathbb{N}}^{k}_{\text{posets under } \geq})$ with objects $\{(r_1, \dots, r_k) \mid \sum r_i \leq n\}$

e.g. $k=2$: $\bullet$, $\bullet \to \bullet$, ...

Define $p_n(FG)(X) = \underset{(r_1,\dots,r_k) \in \Pi_k(n)}{\mathrm{holim}} L(P_{r_1}G(X), \dots, P_{r_k}G(X))$

restriction gives $p_n(FG) \to p_{n-1}(FG)$. Let $d_n(FG) = \mathrm{hofib}(p_n(FG) \to p_{n-1}(FG$

Prop: $\exists$ natural equiv $\alpha: \prod L(D_{r_1}G, \dots, D_{r_k}G) \xrightarrow{\sim} d_n(FG)$

Proof: $L(D_{r_1}G, \to D_{r_k}G) \to L(P_{r_1}G, \to P_{r_k}G)$ $\quad \forall r_1+\dots+r_k = n$

$$\Big\downarrow \text{holim}$$

$$\underbrace{\prod_{r_1+\dots+r_k=n} L(D_{r_1}G, \to D_{r_k}G)}_{\text{on } r_1+\dots+r_k \le n-1 \text{, this is } *,} \to P_n(FG)$$

so upon composing with $P_n(FG) \to P_{n-1}(FG)$ it's trivial.

$\Rightarrow$ factors through $D_n(FG)$ $\rightsquigarrow \alpha$.

To construct inverse, consider

$$\begin{tikzcd}
D_n(FG) \arrow[r] \arrow[dr] & P_n(FG) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & P_{n-1}(FG) \arrow[d] \\
 & L(P_{r_1}G, \to P_{r_k}G) \arrow[r] & L(P_{r_1}G, \to P_{r_k-1}G)
\end{tikzcd}$$

top row $\longrightarrow$ is $*$

$\Rightarrow$ $\searrow$ is $*$.

$\Rightarrow$ $\searrow$ factors via hofib of $\to$ $= L(P_{r_1}G, \to P_{r_k-1}G, D_{r_k}G)$.

Repeat, get $D_n(FG) \to L(D_{r_1}G, \to D_{r_k}G)$

which assemble to $D_n(FG) \to \prod_{r_1+\dots+r_k} L(D_{r_1}G, \to D_{r_k}G)$ □

Cor: $D_n(FG)$ is n-homog, $P_n(FG)$ is n-excisive. □

Proof: Each $L(D_{r_1}G, \to D_{r_k}G)$ is n-homog.

Prop: The $P_n(FG)$ form the Taylor tower of FG. (F kintary, $F(X) = L(X, \to X)$).

Proof: Want $P_n(FG) \xrightarrow{\sim} P_n(P_n(FG))$ (as $\underbrace{P_n(FG)}_{n\text{-excisive}} \xrightarrow{\sim} P_n(P_n FG))$)

Suppose $\underline{G \text{ is } m\text{-excisive}}$. Then for $n \ge mk$,

$$L(P_{r_1}G, \to P_{r_k}G) \xrightarrow{\sim} L(P_{r_1}G, \to P_{r_i-1}G, \to P_{r_k}G) \quad \text{if } r_i > m.$$

Then $\underset{\text{holim } T_k(n)}{P_n(FG)} \simeq \underset{\substack{\text{holim} \\ (r_1,\dots,r_k) \in T_k(n) \\ \sum r_i \le n,\ r_i \le m\ \forall i}}{L(P_mG, \to P_mG)} \simeq L(G, \to G) = FG$

$\Rightarrow P_n(FG) \xrightarrow{\sim} P_n(P_n(FG)) \quad \forall n \ge mk$ (apply P_n to both sides)

For $n < mk$, $P_n(FG) \xrightarrow{\sim} P_n(P_{mk}(FG)) \xrightarrow{\sim} P_n(P_n FG))$.

For $\underline{\text{general } G}$:

$$\begin{tikzcd}
P_n(FG) \arrow[r, "\sim"] \arrow[d] & P_n(FP_nG) \arrow[d, "\sim"] \\
P_n(P_n(FG)) \arrow[r, "\sim"] & P_n(P_n(FP_nG))
\end{tikzcd}$$

removing higher order terms from G won't affect lower order terms of FG.

just shown

diagram for $P_n(FG)$ & $P_n(F(P_nG))$ $\simeq$. □

Thm: $F: Sp \to Sp$ k-homog finitary, $G: Sp \to Sp$ reduced

$$P_n(FG) \simeq \left(\operatorname*{holim}_{r_1+\dots+r_k \le n} \partial_k F \wedge P_{r_1}G \wedge \dots \wedge P_{r_k} G \right)_{h\Sigma_k}$$

$$D_n(FG) \simeq \left(\prod_{r_1+\dots+r_k = n} \partial_k F \wedge D_{r_1}G \wedge \dots \wedge D_{r_k} G \right)_{h\Sigma_k}$$

Proof: Let $\tilde{F}(X) = cr_k F(X, \dots, X)$, so $F(X) = \tilde{F}(X)_{h\Sigma_k}$.

Note $cr_k F(X_1, \dots, X_k) \simeq \partial_k F \wedge X_1 \wedge \dots \wedge X_k$

Prop $\Rightarrow$ $P_n(\tilde{F}G) \simeq P_n(\tilde{F}G) \simeq \operatorname*{holim}_{r_1+\dots+r_k \le n} \partial_k F \wedge P_{r_1} G \wedge \dots \wedge P_{r_k} G$.

Similarly, prev prop $\Rightarrow$ $D_n(\tilde{F}G) \simeq d_n(\tilde{F}G) \simeq \prod_{r_1+\dots+r_k} \partial_k F \wedge D_{r_1} G \wedge \dots \wedge D_{r_k} G$.

Finally, note that htpy quotients commute with everything we've done so far $\square$

Step 3: profit

We want to prove $\partial_*(FG) \simeq \partial_* F \circ \partial_* G$, i.e.

$$\partial_n(FG) = \bigvee_{\lambda \in P(n)} \partial_n((F,G)\lambda).$$

Finite products & coproducts in Sp are equivalent, so we just need to show

Prop: $\Delta: FG \longrightarrow \prod_{\lambda \in P(n)} (F,G)\lambda$ induces

$$D_n\Delta: D_n FG \xrightarrow{\sim} \prod_{\lambda \in P(n)} D_n((F,G)\lambda)$$

Proof: First, if $F(X) = L(\overbrace{X, \dots, X}^{k})$, L multilinear:

L is linear in each var so preserves finite products & coproducts, so

$$L(X_1 \times \dots \times X_r, \dots, X_1 \times \dots \times X_r) = \prod_{s \in [r]^k} L(X_{s(1)}, \dots, X_{s(k)})$$

Then $cr^r(F)(X_1, \dots, X_r) = \operatorname{hofib}\Big(\underbrace{\operatorname*{holim}_{J \subsetneq [r]} \prod_{t \in J^k} L(X_{t(1)}, \dots, X_{t(k)})}_{\displaystyle = \prod_{\substack{s \in [r]^k \\ \text{not surjective}}} L(X_{s(1)}, \dots, X_{s(k)})} \longrightarrow \prod_{s \in [r]^k} L(X_{s(1)}, \dots, X_{s(k)}) \Big)$

$$= \prod_{\text{surj.}} L(X_{s(1)}, \dots, X_{s(k)})$$

So $(F,G)_\lambda = [P_{k_1},\dots,P_{k_r}]\, cr^k(F)(P_{l_1}G,\dots,P_{l_r}G) = \prod_{\substack{s\in[r]^k \\ 1\le|s^{-1}(j)|\le k_j\ \forall j}} L(P_{l_{s(1)}}G,\dots,P_{l_{s(k)}}G)$

$\underbrace{[P_{k_1},\dots,P_{k_r}]}_{\text{kills off higher order terms}}$

Apply D_n to this — want n-homog part

Degree of the term in product is $\sum_{i=1}^{k} l_{s(i)} \le \sum_{j=1}^{r} k_j l_j = n$, equality iff

$|s^{-1}(j)| = k_j \quad \forall j=1,\dots,r$

and in this case, $D_n L(P_{l_{s(1)}}G,\dots,P_{l_{s(k)}}G) \simeq L(D_{l_{s(1)}}G,\dots,D_{l_{s(k)}}G)$

(only look at highest degree terms)

$$\Rightarrow D_n((F,G)_\lambda) \simeq \prod_{\substack{s\in[r]^k \\ |s^{-1}(j)|=k_j}} L(\underbrace{D_{l_{s(1)}}G,\dots,D_{l_{s(k)}}G}_{})$$

↳ so out of these there are k_j of each l_j

so if we $\prod_{\lambda\in P(n)}$ its all the ways to add to n.

i.e. $\prod_\lambda D_n((F,G)_\lambda) \simeq \prod_{r_1+\dots+r_n=n} L(D_{r_1}G,\dots,D_{r_n}G) \simeq D_n FG$

which we showed earlier.

<u>general homog F</u>: take htpy orbits

<u>general F</u>: We have

$$\begin{tikzcd} D_n(FG) \arrow[r] \arrow[d, "\sim"] & \prod_{\lambda\in P(n)} D_n((F,G)_\lambda) \arrow[d, "\sim"] \\ D_n((P_nF)G) \arrow[r] & \prod_{\lambda\in P(n)} D_n((P_nF,G)_\lambda) \end{tikzcd}$$

needs proving but is intuitively true when comparing to polys.

$(\Sigma_{n+} \wedge_{\Sigma_\lambda} \wedge(\partial_n F \wedge \dots))$ = Formula for $\partial_n((F,G)_\lambda)$ & $\partial_n((P_nF,G)_\lambda)$ are the same as $\partial_k F \simeq \partial_k P_n F$ for $k\le n$.

suffices to show bottom is an $\simeq$.

Induct on Taylor tower:

$$\begin{tikzcd} D_n((D_kF)G) \arrow[r, "\sim"] \arrow[d] & \prod_{\lambda\in P(n)} D_n((D_kF,G)_\lambda) \arrow[d] \\ D_n((P_kF)G) \arrow[r] \arrow[d] & \prod_{\lambda\in P(n)} D_n((P_kF,G)_\lambda) \arrow[d] \\ D_n((P_{k-1}F)G) \arrow[r, "\sim"] & \prod_{\lambda\in P(n)} D_n((P_{k-1}F,G)_\lambda) \end{tikzcd}$$

D_kF is k-homog.

induction hyp

$\square$

Generalisation (Arone–Blom 2024)

- Firstly, our ∂_n are now symmetric multilinear functors

$$\partial_n : Sp(\mathcal{C})^n \longrightarrow Sp(\mathcal{D}) \qquad \partial_n \in \mathrm{SymFun}^n_{lin}(Sp(\mathcal{C}), Sp(\mathcal{D}))$$

So $\partial_* F = \{\partial_n F\}_{n \geq 1}$ is a <u>symmetric sequence of functors</u>, $\partial_* F \in \mathrm{SSeq}(Sp(\mathcal{C}), Sp(\mathcal{D}))$

- <u>composition</u>: Let $F_* \in \mathrm{SSeq}(\mathcal{D}, \mathcal{E})$, $G_* \in \mathrm{SSeq}(\mathcal{C}, \mathcal{D})$, $\mathcal{C}, \mathcal{D}, \mathcal{E}$ stable ∞-cats

$$(F \circ G)_n := \coprod_{\substack{\text{unordered} \\ \text{partitions} \\ \text{of } [n]}} F_k(G_{n_1}, \ldots, G_{n_k})$$

But it's not so simple for the general case.

- $\partial_* \mathrm{id}_{\mathcal{C}} \in \mathrm{SSeq}(Sp(\mathcal{C}), Sp(\mathcal{C}))$ form a natural alg structure.

  In fact $\partial_* \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} \mathrm{Cobar}\, \partial_*(\Sigma^\infty_{\mathcal{C}} \Omega^\infty_{\mathcal{C}})$

- each $\partial_* F$, $F: \mathcal{C} \to \mathcal{D}$ is naturally a $(\partial_* \mathrm{id}_{\mathcal{D}}, \partial_* \mathrm{id}_{\mathcal{C}})$-bimod.

- <u>chain rule</u>: $\partial_*(FG) \simeq \partial_* F \circ_{\partial \mathrm{id}_{\mathcal{D}}} \partial_* G$

<u>Rk</u>: In the case for Sp, $\underset{\text{derivative at } X \in Sp}{\partial_n(\mathrm{id}_X)} \simeq \begin{cases} S & n=1 \\ * & \text{else} \end{cases}$

which is exactly the monoidal unit.

small nonsense (in stable ∞-cats)

$A \xrightarrow{f} B$ has splitting $B \xrightarrow{s} A$ ($f \circ s = \mathrm{id}_B$) $\Rightarrow$ $B \vee \underbrace{\mathrm{hofib}(A \xrightarrow{f} B)}_{F} \xrightarrow{\simeq} A$

Proof:

$$\begin{tikzcd}
* & F \arrow[l] \arrow[r] & A \\
* \arrow[u] & * \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d, "\mathrm{id}_B"] \\
* \arrow[u] & B \arrow[l] \arrow[r, "\mathrm{id}_B"] & B
\end{tikzcd}$$

horizontal holims $\rightarrow$

$$\begin{tikzcd}
\mathrm{hocofib}(F \to A) \simeq B \arrow[rr, "\mathrm{holim}"] & & * \arrow[d, "\cong"] \\
B \arrow[u, "\mathrm{id}_B"] \arrow[d] & & \\
* & &
\end{tikzcd}$$

vertical holims $\rightarrow$ $* \leftarrow B \vee F \rightarrow A \xrightarrow{\mathrm{holim}} \mathrm{hocofib}(B \vee F \to A)$ □

$A \xrightarrow{f} B$ has splitting $B \xrightarrow{s} A$ ($f \circ s = \mathrm{id}_B$) $\Rightarrow$ $\mathrm{hocofib}(B \xrightarrow{s} A) \simeq \mathrm{hofib}(A \xrightarrow{f} B)$

Proof:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathrm{hofib}(f) \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\
* \arrow[r] & \mathrm{hocofib}(s) \arrow[r] & *
\end{tikzcd}$$

$\rightsquigarrow \mathrm{hofib}(f) \xrightarrow{\simeq} \mathrm{hocofib}(s)$ □